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CHRISTOL'S CONJECTURE FOR NON-RESONANT GLOBALLY BOUNDED HYPERGEOMETRIC SERIES OF HEIGHT TWO

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Abstract. Christol conjectured that every globally bounded, D-finite power series with rational coefficients is the diagonal of a rational function. We prove this for non-resonant hypergeometric series of arbitrary order $n \geq 3$ with rational parameters, exactly one explicit positive integral lower parameter, and all remaining lower parameters nonintegral. Every globally bounded series in this class is the diagonal of an algebraic power series in two variables over $\mathbb{Q}$, and hence of a rational function in three variables; three is optimal. The result includes Christol's 1987 test case ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}; \frac{1}{3}, 1; x)$ and the series ${}_3F_2(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}; \frac{1}{2}, 1; x)$. For the latter, the Rohrlich–Lang conjecture and a theorem of Rivoal and Roques imply that it is not a Hadamard product of algebraic series. For the septic series we describe a local genus-three curve and differential realizing its period. An explicit diagonal for either series remains open. Christol's integrality criterion and Fedorov's formula identify global boundedness with two adjacent Hodge levels at every conjugate. Restriction of scalars then gives a polarized integral variation of weight one. Deligne's equivalence, specialization over a number field, and regular Riemann–Hilbert comparison realize the series as a period of a family of curves. A logarithmic solution places this period in the span of nodal vanishing-cycle integrals, whose expansions yield algebraic diagonals. An explicit rational differential inverse treats the positive integral lower parameter and proves equivalence of global boundedness with the case in which that parameter equals one. Together with the two extreme heights, this proves the conjecture for every non-resonant rational-parameter ${}_3F_2$ with admissible lower parameters.

Plain-language summary

Expand a rational function of several variables $u_1, \ldots, u_d$ (a ratio of polynomials) as a power series. For each n, keep the coefficient of $u_1^n \cdots u_d^n$, and use that coefficient as the coefficient of x^n. The result is a power series in one variable x, called the *diagonal*. Diagonals satisfy linear differential equations and, after rescaling, have integer coefficients (they are *globally bounded*). In the late 1980s Gilles Christol conjectured the converse: every power series with these two properties is a diagonal. The conjecture is open in general.

Hypergeometric series are a standard test case. We prove the conjecture for every order $n \geq 3$ when exactly one displayed lower parameter is a positive integer and all other lower parameters are nonintegral, under the non-resonance condition stated in Section 1. Counting the factorial in the coefficients as another integral lower parameter, this is the height-two case. Which positive integer appears makes no difference: an explicit differential operator reduces every case to the one in which it equals 1. This family includes Christol's own 1987 test case ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}; \frac{1}{3}, 1; x)$, and the series ${}_3F_2(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}; \frac{1}{2}, 1; x)$, which he suggested might be a counterexample. Each globally bounded series in the family is the diagonal of a rational function in three variables, and three is the least possible number. Combined with earlier results, this settles the conjecture for all non-resonant ${}_3F_2$ series whose lower parameters are

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nonintegral or positive integers. Assuming the Rohrlich–Lang conjecture on values of the Gamma function, earlier work of Rivoal and Roques shows that the second series is not a Hadamard product of algebraic series, the usual elementary way of producing diagonals.

The proof translates the integrality of the coefficients into a condition from Hodge theory. That condition allows the series to be realized as an integral over loops on a family of curves. Christol's principle then applies: an integral over a loop that shrinks to a point is a diagonal, provided the number of logarithms in the solutions of the differential equation matches the dimension of the family. For the curves constructed here this matching condition holds. The usual integral representation has dimension $n-1$, which does not give this matching condition when $n \geq 3$. The argument proves that the rational functions exist but does not construct them; no explicit rational function is known for either of the two series above.

For the series with fractions in sevenths, a companion calculation identifies a particular curve, a loop on it, and the quantity to integrate around that loop. This makes the integral computable near infinity, but finding a rational function whose diagonal gives that series remains open. The paper does not prove Christol's conjecture in general, and the statement about Hadamard products depends on the Rohrlich–Lang conjecture. The argument has not yet been reviewed by independent experts.

Contents

Plain-language summary 1
1. Introduction 2
2. Preliminaries 9
3. Two Hodge levels and global boundedness 13
4. A polarized variation of Hodge structure of weight one 15
5. An abelian scheme and the period 19
6. Curves, semistable models and vanishing cycles 21
7. Expansion at a node and proof of Theorem A 24
8. Positive integral lower parameters 26
9. Examples and further consequences 28
10. Concluding remarks 32
Appendix A. An independent check of the Hodge numbers 33
Appendix B. Computations, verification and imported results 34
Appendix C. A search for explicit representations of H 37
Acknowledgement of AI assistance 41
Data availability 42
References 42

1. Introduction

1.1. **Diagonals and Christol's conjecture.** Let $R = \sum r_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}$ be a power series in n variables with coefficients in $\mathbb{Q}$. Its *diagonal* is the power series in one variable

$$\mathrm{Diag}(R) = \sum_{j \geq 0} r_{j,\dots,j}\, x^j.$$

We are interested in the case where R is the expansion of a rational function that is regular at the origin, that is, of an element of the local ring $\mathbb{Q}[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$. Diagonals of rational functions occur as generating functions in enumerative combinatorics, as periods in algebraic

geometry and as multiple integrals in statistical physics; see Christol [15] and Bostan et al. [9] for surveys and examples.

A power series $f \in \mathbb{Q}[[x]]$ is *D-finite* if it satisfies a nonzero linear differential equation with coefficients in $\mathbb{Q}[x]$. It is *globally bounded* if it has a positive radius of convergence and there are nonzero integers c_1, c_2 with $c_1 f(c_2 x) \in \mathbb{Z}[[x]]$. Every diagonal of a rational function is D-finite [12, 41] and globally bounded [14]. Christol conjectured the converse [13, 14]; see also Christol [15, Conjecture 10].

Christol's conjecture. *Every D-finite, globally bounded power series in $\mathbb{Q}[[x]]$ is the diagonal of a rational function over $\mathbb{Q}$ that is regular at the origin.*

The conjecture is open. Two classical results describe the bottom of the hierarchy. A power series in one variable is algebraic over $\mathbb{Q}(x)$ if and only if it is the diagonal of a rational function in two variables [28, 44]. The diagonal of an algebraic power series in several variables is the diagonal of a rational function in more variables [23]. Beyond these structural results, the evidence for the conjecture consists of families of examples.

1.2. **Hypergeometric series.** Fix an order $n \geq 3$, upper parameters $a = (a_1, \dots, a_n) \in \mathbb{Q}^n$ and nonintegral lower parameters $b = (b_1, \dots, b_{n-2}) \in (\mathbb{Q} \setminus \mathbb{Z})^{n-2}$. We write

$$F_{a,b}(x) = {}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-2}, 1; x) = \sum_{r \geq 0} \frac{\prod_{i=1}^{n} (a_i)_r}{\prod_{j=1}^{n-2} (b_j)_r \, (r!)^2} x^r, \tag{1.1}$$

and, for a positive integer m, set

$$F_m(x) = {}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-2}, m; x). \tag{1.2}$$

Thus $F_1 = F_{a,b}$. Here $(c)_r = c(c+1)\cdots(c+r-1)$, with $(c)_0 = 1$. In order three we also use the scalar notation $b = b_1$.

Hypergeometric series are a natural test class for Christol's conjecture, because global boundedness is decided by an explicit criterion on the parameters [13]; see Lemma 2.7. For ${}_nF_{n-1}$ with rational parameters the problem splits according to the number of integers among the lower parameters. If the only integral lower parameter is the 1 that accounts for $n!$, then a globally bounded series is algebraic [15, Theorem 33], by the interlacing criterion of Beukers and Heckman [7], and hence a diagonal. For nonintegral upper parameters and positive integral lower parameters, the series is a Hadamard product of algebraic series (Remark 9.6). When all lower parameters equal 1, the factors are binomial series. Bostan and Yurkevich [8] describe the remaining cases, of intermediate monodromy weight, as widely open. The first such case, given by Christol in 1987 [13, §VII], is

$$H(x) = {}_3F_2\Big(\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{5}{9}; \tfrac{1}{3}, 1; x\Big). \tag{1.3}$$

Bostan et al. [9] listed many further globally bounded ${}_3F_2$ series that are not obviously reducible to the two extreme cases by closure properties of diagonals, such as Hadamard products. Abdelaziz et al. [2] proved that two of them,

$${}_3F_2\big(\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}; \tfrac{1}{3}, 1; x\big) \qquad \text{and} \qquad {}_3F_2\big(\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}; \tfrac{2}{3}, 1; x\big),$$

are diagonals, and gave two heuristic arguments that H is one as well. In 2026 the series H was still described as not known to be a diagonal [11], and Christol's conjecture as unsolved even for hypergeometric series [10]. Christol also suggested the series

$$S_7(x) = {}_3F_2\Big(\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{4}{7}; \tfrac{1}{2}, 1; x\Big) \tag{1.4}$$

as a potential counterexample to his conjecture, and in 2026 it was not known whether it is a diagonal [34, Remark 4.2]. Assuming the Rohrlich–Lang conjecture on algebraic relations between values of the Gamma function, Rivoal and Roques [45, Proposition 1] proved that S_7 is not a Hadamard product of finitely many algebraic power series with algebraic coefficients. As we observe in Proposition D(a) below, the two series of Abdelaziz, Koutschan and Maillard are in fact Hadamard products of two algebraic power series, by an elementary identity that covers infinitely many further series. The identity does not apply to H, and a computer search over several finite collections of simple candidates found no representation of H among the candidates it enumerated (Appendix C); it says nothing about representations outside them.

1.3. **Main results.** We call the pair (a, b) *non-resonant* if

$$a_i \notin \mathbb{Z}, \qquad b_j \notin \mathbb{Z}, \qquad a_i - b_j \notin \mathbb{Z} \qquad (1 \le i \le n,\ 1 \le j \le n-2). \tag{NR}$$

For a field F we write $F\langle u, v\rangle \subset F[[u, v]]$ for the ring of power series that are algebraic over $F(u, v)$.

Theorem A. *Let $n \ge 3$, let $a \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^{n-2}$ satisfy (NR), and let $m \ge 1$ be an integer. Then F_m is globally bounded if and only if F_1 is globally bounded. If these equivalent conditions hold, there is an algebraic power series $g_m \in \mathbb{Q}\langle u, v\rangle$ with $F_m = \mathrm{Diag}(g_m)$. Consequently F_m is the diagonal of a rational function in three variables over $\mathbb{Q}$, regular at the origin. It is transcendental over $\mathbb{C}(x)$, so fewer than three rational variables do not suffice, even over a number field.*

Internal repetitions among the a_i, or among the nonintegral b_j, are allowed. Only cross-list coincidences modulo $\mathbb{Z}$ are excluded by (NR). The geometric proof treats $m = 1$; Section 8 gives the explicit inverse that treats every positive integral m. Remark 9.6 treats the other extreme, where all displayed lower parameters are positive integers. Intermediate heights $2 < h < n$ and resonant equations lie outside these statements.

Corollary B. *Christol's series $H(x) = {}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}; \frac{1}{3}, 1; x)$ is the diagonal of an algebraic power series in two variables over $\mathbb{Q}$, and of a rational function in three variables over $\mathbb{Q}$. The same holds for the series $S_7(x) = {}_3F_2(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}; \frac{1}{2}, 1; x)$ of (1.4).*

A genus-three period for the septic series. For S_7, the geometric step in the proof can be made explicit near $x = \infty$. Put $x = 64/s^{14}$. A companion reconstruction [1, §§1–5] gives a marked family C_s whose fibre at $s = 0$ is the smooth compactification of $v^2 = u^7 - 1$. On an affine chart the family has equations

$$\begin{aligned} z &= u^2 + h(s)v, \qquad v^2 = G_s(u, z), \\ G_s(u, z) &= uz^3 - 1 + p_1 u + p_2 z + p_3 uz + p_4 z^2 + p_5 uz^2. \end{aligned} \tag{1.5}$$

The functions h and $p_j = p_j(s)$ vanish at zero, with $h = s + O(s^{15})$ and $p_3 = -s^2/2 + O(s^{16})$. For small $s \neq 0$, elimination of v gives the smooth plane quartic $(z - u^2)^2 = h^2 G_s(u, z)$ on this chart. The holomorphic differential and its normalization at the central fibre are

$$\eta_s = \frac{2s^2(b_0 + b_1 u + b_2 z)\, du}{2v - h\, \partial_z G_s}, \qquad \lim_{s \to 0} s^{-2}\eta_s = \frac{u^2\, du}{v}.$$

The nine coefficient functions $h, p_1, \ldots, p_5, b_0, b_1, b_2$ are convergent power series in $\mathbb{Q}[[s]]$, algebraic over $\mathbb{Q}(s)$ and determined recursively. For the transported integral cycle γ_s and nonzero normalization constant K specified in [1, §3], $S_7(64/s^{14}) = K \int_{\gamma_s} \eta_s$ on the branch continued

from $x<0$. In relative de Rham cohomology the class of η_s satisfies the original hypergeometric operator

$$\theta^2(\theta-\tfrac{1}{2})-x(\theta+\tfrac{1}{7})(\theta+\tfrac{2}{7})(\theta+\tfrac{4}{7}),\qquad \theta=x\,d/dx,$$

with θ acting by the Gauss–Manin connection. The smooth fibre at infinity gives a local formula for the period. The diagonal assertion in Corollary B uses the degeneration at $x=0$.

Corollary C. *Let $a_1,a_2,a_3\in\mathbb{Q}\setminus\mathbb{Z}$ and $b_1,b_2\in\mathbb{Q}\setminus\mathbb{Z}_{\le 0}$ with $a_i-b_j\notin\mathbb{Z}$ for all i,j. Then ${}_3F_2(a_1,a_2,a_3;b_1,b_2;x)$ is globally bounded if and only if it is the diagonal of a rational function over $\mathbb{Q}$ that is regular at the origin.*

The rank-three Corollary C combines Theorem A with the two extreme cases recalled above; it is proved in §9.3. Theorem A gives existence and optimal dimension for the diagonal. Even for S_7, converting the curve period above into an explicit diagonal remains open. Theorem A is the principal contribution of this paper. For some neighbours of H, however, the conclusion of Theorem A can be made explicit, and in part elementary. We write $f\star g$ for the Hadamard product of power series, defined in the notation below.

Proposition D below is elementary. We include it because it explains the explicit examples known so far and shows that they are of a different nature from H.

Proposition D. *For $s\in\mathbb{Q}\setminus\mathbb{Z}$ let*

$$\mathcal{T}_s(x)=\sum_{n\ge 0}\frac{(s)_{3n}}{(3n)!}\,x^n=\frac{1}{3}\sum_{\omega^3=1}\bigl(1-\omega\,x^{1/3}\bigr)^{-s},$$

which is algebraic over $\mathbb{Q}(x)$.

(a) *For $c\in\{\frac{1}{3},\frac{2}{3}\}$,*

$${}_3F_2\Bigl(\frac{s}{3},\frac{s+1}{3},\frac{s+2}{3};1-c,1;x\Bigr)=\mathcal{T}_s(x)\star(1-x)^{-c}=\mathrm{Diag}\,\bigl[\mathcal{T}_s(u)\,(1-v)^{-c}\bigr].$$

(b) *Let $\phi_s(y)=\sum_{n\ge 0}\binom{2n+s-1}{n}y^n$. Then ϕ_s is algebraic over $\mathbb{Q}(y)$, and*

$${}_3F_2\Bigl(\frac{s}{3},\frac{s+1}{3},\frac{s+2}{3};s,1;27x\Bigr)=\mathrm{Diag}\,\Bigl[(1-u)^{-s}\,\phi_s\Bigl(\frac{v}{(1-u)^2}\Bigr)\Bigr].$$

The two series of Abdelaziz et al. [2] are the cases $(s,c)=(\frac{1}{3},\frac{2}{3})$ and $(\frac{2}{3},\frac{1}{3})$ of (a)*, and the cases $s=\frac{1}{3}$ and $s=\frac{2}{3}$ of* (b).

Corollary B also bears on the relation between diagonals and Hadamard products. A Hadamard product of k algebraic power series is the diagonal of a rational function in $k+1$ variables [34, Proposition 4.1]. Harder and Kramer-Miller [34, Remark 4.2] observe that the converse fails if both Christol's conjecture and the Rohrlich–Lang conjecture hold. Corollary B removes the first of these assumptions.

Corollary E. *Assume the Rohrlich–Lang conjecture. Then S_7 is the diagonal of a rational function in three variables over $\mathbb{Q}$, but it is not a Hadamard product of finitely many power series in $\overline{\mathbb{Q}}[[x]]$ that are algebraic over $\overline{\mathbb{Q}}(x)$.*

1.4. **Strategy of the proof.** Christol proved that solutions of minimum monodromy weight of Picard–Fuchs equations are diagonals [14, Theorem 12]; see also Christol [12] and Christol [15, §4.2, Corollary 30]. For a family of varieties of relative dimension n the minimum weight is $-n$; it is attained by a solution f, holomorphic at the singular point, for which the equation has a further solution of the form $f\log^n x+\cdots$. The local mechanism is Deligne's integration over the vanishing cycle at a point where the family is given by $x_1\cdots x_{n+1}=x$ [19].

The generalized 1-eigenspace of local monodromy has basis $F_{a,b}$ and $F_{a,b}\log x + \cdots$ (Lemma 2.6); in Christol's terminology, $F_{a,b}$ has monodromy weight -1. Its classical Euler integral realizes $F_{a,b}$ as a period of a family of dimension $n-1$ (surfaces when $n=3$; see [15, §4.3]), for which the minimum weight is $-(n-1)$. This is why Christol's theorem does not apply directly, and why H became the standard test case. We show that a globally bounded $F_{a,b}$ is also a period of a family of *curves*, for which -1 is the minimum weight; see Table 1.

Realization of $F_{a,b}$	relative dimension	minimum weight	weight of $F_{a,b}$
Euler integral	$n-1$	$-(n-1)$	-1
Sections 5–6 (curves)	1	-1	-1

TABLE 1. Monodromy weights in the two geometric realizations of $F_{a,b}$.

The proof has five steps, shown in Figure 1.

(1) *Arithmetic and Hodge theory.* Christol's criterion for global boundedness holds if and only if every Galois conjugate of the hypergeometric local system carries a complex variation of Hodge structure with exactly two adjacent nonzero Hodge levels, with arbitrary multiplicities (Proposition 3.1). The proof uses Fedorov's formula for the Hodge numbers.
(2) *Weight one.* The restriction of scalars of the local system to $\mathbb{Z}$ then carries a polarized variation of Hodge structure of weight one (Theorem 4.3). Over a single point this construction is a half twist in the sense of van Geemen [48].
(3) *An abelian scheme.* By Deligne's equivalence between such variations and abelian schemes, there is an abelian scheme over $\mathbb{P}^1 \setminus \{0,1,\infty\}$, defined over a number field, of which $F_{a,b}$ is a period (Propositions 5.1 and 5.4).
(4) *Curves and vanishing cycles.* Hyperplane sections give a family of curves with the same period. After a base change $x = w^m$ the family has a regular semistable model, and $F_{a,b}(w^m)$ is a linear combination of integrals over the vanishing cycles at its nodes (Proposition 6.2, Lemma 6.4 and Proposition 7.3(a)).
(5) *Expansion at a node.* At a node with local equation $uv = w$, such an integral is a power of w times the diagonal of an algebraic power series in u and v (Proposition 7.1). Elementary closure properties of two-variable algebraic diagonals finish the proof (Proposition 7.3(b) and §7.1).

Proposition 7.3 isolates the passage from step (4) to step (5). The abelian scheme in step (3) is obtained from Riemann's theorem in families and is not explicit. This is the source of the non-constructive nature of the proof. We therefore distinguish four kinds of information: the existence of diagonal representations, algebraic in two variables and rational in three (Theorem A); a proof that three rational variables are necessary (also Theorem A); explicit algebraic representations in two variables (Proposition D, for some neighbours of H); and explicit rational representations, which are known, in six variables, for the two series of Abdelaziz et al. [2]. For H and S_7 the diagonal representations are known by existence. Equation (1.5) makes the intermediate curve realization computable for S_7.

1.5. **Related work.** *Diagonals and periods.* Christol [12] and Deligne [19] connected diagonals of rational functions with integrals over vanishing cycles, and Christol [14, Theorem 12] turned this into the sufficient condition recalled above. Christol [15, §4.2] explains the role of the duality between the monodromy weight filtration and the pole-order filtration. Bostan et al. [9] used these ideas to show that many integrals from the Ising model and many solutions of Calabi–Yau equations are diagonals, and discussed H as a test case of intermediate weight.

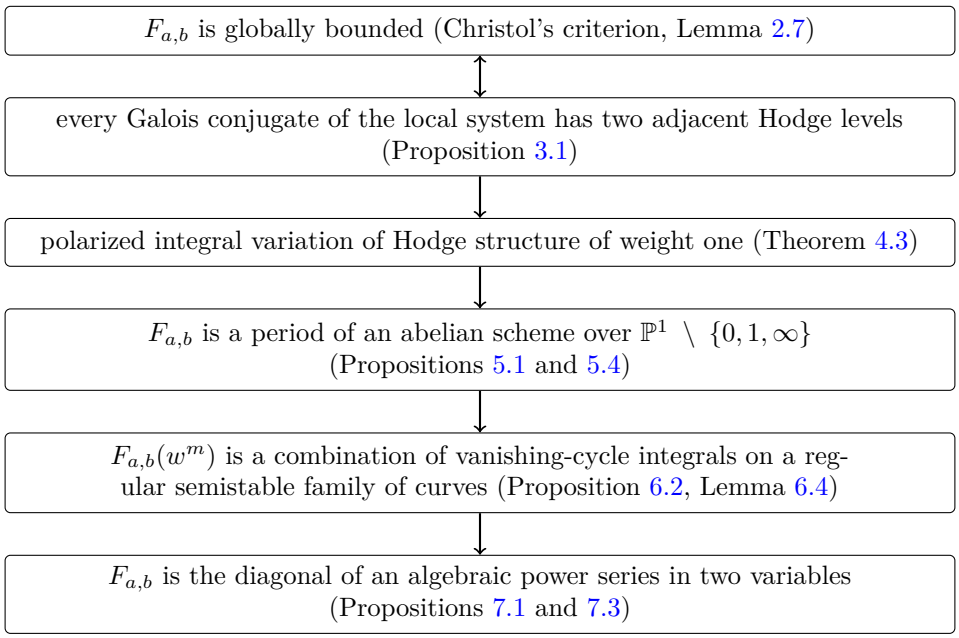


FIGURE 1. The structure of the proof of Theorem A.

Harder and Kramer-Miller [34] relate the minimal number of variables in a diagonal representation to the nilpotence of the local monodromy. Their Lemma 3.2, stated for D-finite series over a number field, identifies diagonals of algebraic series in $n+1$ variables with diagonals of rational functions in $n+2$ variables; we use it to pass from two algebraic to three rational variables. Hassani et al. [35] formulate conjectures on the minimal number of variables in terms of the powers of the logarithm in the formal solutions.

Global boundedness and algebraicity. Christol [13] characterized the globally bounded hypergeometric series with rational parameters by an interlacing condition over all Galois conjugates of the parameters. Beukers and Heckman [7] gave the analogous criterion for finite monodromy, hence for algebraicity, and determined the monodromy groups. Fürnsinn and Yurkevich [27] extended the algebraicity criterion to arbitrary parameters, and Caruso and Fürnsinn [10, 11] implemented these criteria and related arithmetic invariants.

Hypergeometric diagonals. Abdelaziz et al. [2] represented their two series, and the one-parameter family of Proposition D(b), as diagonals of algebraic functions in three variables. Using the algorithm of Denef and Lipshitz [23], they also wrote down rational functions in six variables with these diagonals, and they settled further ${}_3F_2$ and ${}_4F_3$ examples. Bostan and Yurkevich [8] determined the hypergeometric series that arise as diagonals of products of powers of $1-x_1-\cdots-x_n$. Proposition D(a) shows that the two series of Abdelaziz, Koutschan and Maillard are Hadamard products of two algebraic series, and Proposition D(b) represents their one-parameter family with two algebraic variables.

Hodge theory of hypergeometric local systems. Hypergeometric local systems are rigid, and Katz [38] showed that rigid local systems with quasi-unipotent local monodromy are of geometric origin. Fedorov [26, Proposition 2.6] shows that a hypergeometric connection with real parameters underlies a polarizable complex variation of Hodge structure, unique up to a shift of the grading; for existence he cites Simpson [47, Corollary 8.1] and Dettweiler and Sabbah [24, Theorem 2.4.1], and for uniqueness Deligne [20, Proposition 1.13(i)]. He also computes the

Hodge numbers [26, Theorem 1]. This formula is the main input of Proposition 3.1. Restriction of scalars and diagonals are not considered there.

Hadamard and diagonal grades. Allouche and Mendès France [3] introduced the Hadamard grade of a power series. Rivoal and Roques [45] bounded the Hadamard grades of hypergeometric series by comparing the asymptotic expansions of their coefficients with those of algebraic series. Several of their results, including their Proposition 1 on S_7, assume the Rohrlich–Lang conjecture. Harder and Kramer-Miller [34] bound the diagonal and Hadamard grades from below by the nilpotence index of the local monodromy (their Corollary 2.6). For hypergeometric series this gives both grades at least the height (their Theorem 3.8), and they show that both grades of ${}_nF_{n-1}(\frac{1}{2},\dots,\frac{1}{2};1,\dots,1;x)$ equal n. Lemma 2.3 below is their Lemma 3.2.

Geometric realizations. The Euler integral realizes ${}_nF_{n-1}$ as a period of a family of varieties of relative dimension $n-1$; see Christol [15, §4.3] for ${}_3F_2$ and Kelly and Voight [39] for the resulting hypergeometric motives. For ${}_2F_1$ the realization is by curves, and Archinard [5] constructed the corresponding abelian varieties explicitly. Our weight-one realization in arbitrary rank comes instead from the equivalence of Deligne [18, 4.4.3] between abelian schemes and polarizable variations of Hodge structure of weight one, proved in detail by Milne [43, Theorem 7.13]. In rank three, over a point, the passage from the weight-two Hodge structure of the Euler realization to weight one is a half twist [48, 49]. Periods of degenerating abelian varieties over locally invariant cycles were studied by André [4] in connection with G-functions.

To our knowledge, no proof that H or S_7 is a diagonal has appeared; for S_7 this is stated by Harder and Kramer-Miller [34, Remark 4.2]. In the rank-three examples with scalar $b \notin \mathbb{Z}$ that we know to be diagonals, including those of Proposition D, the representation is explicit. For H none is known, and the search of Appendix C found none among the candidates it enumerated. Each ingredient of our proof is classical. To our knowledge, what is new is the equivalence of Proposition 3.1 and its use, through a realization of weight one, to bring Christol's minimum-weight principle to bear. Proposition D consists of elementary identities, which may well be known to specialists, although we have not found them in print.

The arbitrary-rank extension uses the same geometric argument, but requires the multiset endpoint calculation in (3.2), rather than a rank-three ordering table. Section 8 supplies the positive-integral-parameter extension. These changes enlarge the theorem's scope without asserting a new general principle for all heights. The literature discussion records the sources and comparisons already identified; no exhaustive new priority search for the arbitrary-rank statement is claimed here.

1.6. **Organization.** Section 2 collects the facts we need on algebraic power series, hypergeometric local systems, global boundedness and variations of Hodge structure. Sections 3–7 carry out steps (1)–(5); Proposition 7.3 collects the passage from periods to diagonals. The case $m = 1$ of Theorem A is proved in §7.1, and Section 8 completes the theorem for every positive integral m. Section 9 treats the examples and proves Corollaries B, C and E and Proposition D. Section 10 contains concluding remarks. Appendix A gives an independent computational check of the Hodge numbers, Appendix B describes the accompanying computations, how to reproduce them, and the results from the literature on which the proof depends, and Appendix C reports a search for explicit representations of H.

Notation and conventions. We write $\mathbf{e}(t) = e^{2\pi i t}$ and $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, a scheme over $\mathbb{Q}$; base change to a field F is denoted by U_F, and $U(\mathbb{C})$ carries its analytic topology. We put $\theta = x\frac{d}{dx}$. Local systems are on $U(\mathbb{C})$ unless stated otherwise. The local monodromy around 0 of any local system on $U(\mathbb{C})$ is denoted by T_0; on solutions of a differential equation it acts by analytic continuation. For $t \in \mathbb{R}$, $\{t\} \in [0, 1)$ is the fractional part. The letter k

always denotes an element of $(\mathbb{Z}/N\mathbb{Z})^\times$, where N is a common denominator of the parameters. Complex conjugation is denoted by a bar, and $X^* = \overline{X}^T$ for a complex matrix X. For a field F and variables $u_1, \dots, u_n$, $F\langle u_1, \dots, u_n\rangle$ is the ring of power series in $F[[u_1, \dots, u_n]]$ that are algebraic over $F(u_1, \dots, u_n)$. The Hadamard product of $f = \sum a_n x^n$ and $g = \sum b_n x^n$ is $f \star g = \sum a_n b_n x^n$; it equals $\mathrm{Diag}\,(f(u)g(v))$. Following Allouche and Mendès France [3] and Harder and Kramer-Miller [34], the *Hadamard grade* $\mathrm{hg}(f)$ of a power series f with algebraic coefficients is 0 if f is rational, and otherwise the least k such that f is a Hadamard product of k power series that are algebraic over $\overline{\mathbb{Q}}(x)$, or ∞ if there is no such k. The *diagonal grade* $\mathrm{dg}(f)$ is the least n such that f is the diagonal of a rational function in $n+1$ variables over a number field.

2. Preliminaries

2.1. **Algebraic power series and diagonals.** Let F be a field of characteristic 0. The ring $F\langle u_1, \dots, u_n\rangle$ is closed under partial derivatives and under the specializations $u_i = 0$. The following lemma identifies it with a henselization. It is well known; we include a proof because the identification, and its analytic counterpart (ii), are used at the nodes in Section 7.

Lemma 2.1. *Let $\mathcal{O}$ be the local ring of a smooth F-variety Y of dimension n at an F-rational point P, let $\mathcal{O}^h$ be its henselization, with maximal ideal $\mathfrak{m}^h$, and let $u_1, \dots, u_n$ be a regular system of parameters of $\mathcal{O}^h$.*

(i) *The homomorphism $F[[t_1, \dots, t_n]] \to \widehat{\mathcal{O}}$, $t_i \mapsto u_i$, is an isomorphism, and it maps $F\langle t_1, \dots, t_n\rangle$ onto $\mathcal{O}^h$. Thus $\mathcal{O}^h = F\langle u_1, \dots, u_n\rangle$ inside $\widehat{\mathcal{O}} = F[[u_1, \dots, u_n]]$.*

(ii) *If $F \subset \mathbb{C}$, then every element of $\mathcal{O}^h$ is the germ at P of a holomorphic function on $Y(\mathbb{C})$ whose Taylor series in the local coordinates $u_1, \dots, u_n$ is its image in $F[[u_1, \dots, u_n]]$, regarded as an element of $\mathbb{C}[[u_1, \dots, u_n]]$ via the coefficientwise inclusion. In particular, the elements of $F\langle u_1, \dots, u_n\rangle$ converge near the origin.*

Proof. (i) The rings $\mathcal{O}$ and $\mathcal{O}^h$ are regular local rings of dimension n with residue field F and the same completion $\widehat{\mathcal{O}}$, whose maximal ideal is generated by $u_1, \dots, u_n$. The homomorphism is surjective, because its image is complete and contains F and generators of the maximal ideal; it is injective, because both sides are complete regular local rings of dimension n, so that its kernel is a prime ideal of height 0.

Let $y \in \mathcal{O}^h$. The ring $\mathcal{O}^h$ is a filtered colimit of local rings of étale $\mathcal{O}$-algebras, so y is algebraic over the function field $F(Y)$, and so are $u_1, \dots, u_n$. Since the u_i are algebraically independent over F and $F(Y)$ has transcendence degree n, the field $F(Y)(u_1, \dots, u_n)$ is algebraic over $F(u_1, \dots, u_n)$. Hence y is algebraic over $F(u_1, \dots, u_n)$, that is, $\mathcal{O}^h \subseteq F\langle u_1, \dots, u_n\rangle$.

Conversely, let $g \in F[[u_1, \dots, u_n]]$ be algebraic over $F(u_1, \dots, u_n)$, and let $\Phi \in F[u_1, \dots, u_n][Y]$ be an irreducible polynomial with $\Phi(g) = 0$. As F has characteristic 0, $\Phi'(g) \neq 0$, where $\Phi' = \partial\Phi/\partial Y$; let e be the order of $\Phi'(g)$ with respect to the maximal ideal of $\widehat{\mathcal{O}}$. By Artin's approximation theorem [6, Theorem 1.10], which applies to the henselization of a local ring of a scheme of finite type over a field, the equation $\Phi(y) = 0$, which has the solution g in $\widehat{\mathcal{O}}$, has a solution $y \in \mathcal{O}^h$ with $y \equiv g$ modulo $(\mathfrak{m}^h)^{e+1}\widehat{\mathcal{O}}$. Write $\Phi(y) - \Phi(g) = (y-g)(\Phi'(g) + (y-g)\,r)$ with $r \in \widehat{\mathcal{O}}$. If $y \neq g$, the second factor has order exactly e, because $(y-g)r$ has order at least $e+1$, and it is therefore nonzero; as $\widehat{\mathcal{O}}$ is a domain, $\Phi(y) \neq \Phi(g)$, a contradiction. Hence $g = y \in \mathcal{O}^h$.

(ii) Let $\iota : F[[u_1, \dots, u_n]] \to \mathbb{C}[[u_1, \dots, u_n]]$ be the natural coefficientwise map induced by $F \subset \mathbb{C}$. The u_i are holomorphic functions near P with linearly independent differentials at P, hence holomorphic coordinates there, and we write τ for the map sending a holomorphic germ

at P to its Taylor series in $u_1, \dots, u_n$. An étale morphism of smooth complex varieties is a local biholomorphism, so the local ring $\mathcal{O}'$ at P of an étale neighbourhood of (Y, P) embeds into the ring of holomorphic germs at P. The two homomorphisms $\mathcal{O}' \to \mathbb{C}[[u_1, \dots, u_n]]$ obtained from τ and from ι composed with $\mathcal{O}' \to \widehat{\mathcal{O}} = F[[u_1, \dots, u_n]]$ agree: both are local homomorphisms into a complete local ring, so they factor through continuous homomorphisms on $\widehat{\mathcal{O}}$, and these agree on F and on the u_i, hence on the dense subring $F[u_1, \dots, u_n]$ and so everywhere. Passing to the colimit embeds $\mathcal{O}^h$ into the ring of holomorphic germs at P, and the Taylor series of the image of $y \in \mathcal{O}^h$ is $\iota(y)$. Since the Taylor series of a holomorphic germ converges to it, every element of $F\langle u_1, \dots, u_n\rangle = \mathcal{O}^h$ converges near the origin. □

For $g = \sum_{j,l \geq 0} g_{j,l} u^j v^l \in F[[u, v]]$ we put $\mathrm{Diag}(g)(w) = \sum_{n \geq 0} g_{n,n} w^n$. The following closure properties are elementary; we include proofs because we need them for two-variable algebraic series and over number fields.

Lemma 2.2. *Let $F \subset \mathbb{C}$ be a number field.*

(a) *Let $\varphi_1, \dots, \varphi_r \in F[[w]]$ and $z_1, \dots, z_r \in \mathbb{C}$. If $\sum_i z_i \varphi_i$ lies in $F[[w]]$, then it equals $\sum_i c_i \varphi_i$ for some $c_1, \dots, c_r \in F$.*
(b) *If $g \in F\langle u, v\rangle$ and $\mathrm{Diag}(g) \in \mathbb{Q}[[w]]$, then $\mathrm{Diag}(g) = \mathrm{Diag}(g')$ for some $g' \in \mathbb{Q}\langle u, v\rangle$.*
(c) *If $g \in \mathbb{Q}\langle u, v\rangle$ and $\mathrm{Diag}(g) = w^\nu \varphi(w)$ with $\nu \geq 0$ and $\varphi \in \mathbb{Q}[[w]]$, then $\varphi = \mathrm{Diag}(g')$ for some $g' \in \mathbb{Q}\langle u, v\rangle$.*
(d) *If $g \in \mathbb{Q}\langle u, v\rangle$ and $\mathrm{Diag}(g)(w) = \varphi(w^m)$ with $m \geq 1$ and $\varphi \in \mathbb{Q}[[w]]$, then $\varphi = \mathrm{Diag}(g')$ for some $g' \in \mathbb{Q}\langle u, v\rangle$.*

Proof. (a) After renumbering, $\varphi_1, \dots, \varphi_s$ form a basis of the F-span of the φ_i. Then $\sum_i z_i \varphi_i = \sum_{j \leq s} y_j \varphi_j$ with $y_j \in \mathbb{C}$. Since $\varphi_1, \dots, \varphi_s$ are linearly independent over F, there are exponents $n_1, \dots, n_s$ such that the $s \times s$ matrix formed by the coefficients of w^{n_l} in φ_j is invertible. Comparing the coefficients of $w^{n_1}, \dots, w^{n_s}$, which lie in F on both sides, gives $y_j \in F$.

(b) Replacing F by its Galois closure, we may assume that F is Galois over $\mathbb{Q}$, with group Γ. For $\sigma \in \Gamma$ let $\sigma(g)$ be the series obtained by applying σ to the coefficients of g. It is algebraic, since it satisfies the conjugate polynomial equation, and $\mathrm{Diag}(\sigma(g)) = \sigma(\mathrm{Diag}(g)) = \mathrm{Diag}(g)$. The series $g' = |\Gamma|^{-1} \sum_{\sigma \in \Gamma} \sigma(g)$ has rational coefficients, is algebraic over $F(u, v)$ and hence over $\mathbb{Q}(u, v)$, and satisfies $\mathrm{Diag}(g') = \mathrm{Diag}(g)$.

(c) Put $g' = \sum_{j,l \geq \nu} g_{j,l} u^{j-\nu} v^{l-\nu}$, so that $\mathrm{Diag}(g') = \varphi$. The series $g_j(v) = \sum_l g_{j,l} v^l = \frac{1}{j!} \partial_u^j g(0, v)$ and $g^l(u) = \sum_j g_{j,l} u^j$ are algebraic. Hence so is

$$(uv)^\nu g' = g - \sum_{j < \nu} u^j g_j(v) - \sum_{l < \nu} v^l g^l(u) + \sum_{j,l < \nu} g_{j,l} u^j v^l,$$

and therefore so is g'.

(d) Put $g' = \sum_{j,l \geq 0} g_{mj,ml} u^j v^l \in \mathbb{Q}[[u, v]]$, so that $\mathrm{Diag}(g') = \varphi$. Then

$$g'(u^m, v^m) = \frac{1}{m^2} \sum_{j,l=0}^{m-1} g(\mathbf{e}(j/m)u, \mathbf{e}(l/m)v)$$

is algebraic over $\mathbb{Q}(\mathbf{e}(1/m))(u, v)$, hence over $\mathbb{Q}(u, v)$, hence over the subfield $\mathbb{Q}(u^m, v^m)$, over which $\mathbb{Q}(u, v)$ is finite. Renaming u^m, v^m as u, v shows that g' is algebraic. □

Lemma 2.3 (Harder and Kramer-Miller [34, Lemma 3.2])**.** *Let K be a number field, let $n \geq 0$ and let $f \in K[[x]]$ be D-finite. Then f is the diagonal of an algebraic power series in $n + 1$ variables over K if and only if it is the diagonal of a rational function in $n + 2$ variables over K that is regular at the origin.*

The proof of Lemma 2.3 rests on Denef and Lipshitz [23, Theorem 6.2 and Remark 6.4]: the map that sends $\sum a_{i_0\dots i_{n+1}} x_0^{i_0}\cdots x_{n+1}^{i_{n+1}}$ to the sum of the terms with $i_0+\cdots+i_n = i_{n+1}$, with x_{n+1} omitted, maps the rational functions in $n+2$ variables that are regular at the origin onto the algebraic power series in $n+1$ variables; for $n=0$ this is the theorem of Furstenberg [28]. Applied directly, the theorem of Denef and Lipshitz gives $2(n+1)$ rational variables. We use Lemma 2.3 only with $K=\mathbb{Q}$: with $n=1$ in §7.1, where $f=F_{a,b}$ is D-finite because $L_{a,b}F_{a,b}=0$, and in Remark 9.7(4); and in §9.3 for Case 1 and its extension in Remark 9.6. For $n=1$ it produces three rational variables. That three variables are also necessary under (NR) is a separate statement, which follows from Theorem 2.4 and the transcendence of $F_{a,b}$.

Theorem 2.4 (Pólya and Furstenberg [28, 44])**.** *Let F be a field of characteristic 0. The diagonal of a rational function in two variables over F that is regular at the origin is algebraic over $F(x)$. Conversely, every power series in $F[[x]]$ that is algebraic over $F(x)$ is such a diagonal.*

2.2. **Hypergeometric local systems.** The series $F_{a,b}$ is annihilated by

$$L_{a,b} = \theta^2 \prod_{j=1}^{n-2}(\theta+b_j-1) - x\prod_{i=1}^{n}(\theta+a_i). \tag{2.1}$$

This is Fuchsian on $\mathbb{P}^1$, with singularities contained in $0,1,\infty$. Let $\mathcal{S}_{a,b}$ be its solution local system on $U(\mathbb{C})$. Put

$$P_a(t) = \prod_{i=1}^{n}(t-\mathbf{e}(a_i)), \qquad P_b(t) = (t-1)^2\prod_{j=1}^{n-2}(t-\mathbf{e}(b_j)),$$

and let A, B be their companion matrices.

Lemma 2.5 (Levelt; [7, Propositions 3.2–3.3, Theorem 3.5])**.** *Under (NR), $\mathcal{S}_{a,b}$ is irreducible. In a suitable basis, with suitable loops,*

$$T_0 = B^{-1}, \qquad T_1 = A^{-1}B, \qquad T_\infty = A, \qquad T_\infty T_1 T_0 = 1.$$

The operator T_1-1 has rank one. The local system depends only on the parameter multisets modulo $\mathbb{Z}$.

Lemma 2.6. *Assume (NR), and put $d=\operatorname{lcm}_j \operatorname{den}(b_j)$.*

(i) *The generalized 1-eigenspace of T_0 has dimension two and is spanned by $F_{a,b}$ and $Y = F_{a,b}\log x + G$, with $G\in x\mathbb{Q}[[x]]$.*

(ii) *$T_0F_{a,b}=F_{a,b}$ and $(T_0^r-1)Y = 2\pi i r F_{a,b}$ for every integer $r\ge 1$.*

(iii) *The minimal polynomial of T_0 is*

$$(t-1)^2\prod_j (t-\mathbf{e}(-b_j)).$$

There is one Jordan block per distinct eigenvalue, of size its multiplicity in this polynomial. If $F_{a,b}$ is globally bounded, then

$$(T_0^d-1)^2=0, \qquad \operatorname{rank}(T_0^d-1) = 1+\#\{\textit{nonintegral lower classes of multiplicity two}\}.$$

(iv) *$F_{a,b}$ is transcendental over $\mathbb{C}(x)$.*

Proof. A companion matrix and its inverse are non-derogatory. Since no b_j is integral, the eigenvalue 1 has precisely one block of size two. The indicial polynomial $\theta^2\prod_j(\theta+b_j-1)$ has no other integral root. Frobenius recursion therefore gives the solutions in (i), with rational coefficients, and analytic continuation gives (ii). The first part of (iii) follows from the

companion description. Under global boundedness, Proposition 3.1 below bounds every lower multiplicity by two. Taking the d-th power makes all eigenvalues one without changing block sizes; this proves the remaining assertions. Finally, if $F_{a,b}$ were algebraic, its finitely many branches would span a nonzero invariant space with finite monodromy. Irreducibility makes this the whole solution space, contradicting the nontrivial unipotent block at 1. □

The rank-one block used to extract $F_{a,b}$ in (ii) must not be confused with the full monodromy variation in (iii), whose rank can exceed one.

2.3. **Global boundedness.** For $t \in \mathbb{R}$, let $\langle t \rangle \in (0,1]$ represent t modulo $\mathbb{Z}$.

Lemma 2.7 (Christol; [27, Theorem 2.3])**.** *Let $a_1, \dots, a_n \in \mathbb{Q} \setminus \mathbb{Z}$ and $c_1, \dots, c_{n-1} \in (\mathbb{Q} \setminus \mathbb{Z}) \cup \{1\}$, with no cross-list integral differences. Put $c_n = 1$ and let N be a common denominator. Then ${}_nF_{n-1}(a; c; x)$ is globally bounded if and only if, for every $k \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $0 < t < 1$,*

$$\#\{i : \langle k a_i \rangle \le t\} \ge \#\{j : \langle k c_j \rangle \le t\}.$$

The counts include multiplicities.

This is Christol's criterion [13] in the precise general-order form of the cited modern restatement. Cross-list ties are absent here; integral lower entries occur only at the endpoint 1. The source's ordering convention for equal fractional parts therefore creates no further restriction. Integer shifts of the nonintegral parameters do not alter the comparisons.

2.4. **Complex variations of Hodge structure.** We recall the notion of a complex variation of Hodge structure; see Deligne [20, §1] and Simpson [46]. For a complex manifold M we write $\mathcal{E}_M^{p,q}$ for the sheaf of smooth forms of type (p,q) and $\mathcal{E}_M^0$ for the sheaf of smooth functions.

Definition 2.8. Let M be a complex manifold and $\mathcal{S}$ a local system of finite-dimensional complex vector spaces on M, with flat connection ∇ on the smooth bundle $\mathcal{S} \otimes \mathcal{E}_M^0$. A *complex variation of Hodge structure* on $\mathcal{S}$ is a decomposition $\mathcal{S} \otimes \mathcal{E}_M^0 = \bigoplus_{p \in \mathbb{Z}} \mathcal{S}^p$ into smooth subbundles such that

$$\nabla(\mathcal{S}^p) \subset \mathcal{E}_M^{1,0}(\mathcal{S}^p \oplus \mathcal{S}^{p-1}) \oplus \mathcal{E}_M^{0,1}(\mathcal{S}^p \oplus \mathcal{S}^{p+1}) \qquad \text{for all } p. \tag{2.2}$$

The *Hodge numbers* are $h^p = \operatorname{rank} \mathcal{S}^p$. The variation is *polarized* by a flat hermitian form Q on $\mathcal{S}$ if the subbundles $\mathcal{S}^p$ are mutually orthogonal for Q and $(-1)^p Q$ is positive definite on $\mathcal{S}^p$ for every p. It is *polarizable* if such a form exists.

By (2.2), the bundle $\bigoplus_{q \ge p} \mathcal{S}^q$ is a holomorphic subbundle of $\mathcal{S} \otimes \mathcal{O}_M$ for every p. In particular, if only two degrees p and $p-1$ occur, then $\mathcal{S}^p$ is a holomorphic subbundle, and $\mathcal{S}^{p-1}$ is stable under the $(1,0)$-part of ∇. In this case we call $\mathcal{S}^p$ the *top piece* and $\mathcal{S}^{p-1}$ the *bottom piece*, and we say that the variation has *two adjacent Hodge levels*.

Theorem 2.9 (Fedorov [26, Proposition 2.6 and Theorem 1])**.** *Let $0 \le \alpha_1 \le \cdots \le \alpha_n < 1$ and $0 \le \beta_1 \le \cdots \le \beta_n < 1$ be real numbers with $\alpha_i \ne \beta_j$ for all i, j. The connection on U defined by the operator $\prod_j (\theta - \alpha_j) - x \prod_j (\theta - \beta_j)$ underlies a polarizable complex variation of Hodge structure, which is unique up to a shift of the grading. Up to such a shift, its Hodge numbers are*

$$h^p = \#\{l : \rho(l) = p\}, \qquad \text{where } \rho(l) = \#\{j : \alpha_j < \beta_l\} - l \quad (l = 1, \dots, n).$$

The lists may have internal repetitions: Fedorov requires only weak ordering and the absence of cross-list coincidences. We use the existence and uniqueness statement of his Proposition 2.6 and the formula of his Theorem 1 in arbitrary rank. The underlying results of Simpson, Deligne and Dettweiler–Sabbah cited there are credited through Fedorov, rather than treated as additional independently rechecked imports.

Remark 2.10. The dual of a polarized complex variation of Hodge structure is again one, with the Hodge numbers in reverse order. The local system of solutions of a differential operator is dual to the local system of horizontal sections of the corresponding connection. Hence it does not matter, for the property of having two adjacent Hodge levels, whether Theorem 2.9 is read on the connection or on its solutions.

Lemma 2.11. *Let $(\mathcal{S}, \bigoplus_p \mathcal{S}^p, Q)$ be a polarized complex variation of Hodge structure on M, and let $\overline{\mathcal{S}}$ be the complex conjugate local system. Put $\overline{\mathcal{S}}^q = \overline{\mathcal{S}^{-q}}$ and $\overline{Q}(\bar{s}, \bar{s}') = \overline{Q(s, s')}$. Then $(\overline{\mathcal{S}}, \bigoplus_q \overline{\mathcal{S}}^q, \overline{Q})$ is a polarized complex variation of Hodge structure. In particular, if $\mathcal{S}$ has two adjacent Hodge levels, then so has $\overline{\mathcal{S}}$; the complex conjugate of the bottom piece of $\mathcal{S}$ is the top piece of $\overline{\mathcal{S}}$, and it is a holomorphic subbundle of $\overline{\mathcal{S}} \otimes \mathcal{O}_M$.*

Proof. The flat connection of $\overline{\mathcal{S}}$ is given by $\overline{\nabla}(\bar{s}) = \overline{\nabla s}$, and complex conjugation exchanges $\mathcal{E}^{1,0}_M$ and $\mathcal{E}^{0,1}_M$. Conjugating (2.2) gives

$$\overline{\nabla}(\overline{\mathcal{S}^p}) \subset \mathcal{E}^{0,1}_M(\overline{\mathcal{S}^p} \oplus \overline{\mathcal{S}^{p-1}}) \oplus \mathcal{E}^{1,0}_M(\overline{\mathcal{S}^p} \oplus \overline{\mathcal{S}^{p+1}}),$$

which is (2.2) for the decomposition $\overline{\mathcal{S}}^q = \overline{\mathcal{S}^{-q}}$. The form $\overline{Q}$ is flat and hermitian, the subbundles $\overline{\mathcal{S}}^q$ are mutually orthogonal, and $\overline{Q}$ has on $\overline{\mathcal{S}}^q$ the sign of Q on $\mathcal{S}^{-q}$, which is $(-1)^q$. If $\mathcal{S} = \mathcal{S}^p \oplus \mathcal{S}^{p-1}$, then the top piece of $\overline{\mathcal{S}}$ is $\overline{\mathcal{S}}^{1-p} = \overline{\mathcal{S}^{p-1}}$, and it is holomorphic by the remark following Definition 2.8. □

2.5. **Weight one and abelian schemes.**

Definition 2.12. Let M be a complex manifold. A *polarized variation of Hodge structure of type* $\{(1,0),(0,1)\}$ on M consists of a local system $V_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of finite rank, a holomorphic subbundle $F^1 \subset V_{\mathbb{C}} \otimes \mathcal{O}_M$ such that $V_{\mathbb{C}} \otimes \mathcal{E}^0_M = F^1 \oplus \overline{F^1}$, and a flat alternating form $\psi : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ such that $\psi(F^1, F^1) = 0$ and $i\,\psi(v, \bar{v}) > 0$ for all nonzero $v \in F^1$. Here $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$, and ψ is extended bilinearly.

Griffiths transversality is automatic for this type, since the Hodge filtration has only the two steps $F^1 \subset V_{\mathbb{C}} \otimes \mathcal{O}_M$. The Tate twist $V_{\mathbb{Z}}(1) = V_{\mathbb{Z}} \otimes \mathbb{Z}(1)$, where $\mathbb{Z}(1) = 2\pi i\,\mathbb{Z}$ has type $(-1,-1)$, is a polarizable variation of type $\{(-1,0),(0,-1)\}$. Its underlying local system $V_{\mathbb{Z}} \otimes 2\pi i\,\mathbb{Z}$ is isomorphic to $V_{\mathbb{Z}}$, by multiplication by $(2\pi i)^{-1}$; see Remark 4.4(2) for the difference between a Tate twist and a shift of the grading of a single complex variation.

Theorem 2.13 (Deligne [18, 4.4.3]; Milne [43, Theorem 7.13])**.** *Let S be a smooth scheme of finite type over $\mathbb{C}$. The functor $(f : \mathcal{A} \to S) \mapsto R_1 f_* \mathbb{Z}$ is an equivalence between the category of abelian schemes over S and the category of polarizable variations of Hodge structure of type $\{(-1,0),(0,-1)\}$ on $S(\mathbb{C})$ whose underlying local system is torsion-free.*

Both references state the equivalence for the homology $R_1 f_* \mathbb{Z}$ and the type $\{(-1,0),(0,-1)\}$, as above, and Deligne attributes the essential surjectivity to Borel. We use it only through the following consequence: if $V_{\mathbb{Z}}$ is a polarized variation of type $\{(1,0),(0,1)\}$ on $S(\mathbb{C})$, then the abelian scheme $f : \mathcal{A} \to S$ with $R_1 f_* \mathbb{Z} \cong V_{\mathbb{Z}}(1)$ satisfies $R_1 f_* \mathbb{Q} \cong V_{\mathbb{Q}}$ as local systems. Throughout, $R_1 f_*$ denotes the homology and $R^1 f_* = (R_1 f_*)^\vee$ the cohomology of the fibres, and periods are obtained by pairing $R_1 f_*$ with the de Rham cohomology H^1_{dR}.

3. Two Hodge levels and global boundedness

Throughout this section assume (NR). Let N be the least common denominator and $k \in (\mathbb{Z}/N\mathbb{Z})^\times$. Define the step function

$$D_k(t) = \#\{i : \{ka_i\} \le t\} - \#\{j : \{kb_j\} \le t\}, \qquad 0 < t < 1.$$

The supports of its positive and negative jumps are disjoint. The total change is $n-(n-2)=2$.

Proposition 3.1. *The following conditions are equivalent:*

(i) *$F_{a,b}$ is globally bounded;*
(ii) *$0 \le D_k(t) \le 2$ for every k and $0 < t < 1$;*
(iii) *every conjugate $\mathcal{S}_{ka,kb}$ has precisely two adjacent nonzero Hodge levels.*

If they hold, each fractional upper class and each nonintegral lower class has multiplicity at most two.

Proof. Lemma 2.7 says that (i) is equivalent to $D_k \ge 0$ for all k, since both integral lower entries are at 1. Reflection gives

$$D_{-k}(1-t) = 2 - D_k(t^-). \tag{3.1}$$

Values and left limits have the same range, so imposing nonnegativity at k and $-k$ is equivalent to (ii).

For Fedorov's formula take the sorted lists

$$\alpha = (0, 0, \{-kb_1\}, \dots, \{-kb_{n-2}\}), \qquad \beta = (\{-ka_i\})_{i=1}^n.$$

Integer parameter shifts identify the local systems, and passage to the dual reverses the Hodge degrees without changing adjacency (Remark 2.10). Put $t_l = 1 - \beta_l$. Then

$$\rho(l) = n - l - \#\{j : \{kb_j\} \le t_l\}.$$

If $\beta_p = \cdots = \beta_q$ is a maximal equal block, and $t^* = 1 - \beta_p$, the endpoints satisfy

$$D_k((t^*)^-) = \rho(q) \le \rho(l) \le \rho(p) = D_k(t^*) - 1 \qquad (p \le l \le q). \tag{3.2}$$

Thus (ii) forces every $\rho(l)$ into $\{0,1\}$. Since the two zeros in α give $\rho(1) \ge 1$, whereas $\rho(n) \le 0$, both values occur. This proves (iii).

Conversely, if the $\rho(l)$ occupy two adjacent integers, these same endpoint inequalities force that pair to be $\{0,1\}$. Every minimum of D_k is attained immediately before a positive jump, or at its initial value 0; every maximum is attained immediately after such a jump, or at its final value 2. Formula (3.2) gives $\min D_k \ge 0$ and $\max D_k \le 2$. Apply this to every conjugate to obtain (ii). Finally, no jump of a function confined to $[0,2]$ has magnitude greater than two, proving the multiplicity assertion. □

Corollary 3.2 (The rank-three criterion)**.** *For $n = 3$, write $b = b_1$, and sort $a_1^{(k)} \le a_2^{(k)} \le a_3^{(k)}$, the fractional parts of ka_i; put $b^{(k)} = \{kb\}$. For fixed k, the Hodge numbers from highest to lowest degree are*

position	*nonzero Hodge numbers*
$b^{(k)} < a_1^{(k)}$	$(1,1,1)$
$a_1^{(k)} < b^{(k)} < a_2^{(k)}$	$(1,2)$
$a_2^{(k)} < b^{(k)} < a_3^{(k)}$	$(2,1)$
$a_3^{(k)} < b^{(k)}$	$(1,1,1)$.

Consequently $F_{a,b}$ is globally bounded exactly when $a_1^{(k)} < b^{(k)} < a_3^{(k)}$ for every k.

Proof. Substitute $\alpha = (0, 0, 1-b^{(k)})$ and $\beta = (1-a_3^{(k)}, 1-a_2^{(k)}, 1-a_1^{(k)})$ into Fedorov's formula. It gives

$$(\rho(1), \rho(2), \rho(3)) = (1 + [a_3^{(k)} < b^{(k)}],\ [a_2^{(k)} < b^{(k)}],\ -1 + [a_1^{(k)} < b^{(k)}]),$$

where brackets denote indicator functions. This yields the table, including repeated upper entries, and the final assertion follows from Proposition 3.1. Reversing the dual convention reverses the displayed Hodge numbers and leaves the criterion unchanged. □

Appendix A checks this rank-three specialization independently on Euler-integral surfaces. That finite computation is not used to prove the arbitrary-rank proposition.

4. A polarized variation of Hodge structure of weight one

Throughout this section (a, b) satisfies (NR) and $F_{a,b}$ is globally bounded. Let N be the common denominator of the parameters. Put $\zeta_N = \mathbf{e}(1/N)$ and $K = \mathbb{Q}(\zeta_N) \subset \mathbb{C}$, with ring of integers $\mathcal{O}_K = \mathbb{Z}[\zeta_N]$. Let $K^+ = K \cap \mathbb{R}$, and for $k \in (\mathbb{Z}/N\mathbb{Z})^\times$ let σ_k be the automorphism of K with $\sigma_k(\zeta_N) = \zeta_N^k$. Complex conjugation preserves K and equals σ_{-1}.

Lemma 4.1. *We have $N \geq 3$. Consequently K is a CM field of degree $\varphi(N)$ with maximal totally real subfield K^+, complex conjugation commutes with every σ_k, and $\sigma_k \neq \sigma_{-k}$ for all k.*

Proof. If $N \leq 2$, then the non-integers a_1, b_1 all lie in $\frac{1}{2} + \mathbb{Z}$, so that $a_1 - b_1 \in \mathbb{Z}$, contradicting (NR). The remaining statements hold for every cyclotomic field $\mathbb{Q}(\zeta_N)$ with $N \geq 3$. □

We fix a nonzero element $\xi_0 \in K$ with $\overline{\xi_0} = -\xi_0$, for instance $\xi_0 = \zeta_N - \zeta_N^{-1}$. Every $\xi \in K$ with $\bar{\xi} = -\xi$ is of the form $\lambda\xi_0$ with $\lambda \in K^+$.

4.1. **The cyclotomic lattice.** The coefficients of P_a and P_b are symmetric functions of N-th roots of unity, so A and B have entries in $\mathcal{O}_K$. Their determinants $\mathbf{e}(\sum_i a_i)$ and $\mathbf{e}(\sum_j b_j)$ are roots of unity, so $A, B \in \mathrm{GL}_n(\mathcal{O}_K)$. Let Λ be the local system on $U(\mathbb{C})$ with fibre $\mathcal{O}_K^n$ and with the monodromy matrices $T_0 = B^{-1}$, $T_1 = A^{-1}B$, $T_\infty = A$ of Lemma 2.5. Let $V_\mathbb{Z}$ be Λ viewed as a local system of free $\mathbb{Z}$-modules of rank $n\varphi(N)$, and put $V_\mathbb{Q} = V_\mathbb{Z} \otimes \mathbb{Q}$, $V_\mathbb{R} = V_\mathbb{Z} \otimes \mathbb{R}$ and $V_\mathbb{C} = V_\mathbb{Z} \otimes \mathbb{C}$. The fibre of $V_\mathbb{Q}$ is K^n, on which the monodromy acts K-linearly through matrices in $\mathrm{GL}_n(\mathcal{O}_K)$.

For $k \in (\mathbb{Z}/N\mathbb{Z})^\times$ let $\mathcal{S}_k$ be the local system with fibre $\mathbb{C}^n$ and monodromy matrices $\sigma_k(T_0)$, $\sigma_k(T_1)$, $\sigma_k(T_\infty)$. The $\mathbb{C}$-linear isomorphism

$$K^n \otimes_\mathbb{Q} \mathbb{C} \longrightarrow \bigoplus_{k \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathbb{C}^n, \qquad x \otimes z \longmapsto \big(\sigma_k(x)\, z\big)_k, \tag{4.1}$$

where σ_k is applied to the coordinates of x, intertwines $T \otimes 1$ with $(\sigma_k(T))_k$ for every $T \in \mathrm{GL}_n(K)$. It therefore defines an isomorphism of local systems

$$V_\mathbb{C} \cong \bigoplus_{k \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathcal{S}_k. \tag{4.2}$$

Since $\sigma_k(A)$ and $\sigma_k(B)$ are the companion matrices of P_{ka} and P_{kb}, Lemma 2.5 gives $\mathcal{S}_k \cong \mathcal{S}_{ka,kb}$; in particular $\mathcal{S}_1 \cong \mathcal{S}_{a,b}$. We write elements of $V_\mathbb{C}$ as $X = (X_k)_k$ with $X_k \in \mathcal{S}_k$.

Let $X \mapsto \overline{X}$ be complex conjugation on $V_\mathbb{C} = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, that is, $x \otimes z \mapsto x \otimes \bar{z}$ for $x \in V_\mathbb{Q}$. Since complex conjugation on K is σ_{-1} and commutes with every σ_k, we have $\sigma_k(x)\,\bar{z} = \overline{\sigma_{-k}(x)\, z}$. Hence (4.1) transforms complex conjugation into

$$(\overline{X})_k = \overline{X_{-k}}, \tag{4.3}$$

where on the right the bar denotes complex conjugation of the coordinates in $\mathbb{C}^n$. Thus complex conjugation maps $\mathcal{S}_{-k}$ antilinearly onto $\mathcal{S}_k$. It carries flat sections to flat sections, because $\overline{\sigma_{-k}(T)} = \sigma_k(T)$, and so identifies $\mathcal{S}_k$, as a local system, with the complex conjugate local system $\overline{\mathcal{S}_{-k}}$. Under this identification the holomorphic structures of $\mathcal{S}_k \otimes \mathcal{O}$ and $\overline{\mathcal{S}_{-k}} \otimes \mathcal{O}$ agree, since both are determined by the flat sections. For $Z \in \mathcal{S}_{-k}$ we write $\overline{Z} \in \mathcal{S}_k$ for its image.

Even if $\mathcal{S}_k$ and $\mathcal{S}_{-k}$ are isomorphic as abstract local systems in even rank, they are distinct summands in (4.2), since $k \neq -k$. This is why the full cyclotomic CM field suffices uniformly.

4.2. **Invariant hermitian forms.**

Lemma 4.2. *There is a matrix $h \in M_n(K)$ with $h^* = h$ and $\det h \neq 0$ such that $A^*hA = h$ and $B^*hB = h$. It is unique up to multiplication by an element of $(K^+)^\times$. For every k, the form*

$$\langle X, X' \rangle_k = X^* \sigma_k(h) X'$$

is a flat nondegenerate hermitian form on $\mathcal{S}_k$, and every flat hermitian form on $\mathcal{S}_k$ is a real multiple of it.

Proof. The equations $A^*XA = X$ and $B^*XB = X$ form a system of linear equations over K in the entries of a $n \times n$ matrix X. By Proposition 3.1, the local system $\mathcal{S}_1$ carries a polarized complex variation of Hodge structure, hence a flat nondegenerate hermitian form. Its Gram matrix in the basis of Lemma 2.5 is a nonzero complex solution of the system. A system of linear equations over K that has a nonzero solution over $\mathbb{C}$ has a nonzero solution over K; let X be one. Then X^* is also a solution. If $X + X^* \neq 0$ we put $h = X + X^*$, and otherwise $h = \xi_0 X$, which is hermitian because $X^* = -X$. The relation $A^*hA = h$ gives $hAv = (A^*)^{-1}hv$, and similarly for B, so the kernel of h is stable under the monodromy; it is zero because $\mathcal{S}_1$ is irreducible. If h' is a second matrix with these properties, then $h^{-1}h'$ commutes with A and B and is therefore a scalar $\lambda \in K$, by irreducibility, and $\lambda \in K^+$ because h and h' are hermitian. Applying σ_k, which commutes with complex conjugation, shows that $\sigma_k(h)$ is hermitian, nondegenerate and invariant under $\sigma_k(A)$ and $\sigma_k(B)$. Finally, a flat hermitian form on the irreducible local system $\mathcal{S}_k$ is zero or nondegenerate, because its kernel is a sub-local system. Two flat nondegenerate hermitian forms differ by an automorphism of $\mathcal{S}_k$, that is, by a scalar, and this scalar is real. □

4.3. **The variation of weight one.**

Theorem 4.3. *Assume (NR) and that $F_{a,b}$ is globally bounded. Then there are a holomorphic subbundle $F^1 \subset V_{\mathbb{C}} \otimes \mathcal{O}_{U(\mathbb{C})}$ of rank $\frac{n}{2}\varphi(N)$ and an element $\lambda \in (K^+)^\times$ such that the local system $V_{\mathbb{Z}}$ of §4.1, F^1 and the form*

$$\psi(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}\left(\lambda\, \xi_0\, x^* h y\right) \qquad (x, y \in K^n)$$

constitute a polarized variation of Hodge structure of type $\{(1,0),(0,1)\}$ on $U(\mathbb{C})$.

Proof. We verify the conditions of Definition 2.12 in eight steps.

Step 1: the integral local system and its complex decomposition. By construction, $V_{\mathbb{Z}}$ is a local system of free $\mathbb{Z}$-modules of rank $n\varphi(N)$ with $V_{\mathbb{Z}} \otimes \mathbb{Q} = V_{\mathbb{Q}}$, and (4.2) decomposes $V_{\mathbb{C}}$ into the local systems $\mathcal{S}_k \cong \mathcal{S}_{ka,kb}$. Each of them has rank n and is irreducible by Lemma 2.5, since (ka, kb) satisfies (NR).

Step 2: complex variations on the conjugates, compatible with conjugation. Let $\Phi \subset (\mathbb{Z}/N\mathbb{Z})^\times$ be a set of representatives modulo ± 1. By Lemma 4.1, Φ has $\frac{1}{2}\varphi(N)$ elements and $(\mathbb{Z}/N\mathbb{Z})^\times$ is the disjoint union of Φ and $-\Phi$. For $k \in \Phi$ we choose on $\mathcal{S}_k \cong \mathcal{S}_{ka,kb}$ a polarized complex variation of Hodge structure, as in Proposition 3.1. Since $F_{a,b}$ is globally bounded, it has two adjacent Hodge levels; let $\mathcal{S}_k^{\mathrm{top}}$ and $\mathcal{S}_k^{\mathrm{bot}}$ be its top and bottom pieces and Q_k a polarization. For $k \in -\Phi$ we transport the conjugate variation of Lemma 2.11 along the identification $\mathcal{S}_k = \overline{\mathcal{S}_{-k}}$ of (4.3):

$$\mathcal{S}_k^{\mathrm{top}} = \overline{\mathcal{S}_{-k}^{\mathrm{bot}}}, \qquad \mathcal{S}_k^{\mathrm{bot}} = \overline{\mathcal{S}_{-k}^{\mathrm{top}}}, \qquad Q_k(\overline{Z}, \overline{Z'}) = \overline{Q_{-k}(Z, Z')} \qquad (k \in -\Phi,\ Z, Z' \in \mathcal{S}_{-k}).$$

Since (4.3) is an involution, the relations

$$\overline{\mathcal{S}_k^{\mathrm{top}}} = \mathcal{S}_{-k}^{\mathrm{bot}} \qquad \text{and} \qquad \overline{\mathcal{S}_k^{\mathrm{bot}}} = \mathcal{S}_{-k}^{\mathrm{top}} \tag{4.4}$$

then hold for every k. These relations are the only constraint imposed between the gradings of $\mathcal{S}_k$ and $\mathcal{S}_{-k}$. The displayed identity for Q_k concerns the unshifted conjugate variation. A subsequent grading shift may multiply its polarization by -1; no identity between the shifted polarizations is assumed. The signs in Step 7 account for this change. In particular, no Tate twist is involved at this stage; see Remark 4.4.

Step 3: holomorphicity of F^1. Put

$$F^1 = \bigoplus_k \mathcal{S}_k^{\mathrm{top}} \ \subset\ V_{\mathbb{C}} \otimes \mathcal{O}_{U(\mathbb{C})}.$$

For $k \in \Phi$, the top piece $\mathcal{S}_k^{\mathrm{top}}$ is a holomorphic subbundle of $\mathcal{S}_k \otimes \mathcal{O}$ by the remark following Definition 2.8. For $k \in -\Phi$ it is one by Lemma 2.11, because the holomorphic structure of $\mathcal{S}_k \otimes \mathcal{O}$ is that of $\overline{\mathcal{S}_{-k}} \otimes \mathcal{O}$. Hence F^1 is a holomorphic subbundle. By (4.4) its rank is

$$\sum_{k\in\Phi} \left(\operatorname{rank} \mathcal{S}_k^{\mathrm{top}} + \operatorname{rank} \mathcal{S}_{-k}^{\mathrm{top}}\right) = \sum_{k\in\Phi} \left(\operatorname{rank} \mathcal{S}_k^{\mathrm{top}} + \operatorname{rank} \mathcal{S}_k^{\mathrm{bot}}\right) = \tfrac{n}{2}\varphi(N).$$

Step 4: the Hodge decomposition. By (4.3) and (4.4), complex conjugation maps $\mathcal{S}_k^{\mathrm{top}}$ onto $\mathcal{S}_{-k}^{\mathrm{bot}}$ for every k. Hence $\overline{F^1} = \bigoplus_k \mathcal{S}_k^{\mathrm{bot}}$, and

$$V_{\mathbb{C}} \otimes \mathcal{E}^0 = \bigoplus_k \left(\mathcal{S}_k^{\mathrm{top}} \oplus \mathcal{S}_k^{\mathrm{bot}}\right) = F^1 \oplus \overline{F^1}.$$

Step 5: the trace form. For $\xi \in K^\times$ with $\bar\xi = -\xi$ put

$$\psi_\xi(x,y) = \operatorname{Tr}_{K/\mathbb{Q}}\left(\xi\, x^* h y\right) \qquad (x, y \in K^n).$$

This is a $\mathbb{Q}$-bilinear form on the fibre of $V_{\mathbb{Q}}$. It is invariant under the monodromy, because the monodromy group is generated by A and B, and $T^* h T = h$ for $T \in \{A, B\}$ by Lemma 4.2. It is alternating: since $\overline{x^* h y} = y^* h x$ and the trace is invariant under complex conjugation, $\psi_\xi(y,x) = \operatorname{Tr}_{K/\mathbb{Q}}(\bar\xi\, x^* h y) = -\psi_\xi(x,y)$; moreover $\psi_\xi(x,x) = \operatorname{Tr}_{K^+/\mathbb{Q}}\left(x^* h x \cdot \operatorname{Tr}_{K/K^+}(\xi)\right) = 0$, because $x^* h x \in K^+$ and $\operatorname{Tr}_{K/K^+}(\xi) = \xi + \bar\xi = 0$. It is nondegenerate: if $\psi_\xi(x,y) = 0$ for all y, then the K-linear form $y \mapsto \xi\, x^* h y$ on K^n has trace zero everywhere; as the trace form of K is nondegenerate, it vanishes, so $x^* h = 0$ and $x = 0$.

To compute the complex bilinear extension, note that $\operatorname{Tr}_{K/\mathbb{Q}}(c) = \sum_k \sigma_k(c)$ and $\sigma_k(\bar x) = \sigma_{-k}(x)$. For $x, y \in K^n$ with images $X = (\sigma_k(x))_k$ and $Y = (\sigma_k(y))_k$ under (4.1),

$$\psi_\xi(x,y) = \sum_k \sigma_k(\xi)\, \sigma_{-k}(x)^T \sigma_k(h)\, \sigma_k(y) = \sum_k \sigma_k(\xi)\, X_{-k}^T\, \sigma_k(h)\, Y_k.$$

The right-hand side is $\mathbb{C}$-bilinear in (X, Y), so the complex bilinear extension of ψ_ξ is

$$\psi_{\xi,\mathbb{C}}(X, X') = \sum_k \sigma_k(\xi)\, X_{-k}^T\, \sigma_k(h)\, X'_k. \tag{4.5}$$

Step 6: isotropy of F^1. For every k, Q_k is a flat hermitian form on $\mathcal{S}_k$ (for $k \in -\Phi$ by Lemma 2.11), so Lemma 4.2 gives $\langle\cdot,\cdot\rangle_k = c_k Q_k$ with $c_k \in \mathbb{R}^\times$. Hence $\mathcal{S}_k^{\mathrm{top}}$ and $\mathcal{S}_k^{\mathrm{bot}}$ are orthogonal for $\langle\cdot,\cdot\rangle_k$. Let $X, X' \in F^1$, and put $Z = \overline{X_{-k}} \in \mathcal{S}_k$. By (4.3) and Step 4, $Z = (\overline{X})_k$ lies in $\mathcal{S}_k^{\mathrm{bot}}$, and $X_{-k}^T = Z^*$. The k-th summand of (4.5) is therefore $\sigma_k(\xi)\langle Z, X'_k\rangle_k = 0$. Hence $\psi_{\xi,\mathbb{C}}(F^1, F^1) = 0$.

Step 7: positivity. Since Q_k polarizes a variation with two adjacent levels, $\langle\cdot,\cdot\rangle_k$ is definite on $\mathcal{S}_k^{\mathrm{top}}$ and on $\mathcal{S}_k^{\mathrm{bot}}$, with opposite signs. Let $\varepsilon_k \in \{\pm 1\}$ be its sign on $\mathcal{S}_k^{\mathrm{top}}$. Since $\sigma_{-k}(h) = \overline{\sigma_k(h)}$, we have $\langle\overline{Z},\overline{Z}\rangle_{-k} = \overline{\langle Z, Z\rangle_k} = \langle Z, Z\rangle_k$ for $Z \in \mathcal{S}_k$, where $\overline{Z} \in \mathcal{S}_{-k}$. Taking $Z \in \mathcal{S}_k^{\mathrm{bot}}$, so that $\overline{Z} \in \mathcal{S}_{-k}^{\mathrm{top}}$ by (4.4), gives

$$\varepsilon_{-k} = -\varepsilon_k \qquad \text{for all } k. \tag{4.6}$$

Now let $X \in F^1$. By (4.5) and (4.3),

$$\psi_{\xi,\mathbb{C}}(X, \overline{X}) = \sum_k \sigma_k(\xi)\, X_{-k}^T \sigma_k(h) \overline{X_{-k}} = \sum_k \sigma_k(\xi)\, \langle X_{-k}, X_{-k} \rangle_{-k},$$

where the second equality transposes the scalar $X_{-k}^T \sigma_k(h) \overline{X_{-k}}$ and uses $\sigma_k(h)^T = \overline{\sigma_k(h)} = \sigma_{-k}(h)$. Replacing k by $-k$ and using $\sigma_{-k}(\xi) = \overline{\sigma_k(\xi)} = -\sigma_k(\xi)$, we obtain

$$i\,\psi_{\xi,\mathbb{C}}(X, \overline{X}) = -\sum_k i\,\sigma_k(\xi)\, \langle X_k, X_k \rangle_k \qquad \text{with } X_k \in \mathcal{S}_k^{\mathrm{top}}. \tag{4.7}$$

The numbers $i\sigma_k(\xi)$ are real and nonzero. As the components X_k of $X \in F^1$ range independently over the nonzero bundles $\mathcal{S}_k^{\mathrm{top}}$, the form $X \mapsto i\,\psi_{\xi,\mathbb{C}}(X, \overline{X})$ is positive on $F^1 \setminus \{0\}$ if and only if

$$-i\,\sigma_k(\xi)\,\varepsilon_k > 0 \qquad \text{for all } k. \tag{4.8}$$

By (4.6) and $\sigma_{-k}(\xi) = -\sigma_k(\xi)$, the conditions for k and for $-k$ coincide, so it suffices to impose (4.8) for $k \in \Phi$. Write $\xi = \lambda\xi_0$ with $\lambda \in K^+$. Since $i\sigma_k(\xi_0)$ is a nonzero real number, the condition for $k \in \Phi$ prescribes the sign of $\sigma_k(\lambda)$, that is, the sign of λ at the real place of K^+ given by $\sigma_k|_{K^+}$. The map $k \mapsto \sigma_k|_{K^+}$ is a bijection from Φ onto the set of real places of K^+, because σ_k and σ_{-k} have the same restriction to K^+ and $[K^+ : \mathbb{Q}] = \frac{1}{2}\varphi(N)$. By weak approximation, that is, because K^+ is dense in $K^+ \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{\varphi(N)/2}$, there is $\lambda \in K^+$ with the prescribed signs at all real places.

Step 8: integrality. The fibre $\mathcal{O}_K^n$ of $V_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank, and $\psi_{\lambda\xi_0}$ takes rational values on it. Multiplying λ by a common denominator of its values on pairs of elements of a $\mathbb{Z}$-basis, a positive integer, preserves the sign conditions of Step 7 and makes $\psi = \psi_{\lambda\xi_0}$ integral on $V_{\mathbb{Z}}$; it remains invariant, as the monodromy preserves $\mathcal{O}_K^n$. By Steps 5–7, ψ is a flat, alternating, nondegenerate form $V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ with $\psi(F^1, F^1) = 0$ and $i\,\psi(X, \overline{X}) > 0$ for $0 \neq X \in F^1$. Together with Steps 3 and 4 this is Definition 2.12. □

Remark 4.4 (Grading shifts and Tate twists). (1) A complex variation of Hodge structure on an irreducible local system is determined only up to a shift of the grading (Theorem 2.9). Shifting the grading of a polarized complex variation by one replaces a polarization Q by $-Q$, because the sign of Q on $\mathcal{S}^p$ is $(-1)^p$. In the proof of Theorem 4.3 the only constraint between gradings is (4.4), which puts every top piece in degree 1 and every bottom piece in degree 0 of the real variation; the resulting signs are recorded by the ε_k and absorbed by the choice of λ in Step 7.

(2) Such a choice of gradings, made separately for each pair $\{k, -k\}$, is not a Tate twist. For an integral variation V, the Tate twist $V(1) = V \otimes \mathbb{Z}(1)$, where $\mathbb{Z}(1) = 2\pi i\,\mathbb{Z}$ has type $(-1,-1)$, shifts the types of all conjugates simultaneously by $(-1,-1)$; its underlying local system $V \otimes 2\pi i\,\mathbb{Z}$ is isomorphic to that of V by multiplication by $(2\pi i)^{-1}$. We use a Tate twist only in Section 5, to pass from type $\{(1,0),(0,1)\}$ to the type $\{(-1,0),(0,-1)\}$ of Theorem 2.13.

(3) In rank three, the Euler integral realizes the local systems $\mathcal{S}_k$ in the second cohomology of a family of surfaces, with a variation of Hodge structure of weight two; for Christol's series the Hodge numbers of the six conjugates add up to $(3, 12, 3)$ (Appendix A). Relative to that realization, the variation of Theorem 4.3 is obtained by shifting the gradings of the conjugates indexed by Φ and by $-\Phi$ relative to each other by one. Over a point this is the weight-lowering half twist of van Geemen [48]; see also van Geemen and Izadi [49]. It is possible only if every conjugate has two adjacent Hodge levels, and by Proposition 3.1 this is equivalent to global boundedness. In arbitrary rank the same two-level condition

supplies the weight-one construction. Global boundedness enters the geometric proof only through Proposition 3.1.

5. An abelian scheme and the period

In this section and the next two, (a, b) satisfies (NR) and $F_{a,b}$ is globally bounded.

Proposition 5.1. *There are a number field $K_0 \subset \mathbb{C}$ and an abelian scheme $f : \mathcal{A} \to U_{K_0}$ of relative dimension $\frac{n}{2}\varphi(N)$ such that $R_1 f_* \mathbb{Q} \cong V_{\mathbb{Q}}$ as local systems on $U(\mathbb{C})$.*

Proof. By Theorem 4.3, the Tate twist $V_{\mathbb{Z}}(1)$ is a polarizable variation of type $\{(-1,0),(0,-1)\}$ whose underlying local system is isomorphic to $V_{\mathbb{Z}}$ (Remark 4.4(2)). By Theorem 2.13, applied on the smooth curve $U_{\mathbb{C}}$, there is an abelian scheme $f' : \mathcal{A}' \to U_{\mathbb{C}}$ with $R_1 f'_* \mathbb{Z} \cong V_{\mathbb{Z}}(1)$, so that $R_1 f'_* \mathbb{Q} \cong V_{\mathbb{Q}}$. The field $\mathbb{C}$ is the union of its finitely generated $\overline{\mathbb{Q}}$-subalgebras. By finite-presentation descent, including descent of morphisms and their identities, and descent of smoothness and properness [54, Tags 01ZM, 0C0C, 081F], $\mathcal{A}'$ is therefore the base change of an abelian scheme $\mathcal{A}_R \to U \times_{\mathbb{Q}} \operatorname{Spec} R$ for some finitely generated $\overline{\mathbb{Q}}$-algebra $R \subset \mathbb{C}$. Replacing R by $R[1/r]$ for a suitable nonzero $r \in R$, which is again a subring of $\mathbb{C}$, we may assume that $T = \operatorname{Spec} R$ is smooth over $\overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}}$ is algebraically closed and R is an integral domain, T is geometrically integral, so the complex manifold $T(\mathbb{C})$ is connected. The point induced by $R \hookrightarrow \mathbb{C}$ lies over the generic point of T, so every nonzero localization retains it. Geometric connectedness of the fibres can be arranged at the same stage, using [54, Tag 0E0N]. The inclusion $R \subset \mathbb{C}$ defines a point $\tau_0 \in T(\mathbb{C})$, and $T(\overline{\mathbb{Q}})$ is nonempty by Hilbert's Nullstellensatz.

The map $\mathcal{A}_R(\mathbb{C}) \to U(\mathbb{C}) \times T(\mathbb{C})$ is a proper holomorphic submersion. By Ehresmann's theorem [51, §1, proposition, p. 154], the first homology groups of its fibres form a local system on $U(\mathbb{C}) \times T(\mathbb{C})$. The restrictions of this local system to the slices $U(\mathbb{C}) \times \{\tau\}$, $\tau \in T(\mathbb{C})$, are all isomorphic, as one sees by pulling it back to $U(\mathbb{C}) \times [0,1]$ along a path in $T(\mathbb{C})$. For $\tau = \tau_0$ the restriction is $R_1 f'_* \mathbb{Z}$. We choose $\tau \in T(\overline{\mathbb{Q}})$ and let $\mathcal{A}$ be the fibre of $\mathcal{A}_R$ over τ. It is an abelian scheme over $U_{\overline{\mathbb{Q}}}$ with $R_1 f_* \mathbb{Q} \cong R_1 f'_* \mathbb{Q} \cong V_{\mathbb{Q}}$, and, being of finite presentation, it is defined over a number field K_0. □

The specialization from τ_0 to τ preserves the local system, but it need not preserve the Hodge filtration: we make no claim that the Hodge filtration of $R_1 f_* \mathbb{C}$ corresponds to F^1 under the isomorphism with $V_{\mathbb{C}}$. Only the local system $R_1 f_* \mathbb{Q}$ of $\mathcal{A}$ is used below.

Remark 5.2. The field K acts on $V_{\mathbb{Q}}$, and this action preserves F^1, because $x \in K$ acts on $\mathcal{S}_k$ as the scalar $\sigma_k(x)$. By Theorem 2.13, K therefore embeds into $\operatorname{End}(\mathcal{A}') \otimes \mathbb{Q}$. Finitely many generators of an order of K spread out together with $\mathcal{A}'$ and specialize, so $\mathcal{A}$ can be chosen with multiplication by an order of K. We do not use this.

We use the following standard consequences of the theory of regular singular connections.

Lemma 5.3. *Let $K_0 \subset \mathbb{C}$ be a number field, and let M and E be algebraic vector bundles with flat connections on U_{K_0} that have regular singularities at 0, 1 and ∞, that is, whose base changes to $U_{\mathbb{C}}$ are regular in the sense of Deligne [17, Définition II.4.5]. Let $\operatorname{Hom}_{\mathcal{D}}(M,E)$ be the K_0-vector space of $\mathcal{O}$-linear maps $M \to E$ that commute with the connections.*

(a) *$\operatorname{Hom}_{\mathcal{D}}(M,E)$ is finite-dimensional, and the natural map $\operatorname{Hom}_{\mathcal{D}}(M,E) \otimes_{K_0} \mathbb{C} \to \operatorname{Hom}_{\mathcal{D}}(M_{\mathbb{C}}, E_{\mathbb{C}})$ is an isomorphism, where $M_{\mathbb{C}}$ and $E_{\mathbb{C}}$ are the base changes to $U_{\mathbb{C}}$.*

(b) *Sending $\phi \in \operatorname{Hom}_{\mathcal{D}}(M_{\mathbb{C}}, E_{\mathbb{C}})$ to the induced map $M^{\nabla} \to E^{\nabla}$ between the local systems of horizontal sections of the analytifications gives an isomorphism $\operatorname{Hom}_{\mathcal{D}}(M_{\mathbb{C}}, E_{\mathbb{C}}) \cong \operatorname{Hom}(M^{\nabla}, E^{\nabla})$.*

Proof. (a) The connections on M and E induce a flat connection on the vector bundle $\mathcal{H} = \mathcal{H}om_{\mathcal{O}}(M, E)$, and $\mathrm{Hom}_{\mathcal{D}}(M, E)$ is the kernel of the K_0-linear map $\nabla : \Gamma(U_{K_0}, \mathcal{H}) \to \Gamma(U_{K_0}, \mathcal{H} \otimes \Omega^1)$. Since U is affine, taking global sections commutes with the flat base change $K_0 \to \mathbb{C}$, and so does taking kernels; this gives the isomorphism. Over $\mathbb{C}$, a horizontal section of $\mathcal{H}$ is determined by its value at one point of the connected manifold $U(\mathbb{C})$, so $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(M_{\mathbb{C}}, E_{\mathbb{C}}) \leq \mathrm{rank}\, \mathcal{H}$, and hence $\dim_{K_0} \mathrm{Hom}_{\mathcal{D}}(M, E) \leq \mathrm{rank}\, \mathcal{H}$.

(b) By Deligne [17, Théorème II.5.9], applied to the smooth complex algebraic variety $U_{\mathbb{C}}$, analytification is an equivalence between the category of algebraic vector bundles on $U_{\mathbb{C}}$ with a regular integrable connection and the category of holomorphic vector bundles on $U(\mathbb{C})$ with an integrable connection; in particular it is fully faithful, which is proved explicitly there. By Deligne [17, Théorème I.2.17], taking horizontal sections is an equivalence between the latter category and the category of local systems on $U(\mathbb{C})$. Composing the two gives (b). □

Proposition 5.4. *In the situation of Proposition 5.1, let $E = H^1_{\mathrm{dR}}(\mathcal{A}/U_{K_0})$ be the relative algebraic de Rham cohomology, with its Gauss–Manin connection. There is a nonzero global section $\omega \in \Gamma(U_{K_0}, E)$ with $L_{a,b}\,\omega = 0$, where θ acts on E through the Gauss–Manin connection. For every flat section γ of $R_1 f_* \mathbb{C}$ over an open subset of $U(\mathbb{C})$, the function $\Pi(\gamma) = \int_\gamma \omega$ is a solution of $L_{a,b}$, and the resulting map $\Pi : R_1 f_* \mathbb{C} \to \mathcal{S}_{a,b}$ of local systems is surjective.*

The section ω is an algebraic de Rham class defined over K_0. It need not be represented by a relative holomorphic differential, that is, it need not lie in the Hodge subbundle $f_* \Omega^1_{\mathcal{A}/U}$ of E, and the Hodge filtration of E plays no role below.

Proof. The Gauss–Manin connection on E has regular singularities. This is Deligne [17, Théorème II.7.9], applied after base change to $\mathbb{C}$ to the proper smooth morphism f and the trivial connection on $\mathcal{O}_{\mathcal{A}}$; the hypotheses (i)–(iii) of Deligne [17, Théorème II.6.13] required there hold for proper smooth morphisms by Deligne [17, Proposition II.6.14], with empty boundary divisor. See also Katz [37]. As a polynomial in θ, $L_{a,b} = (1-x)\theta^n + \cdots$, and $1 - x$ is invertible on U. Hence $M_L = \mathcal{D}_U / \mathcal{D}_U L_{a,b}$, the quotient of the sheaf of differential operators on U_{K_0} by the left ideal generated by $L_{a,b}$, is a vector bundle of rank n with basis the classes of $1, \theta, \ldots, \theta^{n-1}$, with the connection given by left multiplication; it has regular singularities, since $L_{a,b}$ is Fuchsian. A $\mathcal{D}$-linear map $M_L \to E$ is determined by the image ω of the class of 1, and a section $\omega \in \Gamma(U_{K_0}, E)$ occurs if and only if $L_{a,b}\,\omega = 0$. Thus

$$\mathrm{Hom}_{\mathcal{D}}(M_L, E) \cong \{\omega \in \Gamma(U_{K_0}, E) : L_{a,b}\omega = 0\}.$$

A horizontal section of the dual connection $M_L^\vee$ over an open set is a $\mathcal{D}$-linear map $M_L \to \mathcal{O}^{\mathrm{an}}$, and evaluation at the class of 1 identifies these with the solutions of $L_{a,b}$. Hence $(M_L^\vee)^\nabla = \mathcal{S}_{a,b}$ and $M_L^\nabla = \mathcal{S}_{a,b}^\vee$. By the comparison between algebraic de Rham and singular cohomology, $E^\nabla = R^1 f_* \mathbb{C} = (R_1 f_* \mathbb{C})^\vee$, the duality being given by integration. Lemma 5.3 therefore gives

$$\mathrm{Hom}_{\mathcal{D}}(M_L, E) \otimes_{K_0} \mathbb{C} \cong \mathrm{Hom}\,(\mathcal{S}_{a,b}^\vee, (R_1 f_* \mathbb{C})^\vee) \cong \mathrm{Hom}\,(R_1 f_* \mathbb{C}, \mathcal{S}_{a,b}).$$

Unwinding the identifications, this isomorphism sends ω to the map $\Pi_\omega : \gamma \mapsto \int_\gamma \omega$. Indeed, for a flat section γ the map $E \to \mathcal{O}^{\mathrm{an}}$, $\eta \mapsto \int_\gamma \eta$, is $\mathcal{D}$-linear, so its composite with the map $M_L \to E$ defined by ω is the solution $\int_\gamma \omega$ of $L_{a,b}$; in particular $L_{a,b} \int_\gamma \omega = \int_\gamma L_{a,b}\,\omega = 0$. The right-hand side is nonzero: it contains the projection of $R_1 f_* \mathbb{C} \cong V_{\mathbb{C}}$ onto the summand $\mathcal{S}_1 \cong \mathcal{S}_{a,b}$ of (4.2). Hence the K_0-vector space $\mathrm{Hom}_{\mathcal{D}}(M_L, E)$ is nonzero, without any extension of K_0. Let ω be a nonzero element. Then $\Pi = \Pi_\omega$ is nonzero, since $\mathrm{Hom}_{\mathcal{D}}(M_L, E)$ embeds into its tensor product with $\mathbb{C}$, and it is surjective, because its image is a nonzero sub-local system of the irreducible local system $\mathcal{S}_{a,b}$. □

Corollary 5.5. *There is a flat section γ_Y of $R_1 f_* \mathbb{C}$ over a sector at 0 such that, for every $m \geq 1$,*

$$F_{a,b}(x) = \frac{1}{2\pi i\, m} \int_{(T_0^m - 1)\gamma_Y} \omega.$$

Proof. Let Y be the logarithmic solution of Lemma 2.6. Since Π is a surjective map of local systems and a sector is simply connected, there is a flat section γ_Y over the sector with $\Pi(\gamma_Y) = Y$. The map Π commutes with the local monodromy T_0, so the claim follows from Lemma 2.6(ii). □

6. Curves, semistable models and vanishing cycles

Proposition 6.1. *There are a nonempty Zariski open subset $U_1 \subseteq U_{K_0}$, a smooth projective morphism $q : C \to U_1$ whose fibres are geometrically connected curves of genus at least 3, and a morphism $\iota : C \to \mathcal{A}$ over U_1 such that $\iota_* : R_1 q_* \mathbb{Z} \to R_1 f_* \mathbb{Z}|_{U_1(\mathbb{C})}$ is surjective.*

Proof. The generic fibre $\mathcal{A}_\eta$ of $\mathcal{A}$ is an abelian variety over the field $K_0(x)$, of dimension $\delta = \frac{n}{2}\varphi(N) \geq 3$, and it is projective. By Bertini's theorem [54, Tag 0FD6 and Section 37.32], applied over the infinite field $K_0(x)$, the intersection of $\mathcal{A}_\eta$ with $\delta - 1$ general hyperplanes in a projective embedding is a smooth, geometrically connected curve. The embedding and the curve spread out over a nonempty open subset $U_1 \subseteq U_{K_0}$, and after shrinking U_1 every fibre C_x is a smooth complete intersection of ample divisors in $\mathcal{A}_x$. By the Lefschetz hyperplane theorem [50, Theorem 3], applied successively from dimension δ down to dimension one, the map $H_1(C_x, \mathbb{Z}) \to H_1(\mathcal{A}_x, \mathbb{Z})$ is surjective for every $x \in U_1(\mathbb{C})$. Since $H_1(\mathcal{A}_x, \mathbb{Z})$ has rank $2\delta \geq 6$, the genus of C_x is at least 3. □

Every nonempty Zariski open subset of U contains a punctured disc around 0 in $U(\mathbb{C})$, and this is all we need from U_1.

Proposition 6.2. *There are a number field $K_1 \supseteq K_0$, an integer $m \geq 1$, a Zariski open neighbourhood W of 0 in the affine line $\mathbb{A}^1_{K_1} = \operatorname{Spec} K_1[w]$ such that $w \mapsto w^m$ maps $W \setminus \{0\}$ into U_1, and a projective flat morphism $\pi : \mathcal{X} \to W$ with the following properties.*

(i) *Over $W \setminus \{0\}$, π is smooth, and $\mathcal{X}$ is isomorphic to the base change of C along $x = w^m$.*
(ii) *The surface $\mathcal{X}$ is regular.*
(iii) *The fibre $\mathcal{X}_0$ over $w = 0$ is reduced, and its singular points $P_1, \dots, P_r$ are ordinary double points. Each P_j is K_1-rational, and $\widehat{\mathcal{O}}_{\mathcal{X},P_j} \cong K_1[[\hat{u}, \hat{v}]]$ with $w = \hat{u}\hat{v}$.*

Proof. Step 1: semistable reduction over an algebraically closed residue field. By the monodromy theorem [17, III.2.3], the local monodromy of $R_1 q_* \mathbb{Z}$ around $x = 0$ is quasi-unipotent; let $m \geq 1$ be such that its m-th power is unipotent. Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$, and let $R = \overline{\mathbb{Q}}[w]_{(w)}$ be the local ring of the affine w-line over $\overline{\mathbb{Q}}$ at 0, a discrete valuation ring with algebraically closed residue field $\overline{\mathbb{Q}}$, regarded as an extension of the local ring of the x-line at 0 via $x = w^m$. Let $C_{\overline{\mathbb{Q}}(w)}$ be the base change of the generic fibre of C to the fraction field $\overline{\mathbb{Q}}(w)$ of R, a smooth projective geometrically connected curve of genus $g \geq 3$, and let J be its Jacobian. Since the residue characteristic is 0, the inertia group at $w = 0$ is procyclic, topologically generated by an element σ with $\sigma(w^{1/n}) = \mathbf{e}(1/n)\, w^{1/n}$ for all n. The required comparison can be seen on torsion: over the smooth punctured base, $J[\ell^r]$ is finite étale, and its analytic covering has fibre $H_1(C_{w^m}(\mathbb{C}), \mathbb{Z})/\ell^r$. Riemann existence and invariance under algebraically closed characteristic-zero field extensions [52, Remark 3.3(c), Theorem 3.4] identify these covering monodromies with their algebraic counterparts. At $w = 0$, the local generator acts on a ramified parameter as $w^{1/e} \mapsto \mathbf{e}(1/e) w^{1/e}$, as does a positive complex loop.

Passing to the inverse limit identifies the Tate-module action with the topological monodromy, up to inverse convention, which does not affect unipotence. This generator acts by the m-th power of the original monodromy. The whole inertia image is unipotent because powers of the generator are dense and the unipotent locus is closed. By Grothendieck's criterion in the explicit form of Silverberg and Zarhin [53, Theorem 4.1] (see also [33, Exposé IX, Proposition 3.5 and Corollary 3.8]), J has semi-abelian reduction over R, and by Deligne and Mumford [22, Theorem 2.4], which is stated for discrete valuation rings with algebraically closed residue field, $C_{\overline{\mathbb{Q}}(w)}$ has a stable model $\mathcal{X}^{\mathrm{st}}$ over R.

Step 2: descent to a number field. The ring R is the filtered union of the rings $R_{K'} = K'[w]_{(w)}$, where $K' \subset \overline{\mathbb{Q}}$ runs through the number fields containing K_0. By finite-presentation descent and descent of flatness [54, Tags 01ZM, 04AI], there are such a field K_2 and a projective flat R_{K_2}-scheme $\mathcal{X}'$ with $\mathcal{X}' \otimes_{R_{K_2}} R \cong \mathcal{X}^{\mathrm{st}}$, together with an isomorphism of its generic fibre with $C \otimes K_2(w)$. The special fibre of $\mathcal{X}'$ becomes isomorphic to that of $\mathcal{X}^{\mathrm{st}}$ over $\overline{\mathbb{Q}}$, so it is reduced and its singular points are ordinary double points. After enlarging K_2, every singular point P of the special fibre is K_2-rational and both branches at P are defined over K_2. Then the completed local ring of $\mathcal{X}'$ at P is isomorphic to $K_2[[w]][[u,v]]/(uv-c)$ with $c \in wK_2[[w]]$ [22, §1, especially pp. 84–85]. Writing $c = w^e\varepsilon$ with a unit ε and replacing u by $\varepsilon^{-1}u$, it is $K_2[[u,v,w]]/(uv-w^e)$ for some $e \geq 1$.

Step 3: resolution. For $e \geq 2$ the point P is a rational double point of type A_{e-1} of the surface $\mathcal{X}'$. Its minimal resolution, obtained by successive blow-ups of closed points that are K_2-rational at every stage, replaces P by a chain of $e-1$ projective lines over K_2, each of multiplicity one in the special fibre, meeting each other and the strict transforms of the two branches transversally in K_2-rational points; at each of these points the completed local ring is $K_2[[u,v,w]]/(uv-w) \cong K_2[[u,v]]$ [22, §1], [22, §1, especially pp. 84–85]. Doing this at every singular point with $e \geq 2$ gives a projective flat R_{K_2}-scheme that is regular and whose special fibre is reduced, with only ordinary double points, each K_2-rational with completed local ring $K_2[[u,v]]$, $w = uv$.

Step 4: spreading out. Put $K_1 = K_2$. The scheme of Step 3 and the isomorphism of its generic fibre with $C \otimes K_1(w)$ are defined over some Zariski open neighbourhood W of 0 in $\mathbb{A}^1_{K_1}$ [54, Tag 01ZM]; this gives $\pi : \mathcal{X} \to W$. After shrinking W, π is projective and flat, smooth over $W \setminus \{0\}$, $w \mapsto w^m$ maps $W \setminus \{0\}$ into U_1, and (i) holds. The non-regular locus of $\mathcal{X}$ is closed, because $\mathcal{X}$ is of finite type over a field. It does not meet $\mathcal{X}_0$, because the local rings of $\mathcal{X}$ at points of $\mathcal{X}_0$ are local rings of the regular scheme of Step 3. Its image in W is therefore closed and does not contain 0, and after removing it from W we obtain (ii). Property (iii) holds by Step 3. □

Remark 6.3. Since ι_* is surjective and the monodromy of $R_1\pi_*\mathbb{Z}$ around $w = 0$ is unipotent, T_0^m is unipotent on $V_{\mathbb{Q}}$. By Lemma 2.6(iii), $\operatorname{lcm}_j \operatorname{den}(b_j)$ divides m.

Put $W^\circ = W \setminus \{0\}$. In the next lemma, $\omega_{\mathcal{X}/W}$ denotes the relative dualizing sheaf of π. Since $\mathcal{X}$ and W are regular, π is a local complete intersection morphism, and $\omega_{\mathcal{X}/W}$ is an invertible sheaf that restricts to $\Omega^1_{\mathcal{X}/W}$ on the smooth locus of π [22, §1, pp. 76–77]. The class ω of Proposition 5.4 pulls back along ι and along $x = w^m$ to a section $\omega_{\mathcal{X}}$ of the relative de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{X}_{W^\circ}/W^\circ)$. By functoriality of the comparison isomorphism, for $w \in W^\circ(\mathbb{C})$ and $c \in H_1(\mathcal{X}_w, \mathbb{C}) = H_1(C_{w^m}, \mathbb{C})$ the period of $\omega_{\mathcal{X}}$ over c equals the period of ω over $\iota_* c$.

Lemma 6.4. *After shrinking W, there are a closed subscheme $D \subset \mathcal{X}$, finite over W and containing none of the points P_j, and a rational section ϖ of $\omega_{\mathcal{X}/W}$ whose poles are contained in $D \cup \mathcal{X}_0$, with the following properties for every $w \in W^\circ(\mathbb{C})$.*

(i) *The restriction ϖ_w of ϖ to $\mathcal{X}_w \setminus D_w$ is a holomorphic 1-form, and its residues at the points of D_w vanish.*
(ii) *For every 1-cycle γ on $\mathcal{X}_w \setminus D_w$,*

$$\int_\gamma \varpi = \int_{\iota_*[\gamma]} \omega,$$

where $[\gamma]$ denotes the class of γ in $H_1(\mathcal{X}_w, \mathbb{Z}) = H_1(C_{w^m}, \mathbb{Z})$.

In particular, $\int_\gamma \varpi$ depends only on $[\gamma]$, and we write $\int_c \varpi$ for $c \in H_1(\mathcal{X}_w, \mathbb{C})$, extending linearly.

Proof. Choose a projective embedding of $\mathcal{X}$ over W and two hyperplanes $\mathcal{H}, \mathcal{H}'$ that contain none of the points P_j, no irreducible component of $\mathcal{X}_0$ and no irreducible component of the generic fibre $\mathcal{X}_\eta$, a smooth projective curve over $K_1(w)$, and such that $\mathcal{H} \cap \mathcal{H}' \cap \mathcal{X}_\eta = \emptyset$. Let $D = \mathcal{H} \cap \mathcal{X}$ and $D' = \mathcal{H}' \cap \mathcal{X}$. After shrinking W, neither hyperplane contains a component of a fibre of π and $D \cap D' \cap \mathcal{X}_{W^\circ} = \emptyset$. Then D is proper and quasi-finite, hence finite, over W.

The curves $\mathcal{X}_\eta \setminus D_\eta$ and $\mathcal{X}_\eta \setminus D'_\eta$ are affine, and they cover $\mathcal{X}_\eta$. Their algebraic de Rham cohomology is computed by global algebraic differential forms [31], so $H^1_{\mathrm{dR}}(\mathcal{X}_\eta)$ is the first cohomology of the Čech–de Rham complex of this cover. Hence the generic fibre of $\omega_{\mathcal{X}}$ is represented by a triple (ϖ, ϖ', ϕ), where ϖ and ϖ' are regular 1-forms on $\mathcal{X}_\eta \setminus D_\eta$ and $\mathcal{X}_\eta \setminus D'_\eta$, and ϕ is a regular function on $\mathcal{X}_\eta \setminus (D_\eta \cup D'_\eta)$ with $\varpi - \varpi' = d\phi$. Each of these is defined away from finitely many fibres of π; after shrinking W they are defined on $(\mathcal{X} \setminus D)_{W^\circ}$, $(\mathcal{X} \setminus D')_{W^\circ}$ and $(\mathcal{X} \setminus (D \cup D'))_{W^\circ}$, and the triple defines a section of $H^1_{\mathrm{dR}}(\mathcal{X}_{W^\circ}/W^\circ)$, computed by the same Čech–de Rham complex since these open sets are affine over the affine curve W°. This section agrees with $\omega_{\mathcal{X}}$ at the generic point. The relative de Rham cohomology of the smooth projective family of curves $\mathcal{X}_{W^\circ} \to W^\circ$ is locally free and its formation commutes with base change: it is an extension of $R^1\pi_*\mathcal{O}$ by $\pi_*\Omega^1$, both locally free of rank equal to the genus and compatible with base change by the theorem on cohomology and base change. As W° is integral, the two sections agree on W°, and so, for every $w \in W^\circ(\mathbb{C})$, the restricted triple $(\varpi_w, \varpi'_w, \phi_w)$ represents $(\omega_{\mathcal{X}})_w$. We view ϖ as a rational section of $\omega_{\mathcal{X}/W}$; it is regular on $(\mathcal{X} \setminus D)_{W^\circ}$, so its poles lie in $D \cup \mathcal{X}_0$.

(i) The form ϖ_w is holomorphic on $\mathcal{X}_w \setminus D_w$. Near a point of D_w, which does not lie on D'_w, we have $\varpi_w = \varpi'_w + d\phi_w$ with ϖ'_w holomorphic, and the residue of an exact form is zero.

(ii) On $\mathcal{X}_w \setminus D_w$ the class $(\omega_{\mathcal{X}})_w$ is represented by ϖ_w, so its period over a cycle γ in $\mathcal{X}_w \setminus D_w$ is $\int_\gamma \varpi_w$. By the remark before the lemma, this equals $\int_{\iota_*[\gamma]} \omega$.

Finally, two cycles on $\mathcal{X}_w \setminus D_w$ with the same class in $H_1(\mathcal{X}_w, \mathbb{Z})$ differ in $H_1(\mathcal{X}_w \setminus D_w, \mathbb{Z})$ by a combination of small loops around the points of D_w, over which the integral of ϖ_w is $2\pi i$ times a residue, hence zero by (i). □

Lemma 6.5. *Let $1 \le j \le r$, let $\mathcal{O} = \mathcal{O}_{\mathcal{X},P_j}$, with maximal ideal $\mathfrak{m}$, and let $\mathcal{O}^h$ be its henselization. There are $u, v \in \mathcal{O}^h$ with $uv = w$ that form a regular system of parameters of $\mathcal{O}^h$. For any such u, v:*

(i) *$\mathcal{O}^h = K_1\langle u, v\rangle$ inside $\widehat{\mathcal{O}} = K_1[[u, v]]$;*
(ii) *every element of $\mathcal{O}^h$ is a holomorphic function on a neighbourhood of P_j in $\mathcal{X}(\mathbb{C})$ whose Taylor series in u, v is its image in $K_1[[u, v]] \subset \mathbb{C}[[u, v]]$;*
(iii) *u and v are holomorphic coordinates on a neighbourhood of P_j in $\mathcal{X}(\mathbb{C})$, in which π is given by $(u, v) \mapsto uv$.*

Proof. By Proposition 6.2(iii) there are $\hat{u}, \hat{v} \in \widehat{\mathcal{O}}$ with $\hat{u}\hat{v} = w$ and $\widehat{\mathcal{O}} = K_1[[\hat{u}, \hat{v}]]$. The ring $\mathcal{O}$ is a local ring of the scheme $\mathcal{X}$ of finite type over K_1, and $\widehat{\mathcal{O}}$ is also the completion of $\mathcal{O}^h$. By Artin's approximation theorem [6, Theorem 1.10], applied to the polynomial equation

$Y_1Y_2 - w = 0$ over $\mathcal{O}^h$, which has the solution $(\hat{u}, \hat{v})$ in $\widehat{\mathcal{O}}$, there are $u, v \in \mathcal{O}^h$ with $uv = w$, $u \equiv \hat{u}$ and $v \equiv \hat{v}$ modulo $\mathfrak{m}^2\widehat{\mathcal{O}}$. Then u and v generate $\mathfrak{m}\mathcal{O}^h$ modulo $(\mathfrak{m}\mathcal{O}^h)^2$, so they form a regular system of parameters of the two-dimensional regular local ring $\mathcal{O}^h$. The scheme $\mathcal{X}$ is regular and of finite type over a field of characteristic 0, hence smooth over K_1, and P_j is K_1-rational. Statements (i) and (ii) are therefore Lemma 2.1. Statement (iii) follows from (ii) and the inverse function theorem, because du and dv are linearly independent at P_j and $uv = w$ in $\mathcal{O}^h$. □

For each j we fix u, v as in Lemma 6.5 and a radius $r_0 > 0$ such that u, v are coordinates on the polydisc $\Delta_j = \{|u| < r_0,\ |v| < r_0\}$ around P_j, and $\Delta_j \cap D(\mathbb{C}) = \emptyset$. For $0 < |w| < r_0^2$ let $\delta_j(w) \subset \mathcal{X}_w \cap \Delta_j$ be the circle

$$\delta_j(w) = \{\, |u| = |w|^{1/2},\ v = w/u \,\},$$

oriented by increasing argument of u. This is the *vanishing cycle* at P_j. It lies in $\mathcal{X}_w \setminus D_w$.

Lemma 6.6. *Let $T_{\mathcal{X}}$ be the monodromy of $H_1(\mathcal{X}_w, \mathbb{Z})$ around $w = 0$. Then $T_{\mathcal{X}}\, \delta_j(w) = \delta_j(w)$ for all j, and*

$$(T_{\mathcal{X}} - 1)\, H_1(\mathcal{X}_w, \mathbb{Z}) \subseteq \sum_{j=1}^{r} \mathbb{Z}\, \delta_j(w).$$

Proof. Let Δ be a small disc around 0 in $W(\mathbb{C})$, and let $\mathcal{B}_j \subset \Delta_j$ be the closed ball $|u|^2 + |v|^2 \leq \epsilon^2$ around P_j, for a small $\epsilon > 0$. For $|w|$ small its boundary meets the fibres $uv = w$ transversally. Since the only singular points of π over Δ are the P_j, outside $\bigcup_j \mathcal{B}_j$ the map π is a proper submersion of manifolds with boundary over Δ, hence a trivial fibre bundle. For completeness, the boundary version follows by choosing local lifts of base tangent vectors tangent to the boundary, extending them through a collar, and combining them with a partition of unity. Properness guarantees their flows for each compact base path, giving local trivializations; contraction of the disc gives a global trivialization. Thus the monodromy diffeomorphism of $\mathcal{X}_w$ can be chosen to be the identity on $\mathcal{X}_w \setminus \bigcup_j \mathcal{B}_j$. Each $\mathcal{X}_w \cap \mathcal{B}_j$ is an annulus with core circle $\delta_j(w)$. Write a 1-cycle as $\gamma = \gamma_{\text{out}} + \sum_j \gamma_j$, where γ_j is a chain in the j-th annulus and γ_{out} is a chain outside the balls. Then $T_{\mathcal{X}}\gamma - \gamma = \sum_j (T_{\mathcal{X}}\gamma_j - \gamma_j)$, and $T_{\mathcal{X}}\gamma_j - \gamma_j$ is a cycle in the j-th annulus, because the monodromy fixes its boundary pointwise. It is therefore homologous to a multiple of $\delta_j(w)$. In the local annulus, transport around $w = 0$ preserves the oriented core generator: the coordinate u winds positively on each displayed circle and returns to the same oriented class. Hence each δ_j is invariant under monodromy. □

Lemma 6.6 is the elementary half of the Picard–Lefschetz formula; compare Deligne and Katz [21, Exp. XV] and Clemens [16]. The coefficients in that formula, which are intersection numbers, are not needed here.

7. Expansion at a node and proof of Theorem A

Proposition 7.1. *For each j there are an integer $\nu_j \geq 0$ and an algebraic power series $g_j \in K_1\langle u, v\rangle$ such that $\varpi = w^{-\nu_j} g_j\, du/u$ near P_j and, for $0 < |w| < r_0^2$ after shrinking r_0,*

$$\frac{1}{2\pi i} \int_{\delta_j(w)} \varpi = w^{-\nu_j}\ \mathrm{Diag}(g_j)(w).$$

Proof. Let $\mathcal{O} = \mathcal{O}_{\mathcal{X}, P_j}$ and let $u, v \in \mathcal{O}^h$ be as in Lemma 6.5.

The dualizing sheaf near P_j. The elements u, v lie in the local ring of some étale neighbourhood $(\mathcal{X}', P')$ of $(\mathcal{X}, P_j)$ with trivial residue field extension, and define a morphism $\chi = (u, v)$:

$\mathcal{X}' \to \mathbb{A}^2_{K_1}$. It is étale at P', because it induces an isomorphism $K_1[[u,v]] \cong \widehat{\mathcal{O}}$ of completed local rings. Since $uv = w$, χ is a morphism over $\mathbb{A}^1_{K_1}$ if $\mathbb{A}^2_{K_1}$ is mapped to $\mathbb{A}^1_{K_1}$ by $(u,v) \mapsto uv$. This map is the relative complete intersection $\operatorname{Spec} K_1[w][u,v]/(uv-w) \to \operatorname{Spec} K_1[w]$, whose dualizing sheaf is generated by the form that equals $du/u = -dv/v$ wherever u or v is invertible [22, §1, pp. 76–77]. This generator is also obtained directly by hypersurface adjunction: for $F = uv - w$, the relative residue of $du \wedge dv/F$ is $du/(\partial F/\partial v) = du/u$, with the equivalent expression $-dv/v$. Relative dualizing sheaves are compatible with étale base change. Hence $\omega_{\mathcal{X}/W} \otimes \mathcal{O}^h$ is the free $\mathcal{O}^h$-module generated by du/u.

Poles. Let ω_0 be a generator of $\omega_{\mathcal{X}/W} \otimes \mathcal{O}$. Then $du/u = \vartheta\,\omega_0$ for a unit ϑ of $\mathcal{O}^h$, and $\varpi = \kappa\,\omega_0$ for an element κ of the fraction field of $\mathcal{O}$. By Lemma 6.4, and since $P_j \notin D$, the poles of κ on $\operatorname{Spec}\mathcal{O}$ lie along prime divisors containing w. Let $\nu_j \geq 0$ be the largest order of such a pole. Then $w^{\nu_j}\kappa$ has no poles in codimension one on $\operatorname{Spec}\mathcal{O}$, so $w^{\nu_j}\kappa \in \mathcal{O}$, as $\mathcal{O}$ is normal. Hence, in $\omega_{\mathcal{X}/W} \otimes \mathcal{O}^h$,

$$\varpi = w^{-\nu_j}\, g_j\, \frac{du}{u} \qquad \text{with } g_j = w^{\nu_j}\kappa\,\vartheta^{-1} \in \mathcal{O}^h = K_1\langle u, v\rangle.$$

Integration. By Lemma 6.5(ii), after shrinking r_0, g_j is holomorphic on Δ_j and equals the sum of its Taylor series $\sum_{l,n\geq 0} g_{j;l,n} u^l v^n$ there. On $\delta_j(w)$, $0 < |w| < r_0^2$, we have $|u| = |v| = |w|^{1/2} < r_0$ and $v = w/u$, and the series converges uniformly. Hence

$$\int_{\delta_j(w)} \varpi = w^{-\nu_j} \sum_{l,n\geq 0} g_{j;l,n}\, w^n \oint_{|u|=|w|^{1/2}} u^{l-n}\,\frac{du}{u} = 2\pi i\, w^{-\nu_j} \sum_{n\geq 0} g_{j;n,n}\, w^n. \qquad \square$$

Remark 7.2. The regularity of the model is essential for Proposition 7.1. For the Legendre family $y^2 = t(t-x)(1-t)$, natural coordinates at the node of the fibre over $x = 0$ satisfy $uv = x^2/4$, which describes a non-regular point of the total space. In the chart $u = \frac{x}{2}\,s$ of its blow-up one has $sv = \frac{x}{2}$ and

$$\frac{dt}{y} = -\Big(1 - \tfrac{1}{2}\,v(1+s)^2\Big)^{-1/2}\,\frac{ds}{s}.$$

The diagonal of $(1 - \frac{1}{2}v(1+s)^2)^{-1/2}$ in the variables s, v, evaluated at $x/2$, is ${}_2F_1(\frac{1}{2},\frac{1}{2};1;x)$: the coefficient of $s^n v^n$ is $2^{-n}\binom{2n}{n}\frac{(1/2)_n}{n!}$, and $4^{-n}\binom{2n}{n} = \frac{(1/2)_n}{n!}$.

The following proposition collects the passage from vanishing-cycle periods to diagonals. Its hypotheses are exactly what Proposition 6.2 and Lemma 6.4 provide; it does not assert that algebraic periods are in general diagonals.

Proposition 7.3 (From vanishing-cycle periods to diagonals)**.** *Let $K_1 \subset \mathbb{C}$ be a number field, $W \subseteq \mathbb{A}^1_{K_1}$ a Zariski open neighbourhood of 0, and $\pi : \mathcal{X} \to W$ a projective flat morphism that is smooth over $W \setminus \{0\}$, such that $\mathcal{X}$ is regular and $\mathcal{X}_0$ is reduced with only ordinary double points $P_1, \dots, P_r$, each K_1-rational with $\widehat{\mathcal{O}}_{\mathcal{X},P_j} \cong K_1[[\hat{u},\hat{v}]]$, $w = \hat{u}\hat{v}$. Let $D \subset \mathcal{X}$ be a closed subscheme, finite over W and avoiding the P_j, and let ϖ be a rational section of $\omega_{\mathcal{X}/W}$ with poles in $D \cup \mathcal{X}_0$ (horizontal and vertical poles) whose restrictions to $\mathcal{X}_w \setminus D_w$, $w \neq 0$, have zero residues. Let $T_{\mathcal{X}}$ be the monodromy of $H_1(\mathcal{X}_w, \mathbb{Z})$ around $w = 0$.*

(a) *For every flat section $\tilde{\gamma}$ of $H_1(\mathcal{X}_w, \mathbb{C})$ over a sector at 0 there are $z_1, \dots, z_r \in \mathbb{C}$ such that $(T_{\mathcal{X}} - 1)\tilde{\gamma} = \sum_j z_j\,\delta_j(w)$, where $\delta_j(w)$ is the vanishing cycle at P_j, defined as before Lemma 6.6 with coordinates as in Lemma 6.5, and, on the sector, for small $|w|$,*

$$\frac{1}{2\pi i}\int_{(T_{\mathcal{X}}-1)\tilde{\gamma}} \varpi = \sum_{j=1}^{r} z_j\, w^{-\nu_j}\ \operatorname{Diag}(g_j)(w)$$

with $\nu_j \geq 0$ *and* $g_j \in K_1\langle u, v\rangle$.

(b) *If, moreover,* $f \in \mathbb{Q}[[x]]$ *converges near* 0 *and* $f(w^m) = \frac{1}{2\pi i}\int_{(T_{\mathcal{X}}-1)\tilde{\gamma}} \varpi$ *on the sector, for some* $m \geq 1$, *then* $f = \mathrm{Diag}(g)$ *for some* $g \in \mathbb{Q}\langle u, v\rangle$.

Proof. Lemma 6.5, Lemma 6.6 and Proposition 7.1 use only these hypotheses, and so does the last paragraph of the proof of Lemma 6.4, which shows that $\int_c \varpi$ is a well-defined linear function of $c \in H_1(\mathcal{X}_w, \mathbb{C})$.

(a) By Lemma 6.6, $(T_{\mathcal{X}} - 1)\tilde{\gamma} = \sum_j z_j \delta_j(w)$ with $z_j \in \mathbb{C}$, constant on the sector because both sides are flat. The identity follows from Proposition 7.1 by linearity.

(b) Let $\nu = \max_j \nu_j$ and replace g_j by $(uv)^{\nu-\nu_j} g_j$, which multiplies $\mathrm{Diag}(g_j)$ by $w^{\nu-\nu_j}$. By (a), $w^\nu f(w^m) = \sum_j z_j \,\mathrm{Diag}(g_j)(w)$ on the sector. Both sides are holomorphic on a punctured disc around 0, the right-hand side because each $\mathrm{Diag}(g_j)$ converges near 0; so the identity holds on the punctured disc, and it is an identity of power series. Its left-hand side lies in $\mathbb{Q}[[w]]$. By Lemma 2.2(a), applied to the series $\mathrm{Diag}(g_j) \in K_1[[w]]$, we may replace the z_j by elements of K_1 (descent of the coefficient field), so that $w^\nu f(w^m) = \mathrm{Diag}(\tilde{g})$ with $\tilde{g} = \sum_j z_j g_j \in K_1\langle u, v\rangle$. Lemma 2.2(b) replaces $\tilde{g}$ by an element of $\mathbb{Q}\langle u, v\rangle$ with the same diagonal; Lemma 2.2(c) removes the factor w^ν (removal of a monomial shift); and Lemma 2.2(d) passes from $f(w^m)$ to f (extraction of a section). □

7.1. **Proof of Theorem A for** $m = 1$**.** Let (a, b) satisfy (NR) and assume that $F_{a,b}$ is globally bounded. Theorem 4.3 and Propositions 5.1 and 5.4 give a number field K_0, an abelian scheme $f : \mathcal{A} \to U_{K_0}$ and a de Rham class ω, and Corollary 5.5 gives a flat section γ_Y of $R_1 f_* \mathbb{C}$ over a sector at 0 with

$$F_{a,b}(x) = \frac{1}{2\pi i\, m} \int_{(T_0^m - 1)\gamma_Y} \omega \qquad \text{for every } m \geq 1.$$

Propositions 6.1 and 6.2 give C, ι, K_1, m and $\pi : \mathcal{X} \to W$, and Lemma 6.4 gives D and ϖ. They satisfy the hypotheses of Proposition 7.3.

We pull back along $x = w^m$ to a sector in the w-plane. The map $\iota_* : H_1(\mathcal{X}_w, \mathbb{C}) \to H_1(\mathcal{A}_{w^m}, \mathbb{C})$ is a surjective map of local systems on a punctured disc around 0, by Proposition 6.1, and $\iota_* \circ T_{\mathcal{X}} = T_0^m \circ \iota_*$. As the sector is simply connected, there is a flat section $\tilde{\gamma}$ of $H_1(\mathcal{X}_w, \mathbb{C})$ over it with $\iota_* \tilde{\gamma} = \gamma_Y$, and then $(T_0^m - 1)\gamma_Y = \iota_*(T_{\mathcal{X}} - 1)\tilde{\gamma}$. By Lemma 6.4(ii),

$$F_{a,b}(w^m) = \frac{1}{2\pi i} \int_{(T_{\mathcal{X}} - 1)(\tilde{\gamma}/m)} \varpi$$

on the sector. Proposition 7.3(b), applied to $f = F_{a,b}$ and the flat section $\tilde{\gamma}/m$, gives $g \in \mathbb{Q}\langle u, v\rangle$ with $\mathrm{Diag}(g) = F_{a,b}$.

The series $F_{a,b}$ is D-finite, since $L_{a,b} F_{a,b} = 0$. By Lemma 2.3 with $K = \mathbb{Q}$ and $n = 1$, it is the diagonal of a rational function in three variables over $\mathbb{Q}$ that is regular at the origin. By Lemma 2.6(iv), $F_{a,b}$ is transcendental over $\mathbb{Q}(x)$. By Theorem 2.4, it is therefore not the diagonal of a rational function in two variables, and it is not a rational function, which is the case of one variable. The extension to F_m, and the equivalence of global boundedness, are proved in Section 8. □

8. Positive integral lower parameters

Fix $m \geq 2$ and write $q = m - 1$,

$$B(t) = \prod_{j=1}^{n-2} (t + b_j - 1), \quad R(t) = \prod_{i=1}^{n} (t + a_i), \quad S_q(t) = \frac{R(t) - R(-q)}{t + q}.$$

Here S_q is a polynomial and $R(-q) \neq 0$ because every a_i is nonintegral.

The series F_m satisfies $L_m F_m = 0$, where

$$L_m = \theta(\theta + m - 1)B(\theta) - xR(\theta), \qquad L_1 = L_{a,b}.$$

Here $B(t)$ denotes a scalar polynomial, distinct from the companion matrix B.

Proposition 8.1 (Explicit inverse contiguity)**.** *One has*

$$(\theta + q)F_m = qF_{m-1},$$

and

$$F_m = \frac{q}{R(-q)}\left[x^{-1}\theta B(\theta) - S_q(\theta)\right] F_{m-1}. \tag{8.1}$$

Proof. The first equality follows coefficientwise from the rising factorials. The equation $L_m F_m = 0$ gives

$$R(\theta)F_m = x^{-1}\theta B(\theta)(\theta + q)F_m = qx^{-1}\theta B(\theta)F_{m-1}.$$

Polynomial division also gives

$$R(\theta)F_m = qS_q(\theta)F_{m-1} + R(-q)F_m.$$

Subtract and divide by $R(-q)$. There is no pole at the origin in (8.1): $\theta B(\theta)F_{m-1}$ has zero constant term. □

Proposition 8.2. *Under (NR), global boundedness of F_1 is equivalent to global boundedness of F_m for each $m \geq 1$. If F_1 is a two-variable algebraic diagonal, so is every F_m.*

Proof. The forward contiguity identity iterates to

$$\prod_{r=1}^{m-1}(\theta + r)F_m = (m-1)!F_1.$$

It proves global boundedness of $F_m \Rightarrow$ global boundedness of F_1. Conversely, rational linear combinations, Euler derivatives and a one-place shift of a series with zero constant term preserve global boundedness. For the shift, if $ch(dx) \in \mathbb{Z}[[x]]$ and $h(0) = 0$, then $cd\,(h(x)/x)|_{x=dx} \in \mathbb{Z}[[x]]$. Apply (8.1) successively.

For the diagonal assertion, if $h = \mathrm{Diag}(g)$ with $g \in \mathbb{Q}\langle u, v\rangle$, then

$$\theta h = \mathrm{Diag}(u\partial_u g).$$

Derivatives of algebraic series in characteristic zero are algebraic. If $h(0) = 0$, then

$$\frac{h(x)}{x} = \mathrm{Diag}\left(\frac{g(u,v) - g(u,0) - g(0,v) + g(0,0)}{uv}\right).$$

The numerator is divisible by uv as a power series, and the quotient is algebraic. Thus the operator in (8.1) preserves this diagonal class. Induct on m. □

Theorem A for F_m now follows from Sections 3–7 and Proposition 8.2. If F_m were algebraic over $\mathbb{C}(x)$, its forward-contiguity image F_1 would be algebraic, a contradiction. The conversion in [34, Lemma 3.2] then gives the optimal three-variable rational representation.

No Taylor-series logarithmic companion for the equation of F_m is asserted when $m > 1$. Its indicial roots include $1 - m$, and a logarithmic companion can have a Laurent principal part. Applying the rational differential inverse avoids this issue entirely.

9. Examples and further consequences

Except for the rank-four example and Remark 9.6, this section concerns rank three; we use $a = (a_1, a_2, a_3)$ and the scalar notation $b = b_1$. The trace-field refinement is restricted to rank three.

9.1. **Christol's example.** Let $a = (\frac{1}{9}, \frac{4}{9}, \frac{5}{9})$ and $b = \frac{1}{3}$, so that $F_{a,b} = H$, $N = 9$, $K = \mathbb{Q}(\zeta)$ with $\zeta = \mathbf{e}(1/9)$, and $(\mathbb{Z}/9\mathbb{Z})^\times = \{1, 2, 4, 5, 7, 8\}$. Table 2 lists the data of Corollary 3.2. It shows that H is globally bounded and that every conjugate has two adjacent Hodge levels.

k	$(a_1^{(k)}, a_2^{(k)}, a_3^{(k)})$	$b^{(k)}$	position of $b^{(k)}$	Hodge numbers
1	$(\frac{1}{9}, \frac{4}{9}, \frac{5}{9})$	$\frac{1}{3}$	$a_1^{(k)} < b^{(k)} < a_2^{(k)}$	$(1, 2)$
2	$(\frac{1}{9}, \frac{2}{9}, \frac{8}{9})$	$\frac{2}{3}$	$a_2^{(k)} < b^{(k)} < a_3^{(k)}$	$(2, 1)$
4	$(\frac{2}{9}, \frac{4}{9}, \frac{7}{9})$	$\frac{1}{3}$	$a_1^{(k)} < b^{(k)} < a_2^{(k)}$	$(1, 2)$
5	$(\frac{2}{9}, \frac{5}{9}, \frac{7}{9})$	$\frac{2}{3}$	$a_2^{(k)} < b^{(k)} < a_3^{(k)}$	$(2, 1)$
7	$(\frac{1}{9}, \frac{7}{9}, \frac{8}{9})$	$\frac{1}{3}$	$a_1^{(k)} < b^{(k)} < a_2^{(k)}$	$(1, 2)$
8	$(\frac{4}{9}, \frac{5}{9}, \frac{8}{9})$	$\frac{2}{3}$	$a_2^{(k)} < b^{(k)} < a_3^{(k)}$	$(2, 1)$

Table 2. The six Galois conjugates of Christol's example.

The companion matrices of P_a and P_b are

$$A = \begin{pmatrix} 0 & 0 & \zeta \\ 1 & 0 & \zeta^3 - \zeta^5 \\ 0 & 1 & \zeta + \zeta^4 + \zeta^5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \zeta^3 \\ 1 & 0 & -1 - 2\zeta^3 \\ 0 & 1 & 2 + \zeta^3 \end{pmatrix}.$$

The following statements were verified by exact computation in $\mathbb{Q}(\zeta)$; see Appendix B.

(i) The matrix $T_0 = B^{-1}$ has minimal polynomial $(t-1)^2(t - \mathbf{e}(-\frac{1}{3}))$, and $(T_0^3 - 1)^2 = 0$, $\operatorname{rank}(T_0^3 - 1) = 1$. Moreover $\det T_1 = \mathbf{e}(\frac{2}{9})$ and $T_1^9 = T_\infty^9 = 1$.

(ii) The matrices X with $A^* X A = X$ and $B^* X B = X$ form a one-dimensional K-vector space. It is spanned by the hermitian matrix h with $h_{11} = h_{22} = h_{33} = 1$,

$$h_{12} = h_{23} = 2\zeta + \zeta^2 - \zeta^3 + 2\zeta^5, \qquad h_{13} = -1 + 3\zeta + 2\zeta^2 - 3\zeta^3 + 3\zeta^5,$$

and $\det h = 15 + 12\zeta - 12\zeta^2 + 18\zeta^4 + 6\zeta^5 \neq 0$.

(iii) The hermitian matrix $\sigma_k(h)$ has signature $(2, 1)$ for $k \in \{\pm 1, \pm 4\}$ and $(1, 2)$ for $k = \pm 2$. The signs are certified by rational enclosures of the roots of $8c^3 - 6c + 1$.

(iv) The line $\operatorname{Im}(T_0^3 - 1)$ is isotropic for h. This is automatic, since $T_0^3 - 1$ has square zero and is skew for h; it serves as a check on the computation.

(v) With $\xi_0 = \zeta - \zeta^{-1}$, the form $\psi_0(x, y) = \operatorname{Tr}_{K/\mathbb{Q}}(\xi_0\, x^* h y)$ is alternating, integral, invariant and nondegenerate on $\mathcal{O}_K^3 \cong \mathbb{Z}^{18}$, with Gram determinant 3^{36}.

(vi) With $\Phi = \{1, 4, 7\}$, the sign conditions (4.8) of Step 7 in the proof of Theorem 4.3 require λ to have the same sign at all three real places of K^+. Hence one of the forms $\pm\psi_0$ is a polarization.

Proposition 5.1 and Remark 5.2 thus give an abelian scheme of relative dimension 9 over U, defined over a number field and with multiplication by an order of $\mathbb{Q}(\zeta_9)$, of which H is a period. By (i) and Remark 6.3, the integer m of Proposition 6.2 is a multiple of 3.

9.2. **The field of traces and a second example.** In rank three, the construction of Sections 4 and 5 can be carried out over a smaller field. The following lemma is restricted to that rank.

Lemma 9.1. *Assume $n = 3$ and (NR), and let $F_0 \subseteq K$ be the field generated by the coefficients of P_a and P_b. Then F_0 is a CM field, and $A, B \in \mathrm{GL}_3(\mathcal{O}_{F_0})$. If $F_{a,b}$ is globally bounded, then the results of Sections 4 and 5 hold with F_0 in place of K and with any nonzero $\xi_0 \in F_0$ such that $\overline{\xi_0} = -\xi_0$; in particular, the abelian scheme of Proposition 5.1 can be chosen of relative dimension $\frac{3}{2}[F_0 : \mathbb{Q}]$.*

Proof. As a subfield of the abelian CM field K, the field F_0 is either totally real or a CM field. Suppose that it is totally real. Then complex conjugation fixes P_b and P_a. The first gives $\mathbf{e}(-b) = \mathbf{e}(b)$, hence $b \in \frac{1}{2} + \mathbb{Z}$. The second shows that inversion permutes the three roots $\mathbf{e}(a_j)$ of P_a, counted with multiplicity, so one of them is fixed by inversion; as it is not 1, it equals -1. Then $a_j \in \frac{1}{2} + \mathbb{Z}$ and $a_j - b \in \mathbb{Z}$ for some j, contradicting (NR). Hence F_0 is a CM field. The coefficients of P_a and P_b are algebraic integers in F_0 and the determinants of A and B are roots of unity, which gives the second claim. The arguments of Sections 4 and 5 use only that K is a CM field containing the entries of A and B, that each conjugate of the local system is one of the $\mathcal{S}_{ka,kb}$, and that ξ_0 is a nonzero element with $\overline{\xi_0} = -\xi_0$. They therefore apply to F_0. □

Remark 9.2 (Why the trace-field refinement is not used in arbitrary rank). For $a = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$ and $b = (\frac{1}{2}, \frac{1}{2})$, the companion polynomials are $(t^2+1)^2$ and $(t^2-1)^2$, so $F_0 = \mathbb{Q}$ is totally real. Thus Lemma 9.1 does not extend to even rank. The main proof always uses the full cyclotomic CM field and needs no smaller-field descent.

Example 9.3 (A rank-four case with two monodromy blocks)**.** For the parameters in the preceding remark, consider

$$F(x) = {}_4F_3\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}; \tfrac{1}{2}, \tfrac{1}{2}, 1; x\right).$$

For both units $k = 1, 3$ modulo 4, the function D_k has successive values $0, 2, 0, 2$, with jumps at $1/4, 1/2, 3/4$. Hence F is globally bounded and the two Hodge levels each have dimension two. The minimal polynomial of T_0 is $(t-1)^2(t+1)^2$, so $(T_0^2 - 1)^2 = 0$ but $\mathrm{rank}(T_0^2 - 1) = 2$. Theorem A gives a three-variable rational diagonal, optimally, for F and for every positive integral replacement of its last lower parameter. This illustrates why the full rank-one assertion of the rank-three case cannot be retained in the general proof. The examples that follow return to rank three.

Example 9.4. Let $a = (\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$ and $b = \frac{1}{2}$, so that $N = 14$ and $(\mathbb{Z}/14\mathbb{Z})^\times = \{1, 3, 5, 9, 11, 13\}$. The sorted fractional parts $(a_1^{(k)}, a_2^{(k)}, a_3^{(k)})$ are $(\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$ for $k \in \{1, 9, 11\}$ and $(\frac{3}{7}, \frac{5}{7}, \frac{6}{7})$ for $k \in \{3, 5, 13\}$, and $b^{(k)} = \frac{1}{2}$ in all cases. Hence $a_1^{(k)} < b^{(k)} < a_3^{(k)}$ for every k, and the series ${}_3F_2(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}; \frac{1}{2}, 1; x)$ is globally bounded by Proposition 3.1. Here

$$P_a(t) = t^3 - \tfrac{-1+\sqrt{-7}}{2}\, t^2 + \tfrac{-1-\sqrt{-7}}{2}\, t - 1, \qquad P_b(t) = t^3 - t^2 - t + 1,$$

so that $F_0 = \mathbb{Q}(\sqrt{-7})$. By Lemma 9.1 and Remark 5.2, this series is a period of a family of abelian threefolds with multiplication by an order of $\mathbb{Q}(\sqrt{-7})$. The marked Jacobians of (1.5) realize this geometry near infinity. The unmarked generic curve descends to a plane quartic over $\mathbb{Q}(x)$ by [1, §6]. Determining the coefficients of such a global quartic is a separate problem.

Proof of Corollary B. Both series satisfy (NR) and are globally bounded, by §9.1 and Example 9.4. The claim follows from Theorem A. □

Proof of Corollary E. The first statement is part of Corollary B and does not use the Rohrlich–Lang conjecture. The second is Rivoal and Roques [45, Proposition 1], whose notion of grade is defined over $\overline{\mathbb{Q}}$. □

Remark 9.5 (Grades). (1) For every series $F_{a,b}$ satisfying (NR), Harder and Kramer-Miller [34, Theorem 3.8] show unconditionally that $\mathrm{dg}(F_{a,b}) \geq 2$ and $\mathrm{hg}(F_{a,b}) \geq 2$, since the height of $F_{a,b}$, the number of integral lower parameters when $n!$ is counted as $(1)_n$, is 2. Together with Theorem A, every globally bounded $F_{a,b}$ satisfying (NR) has diagonal grade exactly 2. The lower bound also follows from Lemma 2.6(iv) and Theorem 2.4, as in §7.1.
(2) The series of Proposition D(a) have Hadamard grade exactly 2.
(3) Assuming the Rohrlich–Lang conjecture, $\mathrm{dg}(S_7) = 2$ and $\mathrm{hg}(S_7) = \infty$. Since a series of Hadamard grade k has diagonal grade at most k [34, Proposition 4.1], the series of finite Hadamard grade then form a proper subset of the diagonals of rational functions.
(4) For Christol's series, $\mathrm{dg}(H) = 2$ by Theorem A, and $\mathrm{hg}(H) \geq 2$. Christol [13, §VII] remarked that H does not seem to be the Hadamard product of two functions of height one. Rivoal and Roques [45, §10] explain that this amounts to the statement that H is not the Hadamard product of two algebraic hypergeometric series with rational coefficients. It would follow from $\mathrm{hg}(H) \geq 3$, and under the Rohrlich–Lang conjecture $\mathrm{hg}(H) \geq 3$ would imply $\mathrm{hg}(H) = \infty$. Their method does not decide the question. Family (ii) of Appendix C examines a finite range of such factorizations: it excludes most of them, leaves 3,808 unresolved and does not examine those with reducible parameters.

9.3. **Proof of Corollary C.** Let $F = {}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$ be as in Corollary C. Diagonals of rational functions over $\mathbb{Q}$ are globally bounded [14]. Conversely, assume that F is globally bounded. We distinguish three cases.

Case 1. Both lower parameters are positive integers. For $a \in \mathbb{Q} \setminus \mathbb{Z}$ and an integer $m \geq 1$, put

$$A_{a,m}(x) = \sum_{k \geq 0} \frac{(a)_k}{(m)_k} x^k.$$

The binomial theorem gives $A_{a,1} = (1-x)^{-a}$. For $m \geq 2$, shifting the coefficient index gives

$$A_{a,m}(x) = \frac{m-1}{a-1} \, \frac{A_{a-1,m-1}(x) - 1}{x}.$$

The numerator vanishes at the origin. Since $a - 1$ is again nonintegral, induction on m gives $A_{a,m} \in \mathbb{Q}\langle x \rangle$. Multiplying the coefficients term by term gives

$$F = A_{a_1,b_1} \star A_{a_2,b_2} \star A_{a_3,1}.$$

Thus F is the diagonal of the algebraic series $A_{a_1,b_1}(u) A_{a_2,b_2}(v) A_{a_3,1}(w)$. As F is D-finite, Lemma 2.3 gives a rational diagonal over $\mathbb{Q}$ in four variables, regular at the origin.

Case 2. Exactly one lower parameter is a positive integer. By symmetry of the lower parameters, Theorem A applies with $n = 3$.

Case 3: $b_1, b_2 \notin \mathbb{Z}$. Let N be the common denominator of the parameters. For $k \in (\mathbb{Z}/N\mathbb{Z})^\times$ let $\lambda_i = \{k a_i\}$ and $\mu_j = \{k b_j\}$; these numbers lie in $(0,1)$, and $\lambda_i \neq \mu_j$ for all i, j. Lemma 2.7 at k gives $\#\{i : \lambda_i \leq t\} \geq \#\{j : \mu_j \leq t\}$ for all $t \in (0,1)$. At $-k$ it gives $\#\{i : \lambda_i \geq t\} \geq \#\{j : \mu_j \geq t\}$, that is, $\#\{i : \lambda_i < t\} \leq \#\{j : \mu_j < t\} + 1$, for all $t \in (0,1)$. Hence the number of λ_i below t exceeds the number of μ_j below t by 0 or 1, for every t. This forces

$$\lambda_{(1)} < \mu_{(1)} < \lambda_{(2)} < \mu_{(2)} < \lambda_{(3)}$$

for the values sorted in increasing order. Thus the roots of P_{ka} and of $(t-1)(t - \mathbf{e}(kb_1))(t - \mathbf{e}(kb_2))$ interlace on the unit circle, for every k. By the criterion of Beukers and Heckman [7,

Theorem 4.8], the monodromy group of F is finite. Hence F is algebraic over $\mathbb{Q}(x)$ [compare 15, Theorem 33], and by Theorem 2.4 it is the diagonal of a rational function in two variables. □

Remark 9.6 (Maximal height). Let $n \geq 2$, let $a_1, \dots, a_n \in \mathbb{Q} \setminus \mathbb{Z}$, and let $m_1, \dots, m_{n-1}$ be positive integers. Set $m_n = 1$ for the implicit factorial. The same coefficient identity gives

$$F = {}_nF_{n-1}(a_1, \dots, a_n; m_1, \dots, m_{n-1}; x) = A_{a_1,m_1} \star \cdots \star A_{a_n,m_n}.$$

The n algebraic factors give Hadamard grade at most n and, by taking their product in separate variables and applying Lemma 2.3, a rational diagonal over $\mathbb{Q}$ in $n+1$ variables. In particular F is globally bounded. Its height is n, so the lower bound of Harder and Kramer-Miller [34, Theorem 3.8] gives $\mathrm{hg}(F) = \mathrm{dg}(F) = n$. Hence $n+1$ rational variables are optimal, including four in Case 1 above.

9.4. **Explicit representations of some neighbours of H.**

Proof of Proposition D. (a) Expanding $(1-y)^{-s} = \sum_m (s)_m y^m/m!$ and using that $\sum_{\omega^3=1} \omega^m$ equals 3 if $3 \mid m$ and 0 otherwise gives the two expressions for $\mathcal{T}_s$. Each $(1-\omega y)^{-s}$ is algebraic over $\mathbb{Q}(\omega, y)$, which is a finite extension of $\mathbb{Q}(y^3)$. Hence $\mathcal{T}_s(x)$ is algebraic over $\mathbb{Q}(x)$. By Gauss's multiplication formula,

$$(s)_{3n} = 27^n \Big(\frac{s}{3}\Big)_n \Big(\frac{s+1}{3}\Big)_n \Big(\frac{s+2}{3}\Big)_n \qquad \text{and} \qquad (3n)! = 27^n \Big(\frac{1}{3}\Big)_n \Big(\frac{2}{3}\Big)_n n!.$$

Since $\{c, 1-c\} = \{\frac{1}{3}, \frac{2}{3}\}$, we have $(\frac{1}{3})_n (\frac{2}{3})_n = (1-c)_n (c)_n$, and therefore

$$\frac{(s)_{3n}}{(3n)!} \cdot \frac{(c)_n}{n!} = \frac{(\frac{s}{3})_n (\frac{s+1}{3})_n (\frac{s+2}{3})_n}{(1-c)_n \, n!^2}.$$

As $(1-x)^{-c} = \sum_n (c)_n x^n/n!$, this is the first identity. The second holds for every Hadamard product, because the coefficient of $u^n v^n$ in $f(u)g(v)$ is $a_n b_n$.

(b) Let $\mathcal{C}(y) = \frac{1-\sqrt{1-4y}}{2y}$ be the generating function of the Catalan numbers, so that $\mathcal{C} = 1 + y\,\mathcal{C}^2$. We first show that, for every $\alpha \in \mathbb{Q}$,

$$\sum_{n \geq 0} \binom{2n+\alpha}{n} y^n = \frac{\mathcal{C}(y)^\alpha}{\sqrt{1-4y}}; \tag{9.1}$$

compare Graham et al. [29, §5.4]. Put $\tau = \mathcal{C} - 1$, so that $\tau = y(1+\tau)^2$. The Lagrange inversion formula, in the form

$$\frac{G(\tau)}{1 - y\chi'(\tau)} = \sum_{n \geq 0} y^n \, [\tau^n] \, G(\tau)\chi(\tau)^n \qquad \text{for } \tau = y\chi(\tau),$$

applied with $\chi(\tau) = (1+\tau)^2$ and $G(\tau) = (1+\tau)^\alpha$, gives

$$\frac{\mathcal{C}^\alpha}{1 - 2y\,\mathcal{C}} = \sum_{n \geq 0} y^n \, [\tau^n](1+\tau)^{\alpha+2n} = \sum_{n \geq 0} \binom{2n+\alpha}{n} y^n,$$

and $1 - 2y\,\mathcal{C} = \sqrt{1-4y}$. For $\alpha = s-1$ this shows that $\phi_s = \mathcal{C}^{s-1}(1-4y)^{-1/2}$ is algebraic over $\mathbb{Q}(y)$.

Consequently the series

$$g_s(u,v) = (1-u)^{-s} \, \phi_s\Big(\frac{v}{(1-u)^2}\Big) = \sum_{n \geq 0} \binom{2n+s-1}{n} v^n \, (1-u)^{-s-2n}$$

lies in $\mathbb{Q}\langle u, v\rangle$. The coefficient of u^n in $(1-u)^{-s-2n}$ is $(s+2n)_n/n!$, and $\binom{2n+s-1}{n} = (s+n)_n/n!$. Hence

$$[u^n v^n]\, g_s = \frac{(s+n)_n\,(s+2n)_n}{n!^2} = \frac{(s+n)_{2n}}{n!^2} = \frac{(s)_{3n}}{(s)_n\, n!^2}.$$

By the first formula of Gauss above, $[u^n v^n]\, g_s$ is therefore the coefficient of x^n in

$${}_3F_2\Big(\frac{s}{3}, \frac{s+1}{3}, \frac{s+2}{3}; s, 1; 27x\Big). \qquad \square$$

Remark 9.7. (1) For every $s \in \mathbb{Q} \setminus \mathbb{Z}$ and $c \in \{\frac{1}{3}, \frac{2}{3}\}$ the parameters in part (a) satisfy (NR). So part (a) applies to infinitely many of the series covered by Theorem A. Besides the two series of Abdelaziz et al. [2], these include ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}; \frac{2}{3}, 1; x) = \mathcal{T}_{1/3}(x) \star (1-x)^{-1/3}$ and ${}_3F_2(\frac{2}{9}, \frac{5}{9}, \frac{8}{9}; \frac{1}{3}, 1; x) = \mathcal{T}_{2/3}(x) \star (1-x)^{-2/3}$.

(2) The factorization in part (a) uses the auxiliary parameters $\frac{1}{3}$ and $\frac{2}{3}$, which cancel. It is therefore not among the six factorizations obtained by distributing the given parameters over two factors, which Abdelaziz et al. [2, §2.3] examine and rule out for their two series. In the same paper (footnote on p. 4) it is stated, with a reference to Rivoal and Roques [45], that ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}; \frac{2}{3}, 1; x)$ cannot be written as a Hadamard product. We have not found such a statement in Rivoal and Roques [45]. Their result of this kind, their Proposition 1, concerns the series S_7 (see Corollary E). By part (a), ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}; \frac{2}{3}, 1; x)$ is the Hadamard product of two algebraic power series with rational coefficients. The identity in part (a) is elementary, and it may well be known to specialists; we have not found it in print.

(3) Abdelaziz et al. [2] represent the series of part (b) as diagonals of the algebraic functions $(1-x-y)^{1-s}/(1-x-y-z)$ in three variables. Indeed, the coefficient of z^n in this function is $(1-x-y)^{-s-n}$, whose coefficient of $x^n y^n$ is $(s+n)_{2n}/n!^2$. Part (b) needs only two variables. For $s \notin \{\frac{1}{3}, \frac{2}{3}\}$ we do not know whether the series of part (b) is a Hadamard product of algebraic series.

(4) Parts (a) and (b) give explicit algebraic power series in two variables. By Lemma 2.3, every series in parts (a) and (b) is therefore the diagonal of a rational function in three variables; we have not written such rational functions down, although the construction of Denef and Lipshitz [23] behind Lemma 2.3 is effective. When (NR) holds, three is the least possible number, by Lemma 2.6(iv) and Theorem 2.4.

(5) Neither part applies to H, whose upper parameters $\frac{1}{9}, \frac{4}{9}, \frac{5}{9}$ are not of the form $\frac{s}{3}, \frac{s+1}{3}, \frac{s+2}{3}$. Appendix C reports a computer search over finite collections of candidates, including Hadamard products of two algebraic hypergeometric series; it found no representation of H among the candidates it enumerated, and it records which candidates were excluded rigorously and which were left unresolved.

10. Concluding remarks

(1) *An explicit representation of H.* The proof of Theorem A gives no explicit rational function with diagonal H, and the search of Appendix C found no representation among the candidates it enumerated. This does not indicate that explicit representations must be complicated; it only rules out the candidates actually tested, which in families B2 and B3 are fewer than first described (Appendix C). Christol's question whether H is the Hadamard product of two algebraic hypergeometric series also remains open (Remark 9.5(4)). One way to obtain one would be to find an explicit family of curves, for example a family of cyclic covers of $\mathbb{P}^1$ branched at moving points, whose Jacobians contain the abelian scheme of §9.1 up to isogeny, and to make Sections 6 and 7 explicit for it.

(2) *Higher heights and resonance.* Theorem A treats all ranks $n \geq 3$ at height two, and Section 8 treats the positive integral lower parameter. At larger heights the two-level argument need not give a weight-one variation. At maximal height, Remark 9.6 gives an elementary factorization for every choice of positive integral lower parameters. Resonant equations are also outside the theorem; differential reductions cover some cases, but nontrivial extensions require additional arguments.

(3) *The number of variables.* Every globally bounded F_m of Theorem A has diagonal grade exactly two. The proof uses the distinguished size-two Jordan block at eigenvalue 1, without requiring the full monodromy variation to have rank one. In higher rank the local systems may be self-dual; the non-self-duality argument in Lemma 9.1 is restricted to rank three. Comparisons with conjectures about self-adjoint differential operators [35] must therefore be made case by case.

(4) *Elementary cases.* Proposition D(a) shows that some of the series covered by Theorem A are diagonals for elementary reasons. It would be useful to know which globally bounded series ${}_3F_2(a;b,1;x)$ are Hadamard products of algebraic series. Among the sixteen such series with $a_1 \leq a_2 \leq a_3$ and b in $\{\frac{1}{9}, \frac{2}{9}, \dots, \frac{8}{9}\}$ that satisfy (NR), the search of Appendix C produced surviving candidates for exactly the four covered by Proposition D(a). For the other twelve, H among them, it found no representation among the candidates it enumerated, and we know of no proof that they are diagonals other than Theorem A.

(5) *Moduli.* For H, the abelian scheme of §9.1 has multiplication by an order of $\mathbb{Q}(\zeta_9)$, and its period map takes values in a unitary Shimura variety. An identification of the image of U might lead to an explicit model.

Appendix A. An independent check of the Hodge numbers

Fedorov's formula (Theorem 2.9) is the main input of Proposition 3.1. This appendix concerns its rank-three specialization, Corollary 3.2. As a safeguard against errors of normalization, we checked that output independently, by a count on the surfaces of the Euler integral. This check is not used in the proofs.

On $\mathbb{P}^1 \times \mathbb{P}^1$, with coordinates s, t, consider the Euler integrand of ${}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$,

$$s^{a_2-1}(1-s)^{b_1-a_2-1}\, t^{a_3-1}(1-t)^{b_2-a_3-1}\,(1-xst)^{-a_1}\, ds\, dt.$$

Its branch divisor consists of the six lines $s \in \{0, 1, \infty\}$, $t \in \{0, 1, \infty\}$ and the curve $xst = 1$. It has normal crossings except at the two triple points $(s,t) = (\infty, 0)$ and $(0, \infty)$, and one blow-up at each of them produces a divisor with normal crossings. Let χ be a class of the exponent vector modulo $\mathbb{Z}^5$, and let $(e_1, \dots, e_5) \in (-1, 0)^5$ be its representative. Every form in the corresponding eigenspace is a polynomial in s, t times the form $s^{e_1}(1-s)^{e_2}t^{e_3}(1-t)^{e_4}(1-xst)^{e_5}\, ds\, dt$, and for integers $i, j \geq 0$ the monomial $s^i t^j$ gives a square-integrable form if and only if

$$i < -(e_1+e_2+e_5)-1, \qquad j < -(e_3+e_4+e_5)-1, \qquad j-i > e_1+e_2-e_3, \qquad i-j > e_3+e_4-e_1.$$

These four inequalities say that the order of the form is greater than -1 along $s = \infty$, along $t = \infty$ and along the two exceptional curves; along the remaining components this is automatic. Let $n(\chi)$ be the number of such monomials. Assume that no component of the normal crossings divisor carries an integral exponent. Then the χ-eigenspace of the second cohomology of the complement is pure of weight two and of rank three, and the top step of its Hodge filtration is the space of square-integrable holomorphic 2-forms. Granting these two standard facts, its Hodge numbers are

$$(h^{2,0}, h^{1,1}, h^{0,2}) = \big(n(\chi),\ 3 - n(\chi) - n(-\chi),\ n(-\chi)\big).$$

For all six conjugates of H, and for every assignment of the parameters to the roles in the integrand, this count agrees with Proposition 3.1 up to a shift of the grading and passage to the dual local system (compare Remark 2.10), and $n(\chi)\,n(-\chi) = 0$. Summing over the six conjugates gives the Hodge numbers $(3, 12, 3)$. The same agreement holds in all 221,946 conjugate cases arising from 38,164 irreducible parameter sets, namely the series ${}_3F_2(a; b, 1)$ with denominators 3–10, 12, 14, 15 and 18, and the general ${}_3F_2(a; b_1, b_2)$ with denominators 3–10. There are no exceptions, and the passage to the dual is uniform.

Appendix B. Computations, verification and imported results

Scripts. Apart from the search of Appendix C, the computations quoted in this paper use exact rational arithmetic and only the Python standard library. The field $\mathbb{Q}(\zeta_9)$ is realized as $\mathbb{Q}[z]/(z^6 + z^3 + 1)$. Signs at real places are certified by rational enclosures of the three roots of $8c^3 - 6c + 1$, obtained by 200 bisection steps. The supplementary material is not part of this deposit; it is available in the folder `supplementary` of the repository https://github.com/bbrhuft/christol-3f2-diagonals, and all paths below are relative to that folder. It consists of the following scripts and their outputs.

- `scripts/monodromy_hermitian.py` verifies statements (i)–(vi) of §9.1, including the Gram determinant 3^{36} of the form ψ_0.
- `scripts/hodge_L2_count.py` carries out the comparison of Appendix A.
- `scripts/mechanism_controls.py` verifies, to order 60, the identity of Remark 7.2 and the identities of Proposition D(b) for $s = \frac{1}{3}$ and $s = \frac{2}{3}$, together with polynomial equations satisfied by ϕ_s. It also checks the closed form of ρ in the proof of Proposition 3.1 on 54,264 parameter sets. One floating-point contour integral is included as an illustration and is labelled as such.
- `scripts/verify.py` reruns the three scripts above and compares the results with the stored outputs.
- `scripts/hadamard_identities.py` checks the coefficient identity of Proposition D(a) exactly to order 200 for twelve pairs (s, c). These checks illustrate the identity; its proof is the one given in §9.4.
- `scripts/eisenstein_counterexample.py` checks the identity (C.1) of Appendix C exactly to order 120.
- The folder `search` contains the code and output of Appendix C. It uses NumPy and arithmetic modulo primes, and the complete numerical search takes a few hours on two processor cores. It also contains `test_engine.py`, which requires a positive control to match and a negative control not to match, and `test_symmetry.py`, a regression test for the pools of families B2 and B3 (Appendix C). Both return a nonzero exit status if an expectation fails.
- The folder `audit` contains `audit_exclusions.py`, which recomputes from scratch every count of the exact families of Appendix C and sorts the exclusions according to the cases of Lemma C.1; `audit_numerical.py`, which recomputes the numbers of candidates enumerated in the numerical families and the bound (C.3), and inspects the stored records of the 312 survivors, including the stored outcomes of their 50-coefficient comparisons, without recomputing those comparisons (with the option `-recompute-survivors` it recomputes them, in about 8 minutes); and `rerun_numerical_A2_A4.py`, which recomputes the numerical families A2 and A4 and compares their survivors with the stored output; and `check_failure_propagation.sh`, which injects failed expectations into a temporary copy of the supplementary material and checks that the verification command then fails.

Reproduction. For version 1.5 all scripts listed above were rerun with Python 3.13.18; the scripts in `search` and `audit` also use NumPy 2.5.3. The paper was typeset with TEX Live 2023. The default verification command is

```
sh verify_all.sh
```

run from the top-level folder of the archive that contains the paper and the supplementary material. It checks the file checksums and then runs the script `run_checks.sh` in the folder `supplementary`, which executes every script listed above, apart from the optional recomputation mentioned below, in about 5 minutes. It stops with a nonzero exit status at the first failed expectation; each script compares its results with those stated in the paper. The failure-injection test confirms this behaviour on a temporary copy. All checks passed. The default command does not recompute the 50-coefficient comparisons of the survivors, which it only inspects in the stored records; the optional routine `audit_numerical.py -recompute-survivors` does, and it was run for version 1.5. The numerical families A3, B2 and B3 were not recomputed for versions 1.4 and 1.5. The file `VERIFICATION_LOG.md` records the outputs, and `SHA256SUMS` the checksums of all files.

Higher-order checks in version 1.7. The accompanying revision bundle includes `tools/verify_extension.py`, which uses only the Python standard library. It checks 1,220 coefficients in the forward and inverse contiguity identities across ranks three, four and five, with integral parameters $m = 2, 3, 4, 7, 12$. It also checks 2,000 deterministic rational parameter multisets in ranks three through six, including repetitions: the reflection and equal-block identities, extrema, equivalence with the all-conjugate boundedness criterion, and the multiplicity bound. Forty-seven of these multisets satisfy that criterion. These finite checks support the operator and counting calculations; they do not certify the geometric existence proof. They are separate from the legacy supplementary computations and do not represent a rerun of the numerical searches in Appendix C.

Imported results. Table 3 lists the results from the literature on which the proof of Theorem A depends, and where they are used. All other steps are proved in the paper.

Table 3: Imported statements used in Theorem A; source versions are specified below.

Imported result	Use
Levelt [7, Propositions 3.2–3.3, Theorem 3.5]	Lemmas 2.5 and 2.6
Christol's criterion, through [27, Theorem 2.3]	Lemma 2.7 and Proposition 3.1
Fedorov's general-rank formula [26, Proposition 2.6, Theorem 1]	Proposition 3.1, including repeated parameters
Weight one and abelian schemes [43, Theorem 7.13], crediting [18, 4.4.3]	Proposition 5.1
Finite-presentation descent; smoothness, properness and flatness [54, Tags 01ZM, 0C0C, 081F, 04AI]	Propositions 5.1 and 6.2
Connected components in smooth proper families [54, Tag 0E0N]	Descent of the abelian scheme
Ehresmann [51, §1, p. 154]	Betti local constancy in Proposition 5.1; boundary variant explained in Lemma 6.6
Regular Riemann–Hilbert and Gauss–Manin comparison [17, I.2.17, II.5.9, II.6.13–6.14, II.7.9]	Lemma 5.3 and Proposition 5.4

Imported result	Use
Bertini [54, Tag 0FD6 and Section 37.32]; integral Lefschetz [50, Theorem 3]	Proposition 6.1
Quasi-unipotence [17, III.2.3]; Riemann existence [52, Remark 3.3(c), Theorem 3.4]	Proposition 6.2, Step 1
Grothendieck's semistability criterion, through [53, Theorem 4.1]	Plain unipotence suffices; Proposition 6.2
Jacobians and stable curves [22, Theorem 2.4]	Over $\overline{\mathbb{Q}}[w]_{(w)}$, with algebraically closed residue field and genus at least three
Nodes, dualizing forms and resolution [22, §1, pp. 76–77, 84–85]	Proposition 6.2 and the local generator $du/u = -dv/v$ in Proposition 7.1
de Rham comparison [17, II.6.2]	Lemma 6.4; the affine Čech computation follows from quasi-coherent acyclicity and is explained there
Artin approximation [6, Theorem 1.10]	Lemmas 2.1 and 6.5
Two algebraic to three rational variables [34, Lemma 3.2, v1]	Proof of Theorem A
Two-variable rational diagonals are algebraic [34, Section 1], crediting [28, 44]	Optimality of three variables

Scope of the source checks. For version 1.7, the relevant statements and immediate hypotheses were checked in the primary texts or the explicit modern restatements identified in Table 3. Christol's criterion was checked through Fürnsinn and Yurkevich [27, Theorem 2.3, arXiv:2308.12855v3]; Fedorov through arXiv:1505.01704v2, Proposition 2.6 and Theorem 1; Deligne's abelian-scheme equivalence through Milne [43, Theorem 7.13, arXiv:1105.0887v1]; and algebraic-to-rational diagonal conversion through arXiv:2504.10354v1, Lemma 3.2. All numbering for the last source uses v1: its general height lower bound is Theorem 3.8 and its nilpotence corollary is Corollary 2.6.

The original scan of Deligne [17], including the 1971 erratum, was checked at I.2.17, II.5.9, II.6.2, II.6.13–6.14, II.7.9 and III.2.3. Deligne–Mumford's local calculations and Theorem 2.4 and Artin's Theorem 1.10 were checked in their original articles. The remaining spreading-out and slicing references now have inspected statements in the Stacks Project and Catanese [50]. Ehresmann's statement was checked in the Bourbaki version [51]; the short extension to manifolds with boundary is explained in the proof rather than attributed to that statement alone.

The full original SGA 7 scans and the broad Liu, EGA and Bertini book passages were not all inspected. They are not needed as unchecked sole supports: Proposition 6.2 uses the exact criterion restated in Silverberg and Zarhin [53, Theorem 4.1], whose hypotheses allow residue characteristic zero and ordinary unipotence, and explains the torsion-cover comparison using the inspected Riemann existence statement. The SGA 7 references retain historical attribution. The elementary annulus argument proves the needed vanishing-cycle inclusion without importing the full Picard–Lefschetz formula. Likewise the local dualizing generator follows directly by the relative hypersurface residue calculation for $uv = w$, in agreement with Deligne–Mumford's nodal description. Checking these statements is distinct from reproving their sources, verifying every historical citation, or obtaining independent expert peer review.

Appendix C. A search for explicit representations of H

Theorem A does not produce an explicit representation of H. We searched for one in several finite collections of simple candidates and found none among the candidates enumerated. We report the search because it led to Proposition D(a), and so that it need not be repeated. It says nothing about candidates outside the collections actually enumerated, which in families B2 and B3 are smaller than first described (see below), and it gives no lower bound for the complexity of representations of H. The code and the complete output are part of the supplementary material.

We use three terms. A candidate is *excluded* if a complete mathematical argument, given below, shows that it is not a representation of the target; the argument may rely on exact computations, that is, computations with rational numbers or with residues modulo primes, without rounding. A candidate is *unresolved* if the computations do not decide it, whatever evidence they provide. A candidate *survives* if it passes all tests; survivors are not thereby proved to be representations. Exact arithmetic alone does not make a decision complete: each exclusion below rests on the criterion it is checked against.

A transcendence criterion. For finite multisets T and B of rational numbers with $|T| = |B|$, no element of T or B a non-positive integer, put

$$\Phi_{T,B}(x) = \sum_{n\geq 0} \frac{\prod_{t\in T}(t)_n}{\prod_{b\in B}(b)_n}\, x^n,$$

so that $n! = (1)_n$ counts as a lower parameter 1. We reduce (T, B) by the following two operations, as long as possible:

(R1) remove $t \in T$ and $b \in B$ with $t = b$;
(R2) remove $t \in T$ and $b \in B$ with $t - b \in \mathbb{Z}_{\geq 1}$.

Let (T°, B°) be the result. The operation (R1) does not change the series. A pair removed by (R2) contributes to the n-th coefficient of $\Phi_{T,B}$ the factor $(t)_n/(b)_n = (b+n)_{t-b}/(b)_{t-b}$, which is a polynomial in n. Hence $\Phi_{T,B} = P(\theta)\,\Phi_{T^\circ,B^\circ}$ for a nonzero polynomial P.

Lemma C.1. *Suppose that $T^\circ \neq \emptyset$, that no element of T° differs from an element of B° by an integer, and that B° contains an integer. Suppose further that B° contains at least two integers, or that it contains exactly one and the interlacing condition of Beukers and Heckman [7, Theorem 4.8] fails for T° and B° with this integer replaced by 1. Then $\Phi_{T,B}$ is transcendental over $\mathbb{Q}(x)$.*

Proof. Let $m_0 \geq 1$ be the least integer in B°, and let B^1 be obtained from B° by replacing one copy of m_0 by 1. The contiguity relation $(\theta + b - 1)\Phi_{T,B'\cup\{b\}} = (b-1)\,\Phi_{T,B'\cup\{b-1\}}$, which follows from $(n + b - 1)/(b)_n = (b-1)/(b-1)_n$, applied for $b = m_0, m_0 - 1, \ldots, 2$, gives $\Phi_{T^\circ,B^1} = c\,Q(\theta)\,\Phi_{T^\circ,B^\circ}$ with $Q(\theta) = \prod_{j=1}^{m_0-1}(\theta + j)$ and $c \neq 0$. Since operators in θ commute,

$$c\,Q(\theta)\,\Phi_{T,B} = P(\theta)\,\Phi_{T^\circ,B^1}.$$

As $1 \in B^1$, the series Φ_{T°,B^1} is the solution holomorphic at 0 of the hypergeometric operator L^1 with upper parameters T° and lower parameters B^1. By hypothesis and Beukers and Heckman [7, Proposition 3.3], the local system $\mathcal{S}^1$ of solutions of L^1 is irreducible. Its monodromy group is infinite: if B^1 contains at least two integers, the local monodromy at 0 has the eigenvalue 1 with multiplicity at least two and is conjugate to a companion matrix [7, Theorem 3.5], so it has a nontrivial Jordan block; otherwise this is Beukers and Heckman [7, Theorem 4.8].

The operator $P(\theta)$ commutes with analytic continuation, so its kernel on $\mathcal{S}^1$ is a sub-local system. It is not all of $\mathcal{S}^1$: the solutions of $P(\theta)y = 0$ are holomorphic at $x = 1$, whereas the

local monodromy of $\mathcal{S}^1$ at 1 is a pseudo-reflection, by Levelt's theorem [7, Theorem 3.5], and in particular not trivial. By irreducibility the kernel is zero, so $P(\theta)$ maps $\mathcal{S}^1$ isomorphically onto a local system with infinite irreducible monodromy, which contains $P(\theta)\Phi_{T^\circ,B^1} \neq 0$. If $\Phi_{T,B}$ were algebraic, so would be $Q(\theta)\Phi_{T,B}$, and the analytic continuations of $P(\theta)\Phi_{T^\circ,B^1}$ would span a nonzero sub-local system with finite monodromy, hence the whole local system, a contradiction. □

The criterion does not apply when, after reduction, some lower parameter exceeds an upper parameter by a positive integer, or when no lower parameter is an integer. In those cases we only record the primes that occur in the denominators of the first 80 coefficients but divide no denominator of a parameter. By Eisenstein's theorem, only finitely many distinct primes divide the denominators of the coefficients of a power series in $\mathbb{Q}[[x]]$ that is algebraic over $\mathbb{Q}(x)$; a fixed prime may divide infinitely many of these denominators. The occurrence of a few such primes among the first coefficients is therefore no obstruction to algebraicity. For example,

$$ {}_2F_1\Big(\frac{1}{2}, \frac{116}{11}; \frac{105}{11}; x\Big) = (1-x)^{-1/2} + \frac{11}{210}\, x\,(1-x)^{-3/2} \tag{C.1}$$

is algebraic, although its first coefficient is $\frac{58}{105}$ and the primes $3, 5, 7$, which divide no parameter denominator, occur in the denominators of its coefficients. (This series is decided correctly by Lemma C.1's reduction step (R2), which our classifier applies first, but it shows that three outside primes do not imply transcendence.) We therefore treat all such cases as unresolved.

Exact families. *(i) The shape of Proposition D(b).* Let $g(u,v) = (1-u)^{-A}\varphi(v/(1-u)^k)$ with $k \geq 1$, $A \in \mathbb{Q}$ and φ an algebraic power series. Since $[u^n](1-u)^{-A-kn} = (A+kn)_n/n!$, we have $\mathrm{Diag}(g) = H(\lambda x)$ for some $\lambda \in \mathbb{Q}^\times$ if and only if $\varphi(y) = \Phi_{T,B}(\lambda' y)$ for some $\lambda' \in \mathbb{Q}^\times$, where

$$T = \{\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{5}{9}\} \cup \Big\{\frac{A+i}{k} : 0 \leq i < k\Big\}, \qquad B = \{\tfrac{1}{3}, 1\} \cup \Big\{\frac{A+i}{k+1} : 0 \leq i \leq k\Big\}.$$

We tested $1 \leq k \leq 8$ and all $A = p/q$ with $1 \leq q \leq 36$ and $|A| \leq 4$, excluding non-positive integers: 25,312 pairs (k, A). In 25,266 cases $\Phi_{T,B}$ is transcendental by Lemma C.1, so these candidates are excluded. Of them, 25,221 need only (R1) and have 1 as their only integral lower parameter, so that Beukers and Heckman [7, Theorem 4.8] applies directly; 43 need (R2); and 2 need the shift of an integral lower parameter in the proof of Lemma C.1. The remaining 46 cases are unresolved. In each of them at least three outside primes occur, which is evidence, but not proof, that φ is transcendental.

(ii) Hadamard factorizations. This family concerns Christol's question whether H is the Hadamard product of two algebraic hypergeometric series (Remark 9.5(4)). Suppose that $H = \varphi \star \psi$, where $\psi = \Phi_{S,\{1\}\cup R}$. Then $\varphi = \Phi_{T,B}$ with $T = \{\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\} \cup R$ and $B = \{\frac{1}{3}, 1\} \cup S$. We tested all ψ with $|S| \leq 3$ and parameters in $\{j/18 : 1 \leq j \leq 36\}$: 5,642,388 choices. They fall into three classes.

- For 4,149,117 choices, ψ is transcendental by Lemma C.1: in 3,658,432 cases by Beukers and Heckman [7, Theorem 4.8] directly, in 472,957 because two integral lower parameters remain, and in 17,728 after the shift of an integral lower parameter. These choices are excluded.
- For 1,457,968 choices the parameters of ψ are reducible after (R1), that is, some upper and lower parameter differ by a nonzero integer, or no integral lower parameter remains. These choices were not examined. Reducible parameters can give algebraic series, as (C.1) shows, so these factorizations are not excluded.

- For 35,303 choices, ψ is algebraic: in 34,600 cases by the interlacing criterion, and in 703 cases because all parameters cancel and $\psi = (1-x)^{-1}$. For each of them we examined φ. In 31,495 cases φ is transcendental by Lemma C.1, so the factorization is excluded: in 28,783 cases directly, in 2,009 after (R2), and in the 703 trivial cases $\psi = (1-x)^{-1}$, $\varphi = H$, because two integral lower parameters remain. The remaining 3,808 cases are unresolved, with evidence of the kind described above.

Thus, within this range, no factorization of H into two algebraic hypergeometric series exists with ψ irreducible, except possibly in the 3,808 unresolved cases; factorizations with reducible ψ were not examined.

Numerical families. Let $f = \prod_j F_j^{\alpha_j}$ with polynomials F_j satisfying $F_j(0) = 1$ and $\alpha_j \in \mathbb{Q}$. With $Q = \prod_j F_j$ and $P_v = \sum_j \alpha_j (\partial_v F_j) \prod_{i \neq j} F_i$ we have $Q\,\partial_v f = P_v\, f$ for each variable v. This gives recurrences for the coefficients of f; solving them requires division only by $Q(0) = 1$, by positive integers less than 60 and by the denominator 9 of the exponents, all of which are invertible modulo the primes used below. We computed the first seven coefficients $d_0 = 1, d_1, \dots, d_6$ of $\mathrm{Diag}(f)$ modulo $p = 2^{31} - 1$, simultaneously for a grid of exponent vectors. Table 4 lists the families as they were actually enumerated. The exponents are nonzero multiples of $\frac{1}{9}$.

In families A the factors are distinct, and collections of linear forms are taken up to a simultaneous permutation of x, y, z. Since Diag is unchanged when the same permutation is applied to every factor, this is a genuine symmetry reduction, and families A are complete as described.

Families B2 and B3 were enumerated differently, and they are smaller than the families described in versions 1.2–1.4 of this paper. Each polynomial $1 + au + bv + cuv + du^2 + ev^2$ of a prescribed set was replaced, separately, by the lexicographically smaller of its coefficient vector (a, b, c, d, e) and the vector (b, a, c, e, d) of its image under $u \leftrightarrow v$. Choosing a representative for each factor separately is not a symmetry reduction of a product, because the exchange of u and v must act on all factors at once. For example, $1/((1-u)(1-2v))$ has diagonal $1/(1-2x)$, whereas the representatives $1-u$ and $1-2u$ give $1/((1-u)(1-2u))$, whose diagonal is 1. The procedure therefore restricts the search to products of distinct elements of two explicit pools $\mathcal{P}_2$ and $\mathcal{P}_3$:

- $\mathcal{P}_2$ is obtained in this way from the 1,619 nonconstant polynomials with $a, b \in \{0, 1, -1, -2, -3, -4\}$, $c \in \{0, \pm 1, \pm 2\}$ and $d, e \in \{0, \pm 1\}$. It has 854 elements, all in this set. Family B2 consists of the products $F_1^{\alpha_1} F_2^{\alpha_2}$ with $F_1 \neq F_2$ in $\mathcal{P}_2$: 364,231 factor pairs. They represent 364,231 of the 657,226 classes of unordered pairs of distinct polynomials from the set under the simultaneous exchange $u \leftrightarrow v$. For example, the pair $\{1-u, 1-2v\}$ is not represented.
- $\mathcal{P}_3$ is obtained in this way from the 95 nonconstant polynomials with $a, b \in \{0, -1, -2, -4\}$, $c \in \{0, \pm 1\}$, $d \in \{0, 1\}$ and $e = 0$. It has 77 elements; 30 of them contain v^2 and do not belong to this set. Family B3 consists of the products of three distinct elements of $\mathcal{P}_3$: 73,150 factor triples. Of these, 45,070 are equivalent under the simultaneous exchange to triples from the set, which form 130,489 classes, and 28,080 are not.

The pools are listed in the file `search/pools_B2_B3.json` of the supplementary material, and `search/test_symmetry.py` checks the example above and all of these counts. The enumeration was not changed after the restriction was noticed, so the counts and results reported here describe exactly the candidates that were enumerated.

All candidates were compared with sixteen target series, namely the globally bounded series ${}_3F_2(a; b, 1; x)$ satisfying (NR) whose parameters lie in the grid $\{\frac{1}{9}, \frac{2}{9}, \dots, \frac{8}{9}\}$, with $a_1 \leq a_2 \leq a_3$

	factors	exponents	candidates
A2	two factors $1-L$, $L=ax+by+cz$, $a,b,c\in\{0,1,2,3\}$	$\lvert\alpha\rvert\le 3$	1,140,156
A3	three factors $1-L$, $a,b,c\in\{0,1,2\}$	$\lvert\alpha\rvert\le 2$	23,328,000
A4	four factors $1-L$, $a,b,c\in\{0,1\}$	$\lvert\alpha\rvert\le 1$	1,049,760
B2	two distinct factors from the pool $\mathcal{P}_2$ (854 quadratics)	$\lvert\alpha\rvert\le 2$	472,043,376
B3	three distinct factors from the pool $\mathcal{P}_3$ (77 quadratics)	$\lvert\alpha\rvert\le 1$	426,610,800
	total		924,172,092

TABLE 4. Numerical search families as enumerated. Families A are complete up to simultaneous permutations of the variables; for the pools of B2 and B3 see the text.

and repetitions allowed; H is one of them. Let t_n be the coefficients of such a target, so that $t_0=1$ and $t_1=a_1a_2a_3/b\neq 0$. If $\mathrm{Diag}(f)(x)$ equals the target at λx for some $\lambda\in\mathbb{Q}^\times$, then $\lambda=d_1/t_1\neq 0$ and

$$d_j\,d_1^{-j}=t_j\,t_1^{-j}\qquad(2\le j\le 6). \tag{C.2}$$

The search discarded every candidate with $d_1\equiv 0 \bmod p$ and every candidate for which (C.2) fails modulo p. This excludes it rigorously, for the following reason. The coefficient d_1 is a rational number whose denominator divides $9^3\cdot 3!$ in families A and $9^2\cdot 2!$ in families B. Bounding each coefficient of $F_j^{\alpha_j}$ by the corresponding coefficient of $(1-|M_j|)^{-|\alpha_j|}$, where $F_j=1-M_j$ and $|M_j|$ has the absolute values of the coefficients of M_j, gives

$$|d_1|\cdot\mathrm{den}(d_1)\le 39{,}680{,}928<p \tag{C.3}$$

in all five families (the bound for A2; the others are smaller). For B2 and B3 the bound applies to the pools actually enumerated, whose coefficients satisfy $|a|,|b|\le 4$ and $|c|\le 2$; terms in u^2 or v^2 do not contribute to d_1. Hence $d_1\equiv 0 \bmod p$ only if $d_1=0$, in which case λ would vanish. Otherwise all quantities in (C.2) are p-adic units or p-integral, and (C.2) implies the same congruence modulo p. Thus every candidate that the screen discarded is excluded, provided the code computes $d_0,\dots,d_6$ correctly. As a check on the code, the screen recovers the known representations listed below, the families A2 and A4 were recomputed for version 1.4 with identical results, and `search/test_engine.py` tests a positive and a negative control.

Candidates that passed the screen were recomputed to 50 coefficients modulo the prime 2147483629 during the original run, and the outcomes were stored. There were 312 survivors, all recorded as agreeing with their target in these 50 coefficients. For version 1.5 these 312 comparisons were recomputed from scratch, with the same result; the routine that does this is optional and not part of the default verification command (Appendix B). None of the survivors has H as its target.

Controls and results. The search recovered the representation $(1-x-y)^{2/3}/(1-x-y-z)$ of Abdelaziz et al. [2] in family A2; Remark 9.7(3) proves it. It also recovered the function $(1-u-v)^{-1/3}$, whose diagonal is ${}_2F_1(\frac{1}{6},\frac{2}{3};1;4x)$, in all families containing it. The exact families (i) and (ii), run on the two series of Abdelaziz et al. [2], recovered several representations of each; this is how Proposition D(a) was found.

No candidate among those enumerated in the numerical families represents H. For families A this covers the families as described, up to permutations of the variables. For families B2 and B3 it covers only products of elements of the pools $\mathcal{P}_2$ and $\mathcal{P}_3$; products of the originally

described quadratics outside these pools, such as $(1-u)^{\alpha_1}(1-2v)^{\alpha_2}$, were not tested. In the exact families no candidate represents H except possibly the unresolved ones and, in family (ii), those not examined. Among the sixteen target series, survivors occurred for exactly four, namely those covered by Proposition D(a):

$$ {}_3F_2(\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9};\tfrac{1}{3},1), \quad {}_3F_2(\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9};\tfrac{2}{3},1), \quad {}_3F_2(\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9};\tfrac{1}{3},1), \quad {}_3F_2(\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9};\tfrac{2}{3},1). $$

The survivors were not certified individually as identities; that these four series are diagonals follows from Proposition D(a).

The exact families were also run for the series $S_7 = {}_3F_2(\frac{1}{7},\frac{2}{7},\frac{4}{7};\frac{1}{2},1;x)$ of Corollary B, with the grid $\{j/14 : 1 \le j \le 28\}$ in (ii). In family (i), 25,260 of the 25,312 pairs (k, A) are excluded by Lemma C.1 and 52 are unresolved. In family (ii), 1,128,113 choices of ψ are excluded, 514,632 reducible choices were not examined, and of the 17,011 algebraic choices, 14,539 are excluded and 2,472 are unresolved. This is consistent with Rivoal and Roques [45, Proposition 1], which, under the Rohrlich–Lang conjecture, excludes every factorization of S_7 into finitely many algebraic series. The numerical families were not run for this series.

Acknowledgement of AI assistance

The mathematical argument, computations and exposition of this paper were developed with extensive assistance from large language models, principally Claude models provided by Anthropic and GPT-6.1 Sol provided by OpenAI, under the author's direction. This assistance extended beyond language editing to the formulation of research questions, exploration of possible proof strategies, development of mathematical arguments, preparation of computational scripts and revision of the manuscript. The author determined the research objectives, coordinated the successive investigations and selected the arguments and claims presented here.

Claude Fable 5.2, in particular, supplied the decisive insight on which the development of Theorem A was based. This arose in response to a research prompt prepared with GPT-6.1 Sol, directing Fable to investigate a proof for Christol's 1987 test case

$$ {}_3F_2\left(\frac{1}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1;x\right). $$

The resulting argument was subsequently developed into a detailed proof and incorporated into the manuscript through further AI-assisted mathematical work, computation and exposition. The substantial role of these systems in the mathematical development is acknowledged explicitly; their contribution was not confined to the presentation of results obtained beforehand.

All computational claims can be reproduced using the supplementary scripts described in Appendix B. Reproduction of these computations is distinct from verification of the complete mathematical argument, including the applicability of the imported theorems and the analytic and geometric steps of the proof.

Versions 1.4 and 1.5 incorporate corrections and expansions requested in AI-assisted manuscript reviews of versions 1.3 and 1.4, respectively. Among other contributions, these reviews supplied the example (C.1) and identified the restriction of the factor pools in families B2 and B3 of Appendix C. These reviews formed part of the manuscript's development and revision process; they were not independent journal peer review and should not be interpreted as such.

Google Gemini Deep Research was used to conduct a comprehensive literature review. Several critical papers and book sections that were inaccessible to Deep Research were subsequently located manually by the author and incorporated into the literature audit.

Version 1.7 also incorporates arbitrary-rank sections developed with Google Gemini Neon, followed by an OpenAI-assisted audit and consolidation. That work supplied the multiplicity-safe counting formulation, clarified the use of the full cyclotomic field and the monodromy

blocks, and replaced the integral-parameter sketch by an explicit differential inverse. The finite checks and their limits are recorded in Appendix B. This further AI-assisted review is not independent expert peer review.

For version 1.9, Google Gemini Neon suggested extending Corollary C to arbitrary positive integral lower parameters. The proof via algebraic Hadamard factors was developed with OpenAI assistance.

For version 1.8, the septic reconstruction was developed with OpenAI assistance and checked by a second implementation developed with Claude Opus 5.5. The companion note distinguishes the proof of the differential equation from the finite checks of the period matrix through s^{100} and of the operator through s^{102}.

The author takes responsibility for the mathematical claims, computational evidence, citations and final text, including any errors or omissions that remain. The author welcomes corrections and independent scrutiny.

Data availability

The legacy scripts described in Appendices B and C, and their outputs, are available as supplementary material and at https://github.com/bbrhuft/christol-3f2-diagonals.

The version 1.8 supplement includes the version 1.7 higher-order verification bundle and the version 1.1 septic companion note with its source, verification scripts and check outputs.

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