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# A Self-Dual Vertex Operator Algebra of Central Charge $c = 64$

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**Abstract**

Starting with the standard self-dual $(-1)$-orbifold of the Barnes–Wall lattice VOA of central charge 32, we construct a simple-current extension of a fixed-point subalgebra of its tensor square. The resulting vertex operator algebra is self-dual of central charge 64 and has no nonzero Virasoro highest-weight vector of weight 1 or 2. We give the twisted conformal weights, an elementary proof of the group-cohomological detection statement needed to trivialize the twisted-module associator class, and a permutation-cover argument determining the relevant absolute module weights. This identifies the complete weight-two obstruction set in a 56-dimensional binary quadratic space. An explicit 28-dimensional totally singular graph avoids that set; its exhaustive verifier is included. The final algebra object and its multiplication are also explicit. To my knowledge, this is the first new central-charge value of an extremal self-dual vertex operator algebra since the examples discussed in my 1995 thesis.

## 1 Introduction

### 1.1 Definitions and the main result

All VOAs and their modules are over $\mathbb{C}$. By *strongly rational* we mean simple, rational, $C_2$-cofinite, self-contragredient, and graded as $V = \bigoplus_{n\geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$. For this grading, rationality together with $C_2$-cofiniteness is equivalent to regularity [ABD04, Theorem 4.5]. Write $\mathrm{Rep}(V)$ for its modular tensor category of ordinary modules [Hua07, Theorem 4.6], $\mathrm{Irr}(V)$ for the simple isomorphism classes, and $\boxtimes_V$ for the tensor product. The notation $M^*$ denotes the contragredient module. We write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. A Virasoro highest-weight vector is a nonzero homogeneous vector annihilated by every $L_n$, $n > 0$. As in my thesis [Höh95, Definition 2.1.1], a strongly rational VOA $V$ is called *self-dual* (also called *holomorphic*) if, up to isomorphism, its adjoint module $V$ is its only irreducible ordinary module. Equivalently, $\mathrm{Rep}(V) \simeq \mathrm{Vec}$. This property is stronger than self-contragredience $V \cong V^*$, which is already assumed in the definition of strong rationality.

The notion of *extremality* is that of [Höh95, Definition 5.2.1]: the graded character agrees with the vacuum Virasoro character through weight $\lfloor c/24 \rfloor$, or equivalently there are no nonvacuum Virasoro highest-weight vectors of positive weights at most $\lfloor c/24 \rfloor$. The lowest conformal weight $h(M)$ of an ordinary module $M$ is its lowest $L_0$-eigenvalue; it does not include the character shift by $c/24$. A self-dual VOA of central charge 64 is called extremal here if

$$V_1 = 0, \qquad V_2 = \mathbb{C}\omega.$$

These conditions exclude Virasoro highest-weight vectors of weights 1 and 2; the first possible positive weight is then $3 = \lfloor 64/24 \rfloor + 1$.

Let a finite group $B$ act faithfully on a strongly rational self-dual VOA $T$, and suppose that $T^B$ is strongly rational. The category of twisted $T$-modules is $B$-graded, with one simple object $T(b)$ in each homogeneous component [McR21, Theorems 4.15 and 4.17]. Its neutral component is Vec, and its simple objects are invertible. Choosing isomorphisms $T(b) \boxtimes T(c) \cong T(bc)$ expresses the associativity isomorphisms as a normalized 3-cocycle. Its choice-independent class is denoted by

$$\mathfrak{a}_B(T) \in H^3(B, \mathbb{C}^\times),$$

where the coefficient action is trivial. We call this the *twisted-module associator class*. The group grading is part of this definition. Passage to the homogeneous components indexed by a subgroup restricts the class; an equivalence of braided centers without this grading would not determine it. Under the positivity hypotheses verified in Section 3, the equivalent description using the distinguished function algebra is given in [DNR25, Theorem 6.3 and Remark 7.2] and [DMNO11, Theorem 4.10 and Example 4.11(ii)]. For a pointed VOA category, the conformal-weight quadratic form is

$$q_\Delta([M]) = h(M) \pmod{\mathbb{Z}}.$$

When its values are $0, \frac{1}{2}$, we instead use the binary form

$$q([M]) = 2h(M) \pmod 2, \qquad b(d,e) = q(d+e) + q(d) + q(e). \tag{1.1}$$

Thus $q = 0$ means integral conformal weight. This binary form must not be confused with the cyclic type of an involution:

$$t_g = 4h(V(g)) \pmod 2. \tag{1.2}$$

The two normalizations have different purposes and different factors.

**Theorem 1.1.** *Let $W = \widetilde{V}_{\mathrm{BW}_{32}}$ be the standard self-dual $(-1)$-orbifold, let $A \cong 2^{27}$ be the normal elementary abelian subgroup of $\mathrm{Aut}(W)$, and let $\sigma$ exchange the two factors of $W \otimes W$. Put*

$$U = (W \otimes W)^{\Delta A \times \langle \sigma \rangle}.$$

*The graph subgroup specified in Section 6, interpreted in the actual fusion group of $U$ by Proposition 5.1, defines a strongly rational self-dual extension $V_H$ with*

$$c(V_H) = 64, \qquad (V_H)_1 = 0, \qquad (V_H)_2 = \mathbb{C}\omega.$$

*In particular $V_H$ is extremal.*

## 1.2 Extremal VOAs and related literature

In my thesis [Höh95, Section 5.2], I discussed extremal self-dual VOAs of central charges $8, 16, 24, 32, 40$; at that time self-duality of the indicated rank-32 and rank-40 orbifolds was still conjectural. I explicitly recorded there that no extremal VOAs of central charge $c \geq 48$ were then known. Subsequent constructions have produced non-extremal self-dual VOAs in higher central charges, including 48 and 72 [GK21]. To my knowledge, Theorem 1.1 supplies the first new central-charge value for an *extremal self-dual* VOA since those described in my 1995 thesis. This statement concerns extremality, not the existence of self-dual VOAs at central charge 64 without the extremal condition.

Witten's proposal [Wit07] prompted extensive subsequent study of the specialization $c = 24k$. Three questions must be distinguished: the existence of a modular function with the prescribed initial coefficients, compatibility with further identities involving vertex operators, and existence of a VOA realizing these data. The first two do not by themselves imply the third. Moreover, $64 \notin 24\mathbb{Z}$, so the present central charge lies outside the family originally emphasized by Witten.

**Characters and modular reconstruction.** Manschot [Man07] expressed the proposed genus-one functions as modular sums. Manschot and Moore [MM10] developed convergent reconstruction formulae for vector-valued modular forms of nonpositive weight and constraints on their principal parts. Avramis, Kehagias and Mattheopoulou [AKM07] considered $c = 8m$, computed candidate extremal characters through $c = 88$, and studied genus-two functions through $c = 40$. Their work thus already includes the candidate character at $c = 64$. Ashrafi [Ash20] studied related modular functions and generalized Hecke operators. The calculations of Maloney and Witten [MW10] form part of the broader motivation for this programme, but are not VOA existence theorems.

**Higher-genus identities, including $c = 64$.** Gaiotto and Yin [GY07] determined candidate genus-two functions at $c = 48, 72$ from modular transformation and degeneration conditions. Yin [Yin08] investigated further proposed higher-genus expansions. Gaberdiel, Keller and Volpato [GKV10] systematically studied genus-two vacuum correlation functions and their relation to associativity and genus-one transformation properties. Their construction of a consistent genus-two function for every $c = 24k$ is not a construction of the underlying VOA. Keller, Mathys and Zadeh [KMZ18] obtained additional constraints on multiplicities and structure constants under a positive-definiteness assumption. Their Section 5.6 and Appendix B.2 explicitly treat $c = 64$. Writing $N_i$ for the dimension of the space of Virasoro highest-weight vectors of weight $i$, their formula (5.31) is

$$N_3 = 3(7038N_1 + 165N_2 + 92837).$$

The extremal specialization $N_1 = N_2 = 0$ gives $N_3 = 278511$, agreeing with the character check in Section 8. These calculations concern necessary identities and do not provide an existence proof at $c = 64$.

**Obstructions and restrictions on automorphisms.** Gaberdiel [Gab07] studied constraints obtained from modular differential equations and the $C_2$-quotient. The proposed obstruction for $c = 24k$, $k \geq 42$, is not an unconditional nonexistence theorem: Gaiotto [Gai08] gave a counterexample to an implication used in the original argument, and the subsequent analysis of Gaberdiel and Keller [GK08] retained a holomorphy condition on coefficient functions that was not established in the required generality. Restrictions on possible Monster actions at $c = 48$ were considered by Gaiotto [Gai08] and in my work [Höh08b]; the orbifold hypotheses attached to these statements must be retained. Lin and Pei [LP23] obtained divisibility obstructions using the conjectural Segal–Stolz–Teichner correspondence. Their resulting exclusions, including $c = 48, 72, 96, 120$, are consequently not unconditional nonexistence theorems for VOAs under the hypotheses imposed here.

**Superconformal variants.** Extremality relative to a superconformal algebra is a different condition from Virasoro extremality for an ordinary VOA. Gaberdiel, Gukov, Keller, Moore and Ooguri [GGK+08] studied restrictions imposed by weak Jacobi forms on extremal $\mathcal{N} = (2, 2)$ structures. Benjamin, Dyer, Fitzpatrick and Kachru [BDFK15], and Harrison [Har16], described lattice-orbifold examples at $c = 24$ with, respectively, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ structures. These are separate extremality problems; their examples do not resolve the ordinary self-dual VOA existence question in higher central charge.

**Lattice-orbifold constructions.** Gemünden and Keller [GK19, GK21] constructed self-dual lattice orbifolds at $c = 48, 72$ with small low-weight subspaces, but these examples are not extremal. More recently, Keller, Ashley Winter Roberts and Jeremy Roberts [KRR26] studied orbifolds associated with Barnes–Wall lattices. Their proposed self-dual extension at $c = 128$

has minimum nonvacuum Virasoro highest weight 4, conditional on triviality of a specified associator class in third group cohomology. This assumption is explicit also in the published version. Even conditional on existence, that example is not extremal, since $\lfloor 128/24 \rfloor + 1 = 6$. None of the conjectural assertions surveyed in this subsection is used as an input to the proof below.

### 1.3 Extremal even unimodular lattices

The analogous lattice existence problem has also seen substantial progress. A positive-definite even unimodular lattice $L$ of rank $n$ is extremal if it attains the modular-form bound

$$\min(L) = 2\left\lfloor \frac{n}{24} \right\rfloor + 2, \qquad \min(L) := \min\{(x,x) : 0 \neq x \in L\}.$$

Relevant developments include Nebe's rank-72 lattice of minimum 8, announced in 2010 and published in 2012 [Neb12], her fourth extremal rank-48 lattice [Neb14], and Shimada's further rank-64 constructions [Shi18, Shi22]. Sakamoto's recent preprint [Sak26, Theorem 1.1] reports two nonisomorphic extremal rank-88 lattices, obtained using Hermitian tensor products over $\mathbb{Q}(\sqrt{-23})$ and computer-assisted minimum calculations. These lattice results do not by themselves produce extremal VOAs: $V_L$ contains an $n$-dimensional Heisenberg subspace in degree 1, so it is not extremal for $n \geq 24$.

### 1.4 Structure of the proof

The proof has three distinct parts. First we determine the actual orbifold category and its low-weight modules. Second we give an exhaustive finite calculation excluding those modules from an explicit maximal isotropic subgroup. Third the simple-current extension theorem produces the VOA. Character calculations are included only as a check, not as a substitute for these steps. For imported results used in the proof below, theorem numbers refer to the explicitly listed arXiv versions in the bibliography.

## 2 The central-charge-32 input and its twisted weights

### 2.1 The Barnes–Wall input and the Albert module

I recorded the expected automorphism group of $W$ in my conformal-design paper [Höh08a, Example 3.2] as a joint observation with Griess; the identification was explicitly conjectural. Shimakura [Shi14, Theorem 2.30] subsequently proved the group shape $2^{27}.E_6(2)$; his introduction also records Griess's conjecture at the Edinburgh workshop in July 2004. For the detailed structural input used here we refer to Shimakura [Shi14, Proposition 1.8, (2.1), Proposition 2.11, Theorems 2.19 and 2.30]:

$$c(W) = 32, \quad W_1 = 0, \quad \dim W_2 = 139504, \quad A = O_2(\operatorname{Aut} W) \cong 2^{27}, \quad \operatorname{Aut}(W)/A \cong E_6(2). \tag{2.1}$$

The VOA $W$ is strongly rational and self-dual: the lattice VOA is strongly rational, and its $(-1)$-twisted minimum is $32/16 = 2$. The three nontrivial $V_{\mathrm{BW}_{32}}^{+}$-modules have lowest weights $1, 2, 5/2$, respectively. Thus the cyclic type is $4 \cdot 2 \equiv 0 \pmod 2$, all nontrivial minima are positive, and the integral twisted summand is isotropic. The cyclic orbifold theorem [vEMS20, Theorem 5.16] therefore applies. The quotient acts irreducibly on $A$, and

$$W_2 = \mathbb{C}\omega_W \oplus \bigoplus_{\chi \in S} \mathbb{C}_\chi, \qquad |S| = 139503, \tag{2.2}$$

where the indicated nontrivial $A$-characters are distinct and form one orbit. Here and below $2^{27}.E_6(2)$ denotes group shape; no splitting of this extension is assumed.

We use the split Albert module

$$J = M_3(\mathbb{F}_2) \oplus M_3(\mathbb{F}_2) \oplus M_3(\mathbb{F}_2)$$

with invariant cubic

$$N(X, Y, Z) = \det X + \det Y + \det Z + \operatorname{tr}(XYZ). \tag{2.3}$$

This is the three-matrix model in Cohen–Cooperstein [CC88]; their minus sign equals a plus sign in characteristic two. The differential

$$a^{\#} = dN_a \in J^*, \qquad H_u(a) = (a+u)^{\#} + a^{\#} + u^{\#}$$

is always regarded as a *covector*. Pairings are the ordinary coordinate pairing $J^* \times J \to \mathbb{F}_2$. No identification of $J$ with $J^*$, and no dual-adjoint convention, is needed.

The identification $A \cong J$ requires more than a group-order comparison. In characteristic two the only nontrivial irreducible $E_6(2)$-modules of degree at most 27 are the dual modules with highest weights $\lambda_1, \lambda_6$; see [Lüb01, Theorems 2.3 and 4.4, Appendix A.51]. The same degree bound shows that the irreducible 27-dimensional $\mathbb{F}_2$-module $A$ is absolutely irreducible: a nontrivial scalar extension decomposition would have constituents of smaller nontrivial degree. After possibly composing the chosen quotient identification with the graph automorphism, we may therefore identify $A$ with $J$ and $A^*$ with $J^*$.

There are three nonzero orbits, described by

| rank | condition | orbit size |
|---|---|---|
| 1 | $a \neq 0,\ a^{\#} = 0$ | 139503 |
| 2 | $a^{\#} \neq 0,\ N(a) = 0$ | 69193488 |
| 3 | $N(a) = 1$ | 64884736. |

(2.4)

The rank-one stabilizer has shape $2^{16}.\Omega_{10}^{+}(2)$, and the rank-three stabilizer is $F_4(2)$ [CC88]. The dual module has the same orbit sizes. Consequently $S$ in (2.2) is precisely the dual rank-one orbit.

## 2.2 A flip calculation fixing the absolute weights

Put $R = V^{+}_{\mathrm{BW}_{16}}$. Its irreducible modules $R_s$ form a ten-dimensional plus-type quadratic fusion space $(S_{10}, q_{10})$. Their lowest weights are

$$h(R_s) = \begin{cases} 0, & s = 0, \\ 1, & s \neq 0,\ q_{10}(s) = 0, \\ 3/2, & q_{10}(s) = 1. \end{cases} \tag{2.5}$$

There are 527 nonzero singular labels and 496 nonsingular labels; the singular nonvacuum modules have 16-dimensional weight-one spaces. The latter dimension also follows from transitivity on nonzero singular labels and $(V^{-}_{\mathrm{BW}_{16}})_1 \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{BW}_{16}$; see [Shi14, Proposition 1.8]. That proposition also gives rationality and $C_2$-cofiniteness of $R$. Its self-contragredience follows from the nondegenerate invariant bilinear form on the lattice VOA: averaging over the involution shows that the restriction to $R$ is nondegenerate. Thus $R$ is strongly rational, and all its nontrivial irreducible modules have positive lowest weight. There is a full diagonal extension

$$W = \bigoplus_{s \in S_{10}} R_s \otimes R_s. \tag{2.6}$$

Let $\tau$ be the literal flip on each summand in (2.6). It is a genuine VOA automorphism, with no unspecified lift signs. Indeed, after choosing the same intertwining operator in the two identical factors, every projected component of the vertex operator is a scalar multiple of $I_{s,t} \otimes I_{s,t}$, because the relevant fusion multiplicity is one. Swapping the two tensor positions commutes with this expression. Shimakura's descriptions (2.2) and (2.4) place this flip in $A$.

For $t \in S_{10}$, let $\chi_t \in A$ act on the $s$-summand by $(-1)^{b_{10}(s,t)}$. Write

$$\mathrm{ch}_s(z) = \mathrm{tr}_{R_s} \exp\big(2\pi i z(L_0 - 16/24)\big).$$

For $g \in \mathrm{Aut}(W)$, write $Z_W(g;z) = \mathrm{tr}_W\big(g \exp(2\pi i z(L_0 - 32/24))\big)$. The trace function and its modular transform are

$$Z_W(\tau\chi_t; z) = \sum_s (-1)^{b_{10}(s,t)} \,\mathrm{ch}_s(2z), \tag{2.7}$$

$$Z_W(\tau\chi_t; -1/z) = 32\ \mathrm{ch}_t(z/2). \tag{2.8}$$

For the second identity, use $S_{s,t} = 2^{-5}(-1)^{b_{10}(s,t)}$ [vEMS20, Theorem 3.4] and the Fourier orthogonality sum of order $2^{10}$. Modular covariance of twisted trace functions in the normalization $q^{L_0 - c/24}$, [vEMS20, Theorem 2.3], identifies the leading exponent with that of the unique $(\tau\chi_t)$-twisted $W$-module, up to a nonzero scalar. Equating exponents gives

$$\rho_{\tau\chi_t} - \frac{32}{24} = \frac{1}{2}\left(h(R_t) - \frac{16}{24}\right), \qquad \rho_{\tau\chi_t} = 1 + \frac{h(R_t)}{2}.$$

Thus the three possible absolute weights are $1, 3/2, 7/4$. This calculation determines lowest weights, but does not determine the dimensions of the corresponding lowest-weight spaces.

**Proposition 2.1.** *For the unique $a$-twisted $W$-module $W(a)$, set $\rho_a = h(W(a))$, with $\rho_0 = 0$. Then*

$$\rho_a = \begin{cases} 1, & \mathrm{rank}(a) = 1, \\ 3/2, & \mathrm{rank}(a) = 2, \\ 7/4, & \mathrm{rank}(a) = 3. \end{cases} \qquad 4\rho_a \equiv N(a) \pmod 2.$$

*For rank-one $a$, $\dim W(a)_1 = 32$.*

*Proof.* The traces of $\tau\chi_t$ on $W_2$, for the three types of $t$ in (2.5), are respectively

$$8432, \qquad 240, \qquad -272.$$

The vacuum summand contributes zero flip trace in total weight two. Each nonzero singular summand contributes 16 times its sign. For $t \neq 0$, the singular Fourier sum is $16(-1)^{q_{10}(t)} - 1$; for $t = 0$, it is 527. The three distinct traces must represent the three nonzero $E_6(2)$-orbits in $A$.

Let $z_0$ be the reverse-orbifold grading involution of

$$W = V^+_{\mathrm{BW}_{32}} \oplus V^{T,+}_{\mathrm{BW}_{32}}.$$

The centralizer order in [Shi14, Proposition 2.1] gives

$$|C_{\mathrm{Aut}(W)}(z_0)/A| = 2^{16}|\Omega^+_{10}(2)|,$$

so its $E_6(2)$-orbit has size 139503 and it has rank one. By [Shi14, Proposition 2.1 and Lemma 2.3], its weight-two trace is

$$73968 - 65536 = 8432.$$

The 8432 class therefore has rank one and twisted weight 1.

Since $\dim W_2^A = 1$, character averaging gives

$$139504 + 139503 \cdot 8432 + 240n_{240} - 272n_{-272} = 2^{27},$$
$$n_{240} + n_{-272} = 2^{27} - 1 - 139503.$$

Hence $n_{240} = 69193488$ and $n_{-272} = 64884736$. The latter equals

$$|E_6(2) : F_4(2)| = 2^{12}(2^9 - 1)(2^5 - 1),$$

so the classes of weights $3/2, 7/4$ have ranks two and three.

Finally the reverse-orbifold construction after [vEMS20, Theorem 5.16] gives

$$W(z_0) = V^-_{\mathrm{BW}_{32}} \oplus V^{T,-}_{\mathrm{BW}_{32}}.$$

The lattice is rootless, so $(V^-_{\mathrm{BW}_{32}})_1$ is precisely the 32-dimensional Heisenberg space. The $(-1)$-twisted lattice Fock space has minimum $32/16 = 2$ and half-integer oscillator increments [vEMS20, Theorem 7.6]. With the integral-weight eigenspace denoted by $T, +$, the other eigenspace $T, -$ starts at $5/2$. Hence $\dim W(z_0)_1 = 32$, without using the unknown scalar in the modular transformation. Conjugacy gives the same dimension throughout the rank-one orbit. □

# 3 Group cohomology and the twisted-module associator

## 3.1 Cyclic detection for elementary abelian 2-groups

The following statement is essential. Vanishing of cyclic restrictions does *not* detect $H^3$ for arbitrary groups.

**Lemma 3.1.** *Let $E = (\mathbb{Z}/2)^r$. Restriction to all subgroups of order two gives an injective map*

$$H^3(E, \mathbb{C}^\times) \longrightarrow \prod_{0 \neq v \in E} H^3(\langle v \rangle, \mathbb{C}^\times).$$

*With the latter factors identified with $\mathbb{F}_2$, its image consists of the functions*

$$f(v) = \sum_i c_i v_i + \sum_{i<j} c_{ij} v_i v_j + \sum_{i<j<k} c_{ijk} v_i v_j v_k. \tag{3.1}$$

*Proof.* The split integral homology Künneth theorem gives

$$H_1(E, \mathbb{Z}) \cong (\mathbb{Z}/2)^r, \qquad H_2(E, \mathbb{Z}) \cong (\mathbb{Z}/2)^{\binom{r}{2}}, \qquad H_3(E, \mathbb{Z}) \cong (\mathbb{Z}/2)^{\,r + \binom{r}{2} + \binom{r}{3}}.$$

For clarity, the recurrence for the last exponent is obtained from $E_{r-1} \times C_2$: the contributions are $H_3(E_{r-1})$, $H_3(C_2)$, $H_2(E_{r-1}) \otimes H_1(C_2)$, and $\mathrm{Tor}(H_1(E_{r-1}), H_1(C_2))$. They add $1 + \binom{r-1}{2} + r - 1$. The splitting excludes hidden $\mathbb{Z}/4$-extensions.

Since $\mathbb{C}^\times$ is divisible, universal coefficients identifies $H^3(E, \mathbb{C}^\times)$ with $\mathrm{Hom}(H_3(E, \mathbb{Z}), \mathbb{C}^\times)$. It is therefore elementary abelian of the displayed rank.

For $x, y, z \in E$, consider the normalized multiplicative cocycles

$$(-1)^{x_i y_i z_i}, \qquad (-1)^{x_i y_j z_j} \quad (i < j), \qquad (-1)^{x_i y_j z_k} \quad (i < j < k).$$

Their exponents are trilinear over $\mathbb{F}_2$, so the cocycle equation holds directly. On the cyclic subgroup generated by $v$, they restrict to the standard nontrivial $C_2$-cocycle to the powers $v_i, v_i v_j, v_i v_j v_k$, respectively. In particular a normalized coboundary evaluates to 1 on $(v, v, v)$, whereas the nontrivial normalized $C_2$-cocycle evaluates to $-1$.

The squarefree Boolean monomials of degrees one, two, and three are linearly independent as functions on $\mathbb{F}_2^r$. Thus these cocycle classes are independent; their number equals the computed dimension of $H^3$, so they form a basis. Both claims follow. □

For an involution $g$ of a strongly rational self-dual VOA, cyclic orbifold theory gives $4h(V(g)) \in \mathbb{Z}$ and assigns the type $t_g = 4h(V(g)) \bmod 2$, as in (1.2); see [vEMS20, Theorem 5.11, definition of type, and Theorem 5.13]. Its identification with $\mathfrak{a}_{\langle g\rangle}(V)$ can be checked directly in the order-two case. Write that restriction as $s[\epsilon]$, where $\epsilon(i,j,k) = (-1)^{ijk}$. A simple object of nontrivial $C_2$-degree in $Z(\mathrm{Vec}_{C_2}^{\epsilon^s})$ has a projective centralizer scalar $u$ satisfying $u^2 = (-1)^s$, by the half-braiding hexagon. Its twist square is therefore $(-1)^s$. The VOA twist square, for a constituent of $V(g)$, is

$$\exp(4\pi i h) = \exp(4\pi i h(V(g))) = (-1)^{4h(V(g))},$$

since constituent weights differ by half-integers and $4h(V(g)) \in \mathbb{Z}$ [vEMS20, Theorem 5.11]. This proves $s = t_g$. A possible inverse-cocycle convention has no effect on a class of order two. The ribbon twist here is $\exp(2\pi i h)$, not the character $T$-eigenvalue $\exp(2\pi i(h - c/24))$.

In particular, $\mathfrak{a}_A(W) \neq 0$. Proposition 2.1 and Lemma 3.1 identify its restriction function as the cubic $N$. In squarefree coordinates, if $N(a) = \sum_{i<j<k} n_{ijk} a_i a_j a_k$, a representative is

$$\omega_A(a,b,c) = (-1)^{\sum_{i<j<k} n_{ijk} a_i b_j c_k}. \tag{3.2}$$

One must use ordered monomials here, not the full six-term polarization of $N$, whose diagonal is zero in characteristic two. The representative need not be $E_6(2)$-invariant as a cochain; its cohomology class is invariant, by cyclic detection.

### 3.2 The doubled action

Put

$$V = W \otimes W, \qquad G = \Delta A \times \langle \sigma \rangle \cong (\mathbb{Z}/2)^{28}, \qquad U = V^G.$$

Write $d_a = (a,a)$, $p_a = (a,a)\sigma$. The nonidentity elements are $d_a$ for $a \neq 0$ and $p_a$ for all $a$. Their twisted lowest weights are

$$h(V(d_a)) = 2\rho_a, \qquad h(V(p_a)) = \frac{c(W)}{16} = 2. \tag{3.3}$$

Indeed $V(d_a) = W(a) \otimes W(a)$. Also $p_a$ is conjugate to $\sigma$ by $(a,1)$, and a transposition twist satisfies $L_0^{\mathrm{tw}} = \frac{1}{2}L_0 + c(W)/16$. More generally the $k$-cycle formula is

$$L_0^{\mathrm{tw}} = \frac{1}{k}L_0 + \frac{(k^2-1)c}{24k},$$

with $c$ the central charge of *one* tensor factor [BDM02, proof of Theorem 3.9(3), Theorem 4.4]. The lowest-weight space of each $V(p_a)$ is one-dimensional.

| $a$ | $\rho_a$ | $4\rho_a \bmod 2$ | $h(V(d_a))$ | $4h(V(d_a)) \bmod 2$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| rank 1 | 1 | 0 | 2 | 0 |
| rank 2 | 3/2 | 0 | 3 | 0 |
| rank 3 | 7/4 | 1 | 7/2 | 0 |

Every $p_a$ has weight 2 and type 0.

**Proposition 3.2.** *The full associator class $\mathfrak{a}_G(V)$ is zero.*

*Proof.* For $d_a$, its cyclic type is $8\rho_a \equiv 0 \pmod 2$. For $p_a$, it is $4 \cdot 2 \equiv 0 \pmod 2$. Thus every cyclic restriction vanishes, and Lemma 3.1 applies. □

Checking $\Delta A$ and $\langle \sigma \rangle$ separately would not by itself exclude mixed cohomology terms. The calculation checks *every* $p_a$, and the detection lemma is what excludes those terms. In particular, the calculation retains the contribution of $\mathfrak{a}_A(W)$, represented by (3.2).

### 3.3 Regularity, positivity, and the actual category

The groups used here are finite 2-groups, hence solvable. Carnahan–Miyamoto [CM18, Theorem 5.24 and Corollary 5.25] gives regularity and self-contragredience of the fixed-point VOAs

$$X = W^A, \qquad U = V^G, \qquad Y = (X \otimes X)^\sigma = V^{\mathcal{K}}, \quad \mathcal{K} = (A \times A) \rtimes \langle\sigma\rangle.$$

The nonnegative grading and the one-dimensional degree-zero space are inherited from the original VOA. By [ABD04, Theorem 4.5], these fixed-point VOAs are strongly rational. Completeness of the irreducible constituents of twisted modules, and the corresponding equivariantization equivalence, follow from [McR21, Theorems 4.15 and 4.17], without first imposing positivity on every irreducible fixed-point module.

Every $a$-twisted $W$-module for $a \neq 0$ has positive minimum by Proposition 2.1; among the constituents of $W|_X$, only $X$ contains the vacuum vector. Hence every nonvacuum $X$-module has positive weight. The same argument using (3.3) works for $U$. For $Y$, untwisted constituents come from $P \otimes Q$, with $P, Q \in \mathrm{Irr}(X)$. Only the symmetric vacuum constituent contains weight zero. Permutation-twisted constituents have weights at least $2 + h(P)/2 > 0$.

Thus the positivity hypotheses in the center-description theorem [DNR25, Theorem 6.2(4) and Remark 7.2] are now satisfied. Together with Proposition 3.2, it gives

$$\mathrm{Rep}(U) \simeq Z(\mathrm{Vec}_G), \qquad D = \mathrm{Irr}(U) \cong \mathbb{F}_2^{56}. \tag{3.4}$$

Positivity also identifies the categorical dimensions with the positive Frobenius–Perron dimensions, so the equivalence respects the positive spherical structure and the conformal-weight ribbon twist. All simples are invertible, and $(D, q)$ is a nondegenerate hyperbolic quadratic space. Define the *twisting-label homomorphism*

$$\nu : D \longrightarrow G, \qquad \nu([M]) = g \quad \Longleftrightarrow \quad M \text{ is a constituent of } V(g)|_U.$$

Its kernel consists of the irreducible constituents of $V|_U$ and is canonically indexed by $G^*$. For $\chi \in G^*$ and $d \in D$, their polar pairing satisfies $(-1)^{b(\chi,d)} = \chi(\nu(d))$. The same theory gives $\mathrm{FPdim}\,\mathrm{Rep}(X) = |A|^2$, even though the class $\mathfrak{a}_A(W)$ is nonzero.

Finally,

$$U_1 = 0, \qquad U_2 = \mathbb{C}(\omega_W \otimes \mathbf{1} + \mathbf{1} \otimes \omega_W). \tag{3.5}$$

Indeed $\Delta A$-invariance leaves just the two conformal vectors in $V_2$, and $\sigma$-invariance leaves their sum.

## 4 Permutation-twisted modules and exact fusion weights

The category alone sees weights modulo $\mathbb{Z}$, not their absolute minima. The following argument supplies the additional information.

**Lemma 4.1** (Lowest-weight summands of a tensor square)**.** *Let $X$ be strongly rational, and put $Y = (X \otimes X)^\sigma$. Let $T$ be the simple $Y$-constituent of $T_\sigma(X)$ containing its one-dimensional homogeneous subspace of weight $c(X)/16$. For every $P \in \mathrm{Irr}(X)$, the fusion product $T \boxtimes_Y T$ contains a simple summand of lowest weight $2h(P)$, arising from $(P \otimes P^*)|_Y$.*

*Proof.* The fixed-point theorem [CM18, Corollary 5.25], together with [ABD04, Theorem 4.5], makes $Y$ strongly rational. In particular $\mathrm{Rep}(Y)$ is a rigid semisimple vertex tensor category. All modules used below restrict to finite-length ordinary $Y$-modules.

Assign the modules $T_\sigma(X), P \otimes P^*, T_\sigma(X)$ to the marked points $0, 1, \infty$ of the base sphere, with monodromies $\sigma, 1, \sigma$. The connected covering is $z = w^2$; the modules at the marked

points $0, 1, -1, \infty$ of the covering sphere are $X, P, P^*, X$. Propagation of the vacuum modules reduces the conformal block to the contragredient pairing between $P$ and $P^*$. Here conformal blocks and propagation are used in the algebraic-analytic definitions of [Gui26, Sections 1.2 and 1.3].

Choose lowest-weight vectors $p, p^*$ with $\langle p^*, p\rangle \neq 0$. The cover block is nonzero on $(\mathbf{1}, p, p^*, \mathbf{1})$. At the two unramified marked points the changes to lifted coordinates have nonzero derivatives; positive Virasoro modes annihilate lowest-weight vectors, so these changes only multiply the value by nonzero scalars. Propagation of the vacuum vectors also preserves its nonzero value.

Gui's cover theorem [Gui26, Theorem 2.25] identifies the base and cover blocks as the *same underlying linear functional*, not merely as abstract vector spaces. Therefore the base block is nonzero on $(t, p \otimes p^*, t)$, where $t$ spans the lowest-weight space of $T_\sigma(X)$.

By [Gui26, Theorem 4.10], this block determines a twisted intertwining operator. Restricting the vertex operators to $Y$ gives an ordinary $Y$-intertwining operator by [Gui26, Remark 4.6]: the relevant $L_0$-eigenspaces are finite-dimensional and the spectrum is bounded below. The conformal vector is unchanged. Restrict the first and third entries to $T$. The displayed lowest-weight coefficient remains nonzero. Decomposing $p \otimes p^*$ among simple $Y$-constituents gives one $M$ with a nonzero three-point block with the other two modules $T, T$. Its lowest weight is exactly $2h(P)$: this weight occurs in $M$, whereas the entire ambient module $P \otimes P^*$ has no smaller weight.

Contragredient compatibility [Gui26, Theorem 2.23] makes $T_\sigma(X)$ self-contragredient. Its nondegenerate graded pairing pairs its unique lowest-weight line with itself; hence $T^* \cong T$. Rigidity of $\mathrm{Rep}(Y)$ now implies $M^* \subset T \boxtimes_Y T$. The dual has the same minimum. Moreover $(P \otimes P^*)^* = P^* \otimes P$, and flip identifies the latter with $P \otimes P^*$ as $Y$-modules. Thus $M^*$ also arises from the specified middle module. □

Return to $X = W^A$. For $a \in A$, let $F_a$ be the unique simple $U$-module containing the one-dimensional subspace $V(p_a)_2$ of $V(p_a)$. Then

$$h(F_a) = 2, \quad q(F_a) = 0, \quad \nu(F_a) = p_a.$$

The automorphism $r_c = (c, 1)$ normalizes $G$, fixes $Y$ pointwise, and sends $F_a$ to $F_{a+c}$.

**Proposition 4.2.** *For all $a, b \in A$,*

$$h(F_a \boxtimes_U F_b) = 2\rho_{a+b}. \tag{4.1}$$

*Proof.* Write $W = \bigoplus_{\chi \in A^*} W_\chi$ as an $X$-module. Its summands are simple currents indexed by $A^*$, and

$$U = \bigoplus_{\chi \in A^*} (W_\chi \otimes W_\chi)^+ \tag{4.2}$$

is a simple-current extension of $Y$ of dimension $|A|$. For a direct dimension justification, $\mathcal{K}/G \cong A$ acts on $U$, its fixed algebra is $Y$, and $(c, 1)$ acts on the $\chi$-summand by $\chi(c)$. The tensor subcategory generated by these summands is $\mathrm{Rep}(A)$ [DNR25, Theorem 6.1]. Consequently each summand in (4.2) is invertible in $\mathrm{Rep}(Y)$ and has Frobenius–Perron dimension one. Moreover $U$, as an algebra object in this subcategory, corresponds to $\mathrm{Fun}(A)$ [DNR25, Theorem 6.3]. In particular the multiplication of any two character summands is nonzero.

The restriction $F_0|_Y$ contains the module $T$ in Lemma 4.1: the permutation-twisted $W \otimes W$ module has underlying vector space $W$, and its subspace $X$ has the permutation-twisted $X \otimes X$-module structure $T_\sigma(X)$. Since the $r_c$ fix $Y$, every $F_c|_Y$ contains $T$.

Let $\mathcal{D}_\sigma$ be the $C_2$-graded fusion category of untwisted and $\sigma$-twisted $X \otimes X$-modules, and put $S_\sigma = T_\sigma(X)$. The covering calculation gives

$$\dim \mathrm{Hom}_{\mathcal{D}_\sigma}(S_\sigma \boxtimes S_\sigma, P \otimes Q) = \delta_{Q,P^*} \qquad (P, Q \in \mathrm{Irr}(X)).$$

Indeed, the covering sphere has the labels $X, P^*, Q^*, X$, where the duals arise from the target $P \otimes Q$. Removing the two vacuum labels leaves the contragredient pairing. The dimension formula is [Gui26, Corollary 2.28]. To interpret it as the displayed categorical Hom-space, use [Gui26, Remark 4.6 and the discussion preceding Proposition 4.7] and the tensor-product universal property in [McR21, discussion following Remark 4.16]. The required $Y$-modules are finite-length ordinary modules, hence finitely generated, and $Y$ is $C_2$-cofinite. Consequently

$$S_\sigma \boxtimes S_\sigma \cong \bigoplus_{P \in \mathrm{Irr}(X)} P \otimes P^*, \qquad \mathrm{FPdim}(S_\sigma)^2 = \sum_P \mathrm{FPdim}(P)^2 = |A|^2.$$

The stabilizer of $S_\sigma$ is $C_2$, whose irreducible projective representations are one-dimensional because $H^2(C_2, \mathbb{C}^\times) = 0$. Thus the equivariant simple object corresponding to $T$ forgets to $S_\sigma$. The forgetful tensor functor preserves Frobenius–Perron dimension, so $\mathrm{FPdim}_Y(T) = |A|$.

For the extension $Y \subset U$, the identification of local algebra modules with VOA modules is braided monoidal by [CKM24, Theorem 3.65]. Both VOAs are strongly rational, and $U$ is a finite object of $\mathrm{Rep}(Y)$, as required there. The algebra $\mathrm{Fun}(A)$ has a nondegenerate invariant Frobenius pairing and trivial twist. The dimension formula [KO01, Theorem 1.18], together with the positive categorical dimensions already verified above, gives

$$\mathrm{FPdim}_Y(\mathrm{Res}\, F_a) = \mathrm{FPdim}_Y(U)\, \mathrm{FPdim}_U(F_a) = |A|.$$

Positivity of Frobenius–Perron dimensions forces

$$\mathrm{Res}^U_Y F_a = T. \tag{4.3}$$

We next justify locality of free induction before using it. Every simple-current summand $K$ of $U$ acts nontrivially on the simple underlying object $T$. Otherwise the module axiom, the nonzero product $K^{-1} \boxtimes K \to \mathbf{1}$ in $U$, and the unit action give a contradiction. Thus $K \boxtimes T \cong T$. Since $\theta_K = 1$, the balancing identity shows that the double braiding of $K$ and $T$ is the identity. Hence $T$ centralizes the algebra $U$, as do $T \boxtimes_Y T$ and all its simple summands. By [CKM24, Proposition 2.65 and Theorem 2.67], free induction is local and braided monoidal on this centralizer. The preceding braided equivalence identifies its local tensor product with VOA fusion. Frobenius reciprocity [KO01, Theorem 1.6], (4.3), and dimensions therefore give

$$\mathrm{Ind}^U_Y T = \bigoplus_{a \in A} F_a, \qquad \mathrm{Ind}(T \boxtimes_Y T) = \left(\bigoplus_a F_a\right) \boxtimes_U \left(\bigoplus_b F_b\right).$$

The monoidality is the usual free-module identity

$$(U \boxtimes M) \boxtimes_U (U \boxtimes N) \cong U \boxtimes (M \boxtimes N)$$

for objects centralizing the commutative algebra $U$.

Fix $u \neq 0$, and choose an $X$-constituent $P$ of $W(u)$ containing a lowest-weight vector. Then $h(P) = \rho_u$, and $P^*$ also occurs in $W(u)|_X$, since $u^{-1} = u$. Lemma 4.1 gives a constituent $M$ of $T \boxtimes_Y T$, also arising in $W(u) \otimes W(u)$, with $h(M) = 2\rho_u$. The unit summand of $U$ gives an injective $Y$-map $M \to \mathrm{Res}\,\mathrm{Ind}\, M$; thus some simple $U$-summand $N$ of $\mathrm{Ind}\, M$ contains a vector of weight $2\rho_u$.

We claim that $\nu(N) = d_u$. Using $Y = V^{\mathcal{K}}$, put $g = \nu(N)$. Then

$$\mathrm{Hom}_U(\mathrm{Ind}\, M, N) \cong \mathrm{Hom}_Y(M, \mathrm{Res}\, N) \neq 0.$$

Thus $M$ occurs in both $V(g)|_Y$ and $V(d_u)|_Y$. Twisted modules labeled by different $\mathcal{K}$-conjugacy classes have disjoint simple $Y$-constituents [DNR25, Theorem 5.1(1)]. Since $d_u$ is central in $\mathcal{K}$, its conjugacy class is a singleton, so $g = d_u$. The displayed induced product decomposition consequently makes $N$ one of $F_a \boxtimes F_{a+u}$.

Every module with twisting label $d_u$ occurs in $W(u) \otimes W(u)$, whose minimum is $2\rho_u$. Hence $h(N) = 2\rho_u$. The normalizer translations act transitively on the pairs $(a, a+u)$ and preserve conformal weight, proving the equality for every pair. For $u = 0$, it is $F_a^2 = U$, since $D$ has exponent two. □

## 5 Coherent coordinates and all weight-two obstructions

Define

$$\mathcal{E} = \mathbb{F}_2 \oplus J \oplus \mathbb{F}_2 \oplus J^*, \qquad Q(\alpha, a; \beta, B) = \alpha\beta + \langle B, a\rangle.$$

Its polar form is denoted by $B_{\mathcal{E}}$. For $a \in J$, put

$$f_a = (1, a; N(a), a^{\#}). \tag{5.1}$$

Euler's identity for a cubic in characteristic two gives $\langle a^{\#}, a\rangle = N(a)$. Cubic expansion then gives

$$Q(f_a) = 0, \qquad B_{\mathcal{E}}(f_a, f_b) = N(a+b). \tag{5.2}$$

On the actual fusion group, Propositions 2.1 and 4.2 give

$$b(F_a, F_b) = 4\rho_{a+b} \pmod 2 = N(a+b).$$

**Proposition 5.1.** *There is a unique linear isometry $\Phi : (D, q) \to (\mathcal{E}, Q)$ sending $F_a$ to $f_a$. For the character*

$$E_{\beta,B} : \quad d_a\sigma^s \longmapsto (-1)^{\beta s + \langle B, a\rangle}, \tag{5.3}$$

*its corresponding simple module has image $(0, 0; \beta, B)$.*

*Proof.* The vectors $f_a$ span $\mathcal{E}$. Indeed, a linear functional vanishing on all of them would give an identically zero Boolean polynomial

$$c + \ell(a) + \langle a^{\#}, v\rangle + dN(a).$$

In the coordinates (2.3), $N$ is a sum of 45 distinct squarefree cubic monomials. Each of its 27 derivatives has five quadratic monomials, and these 135 monomials are mutually distinct. The derivative polynomials are therefore independent. Constant, linear, quadratic, and cubic squarefree monomials are independent as functions on $\mathbb{F}_2^{27}$, so all coefficients vanish.

Because the form on $\mathcal{E}$ is nondegenerate, the Gram matrix of the $f_a$ has rank 56. The equal Gram matrix of the actual $F_a$ forces these to span the 56-dimensional space $D$. A relation in either family pairs to zero with the full other family, so nondegeneracy gives the same relation there. Thus the assignment extends to an isometry of polar forms. Both quadratic forms vanish on the spanning family; their difference is linear and hence zero. This proves existence and uniqueness.

The module indexed by (5.3) pairs with $F_a$ as $\beta + \langle B, a\rangle$. The vector $(0, 0; \beta, B)$ has exactly these pairings with every $f_a$. Nondegeneracy proves the final assertion. □

### 5.1 Lowest-weight spaces for rank-one diagonal elements

For rank-one $u$, put

$$\beta_u(a) = \langle a^{\#}, u\rangle.$$

Since $u^{\#} = 0$, cubic expansion yields

$$f_a + f_{a+u} = (0, u; \beta_u(a), H_u a). \tag{5.4}$$

**Lemma 5.2.** *For every nonzero rank-one $u$, the linear map $H_u$ has rank 10, and $\beta_u$ is constant on its fibers. Consequently (5.4) gives exactly $2^{10}$ distinct vectors as $a$ varies.*

*Proof.* The $E_6(2)$-action preserves $N$ and is transitive on rank-one vectors. It suffices to take $u = (E_{11}, 0, 0)$ in (2.3). Direct differentiation gives

$$\beta_u(X, Y, Z) = x_{22}x_{33} + x_{23}x_{32} + y_{11}z_{11} + y_{12}z_{21} + y_{13}z_{31}. \tag{5.5}$$

This consists of five disjoint hyperbolic pairs; its remaining seventeen coordinates form a radical on which it is identically zero. Its polar map is $H_u$, proving the claims for this $u$. Norm invariance gives

$$\beta_{gu}(ga) = \beta_u(a), \qquad H_{gu}(ga) = (g^{-1})^* H_u(a),$$

which transports the assertions to every rank-one vector. □

By Proposition 4.2, all 1024 corresponding distinct $U$-modules have weight two. The full $d_u$-twisted module has a lowest-weight space of dimension

$$\dim(W(u)_1 \otimes W(u)_1) = 32^2 = 1024.$$

Each of these distinct modules contributes at least one line to this space. Equality proves that they exhaust it and that each has a one-dimensional lowest-weight space.

**Proposition 5.3.** *The nonvacuum simple $U$-modules of integral lowest weight at most two all have weight two. Their images under $\Phi$ form the disjoint union*

$$\mathcal{B}_{\text{unt}} = \{(0, 0; 1, 0)\} \cup \{(0, 0; \beta, B) : B \in S,\ \beta \in \mathbb{F}_2\}, \tag{5.6}$$

$$\mathcal{B}_{\text{diag}} = \{(0, u; \beta_u(a), H_u a) : u \neq 0,\ u^{\#} = 0,\ a \in J\}, \tag{5.7}$$

$$\mathcal{B}_{\text{perm}} = \{f_a : a \in J\}. \tag{5.8}$$

*Their respective sizes are*

$$279007, \qquad 142851072, \qquad 134217728,$$

*and their total size is* 277347807*.*

*Proof.* The ordinary module $V$ has $V_2 = W_2 \otimes \mathbf{1} + \mathbf{1} \otimes W_2$. Its $\sigma$-eigenspaces and (2.2) give (5.6), with the vacuum constituent omitted. The preceding dimension argument gives exactly (5.7) for rank-one diagonal twisting labels. The other diagonal minima are 3 and 7/2. Every $V(p_a)$ has a one-dimensional weight-two subspace, giving (5.8). The ordinary module $V$ has no weight-one vectors, and every nontrivially twisted module has minimum at least two. Completeness of their irreducible $U$-constituents excludes additional cases. □

# 6 An explicit maximal isotropic graph

For $\ell \in J^*$ and an alternating linear map $M : J \to J^*$, define

$$H_{\ell,M} = \{(\alpha, a; \ell(a), \alpha\ell + Ma) : \alpha \in \mathbb{F}_2,\ a \in J\}. \tag{6.1}$$

The first two coordinates show that this is 28-dimensional. Moreover

$$Q(\alpha, a; \ell(a), \alpha\ell + Ma) = \alpha\ell(a) + \alpha\ell(a) + \langle Ma, a\rangle = 0.$$

Thus $H_{\ell,M} = H_{\ell,M}^{\perp}$. Its intersection with $\ker \nu$ is zero, where the coordinates of Proposition 5.1 are understood.

Use row-major coordinates in $X$, then $Y$, then $Z$. The entry $(i, j)$ of these matrices has index $3(i-1)+j-1$, $9+3(i-1)+j-1$, and $18+3(i-1)+j-1$, respectively. Encode a vector as $\sum_{i=0}^{26} a_i 2^i$. Covectors use the dual basis and the same encoding. All the following numbers are decimal:

$$\ell = 1701044. \tag{6.2}$$

The 27 columns $Me_0, \ldots, Me_{26}$, in order, are

$$\begin{aligned}(&8745196,\ 110867808,\ 95730417,\ 99134097,\ 22725868,\\ &41716247,\ 128519191,\ 37605405,\ 89711618,\ 58241068,\\ &12720466,\ 113755244,\ 38528647,\ 78467855,\ 11123089,\\ &36920254,\ 91945155,\ 99458066,\ 25831589,\ 111637498,\\ &6390710,\ 131850988,\ 3350328,\ 53374031,\ 44499804,\\ &27826914,\ 2828622).\end{aligned} \tag{6.3}$$

**Proposition 6.1** (Exact finite certificate)**.** *The map $M$ in (6.3) is alternating, and the graph (6.1) is disjoint from the three obstruction sets in Proposition 5.3.*

*Proof by exhaustive binary arithmetic.* Symmetry and zero diagonal of $M$ are checked entry-wise. The set $\mathcal{B}_{\mathrm{unt}} \subset \ker \nu \setminus \{0\}$ has already been excluded.

An intersection with $f_a$ is equivalent to

$$a^{\#} = \ell + Ma. \tag{6.4}$$

Indeed contraction with $a$ and Euler's identity then force $N(a) = \ell(a)$, the remaining required coordinate equality. Exhaustion of all $2^{27} = 134217728$ values of $a$ gives no solution of (6.4).

For the diagonal obstruction, enumerate every nonzero $u$ with $u^{\#} = 0$. There are 139503. Solve

$$H_u a = Mu. \tag{6.5}$$

If no solution exists, that $u$ gives no intersection. If $a_0$ is a particular solution, Lemma 5.2 shows that the entire solution fiber has the same value $\beta_u(a_0)$. The solution fiber is disjoint from the graph precisely when $\beta_u(a_0) \neq \ell(u)$.

Exactly 139500 systems are inconsistent. The other three are:

| $u$ | $a_0$ | $\beta_u(a_0)$ | $\ell(u)$ |
|---|---|---|---|
| 37467505 | 847 | 0 | 1 |
| 102838768 | 16442 | 1 | 0 |
| 122627027 | 787 | 0 | 1 |

Thus every solution fiber is disjoint from the graph. The complete program in Appendix A implements these tests using only (2.3), its coordinate gradient, and Gaussian elimination over $\mathbb{F}_2$. □

The verifier additionally confirms rank 10 for every rank-one Hessian and checks $\beta_u = 0$ on all seventeen kernel basis vectors, for a total of 2371551 such checks. Since the restriction of a quadratic form to its polar radical is linear, these basis tests independently establish constancy on every solution fiber. The optimized and undefined-behavior-sanitized builds both passed on 19 November 2026 with GCC 14.2.0. The source in Appendix A has SHA-256

```
76d8926a1661f79cc0abc38a6ab73313977d994fc88d5d6706d1d523926fafcc.
```

The finite check is exhaustive, not a random search or a sample of rank-one representatives.

### 6.1 Twenty-eight actual simple-current generators

Let $E_{\beta,B}$ denote the irreducible constituent of $V|_U$ indexed by the character in (5.3), and let $e_i$ be the row-major coordinate basis of $J$. Since the cubic is squarefree, $N(e_i) = 0$ and $e_i^{\#} = 0$. The following 28 simple currents generate $\Phi^{-1}(H_{\ell,M})$:

$$\begin{aligned} K_0 &= F_0 \boxtimes E_{0,\ell}, \\ K_{i+1} &= F_{e_i} \boxtimes F_0 \boxtimes E_{\ell_i, Me_i}, \quad 0 \le i < 27. \end{aligned} \tag{6.6}$$

Their images are $(1,0;0,\ell)$ and $(0,e_i;\ell_i,Me_i)$, respectively. They are independent, pairwise orthogonal, and singular.

## 7 The algebra object and the self-dual extension

Two cohomology classes occur here. The ordinary class $\mathfrak{a}_G(V)$ vanishes by Proposition 3.2. Separately, $\bigoplus_{d\in D} U_d$ has an abelian intertwining algebra structure [vEMS20, Theorem 4.1]. Its associativity and commutativity data form an abelian 3-cocycle $(F,\Omega)$, whose diagonal values, in the convention of that theorem, are

$$\Omega(d,d) = \exp(-2\pi i h(U_d)), \qquad q_\Omega = -q_\Delta.$$

Here $q_\Delta$ takes only the values $0, 1/2$, so $-q_\Delta = q_\Delta$. Hence these diagonal values determine the same pre-metric group as the pointed braided category. This abelian class is nontrivial on the full fusion group $D$, although $\mathfrak{a}_G(V) = 0$. Its restriction to an isotropic subgroup is trivial in abelian third cohomology, so both coherences can be corrected at once. The trivialization and the resulting extension are the assertions of [vEMS20, Theorems 4.1 and 4.2].

An explicit algebra object is obtained as follows. Write

$$P = \mathbb{F}_2 \oplus J, \qquad K = \begin{pmatrix} 0 & \ell^{\mathsf{t}} \\ \ell & M \end{pmatrix}, \qquad H_{\ell,M} = \{(x, Kx) : x \in P\}.$$

In the standard hyperbolic pointed model, choose trivial associator and braiding

$$c_{(x,\xi),(y,\eta)} = (-1)^{\xi(y)}.$$

Let $B$ be the strictly upper-triangular part of $K$, so $B + B^{\mathsf{t}} = K$. On the object

$$\mathcal{A}_H = \bigoplus_{x\in P} X_{(x,Kx)}$$

define multiplication on the homogeneous summands, relative to the standard fusion identifications, by

$$\mu(x,y) = (-1)^{x^{\mathsf{t}} B y}. \tag{7.1}$$

Bilinearity makes $\mu$ a normalized 2-cocycle:

$$\mu(x,y)\mu(x+y,z) = \mu(y,z)\mu(x,y+z).$$

Furthermore

$$\frac{\mu(x,y)}{\mu(y,x)} = (-1)^{x^{\mathrm{t}}Ky} = c_{(x,Kx),(y,Ky)},$$

which is braided commutativity. The unit is the $x = 0$ summand. This finite twisted group-algebra object is separable over $\mathbb{C}$ (the usual separability idempotent is divided by $|P|$). It is connected, has trivial twist, and

$$\operatorname{FPdim} \mathcal{A}_H = 2^{28}, \qquad \operatorname{FPdim} \operatorname{Rep}(U) = 2^{56}.$$

Thus it is Lagrangian. The quadratic isometry of Proposition 5.1 lifts to a braided equivalence of pointed categories, by their classification by pre-metric groups [DMNO11, Section 5.3]. Thus this algebra transports to $\operatorname{Rep}(U)$. Alternatively the simple-current extension theorem applies directly, without making such a skeletal choice.

*Proof of Theorem 1.1.* The fixed-point VOA $U$ is strongly rational and every nonvacuum irreducible $U$-module has positive lowest weight, as checked before (3.4). The group $\Phi^{-1}(H_{\ell,M})$ is isotropic and self-perpendicular. The extension theorem [vEMS20, Theorem 4.2] therefore gives a strongly rational self-dual VOA

$$V_H = \bigoplus_{d \in \Phi^{-1}(H_{\ell,M})} U_d$$

extending $U$ with the same conformal vector and central charge 64. Every included simple module has integral conformal weight. Propositions 5.3 and 6.1 exclude all nonvacuum modules of weight at most two. Equation (3.5) consequently gives $(V_H)_1 = 0$ and $(V_H)_2 = \mathbb{C}\omega$, as asserted. □

## 8 The graded character

The candidate extremal character at $c = 64$ and its first highest-weight multiplicities already occur in [AKM07, KMZ18]. The calculation here is a check on the constructed extension, not an independent existence argument. In this section only, $q = \mathrm{e}^{2\pi i\tau}$ denotes the Fourier variable. Let

$$Z_{V_H}(\tau) = \operatorname{tr}_{V_H} q^{L_0 - 8/3}.$$

The modular-invariance theorem for rational $C_2$-cofinite VOAs [vEMS20, Theorems 2.1 and 2.2] implies

$$Z_{V_H}(-1/\tau) = Z_{V_H}(\tau), \qquad Z_{V_H}(\tau+1) = e^{-2\pi i\, 8/3} Z_{V_H}(\tau).$$

Here the $S$-scalar is 1: it is $\pm 1$ by the contragredience relation, and evaluation at $\tau = i$, where the graded character is strictly positive, fixes the sign. The transformation law of $\eta$ therefore shows that $\eta^{64} Z_{V_H}$ is a modular form of weight 32 for $\mathrm{SL}_2(\mathbb{Z})$, with no pole at the cusp. The space of such forms has basis

$$E_4^8, \qquad E_4^5\Delta, \qquad E_4^2\Delta^2, \qquad \Delta = \eta^{24}.$$

Equivalently, set $x = E_4/\eta^8 = q^{-1/3}(1 + 248q + \cdots)$, so $x^3 = j$. The three coefficients are uniquely determined by $\dim(V_H)_0 = 1$, $\dim(V_H)_1 = 0$, and $\dim(V_H)_2 = 1$. This gives

$$\begin{aligned} Z_{V_H} &= x^8 - 1984x^5 + 705057x^2 \\ &= j^{2/3}(j^2 - 1984j + 705057) \\ &= q^{-8/3}(1 + q^2 + 278512q^3 + 13996663144q^4 + 19414403055040q^5 + \cdots). \end{aligned}$$

The branch of $j^{2/3}$ is fixed by $x^2$.

The dimension of the weight-three Virasoro highest-weight space is 278511. Indeed, since $(V_H)_2 = \mathbb{C}\omega$ and $L_1L_{-1}\omega = 4\omega$,

$$(V_H)_3 = \ker\bigl(L_1|_{(V_H)_3}\bigr) \oplus \mathbb{C}L_{-1}\omega.$$

For a vector in this kernel, $L_2$ vanishes because $(V_H)_1 = 0$, and $[L_2, L_1] = L_3$ gives $L_3 = 0$. All higher positive modes vanish by the grading. The kernel is therefore exactly the highest-weight space, of dimension $278512 - 1$.

For an exact formal-series check, put

$$P(q) = \left(1 + 240\sum_{n\geq 1}\sigma_3(n)q^n\right)\prod_{n\geq 1}(1-q^n)^{-8},$$

then $q^{8/3}Z_{V_H} = P^8 - 1984qP^5 + 705057q^2P^2$. Also define the auxiliary formal series

$$A(q) = (q^{8/3}Z_{V_H})\prod_{n\geq 1}(1-q^n) - (1-q).$$

Exact expansion gives nonnegative integral coefficients of both $q^{8/3}Z_{V_H}$ and $A(q)$ through degree 500. This finite calculation is not used in the existence proof. Apart from the weight-three calculation above, no interpretation of the coefficients of $A(q)$ as highest-weight multiplicities is needed or asserted here.

## References


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# A Standalone exhaustive verifier

The following source is self-contained and uses no nonstandard libraries. Save it as `bw64_matrix_certificate.cpp`, then run

```
c++ -O3 -std=c++17 bw64_matrix_certificate.cpp -o c64_check
./c64_check
```

An additional diagnostic build is

```
c++ -O1 -g -std=c++17 -fsanitize=undefined \
    -fno-sanitize-recover=undefined \
    bw64_matrix_certificate.cpp -o c64_check_ubsan
./c64_check_ubsan
```

The coefficient arrays, bit convention, and norm are exactly those in Section 6. The polynomial formula checks on supports of size at most three (respectively two) are exhaustive coefficient checks for squarefree cubics (respectively quadratics). The subsequent open-cell loop ranges over all $2^{27}$ vectors.

```
// Standalone exact verifier for the c=64 finite certificate.
// Build: c++ -O3 -std=c++17 bw64_matrix_certificate.cpp -o bw64_matrix_certificate
// Only N(X,Y,Z)=det(X)+det(Y)+det(Z)+tr(XYZ) over F_2 is used.
// Bits 0..8, 9..17, 18..26 are the row-major entries of X,Y,Z.
// No octonions, change of coordinates, or dual-adjoint identities are used.
#include <algorithm>
#include <array>
#include <chrono>
#include <cstdint>
#include <cstdlib>
#include <iostream>
#include <vector>

using U = std::uint32_t;
constexpr int n = 27;
constexpr U limit = U(1) << n;
constexpr U ell = 1701044;
constexpr std::array<U,n> M = {
    8745196,110867808,95730417,99134097,22725868,41716247,128519191,
    37605405,89711618,58241068,12720466,113755244,38528647,78467855,
    11123089,36920254,91945155,99458066,25831589,111637498,6390710,
    131850988,3350328,53374031,44499804,27826914,2828622
};
std::array<std::uint16_t,512> cof;
std::array<unsigned char,512> det;
std::uint16_t tp[512][512]; // transposed ordinary matrix product
U mt[3][512];
std::vector<std::array<int,3>> terms;

void require(bool ok, const char* message) {
    if (!ok) { std::cerr << "FAIL: " << message << '\n'; std::exit(1); }
}
int bit(U a, int i) { return (a >> i) & 1; }
int parity(U a) { return __builtin_parity(a); }
U linear(const std::array<U,n>& cols, U a) {
    U b=0;
    while (a) { int i=__builtin_ctz(a); b^=cols[i]; a&=a-1; }
    return b;
}
U gradient(U a) {
    U x=a&511, y=(a>>9)&511, z=a>>18;
    return U(cof[x]^tp[y][z]) | (U(cof[y]^tp[z][x])<<9)
         | (U(cof[z]^tp[x][y])<<18);
}
int norm(U a) {
    U x=a&511, y=(a>>9)&511, z=a>>18;
    return det[x]^det[y]^det[z]^parity(tp[x][y]&z);
}
U gradient_by_terms(U a) {
    U b=0;
    for (auto t:terms) for (int j=0;j<3;++j)
```

```
52          if (bit(a,t[(j+1)%3]) && bit(a,t[(j+2)%3])) b^=U(1)<<t[j];
53      return b;
54  }
55  int norm_by_terms(U a) {
56      int b=0;
57      for (auto t:terms) b^=bit(a,t[0])&bit(a,t[1])&bit(a,t[2]);
58      return b;
59  }
60  void initialize() {
61      require(ell<limit,"ell outside 27 bits");
62      for (int i=0;i<n;++i) {
63          require(M[i]<limit && !bit(M[i],i),"M not alternating");
64          for (int j=0;j<n;++j) require(bit(M[i],j)==bit(M[j],i),"M not symmetric");
65      }
66      for (int base:{0,9,18}) {
67          std::array<int,3> p={0,1,2};
68          do { terms.push_back({base+p[0],base+3+p[1],base+6+p[2]}); }
69          while (std::next_permutation(p.begin(),p.end()));
70      }
71      for (int i=0;i<3;++i) for (int j=0;j<3;++j) for (int k=0;k<3;++k)
72          terms.push_back({3*i+j,9+3*j+k,18+3*k+i});
73      require(terms.size()==45,"wrong cubic monomial count");
74      for (U x=0;x<512;++x) {
75          std::array<int,3> p={0,1,2};
76          do { det[x]^=bit(x,p[0])&bit(x,3+p[1])&bit(x,6+p[2]); }
77          while (std::next_permutation(p.begin(),p.end()));
78          for (int i=0;i<3;++i) for (int j=0;j<3;++j) {
79              int r[2],c[2],nr=0,nc=0;
80              for (int k=0;k<3;++k) { if(k!=i)r[nr++]=k; if(k!=j)c[nc++]=k; }
81              int v=(bit(x,3*r[0]+c[0])&bit(x,3*r[1]+c[1]))
82                    ^(bit(x,3*r[0]+c[1])&bit(x,3*r[1]+c[0]));
83              cof[x]|=U(v)<<(3*i+j);
84          }
85          for (U y=0;y<512;++y) for (int i=0;i<3;++i) for (int j=0;j<3;++j) {
86              int v=0;
87              for (int k=0;k<3;++k) v^=bit(x,3*i+k)&bit(y,3*k+j);
88              tp[x][y]|=U(v)<<(3*j+i);
89          }
90          for (int b=0;b<3;++b) mt[b][x]=linear(M,x<<(9*b));
91      }
92      // The two norm formulas are squarefree cubics, and the two gradient
93      // formulas are squarefree quadratics. Values on supports of size <=d
94      // determine every squarefree Boolean polynomial of degree <=d exactly.
95      unsigned nc=0,gc=0;
96      auto check=[&](U a,int weight) {
97          require(norm(a)==norm_by_terms(a),"cubic formula mismatch"); ++nc;
98          if(weight<=2) { require(gradient(a)==gradient_by_terms(a),"gradient mismatch"); ++gc; }
99      };
100     check(0,0);
101     for(int i=0;i<n;++i) {
102         check(U(1)<<i,1);
103         for(int j=i+1;j<n;++j) {
104             check((U(1)<<i)|(U(1)<<j),2);
105             for(int k=j+1;k<n;++k) check((U(1)<<i)|(U(1)<<j)|(U(1)<<k),3);
106         }
107     }
108     require(nc==3304 && gc==379,"incomplete polynomial checks");
109     std::cout<<"Exact formula checks: 3304 cubic, 379 gradient; M alternating.\n";
110 }
111
112 int main() {
113     const auto start=std::chrono::steady_clock::now();
114     initialize();
115     std::vector<U> rankone;
116     std::uint64_t tested=0,bigbad=0;
117     for(U z=0;z<512;++z) for(U y=0;y<512;++y) {
118         U yz=(y<<9)|(z<<18), myz=mt[1][y]^mt[2][z];
119         for(U x=0;x<512;++x) {
120             U a=x|yz, ga=gradient(a); ++tested;
121             bigbad+=(ga==(ell^mt[0][x]^myz));
122             if(a && !ga) rankone.push_back(a);
123         }
```

```
    }
    require(tested==limit,"incomplete big-cell enumeration");
    require(bigbad==0,"big-cell collision");
    require(rankone.size()==139503,"wrong rank-one count");
    std::cout<<"Big cell: "<<tested<<" vectors, "<<bigbad<<" collisions; rank one: "
             <<rankone.size()<<'.'<<std::endl;

    std::uint64_t kernelchecks=0,solvable=0,outside=0,boundarybad=0;
    for(U u:rankone) {
        require(!norm(u),"rank-one norm nonzero");
        std::array<U,n> h{},piv{},pre{};
        std::vector<U> kernel;
        int rank=0;
        for(int i=0;i<n;++i) h[i]=gradient(u^(U(1)<<i)); // grad(u)=grad(e_i)=0
        for(int i=0;i<n;++i) {
            require(!bit(h[i],i),"Hessian diagonal nonzero");
            for(int j=0;j<n;++j) require(bit(h[i],j)==bit(h[j],i),"Hessian not symmetric");
            U v=h[i],a=U(1)<<i;
            while(v) {
                int k=31-__builtin_clz(v);
                if(piv[k]) { v^=piv[k]; a^=pre[k]; }
                else { piv[k]=v; pre[k]=a; ++rank; break; }
            }
            if(!v) {
                require((a>>i)==1,"kernel generators not triangular independent");
                kernel.push_back(a);
            }
        }
        require(rank==10 && kernel.size()==17,"rank-one Hessian rank not 10");
        // beta_u(a)=<grad(a),u> is quadratic with polar form H_u.
        // For k in ker(H_u), beta_u(a+k)=beta_u(a)+beta_u(k).
        // Its restriction to ker(H_u) is linear. Checking beta on this
        // complete kernel basis therefore proves constancy on EVERY fiber.
        for(U k:kernel) {
            require(k && !linear(h,k),"invalid kernel vector");
            require(!parity(gradient(k)&u),"beta not constant on a Hessian fiber");
            ++kernelchecks;
        }
        U rhs=linear(M,u),v=rhs,a=0;
        while(v) {
            int k=31-__builtin_clz(v);
            if(!piv[k]) break;
            v^=piv[k]; a^=pre[k];
        }
        if(v) { ++outside; continue; }
        ++solvable;
        require(linear(h,a)==rhs,"incorrect particular solution");
        boundarybad+=(parity(gradient(a)&u)==parity(ell&u));
        std::cout<<"Solvable boundary u="<<u<<", witness a="<<a
                 <<", beta="<<parity(gradient(a)&u)<<", ell(u)="<<parity(ell&u)<<'.'<<'\n';
    }
    require(kernelchecks==std::uint64_t(139503)*17,"incomplete kernel checks");
    require(solvable+outside==139503,"incomplete boundary enumeration");
    require(boundarybad==0,"boundary collision");
    std::cout<<"Boundary: "<<outside<<" outside Hessian image, "<<solvable
             <<" solvable with opposite beta, "<<boundarybad<<" collisions.\n"
             <<"Every rank-one Hessian has rank 10; "<<kernelchecks
             <<" kernel-basis beta checks passed.\n";
    // K={(alpha,a;ell(a),alpha*ell+M*a)} has dimension 28 by its first
    // coordinates and Q|K=0 because M is alternating. It contains no nonzero
    // vector in the label-map kernel. Equality grad(a)=ell+M*a automatically
    // implies ell(a)=N(a), by cubic Euler and a.M.a=0. For rank-one u,
    // the entire boundary is (0,u;beta_u(a),H_u*a), exactly the test above.
    std::cout<<"PASS: the 28-dimensional totally singular graph avoids all three bad sets.\n"
             <<"Elapsed seconds: "<<std::chrono::duration<double>(std::chrono::steady_clock::now()-
 start).count()
             <<std::endl;
}
```