The equilateral triangle maximises the first Steklov eigenvalue among triangles

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Abstract

Let $\sigma_1(T)$ be the first nonzero Steklov eigenvalue of a triangle T and $L(T)$ its perimeter. Among triangles the equilateral triangle E uniquely maximises $\sigma_1 L$. If the shortest and longest sides of T have lengths $a \le b$, then $\sigma_1(E)L(E) - \sigma_1(T)L(T) \ge (b-a)/(2b)$. Since $|T|/L(T)^2$ is also maximal at E, the equilateral triangle uniquely maximises $\sigma_1\sqrt{|T|}$ as well. This settles the case $n = 3$ of the Steklov analogue of the Pólya–Szegő conjecture for n-gons, for both normalisations. The extremal value satisfies $3.8724615957722 \le \sigma_1(E)L(E) \le 3.8724615957734$. For side length one, $\sigma_1(E) = \sigma_2(E) < \sigma_3(E) = 2\sqrt{3} < \sigma_4(E)$. Under displacement of the apex over a fixed unit base, the first two perimeter-normalised eigenvalues split linearly with equal and opposite slopes independent of direction.

The proof is computer-assisted. Near E we transplant the first eigenspace of E affinely; away from E the trial function $\cos(\pi x)$ suffices. We verify the finite matrix and interval inequalities in ball arithmetic.

AI transparency statement. The main result, its computer-assisted proof, the partial Lean formalisation and the manuscript were produced by artificial intelligence. The human author initiated the broader research and directed the AI systems, but did not write the text and has not checked the arguments. Details are given at the end of the paper.

1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, the Steklov problem $\Delta u = 0$ in Ω, $\partial_\nu u = \sigma u$ on $\partial\Omega$ has eigenvalues $0 = \sigma_0 < \sigma_1(\Omega) \le \sigma_2(\Omega) \le \cdots$. They satisfy

$$\sigma_1(\Omega) = \min_{u \in H^1(\Omega)\setminus\mathbb{R}} \mathcal{R}_\Omega(u), \qquad \mathcal{R}_\Omega(u) = \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2 - \frac{1}{|\partial\Omega|}\left(\int_{\partial\Omega} u\right)^2}. \tag{1}$$

The scale-invariant quantity is

$$\Sigma(\Omega) := \sigma_1(\Omega)\, L(\Omega), \qquad L(\Omega) = |\partial\Omega|.$$

Weinstock's inequality [16] states that $\Sigma(\Omega) \le 2\pi$ for simply connected planar domains, with equality only for the disk; see the survey [10]. For polygons with a prescribed number of sides, the natural conjecture is the analogue of Pólya–Szegő's conjecture for the Dirichlet Laplacian:

Problem (Oudet; AIM problem list *Steklov eigenproblems*, Problem 4.6 [1]). *Do regular n-gons maximise σ_1 among n-gons of given area, or of given perimeter?*

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Dominguez, Nigam and Shahriari [7] gave numerical evidence for triangles using finite elements and Bayesian optimisation. Nigam also posed the perimeter question as Open Problem 8 at the 2020 miniconference on sharp eigenvalue estimates [13]. Among rectangles of given perimeter or area, the square maximises σ_1: this is the planar case of the cuboid inequality of Girouard, Lagacé, Polterovich and Savo [9, Thm. 1.6]. Cheng, Gui, Hu, Li and Yao [4] state the conjecture for convex polygons with at most N sides at fixed perimeter (their Conjecture 1.1), and leave it open for every $N \geq 3$. They prove that σ_1 of the perimeter-normalised regular N-gon increases strictly with N, and that its first eigenspace is two-dimensional. Dryden, Gordon, Moreno, Rowlett and Villegas-Blas [8] prove $\sigma_k(T)L(T) \leq \frac{8\sqrt{3}}{3}\pi^2 k^2 \vartheta(T)$ for triangles, where $\vartheta(T)$ is the smallest angle. This shows that degenerate triangles are far from optimal, but it does not compare triangles with E.

For the Neumann problem, Laugesen and Siudeja [11] proved that the equilateral triangle maximises the first nonzero Neumann eigenvalue times L^2 among triangles, and hence times the area. Near the equilateral triangle their trial functions are the eigenfunctions of the equilateral triangle, transplanted by the linear map onto the given triangle. This affine transplantation is the precedent for our local argument. The Steklov problem adds two difficulties: the first eigenvalue of E is double, so the maximum is a cone and the comparison degenerates at E; and a rigorous lower bound for $\sigma_1(E)$ is needed. We are not aware of a proof of the Steklov case for any $n \geq 3$.

Theorem 1.1. *Let E be an equilateral triangle. For every triangle T,*

$$\Sigma(T) \leq \Sigma(E),$$

with equality if and only if T is equilateral. More precisely, normalise a longest side to $[0,1]$ and write the apex as (p,q) with $p \leq 1/2$ and $q > 0$. If $\rho = |(p,q) - (1/2, \sqrt{3}/2)|$, then

$$\Sigma(E) - \Sigma(T) \;\geq\; \frac{\rho}{2} \;\geq\; \frac{\ell_{\max} - \ell_{\min}}{2\ell_{\max}},$$

where $\ell_{\max}$ and $\ell_{\min}$ are the extreme side lengths of T. Moreover

$$3.8724615957722 \leq \Sigma(E) \leq 3.8724615957734.$$

Girouard and Polterovich computed the first sixty normalised eigenvalues of the equilateral triangle numerically [10, Example 3.2.1]. For side length one we obtain $\sigma_1(E) = \sigma_2(E) < \sigma_3(E) = 2\sqrt{3} < \sigma_4(E)$ (Proposition 3.4). The double eigenvalue agrees with [4, Thm. 1.2] for $N = 3$.

Corollary 1.2. *For every triangle T,*

$$\sigma_1(T)\sqrt{|T|} \leq \sigma_1(E)\sqrt{|E|} \in [0.8494076787253,\ 0.8494076787257],$$

with equality if and only if T is equilateral.

Proof. $\sigma_1\sqrt{|T|} = \Sigma(T)\,(|T|/L^2)^{1/2}$. By Heron's formula and the AM–GM inequality, $|T|/L^2 \leq \sqrt{3}/36$, with equality only for equilateral triangles. Now apply Theorem 1.1. □

Method

Upper bounds for $\sigma_1(T)$ come directly from (1); a lower bound is needed only for the single domain E. We parametrise triangles by the apex (p,q) over the base $[0,1]$ (Lemma 2.3), and let ρ be the distance of (p,q) from the equilateral apex.

1. *Near E* ($\rho \le \rho_1 \approx 0.6$). The trial space is the exact first eigenspace of E, transplanted by the affine map $E \to T$. The comparison involves only a few moments of the eigenspace and two exact identities. These identities let us divide out the factor ρ (Lemma 4.2), which turns the degenerate comparison at $\rho = 0$ into a sign condition. We check the sign condition on boxes in ball arithmetic. The moments are enclosed using a harmonic-polynomial approximation, a residual bound for the Dirichlet-to-Neumann map, and spectral isolation of a cubic eigenfunction by a Crouzeix–Raviart bound for $\sigma_4(E)$.

2. *Away from E* ($\rho \ge \rho_1$). The single trial function $\cos(\pi x)$ gives $\Sigma(T) \le 3.114382798607$, well below $\Sigma(E) > 3.8724$. A one-dimensional check on 16 intervals suffices.

2 Preliminaries

2.1 The Dirichlet-to-Neumann map

Let Ω be a bounded Lipschitz domain. The Dirichlet-to-Neumann map Λ is the self-adjoint operator on $L^2(\partial\Omega)$ associated with the closed form $f \mapsto \int_\Omega |\nabla \mathcal{H} f|^2$ on $H^{1/2}(\partial\Omega)$, where $\mathcal{H}f$ is the harmonic extension. It has compact resolvent and eigenvalues σ_k, with an orthonormal basis of eigenvectors χ_k of $L^2(\partial\Omega)$; χ_0 is constant. If u is a harmonic polynomial with trace f, Green's formula gives $f \in \operatorname{dom}\Lambda$ and $\Lambda f = \partial_\nu u$. For $\tilde\sigma \in \mathbb{R}$ and $\eta > 0$ write $J = [\tilde\sigma - \eta, \tilde\sigma + \eta]$.

Lemma 2.1 (Inclusion)**.** *Let $W \subset \operatorname{dom}\Lambda$ have dimension k, and suppose $\|(\Lambda - \tilde\sigma)f\| \le \eta\|f\|$ for all $f \in W$. Then Λ has at least k eigenvalues, counted with multiplicity, in J.*

Proof. Suppose not. Then some $f \in W \setminus 0$ is orthogonal to the spectral subspace of J, and $\|(\Lambda - \tilde\sigma)f\|^2 = \sum_{|\sigma_j - \tilde\sigma| > \eta} |\sigma_j - \tilde\sigma|^2 |\langle f, \chi_j\rangle|^2 > \eta^2\|f\|^2$, a contradiction. □

Lemma 2.2 (Eigenvector enclosure; cf. [6])**.** *Let P be the spectral projection of Λ onto J. Suppose every eigenvalue of Λ outside J is either 0 or at least $\sigma^{\mathrm{lo}} > \tilde\sigma \ge 0$, and put $\gamma = \sigma^{\mathrm{lo}} - \tilde\sigma$. Let $f \in \operatorname{dom}\Lambda$ be orthogonal to the constants, and $r = (\Lambda - \tilde\sigma)f$. Then*

$$\|(I - P)f\|_{L^2(\partial\Omega)} \le \frac{\|r\|}{\gamma}, \qquad \int_\Omega \left|\nabla \mathcal{H}(I - P)f\right|^2 \le \frac{\sigma^{\mathrm{lo}}}{\gamma^2}\|r\|^2.$$

Proof. $(\sigma_j - \tilde\sigma)\langle f, \chi_j\rangle = \langle r, \chi_j\rangle$ for every j, and the component along χ_0 vanishes. The remaining j with $\sigma_j \notin J$ have $\sigma_j \ge \sigma^{\mathrm{lo}}$, so $\|(I - P)f\|^2 \le \|r\|^2/\gamma^2$, and $\int |\nabla\mathcal{H}(I - P)f|^2 = \sum \sigma_j |\langle r, \chi_j\rangle|^2/(\sigma_j - \tilde\sigma)^2 \le \sigma^{\mathrm{lo}}\|r\|^2/\gamma^2$, because $\sigma/(\sigma - \tilde\sigma)^2$ is decreasing for $\sigma > \tilde\sigma$ when $\tilde\sigma \ge 0$ (its derivative is $-(\sigma + \tilde\sigma)/(\sigma - \tilde\sigma)^3$). □

2.2 Parametrisation

For (p, q) with $q > 0$, write

$$T(p, q) = \operatorname{conv}\{(0, 0), (1, 0), (p, q)\},$$

and set

$$F = \{(p, q) : 0 < p \le \tfrac{1}{2},\ q > 0,\ (1 - p)^2 + q^2 \le 1\}.$$

Since Σ is invariant under similarities, it suffices to treat the triangles $T(p, q)$, $(p, q) \in F$.

Lemma 2.3. *Every triangle is similar to $T(p, q)$ for some $(p, q) \in F$, and $T(p, q)$ is equilateral if and only if $(p, q) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.*

Proof. Rescale so that a longest side has length 1. Move it to $[0,1]\times\{0\}$ with the third vertex (p,q) in the upper half-plane, and reflect in $x=\frac12$ if necessary so that $p\le\frac12$. The side from $(1,0)$ to the apex is not longer than the base, so $(1-p)^2+q^2\le 1$; this forces $p>0$. The other side then satisfies $p^2+q^2\le(1-p)^2+q^2\le 1$. The equilateral case is clear. □

We write $p=\frac12+\rho\cos\psi$, $q=\frac{\sqrt3}{2}+\rho\sin\psi$ for polar coordinates about the equilateral apex; ρ is the Euclidean distance in the (p,q)-plane.

2.3 Affine transplantation

Let $A\in GL_2^+(\mathbb{R})$, $u\in H^1(\Omega)$ and $u_A=u\circ A^{-1}\in H^1(A\Omega)$. Write $G=A^{-1}A^{-T}$. For an edge e of a polygon Ω with unit tangent τ_e, put $s_e=|A\tau_e|$. Changing variables gives

$$\begin{aligned}\int_{A\Omega}|\nabla u_A|^2&=\det A\sum_{i,j}G_{ij}\int_\Omega \partial_i u\,\partial_j u,\\ \int_{Ae}u_A^2=s_e\int_e u^2,\qquad &\int_{Ae}u_A=s_e\int_e u,\end{aligned}\tag{2}$$

because the arc length on Ae is s_e times that on e. For a finite-dimensional space $\mathcal{V}$ of trial functions, define the forms $\mathcal{N}_A(u,u')=\int\nabla u_A\cdot\nabla u'_A$ and $\mathcal{D}_A(u,u')=\sum_e s_e\int_e uu'-\frac{1}{L(A\Omega)}(\sum_e s_e\int_e u)(\sum_e s_e\int_e u')$ on $\mathcal{V}$. If $\mathcal{D}_A$ is positive definite on $\mathcal{V}$, then (1) gives $\sigma_1(A\Omega)\le\min_{\mathcal{V}\setminus0}\mathcal{N}_A/\mathcal{D}_A$.

3 The equilateral triangle

Let $E=T(\frac12,\frac{\sqrt3}{2})$, with side 1 and $L(E)=3$. Let z_0 be its centroid and $w=-i\sqrt3\,(z-z_0)$, so that the circumradius is 1 in w and the apex lies on the positive real w-axis. The dihedral group D_3 acts by $w\mapsto e^{2\pi i/3}w$ and $w\mapsto\bar w$.

3.1 An exactly equivariant trial pair

Given real coefficients c_k ($k\le20$, $k\not\equiv0\bmod3$, fourteen in all), put $\mathfrak{f}=\sum_{k\equiv1}c_kw^k+\sum_{k\equiv2}c_k\bar w^k$, $u_1=\operatorname{Re}\mathfrak{f}$ and $u_2=\operatorname{Im}\mathfrak{f}$. Rotation by $2\pi/3$ multiplies $\mathfrak{f}$ by $e^{2\pi i/3}$, and reflection conjugates it. Hence (u_1,u_2) transforms under D_3 exactly as the coordinates $(\operatorname{Re}w,\operatorname{Im}w)$ do, whatever the c_k are. In particular $\int_{\partial E}u_a=0$; each u_a is the real part of a holomorphic polynomial $\mathfrak{f}_a$ (since $\operatorname{Re}\bar w^k=\operatorname{Re}w^k$ and $\operatorname{Im}\bar w^k=-\operatorname{Im}w^k$), and by Schur's lemma all their Gram matrices are scalar. The c_k come from a floating-point Ritz computation and are then fixed as exact binary numbers.

Let $\tilde\sigma\approx1.290820531924$ be the Rayleigh quotient of u_1, rounded to binary64. All boundary integrals of products of u_a and $\partial_\nu u_a$ are integrals of polynomials along segments, and we compute them exactly in ball arithmetic. This gives the relative residual

$$\eta:=\max_{f\in\operatorname{span}(u_1,u_2)\setminus0}\frac{\|\partial_\nu f-\tilde\sigma f\|_{L^2(\partial E)}}{\|f\|_{L^2(\partial E)}}\le1.9559\times10^{-13}.$$

By Lemma 2.1, Λ_E has at least two eigenvalues in $J=[\tilde\sigma-\eta,\tilde\sigma+\eta]$.

3.2 A Crouzeix–Raviart lower bound for $\sigma_4(E)$

Subdivide E uniformly into n_h^2 equilateral cells K of side $h = 1/n_h$. Let $\mathcal{C}_h$ be the Crouzeix–Raviart space [5]: piecewise linear functions that are continuous at the midpoints x_e of interior edges. Let Π_h be the interpolation by edge means, $(\Pi_h u)(x_e) = |e|^{-1}\int_e u$. Write $\mathfrak{a}_h(u,y) = \sum_K \int_K \nabla u \cdot \nabla y$ and $\mathfrak{b}(u,y) = \int_{\partial E} uy$, and let $\sigma_{h,0} \le \sigma_{h,1} \le \cdots$ be the finite eigenvalues of the pencil $(\mathfrak{a}_h, \mathfrak{b})$ on $\mathcal{C}_h$. The min–max principle holds over subspaces on which $\mathfrak{b}$ is positive definite.

Lemma 3.1 (Trace constant)**.** *Let K be an equilateral triangle of side h, e an edge of K, and $\varphi \in H^1(K)$ with $\int_e \varphi = 0$. Then*

$$\|\varphi\|_{L^2(e)}^2 \le \frac{4}{\sqrt{3}}\Big(\frac{1}{\pi^2} + \frac{1}{\pi}\Big) h\, \|\nabla\varphi\|_{L^2(K)}^2.$$

Proof. Let $\hat\varphi = \varphi - \bar\varphi_K$. Since φ has mean zero on e, $\|\varphi\|_e \le \|\varphi - \zeta\|_e$ for every constant ζ; in particular $\|\varphi\|_e \le \|\hat\varphi\|_e$. Let O_e be the vertex opposite e. The field $(x - O_e)\hat\varphi^2$ has normal component 0 on the two edges through O_e, and $2|K|/|e|$ times $\hat\varphi^2$ on e. Its divergence is $2\hat\varphi^2 + 2\hat\varphi\nabla\hat\varphi\cdot(x-O_e)$. The divergence theorem therefore gives

$$\|\hat\varphi\|_e^2 = \frac{|e|}{|K|}\Big(\|\hat\varphi\|_K^2 + \int_K \hat\varphi\,\nabla\hat\varphi\cdot(x-O_e)\Big) \le \frac{|e|}{|K|}\big(\|\hat\varphi\|_K^2 + h\|\hat\varphi\|_K\|\nabla\varphi\|_K\big).$$

By the Payne–Weinberger inequality [15, 3] on the convex set K of diameter h, $\|\hat\varphi\|_K \le \frac{h}{\pi}\|\nabla\varphi\|_K$. Finally $|e|/|K| = 4/(\sqrt{3}h)$. □

Lemma 3.2 (Projection bound; cf. [14, 17])**.** *Let $n_h \ge 2$ and $C_h^2 = 2\cdot\frac{4}{\sqrt{3}}(\frac{1}{\pi^2}+\frac{1}{\pi})h$. Then for each k below the number of boundary edges,*

$$\sigma_k(E) \ge \frac{\sigma_{h,k}}{1 + C_h^2\sigma_{h,k}}.$$

Proof. (i) For every $y_h \in \mathcal{C}_h$, $\mathfrak{a}_h(u - \Pi_h u, y_h) = 0$. Indeed ∇y_h is constant on each cell K, and $\int_K \nabla(u - \Pi_h u) = \int_{\partial K}(u - \Pi_h u)\nu = 0$ because $\Pi_h u|_K$ is linear and has the same edge means as u.

(ii) Write $\varphi = u - \Pi_h u$. It has mean zero on every edge of every cell. Each boundary edge lies in one cell, and for $n_h \ge 2$ a cell has at most two boundary edges (exactly two at the corners). Lemma 3.1 then gives $\mathfrak{b}(\varphi,\varphi) \le C_h^2\,\mathfrak{a}_h(\varphi,\varphi)$.

(iii) Let $\mathcal{Z}_k$ be the span of the eigenfunctions of $\sigma_0,\dots,\sigma_k$, and take $u \in \mathcal{Z}_k$ with $\mathfrak{b}(u,u) = 1$. Put $\upsilon = \mathfrak{a}_h(u - \Pi_h u, u - \Pi_h u)$. By (i), $\mathfrak{a}_h(\Pi_h u, \Pi_h u) = \int|\nabla u|^2 - \upsilon \le \sigma_k - \upsilon$, and by (ii), $\mathfrak{b}(\Pi_h u, \Pi_h u)^{1/2} \ge 1 - C_h\sqrt{\upsilon}$. If $C_h^2\sigma_k \ge 1$ the claim is trivial, since $x/(1+C_h^2 x) < C_h^{-2}$. Otherwise Π_h is injective on $\mathcal{Z}_k$, and

$$\frac{\mathfrak{a}_h(\Pi_h u,\Pi_h u)}{\mathfrak{b}(\Pi_h u,\Pi_h u)} \le \max_{0\le\upsilon\le\sigma_k} \frac{\sigma_k - \upsilon}{(1 - C_h\sqrt{\upsilon})^2} = \frac{\sigma_k}{1 - C_h^2\sigma_k}.$$

The maximum is attained at $\sqrt{\upsilon} = C_h\sigma_k$. The min–max principle on the $(k+1)$-dimensional space $\Pi_h\mathcal{Z}_k$ gives $\sigma_{h,k} \le \sigma_k/(1 - C_h^2\sigma_k)$, which rearranges to the claim. □

Proposition 3.3. $\sigma_4(E) > 3.5617858371486 > 2\sqrt{3}$.

Proof. Take $n_h = 24$, which gives 900 degrees of freedom and $C_h^2 \approx 0.080758$. Let $\mathbf{S}_h$ and $\mathbf{B}_h$ be the matrices of $\mathfrak{a}_h$ and $\mathfrak{b}$ in the edge basis of $\mathcal{C}_h$. On an equilateral cell the element stiffness matrix is

$$\frac{2}{\sqrt{3}}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

On a boundary edge, the basis function of that edge is $\equiv 1$ and the other two are $\pm$ the linear function rising from -1 to 1 along the edge, so the element boundary-mass entries are h, $h/3$ and $-h/3$. Let $\mathbf{O}$ be a fixed matrix with binary64 entries (computed once as approximate eigenvectors of $\mathbf{S}_h - 5\mathbf{B}_h$). In ball arithmetic we check two things:

- the Gershgorin discs of $\mathbf{O}^T\mathbf{O}$ exclude 0, so $\mathbf{O}$ is nonsingular;
- the Gershgorin discs of $\mathbf{O}^T(\mathbf{S}_h - 5\mathbf{B}_h)\mathbf{O}$ consist of 4 discs in $(-\infty, 0)$ and 896 discs in $(0, \infty)$.

Gershgorin's counting theorem and Sylvester's law of inertia give exactly four negative eigenvalues of $\mathbf{S}_h - 5\mathbf{B}_h$. If $\sigma_{h,4} < 5$, the pencil would be negative definite at this shift on the span of five discrete eigenfunctions, including the constant. Thus $\sigma_{h,4} \geq 5$, and Lemma 3.2 gives $\sigma_4(E) \geq 5/(1 + 5C_h^2) > 3.5617858371486$. □

Proposition 3.4. *For the side-one equilateral triangle,*

$$\sigma_1(E) = \sigma_2(E) =: \sigma^* \in J, \qquad \sigma_3(E) = 2\sqrt{3} < \sigma_4(E).$$

In particular the third positive eigenvalue is simple and

$$\Sigma(E) = 3\sigma^* \in [3.8724615957722,\ 3.8724615957734].$$

Proof. Use Cartesian coordinates centred at the centroid and put $r = 1/(2\sqrt{3})$. The harmonic polynomial $u = \mathrm{Im}((x+iy)^3) + 2r^3$ satisfies $\partial_\nu u = -3x^2 + 3r^2 = u/r$ on $y = -r$. Rotation by $2\pi/3$ preserves u, so the same boundary condition holds on every edge. Thus $1/r$ is an eigenvalue. The inclusion interval J lies strictly between 0 and $1/r$ and contains at least two eigenvalues counted with multiplicity. Together with 0 and $1/r$ these account, with multiplicity, for all eigenvalues below the lower bound in Proposition 3.3. Hence J contains precisely σ_1, σ_2, and $\sigma_3 = 1/r$ is simple. Take $\sigma^{\mathrm{lo}} = 2\sqrt{3}$ in Lemma 2.2. Let P be the spectral projection onto J. It commutes with the D_3-action, so (Pu_1, Pu_2) is again an equivariant pair. Since $\tilde{\sigma} > 0$, Lemma 2.2 gives $\|(I-P)u_a\| \leq \eta\|u_a\|/(\sigma^{\mathrm{lo}} - \tilde{\sigma}) < 10^{-12}\|u_a\|$; since $u_1 \perp u_2$ with equal norms, Pu_1, Pu_2 are linearly independent. They therefore span the two-dimensional range of P, which carries the irreducible representation of D_3. By Schur's lemma Λ acts on this range as a scalar. □

4 Local maximality

Let $\mathcal{U}$ be the eigenspace of σ^*, with basis $u_a^* = Pu_a$ $(a = 1, 2)$. For $T = T(p, q)$, $p = \frac{1}{2} + \rho\cos\psi$, $q = \frac{\sqrt{3}}{2} + \rho\sin\psi$, we have $T = A(E)$ with

$$A = I + \rho\begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix}, \quad \alpha = \tfrac{2\cos\psi}{\sqrt{3}},\ \beta = \tfrac{2\sin\psi}{\sqrt{3}}, \quad d := \det A = 1 + \rho\beta.$$

Number the edges $e_0 = [(0,0),(1,0)]$, $e_1 = [(1,0), \mathrm{apex}]$ and $e_2 = [\mathrm{apex}, (0,0)]$. Then $s_0 = 1$, while

$$s_1 = \left|(\rho\cos\psi - \tfrac{1}{2},\ \tfrac{\sqrt{3}}{2} + \rho\sin\psi)\right|, \qquad s_2 = \left|(\rho\cos\psi + \tfrac{1}{2},\ \tfrac{\sqrt{3}}{2} + \rho\sin\psi)\right|$$

are the side lengths of T, and $L_T = 1 + s_1 + s_2$.

The *moments* of $\mathcal{U}$ are the following 2×2 matrices and vectors, indexed by a, b: the energy matrices $Q_{ij} = (\int_E \partial_i u_a^* \partial_j u_b^*)$, the edge Gram matrices $g^e = (\int_e u_a^* u_b^*)$, and the edge-mean vectors $\mu^e = (\int_e u_a^*)$. They satisfy two exact identities:

$$\textstyle\sum_e \mu^e = 0, \qquad Q_{11} + Q_{22} = \sigma^* \sum_e g^e. \tag{3}$$

The first holds because eigenfunctions of Λ with $\sigma^* \neq 0$ are orthogonal to constants. The second is the weak eigen-equation on $\mathcal{U}$.

Lemma 4.1 (Variational comparison)**.** *Let a trial space on a polygon T have energy matrix N and positive definite boundary-variance matrix D. For a reference value S put $X = (S/L(T))D - N$. Then*

$$S - \Sigma(T) \geq L(T) \max_{\xi \neq 0} \frac{\xi^T X \xi}{\xi^T D \xi}.$$

In particular, if $X - tD$ has a positive direction, the deficit exceeds $L(T)t$.

Proof. Apply (1) to each trial function and subtract the resulting bound from S. □

Apply the lemma with $S = 3\sigma^*$, $N = \mathcal{N}_A$ and $D = \mathcal{D}_A$ on $\mathcal{U}$. At $\rho = 0$ the eigen-equation gives $X = 0$. The following factorisation isolates the leading term.

Lemma 4.2 ($X = \rho Y$)**.** *Let $\lambda_e = (s_e - 1)/\rho$. Then $\lambda_e = (\kappa_e + \rho)/(1 + s_e)$, where $\kappa_1 = -\cos\psi + \sqrt{3}\sin\psi$ and $\kappa_2 = \cos\psi + \sqrt{3}\sin\psi$. Assume (3). Then $X = \rho Y$, where*

$$Y = \sigma^*\Big(\sum_e \omega_e g^e - \frac{3\rho}{L_T^2} v v^T\Big) - \sum_{i,j} H_{ij} Q_{ij}, \qquad H = \begin{pmatrix} \beta + \rho\alpha^2/d & -\alpha/d \\ -\alpha/d & -\beta/d \end{pmatrix},$$

with $\omega_0 = -(\lambda_1 + \lambda_2)/L_T$, $\omega_1 = (2\lambda_1 - \lambda_2)/L_T$, $\omega_2 = (2\lambda_2 - \lambda_1)/L_T$ and $v = \lambda_1 \mu^1 + \lambda_2 \mu^2$.

Proof. First, $s_e^2 = 1 + \rho\kappa_e + \rho^2$, so $s_e - 1 = \rho(\kappa_e + \rho)/(1 + s_e)$.

Next, $A^{-1} = \left(\begin{smallmatrix} 1 & -\rho\alpha/d \\ 0 & 1/d \end{smallmatrix}\right)$, which gives $dG - I = \rho H$.

For the boundary terms, $\sum_e s_e \mu^e = \sum_e (s_e - 1)\mu^e = \rho v$ by (3). Also $(3/L_T)s_e - 1 = (3s_e - L_T)/L_T = \rho\,\omega_e$, using $s_0 = 1$.

Hence $\sigma^*(3/L_T)\mathcal{D}_A - \sigma^* \sum_e g^e = \rho\,\sigma^*(\sum \omega_e g^e - 3\rho v v^T/L_T^2)$, and $\mathcal{N}_A - (Q_{11} + Q_{22}) = \rho \sum H_{ij} Q_{ij}$. Subtracting these two equations and using (3) gives $X = \rho Y$. □

Lemma 4.3 (Moment enclosures)**.** *Let $\delta_a = u_a - u_a^*$ and $\gamma = 2\sqrt{3} - \tilde{\sigma}$. Then*

$$\|\delta_a\|_{L^2(\partial E)} \leq \varepsilon_\partial := \eta \|u_a\|_{\partial E}/\gamma, \qquad \|\nabla \delta_a\|_{L^2(E)} \leq \varepsilon_\nabla := \sqrt{2\sqrt{3}}\,\eta \|u_a\|_{\partial E}/\gamma.$$

Consequently each moment of $\mathcal{U}$ differs from the corresponding moment of (u_1, u_2) by at most the following amounts:

- $\varepsilon_\nabla(\|\nabla u_a\| + \|\nabla u_b\| + \varepsilon_\nabla)$ *for the entry (a, b) of Q_{ij};*
- $\varepsilon_\partial(\|u_a\|_{\partial E} + \|u_b\|_{\partial E} + \varepsilon_\partial)$ *for the entry (a, b) of g^e;*
- $|e|^{1/2}\varepsilon_\partial$ *for the entry a of μ^e.*

Proof. Apply Lemma 2.2 to $f = u_a$, with $\|r\| \leq \eta\|u_a\|$; its hypotheses were verified in the proof of Proposition 3.4. The moment bounds follow from Cauchy–Schwarz. □

Here $\|u_a\|_{\partial E}^2 \approx 1.577442$, so $\varepsilon_\partial \leq 1.14 \times 10^{-13}$ and $\varepsilon_\nabla \leq 2.11 \times 10^{-13}$. We compute the moments of u_a exactly by Green's formula $\int_E \mathfrak{g}\,\mathrm{d}A = \frac{1}{2i}\oint_{\partial E} \bar{z}\,\mathfrak{g}\,\mathrm{d}z$ and $\int_E f_a' \overline{f_b'}\,\mathrm{d}A = \frac{1}{2i}\oint_{\partial E} \overline{f_b} f_a'\,\mathrm{d}z$ for holomorphic $\mathfrak{g}$. This turns every area integral into exact polynomial line integrals.

Proposition 4.4. *Let ρ_1 be the binary64 number nearest 0.6. For $0 < \rho \leq \rho_1$ and every direction, $\Sigma(E) - \Sigma(T(p, q)) > \rho/2$.*

Proof. We cover $[0,\rho_1]\times[0,\psi_{\max}]$ by 907 closed boxes, where $\psi_{\max}$ is the binary64 number nearest 6.3 and exceeds 2π. The boxes lie in the rectangle, have disjoint interiors, and their exact rational areas sum to its area. They cover the rectangle since a nonempty relatively open complement would have positive area. We evaluate Y and $\mathcal{D}_A$ on every box using Lemma 4.3. For $Z = 2Y - \mathcal{D}_A/2$ the checks give

$$(\mathcal{D}_A)_{11} > 0, \qquad \det \mathcal{D}_A > 0, \qquad \operatorname{tr} Z > 0 \quad \text{or} \quad \det Z < 0.$$

Thus $Y - \mathcal{D}_A/4$ has a positive direction. Since $X = \rho Y$, Lemma 4.1 gives a deficit greater than $L_T\rho/4$. The other two sides have total length greater than the unit base, so $L_T > 2$. □

Theorem 4.5 (Splitting of the first pair)**.** *For a fixed unit base let the apex move from* $(1/2, \sqrt{3}/2)$ *by* $h = \rho(\cos\psi, \sin\psi)$*. As* $\rho \downarrow 0$*, uniformly in* ψ*,*

$$L_T\sigma_1(T) = \Sigma(E) - c\rho + O(\rho^2), \qquad L_T\sigma_2(T) = \Sigma(E) + c\rho + O(\rho^2),$$

where the exact constant satisfies $3.5332113353 < c < 3.5332113355$*.*

Proof. Work on $H = H^1(E)/\mathbb{R}$ with the Dirichlet inner product. On this fixed space let $a_h = L_T\mathcal{N}_A$ and $b_h = \mathcal{D}_A$ be the pulled-back forms of (2), now acting on all of H. Their representing operators P_h, B_h are real analytic in the Cartesian components of h: the energy coefficients are rational in A, and the edge lengths and perimeter are positive near $h = 0$. The operator P_h is uniformly positive there. The boundary variance is unchanged by adding constants and the trace is compact into $L^2(\partial E)$, so B_h is compact. Positive eigenvalues of $K_h = P_h^{-1/2} B_h P_h^{-1/2}$ are the reciprocals of the normalised positive Steklov eigenvalues, by the weak equation on the quotient (choose the representative of zero weighted boundary mean). The inverse square root is analytic near $P_0 = 3I$ by its power series.

At $h = 0$ the top eigenvalue of K_0 is isolated and double by Proposition 3.4. The resolvent on a small enclosing circle has a convergent Neumann series for small $|h|$. The Riesz projection defined by this contour is real analytic and has rank two by continuity. Project a basis at 0 and apply $P_h^{-1/2}$ to obtain an analytic frame of the corresponding pencil subspace. Multiplication by the inverse square root of its positive b_h-Gram matrix makes the frame b_h-orthonormal. In this frame the matrix $M(h)$ of a_h is symmetric and real analytic, with eigenvalues $L_T\sigma_1(T), L_T\sigma_2(T)$. Write

$$M(h) = \Sigma(E)I + F(h) + O(|h|^2),$$

where F is linear and the remainder is in matrix norm, uniformly in direction.

For a unit vector $e = (\cos\psi, \sin\psi)$ the matrix $F(e)$ represents $\partial_e(a_h - \Sigma(E)b_h)|_0$ on the unperturbed eigenspace in a b_0-orthonormal basis. Indeed, differentiating the moving frame introduces two further terms, each zero by the weak eigen-equation against every vector of H. In the original basis u_a^* the same derivative is $-3Y(0,\psi)$, since $a_h - \Sigma(E)b_h = -L_T\rho Y$ on that fixed space. Consequently

$$\det F(e) = 9\,\frac{\det Y(0,\psi)}{\det \mathcal{D}_I}.$$

To see why the slope is independent of e, identify the apex with $z = p + iq$. Cyclic relabelling of the vertices followed by a similarity replaces z by $f(z) = 1/(1-z)$. It fixes $z_* = e^{i\pi/3}$ and $f'(z_*) = e^{2\pi i/3}$. Thus the two normalised eigenvalues agree as an unordered pair at z and $f(z)$. Both the trace and the determinant of $M(h) - \Sigma(E)I$ are therefore invariant under $h \mapsto f(z_* + h) - z_*$. Comparing the terms of degree one in the trace and degree two in the determinant shows that $\operatorname{tr} F(h)$ and $\det F(h)$ are invariant under rotation by $2\pi/3$. The only invariant linear form is zero,

and every invariant real quadratic form is a multiple of $|h|^2$ (its symmetric matrix commutes with that rotation). Hence $F(h)$ has eigenvalues $\pm c|h|$, where

$$c = 3\sqrt{-\det Y(0,0)/\det \mathcal{D}_I}.$$

Ball evaluation using the enclosed moments of the projected eigenfunctions gives the stated positive enclosure for c. The min–max principle bounds the change in each ordered eigenvalue by the norm of the matrix remainder, proving both expansions. □

5 The far region and the proof of the theorem

For $0 \le p \le \frac{1}{2}$ put

$$q_{\max}(p) = \min\Big(\sqrt{2p - p^2},\ \tfrac{\sqrt{3}}{2} - \sqrt{\rho_1^2 - (p - \tfrac{1}{2})^2}\Big).$$

Lemma 5.1. (i) $F \cap \{\rho \ge \rho_1\} = \{(p,q) : 0 < p \le \frac{1}{2},\ 0 < q \le q_{\max}(p)\}$.

(ii) *For $(p,q) \in F$, with $\ell_1 = |(p,q)|$ and $\ell_2 = |(1-p,q)|$,*

$$\Sigma(T(p,q)) \le \pi^2 q\, \Xi(p)\,(1 + \ell_1 + \ell_2), \qquad \Xi(p) = \mathcal{I}(p) + \mathcal{I}(1-p),$$

where $\mathcal{I}(t) = \frac{1}{t}\int_0^t x \sin^2(\pi x)\,\mathrm{d}x = \frac{t}{4} - \frac{\sin 2\pi t}{4\pi} + \frac{\sin(\pi t)\,\operatorname{sinc}(\pi t)}{4\pi}$ and $\operatorname{sinc} x = \sin x / x$.

Proof. (i) For $(p,q) \in F$ we have $q \le \sqrt{2p - p^2} \le \frac{\sqrt{3}}{2}$, and $\rho \ge \rho_1$ means $(q - \frac{\sqrt{3}}{2})^2 \ge \rho_1^2 - (p - \frac{1}{2})^2$, where the right side is positive because $|p - \frac{1}{2}| \le \frac{1}{2} < \rho_1$. Since $q < \frac{\sqrt{3}}{2} + \sqrt{\rho_1^2 - (p - \frac{1}{2})^2}$, this is equivalent to $q \le \frac{\sqrt{3}}{2} - \sqrt{\rho_1^2 - (p - \frac{1}{2})^2}$.

(ii) Take $u = \cos(\pi x)$ in (1). The vertical section of T at abscissa x has length qx/p for $x \le p$ and $q(1-x)/(1-p)$ for $x \ge p$, so the substitution $x \mapsto 1 - x$ on $[p,1]$ gives

$$\int_T |\nabla u|^2 = \pi^2 q\Big(\frac{1}{p}\int_0^p x \sin^2 \pi x\,\mathrm{d}x + \frac{1}{1-p}\int_0^{1-p} x \sin^2 \pi x\,\mathrm{d}x\Big) = \pi^2 q\, \Xi(p).$$

The antiderivative $\frac{x^2}{4} - \frac{x \sin 2\pi x}{4\pi} - \frac{\cos 2\pi x}{8\pi^2}$ and the identity $(1 - \cos 2\pi t)/t = 2\pi \sin(\pi t)\operatorname{sinc}(\pi t)$ give the closed form of $\mathcal{I}$. The denominator of (1) is $\min_\zeta \int_{\partial T}(u - \zeta)^2$. On the side from $(0,0)$ to (p,q) arc length is $(\ell_1/p)\,\mathrm{d}x$ with $\ell_1/p \ge 1$, and similarly on the third side. So for every constant ζ,

$$\int_{\partial T}(u - \zeta)^2 \ge 2\int_0^1 (\cos \pi x - \zeta)^2\,\mathrm{d}x = 1 + 2\zeta^2 \ge 1.$$

Hence $\Sigma(T) \le L \int_T |\nabla u|^2$ with $L = 1 + \ell_1 + \ell_2$. □

Proposition 5.2. *For every $(p,q) \in F$ with $\rho \ge \rho_1$, $\Sigma(T(p,q)) \le 3.114382798607 < \Sigma(E)$.*

Proof. By Lemma 5.1(i) these are the points with $0 < p \le \frac{1}{2}$, $0 < q \le q_{\max}(p)$. The bound of Lemma 5.1(ii) is nondecreasing in q, and $\Xi \ge 0$. On an interval $p_- \le p \le p_+$, the first entry of $q_{\max}$ is increasing in p and the second is decreasing, so

$$q_{\max}(p) \le q_+ := \min\Big(\sqrt{2p_+ - p_+^2},\ \tfrac{\sqrt{3}}{2} - \sqrt{\rho_1^2 - (p_- - \tfrac{1}{2})^2}\Big).$$

Moreover $\ell_1 \le (p_+^2 + q_+^2)^{1/2}$ and $\ell_2 \le ((1-p_-)^2 + q_+^2)^{1/2}$. Hence on that interval

$$\Sigma(T) \le \pi^2 q_+ \sup_{[p_-,p_+]} \Xi\,\Big(1 + \sqrt{p_+^2 + q_+^2} + \sqrt{(1-p_-)^2 + q_+^2}\Big).$$

We evaluate the right side in ball arithmetic on the 16 intervals $[\frac{k}{32}, \frac{k+1}{32}]$, $0 \le k \le 15$, which cover $[0, \frac{1}{2}]$. Each upper endpoint is below 3.114382798607. □

Proof of Theorem 1.1

Use the normalisation of Lemma 2.3. The deficit vanishes at $\rho = 0$ and exceeds $\rho/2$ for $0 < \rho \le \rho_1$ by Proposition 4.4. For $\rho \ge \rho_1$, Proposition 5.2 gives a deficit at least $3.8724615957722 - 3.1143827986\overline{07} > 1/2$. Throughout F we have $0 < p \le 1/2$ and $0 < q \le \sqrt{3}/2$, so $\rho < 1$. Finally $|(p,q)| = \ell_{\min}/\ell_{\max}$ in these coordinates, and the reverse triangle inequality gives $1 - |(p,q)| \le \rho$. This proves both deficit bounds and the equality case. The extremal value is enclosed in Proposition 3.4. □

Corollary 5.3 (Fixed-profile thin limit)**.** *Fix $0 < p < 1$, and set $\Phi_p(x) = \min\{x/p, (1-x)/(1-p)\}$ on $[0,1]$. Define*

$$\theta(p) = \inf_{\substack{0 \ne v \in H^1(0,1) \\ \int_0^1 v = 0}} \frac{\int_0^1 \Phi_p(x)|v'(x)|^2\,\mathrm{d}x}{2\int_0^1 |v(x)|^2\,\mathrm{d}x}.$$

Then $\sigma_1(T(p,q)) = q\theta(p) + o(q)$ and $\Sigma(T(p,q)) = 2q\theta(p) + o(q)$ as $q \downarrow 0$.

Proof. In Henrot and Michetti's thin-domain lemma [12, Lemma 3.2] take the upper profile Φ_p, the lower profile zero, and thickness q. Their lower-growth hypothesis holds since $\Phi_p(x) \ge x(1-x)$. Their first one-dimensional eigenvalue is $2\theta(p)$, and their asymptotic formula multiplies it by $q/2$. Now use $L(T(p,q)) \to 2$. □

6 Implementation

All certified steps use Arb [2] (python-flint 0.9.0), at 64–200 bits.

Certificate data. Floating-point arithmetic is used only to produce candidate data, which are then fixed and treated as exact binary numbers: the coefficients c_k and the congruence matrix $\mathbf{O}$. These are committed to the repository, and the certified steps read only them. The shift $\tilde{\sigma}$ is computed from the c_k in ball arithmetic and rounded to binary64. Hence the certified results do not depend on the linear-algebra library or the processor; regenerating the data is a separate, optional step. Every number of this paper that a computation checks is kept in a single claims file. The checking script compares each computed enclosure with these claims in exact rational arithmetic, and the paper's macros are generated from the same file.

Boxes. Box endpoints are exact binary numbers. A parameter box $[x_0, x_1]$ enters as the Arb union of its two exact endpoints, which contains the whole interval. Square roots of nonnegative quantities whose balls straddle 0 are clamped to $[0, \sqrt{\cdot}]$. In the code, ρ_1 is the binary double nearest 0.6, which is $0.5999999999999999777\ldots$, slightly below 0.6. The local and the far computations use this same value, so the two regions meet exactly. The tiling of the local boxes is checked with exact float comparisons and an exact rational area sum.

Consistency checks (not part of the proof).

- The Green-formula moments agree with brute-force quadrature on random harmonic polynomials to 5×10^{-14}.
- The transplant bound of §4, and the full Rayleigh quotient of $\cos(\pi x)$, agree with direct quadrature on T to about 10^{-14}.
- At 20 sample triangles both bounds lie above the floating-point Ritz value of Σ, as upper bounds must.

- The Crouzeix–Raviart discrete values are $\sigma_{h,1} = 1.28042$ for $n_h = 8$ and 1.28790 for $n_h = 16$, increasing towards $\sigma^* = 1.29082$.

Code names. Some identifiers differ from the notation here. In the Python code, $\tilde{\sigma}$ is `sig`, the edge means μ^e are `mm`, λ_1, λ_2 are `d1`, `d2`, the ω_e are `ce`, and $\mathcal{D}_A$ is `D`. In Lean, the μ^e are `m0x`, …, `m2y`, the ω_e are `c0`, `c1`, `c2`, and $\mathcal{N}_A, \mathcal{D}_A$ are `NA`, `DA`.

Lean. A Lean 4/Mathlib development formalises the elementary steps:

- the triangle isoperimetric inequality in Heron form, with its equality case, and the deduction of Corollary 1.2;
- the algebra of Lemma 4.2;
- the integral defining $\mathcal{I}$ in Lemma 5.1, and the two edge integrals of $\cos(\pi x)$ used in the consistency checks.

It uses no `sorry`, and its only axioms are `propext`, `Classical.choice` and `Quot.sound`. It does not formalise the spectral theory, the trace and Payne–Weinberger inequalities, or the ball-arithmetic computations.

Availability. The code, the logs, the box certificates and the Lean files are in the accompanying repository[1]. The single command `scripts/verify.py` reruns all of them and writes a PASS/FAIL summary.

7 Numerical observations (not rigorous)

Except for the certified upper bounds in Table 1, the statements in this section are numerical. They use the harmonic-polynomial Ritz method, with trial space $\{\operatorname{Re} w^k, \operatorname{Im} w^k\}_{k \le n}$ and all integrals computed on the boundary.

Observation 7.1 (Global picture). On a 25×30 grid over F (double precision, $n = 30$) the largest value of Σ is attained at the equilateral apex. Table 1 lists selected values, sorted by Σ, together with the regular N-gons for $N = 4, 5, 6$. They are computed as follows:

- the Ritz matrices are exact polynomial line integrals, evaluated in ball arithmetic;
- the eigenproblem is solved in 50- and 80-digit arithmetic with the full basis of $2n$ directions, for $n = 30, 40, 50$;
- the estimate is the value for $n = 50$;
- its error is estimated (not rigorously) as twice the largest of the last change in n, a geometric and a power-law extrapolation of the remaining changes, and the change between the two precisions; the estimate is printed only to the digits this error justifies;
- the certified upper bound is the Rayleigh quotient of the fixed $n = 50$ trial function, evaluated in ball arithmetic on the exact polygon.

Convergence is super-geometric for E, the square, and the right-angled triangles. It is slowest for the polygons with an angle of 108° or 120° (the 30–30–120 triangle, the pentagon and the hexagon), whose corners produce the strongest singularities.

The regular N-gon values increase with N, consistent with [4].

[1] https://github.com/jasondavies/steklov-triangles.

polygon	certified upper bound for Σ	estimate of Σ
60–60–60	3.8724615957729	$3.8724615957728 \pm 1 \times 10^{-13}$
50–50–80	2.8777309857377	$2.87773098574 \pm 3 \times 10^{-11}$
70–70–40	2.5730765728918	$2.573076572892 \pm 2 \times 10^{-12}$
40–60–80	2.5350358150935	$2.53503581509 \pm 2 \times 10^{-11}$
45–45–90	2.4960723700476	$2.4960723700475 \pm 1 \times 10^{-13}$
30–60–90	1.9013817577256	$1.9013817577255 \pm 1 \times 10^{-13}$
30–30–120	1.5603440800120	$1.5603441 \pm 3 \times 10^{-7}$
square	5.5060219386902	$5.5060219386901 \pm 1 \times 10^{-13}$
regular pentagon	5.9774725550439	$5.9774726 \pm 4 \times 10^{-7}$
regular hexagon	6.1355701455936	$6.135570 \pm 5 \times 10^{-6}$

Table 1: Values of $\Sigma = \sigma_1 L$. The $\pm$ values are heuristic error estimates. Rounding the central estimate to the displayed precision can place it slightly above the certified upper bound.

Observation 7.2 (Thin triangles). For the limiting endpoint profile $\Phi_0(x) = 1 - x$ in the quotient of Corollary 5.3, the Euler–Lagrange equation $((1-x)u')' + 2\theta u = 0$ gives $\theta(0) = j_{1,1}^2/8 \approx 1.83525$. Numerically $\theta(0.05) = 1.93168$, $\theta(0.25) = 2.40957$ and $\theta(\frac{1}{2}) = 2.89159$. The double-precision Ritz values ($n = 40$) of $\Sigma/(2q)$ at $q = 0.01$ agree with these to 4×10^{-4} relative. In particular $\Sigma(T) \approx 2.89\,\vartheta(T)$ for thin isosceles triangles, compared with the rigorous constant 45.6 of [8].

8 Remarks

Lemma 4.1 applies to any polygonal trial space for which the energy and boundary-variance forms can be enclosed. At a multiple reference eigenvalue the spectral gap controls the projection error, and the identity $X = \rho Y$ makes verification of a linear deficit possible up to $\rho = 0$.

For $n \geq 4$ the affine group no longer acts transitively on similarity classes of n-gons, so one needs a richer family of transplantations, for example piecewise affine or Schwarz–Christoffel maps. Parallelograms form a two-parameter affine family of the square. General quadrilaterals have four parameters modulo similarities, so a first-order comparison of a double eigenvalue would also have to address deformations in the kernel of the linear map into symmetric 2×2 matrices.

Declaration of AI-generated research and authorship

This manuscript is the product of an AI-driven research process. During a broader investigation of open problems from the AIM problem lists, initiated by the human author, an AI system identified this problem as a candidate, found the proof, and wrote the code for the certified computations. The author then directed AI systems to package the result, formalise parts of it in Lean and prepare this manuscript. The AI systems developed the results and their presentation, including the mathematical exposition, the computations, the formalisation, and the organisation and writing of the text; the human author wrote no part of the manuscript. The system was Claude Opus 5.5 (Anthropic), used through Claude Code. OpenAI Codex subsequently revised the proof and computational certificates for stability, spectral isolation and splitting, and incorporated the thin-domain corollary.

The proof is computer-assisted and is reproduced by `scripts/verify.py`; the Lean development covers only the elementary steps listed in §6.

The human author has not read this manuscript or checked its arguments. At the time of writing, the mathematics, the code and the formalisation have been checked only by AI systems and by the computer verification described above; no independent human expert has reviewed them. The author takes responsibility for the accuracy of this disclosure.

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