Thermodynamic saturation and prescribed window barriers for Kleinian-code graded spaces

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Abstract

We investigate prescribed graded-window dimensions for Kleinian-code characters, motivated by the four-leg construction of a subexponential 3-SAT algorithm, but without claiming new admissible algebraic operators. For every compact $\mathcal{J} \Subset (0,\infty)$ we prove, uniformly for isotropic additive $S \subset \mathbb{K}^m$,

$$\rho_{\mathrm{win}}(V_S; h, \lambda) \geq \frac{1}{\sqrt{2}} + \eta_{\mathcal{J}} - C_{\mathcal{J}} m^{-1/2}, \quad h/m \in \mathcal{J}, \quad 2^{\lambda} \leq \dim(V_S)_h,$$

where $\eta_{\mathcal{J}} > 0$ is specified by the proof but not yet evaluated as a certified numerical constant for any given compact. The fixed-density proof combines the central-window inequality, sectorwise thermodynamic estimates based on the isotropic local weight bound, finite-length concentration, and certified interval arithmetic (Arb, cross-checked with mpmath.iv) for specified sign inequalities. These numerical certificates do not formalize the complete proof. No concavity or convergence assumption on the code-weight spectrum is needed. For additive codes of 2-rank at most m, we also prove that the maximally encoded prescribed-window cost converges uniformly to $1/\sqrt{2}$ as $h/m \to \infty$, with two-sided explicit errors of order $O(\sqrt{m/h} + (mh)^{-1/4})$ for admissible $h \geq 15m$ and $mh \geq 4000$. These are statements about window dimensions, not intrinsic matrix ranks or implementations within a SAT algorithm.

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1 Introduction

1.1 Background

For classical background on exponential-time hypotheses and sparsification, see Impagliazzo–Paturi and Impagliazzo–Paturi–Zane [1, 2]. A related preprint by OpenAI [3] investigates the Exponential Time Hypothesis; no claim from that work is used in the proofs below. The present analysis is restricted to the indicated prescribed graded-window geometry.

Let $\lambda \geq 1$. The crossing tensor is

$$X_\lambda(x, y; u, v) = [x = v][y = u], \qquad x, y, u, v \in \{0, 1\}^\lambda. \tag{1.1}$$

The source algorithm [4] represents a nonzero multiple of X_λ by efficiently computable pieces with bounded row- and column-factorization bit costs. Its principal construction gives $\rho = 11/12$; Appendix B gives the window-based value $\rho = 1/\sqrt{2}$ on a polynomial ring whose weight-E dimension is the partition number $p(E)$, classically governed at large weight by the Hardy–Ramanujan asymptotic [8]. The complexity exponent is $\alpha = \log_2(1/\rho)$ in the source model.

1.2 Main result and interpretation

The original partition-number example in [4] has a window cost approaching $1/\sqrt{2}$. The question considered here is whether natural graded code families can improve the prescribed window dimensions in the analogous geometry. Theorem 6.5 answers this in the negative at any fixed compact range of densities: not only is the $1/\sqrt{2}$ threshold inaccessible asymptotically, but the cost is separated from it by a positive interval-dependent gap. Theorem 6.4 is the local analytic mechanism, while Lemma 4.7 transfers that sign to dimension growth. Theorem 7.1 complements the result by giving a code-uniform high-density limit, not a new strict fixed-density gap.

The analytic heart of the fixed-density proof is an entropy-growth inequality for cumulative dimensions, stated explicitly after Theorem 6.5. The positive margin is not yet assigned a certified numerical value. An illustrative, non-certified numerical evaluation of the proof's choices suggests $\eta_{[1,2]} \approx 0.005$, which is considerably smaller than the observed limiting margins in simple example codes; this estimate is not used in any theorem.

The argument uses the single central window $t = 0$: $\rho_{\mathrm{win}} \geq D(2h)/(2\lambda)$. This is a statement about the prescribed intermediate graded spaces. It neither bounds the intrinsic row/column ranks of a different factorization nor constructs the operators or identities required by the source algorithm. The gap depends on $\mathcal{J}$ and does not survive uniformly as $h/m \to \infty$: Theorem 7.1 proves quantitative high-density saturation for maximal admissible encodings.

1.3 Related work and scope

The source preprint by Claude/Anthropic [4] reports an exact algebraic decomposition of the crossing tensor. Its main-text piece construction has $\rho = 11/12$, while Appendix B provides $\rho = 1/\sqrt{2}$ through partition growth; Appendix H discusses why square-root growth arises in that construction. Neither the value $1/\sqrt{2}$ nor its partition-growth mechanism is claimed as new here.

Brito [5] proves obstructions for power-growth graded pieces of Appendix-B shape, including the competing rank floor that rules out simply taking a smaller power exponent. Majdoddin [6, Section 4.7, Proposition 4.24] gives a $2/3 - o(1)$ lower bound for concave profiles under the corresponding four-node hypotheses, including $W, H \geq 2h$, and discusses nonconcave ("kinked") profiles as an open escape. These are not assertions about all possible algebraic factorizations. Our strict fixed-density lower bound is numerically stronger at leading order within the particular Kleinian-code family, without assuming concavity of its code-weight spectrum; it is not a universal strengthening of either paper's theorem.

Tulsyan [7, Section 4.2, Theorem 4.3 and Proposition 4.4] already studies the same central-spread mechanism in graded two-order pieces. In his convention $f(E) = \log_2 \dim V_E$, Proposition 4.4 states a lower bound of the form $\rho \geq f(2h)/(2f(h)) - c\log(2\lambda)/(4\lambda)$ for prescribed graded costs with spread $H = 2h$, and obtains $1/\sqrt{2} - o(1)$ for Fock-space examples and a

stated class of vertex-operator algebras. The dimension inequalities behind his Theorem 4.3 are machine-checked; Proposition 4.4 is a paper-only statement. Accordingly, neither the central-window idea nor the leading $1/\sqrt{2}$ barrier is claimed as novel here. What is specific to the present manuscript is the *strict fixed-compact* code-uniform gap for isotropic Kleinian Hilbert characters, without a concavity or limiting weight-spectrum hypothesis, together with quantitative two-sided high-density errors for maximal encodings. Our $D_S(E) = \log_2 \dim(V_S)_{\le E}$ is a cumulative profile, whereas Tulsyan's displayed $f(E)$ counts exact levels; the two statements should not be silently identified. These prescribed window bounds are not intrinsic lower bounds for the ranks of the actual crossing-piece matrices.

Work	Setting and hypotheses	Relevant conclusion
Brito [5]	Graded pieces of the Appendix-B shape; power-growth profiles and encoding/rank restrictions.	A $1/\sqrt{2}$ obstruction for the specified power-profile route.
Majdoddin [6]	Four-leg dimension profiles satisfying the stated concavity and window-size hypotheses.	A $2/3 - o(1)$ bound in that regime.
Tulsyan [7]	Graded two-order pieces with stated spread and pairing conditions.	Central-spread and $1/\sqrt{2}$-scale obstructions for specified graded models.
This work	Isotropic Kleinian-code characters at fixed $h/m \in \mathcal{J} \Subset (0,\infty)$; no limiting or concavity hypothesis on code-weight spectra.	Strict prescribed-window bound $1/\sqrt{2} + \eta_{\mathcal{J}} - O_{\mathcal{J}}(m^{-1/2})$.
This work	Additive Kleinian codes of 2-rank at most m and maximally encoded levels with $h/m \to \infty$.	Uniform prescribed-window saturation at $1/\sqrt{2}$ with explicit two-sided errors.

Table 1: Comparison of the stated settings and conclusions. This table does not establish independent priority or logical non-implication.

The leading high-energy term used in the present note is consistent with the classical Hardy–Ramanujan partition asymptotics [8] and the Cardy square-root entropy law [13]. The coefficient-monotonicity/tauberian viewpoint has precedents in Ingham [9], while modular-invariance bounds for microcanonical entropy have been developed by Mukhametzhanov–Zhiboedov [10]. The universal large-central-charge Cardy regime studied by Hartman–Keller–Stoica [17] uses a sparse-light-spectrum hypothesis; we do *not* apply that theorem to V_S. Instead, our family-specific estimates are derived directly from its explicit characters. The background relation between codes, lattices and conformal field theory is discussed by Höhn [18] and, for distinct Lorentzian/Narain code constructions, by Dymarsky–Shapere [15]; these should not be treated as an identification of the two constructions.

Precise scope of the contribution. The claim of this note is a proved *quantitative formulation for the specified Kleinian-code characters*: a strict lower gap on each fixed positive compact density interval, uniform in m and the isotropic code S with an $O_{\mathcal{J}}(m^{-1/2})$ correction and without any concavity or limiting weight-spectrum assumption; and a complementary uniform high-density limit for maximal admissible encodings, with explicit two-sided errors. In the specific statements of the papers compared above, we have not identified that exact pair of code-uniform estimates. This limited comparison is neither an exhaustive priority search nor a claim that the estimates cannot be derived from broader existing principles. In particular, the high-density leading constant is classical, the quantitative bounds are conservative, the window cost is not an intrinsic matrix rank, and no new admissible piece system or SAT algorithm is constructed.

2 Window factorizations

A graded space $V = \bigoplus_{E\in\Gamma} V_E$ has discrete $\Gamma \subset [0,\infty)$ and finite-dimensional components. Put

$$V_{\le E} = \bigoplus_{e\le E} V_e, \quad D(E) = \log_2 \dim V_{\le E}, \quad V_{[a,b]} = \bigoplus_{e\in[a,b]} V_e, \quad W[a,b] = \log_2 \max\{1, \dim V_{[a,b]}\}.$$

Notation. We use t for a freely varying window shift, reserving s for codewords in S when code sectors are discussed. The source notation A_s for the indexed crossing-tensor pieces is retained verbatim to permit comparison with Anthropic's Appendix B; its index s is not a codeword. In Section 6, w denotes a sector weight; in Section 7 and Appendix C, $r_S = \log_2 |S|$ denotes the code 2-rank. The analytic remainder $r(\beta)$ in Appendix C is a separate function.

Definition 2.1 (Window cost)**.** *For $h > 0$, $\lambda \geq 1$ and $|t| \leq 2h$ set*

$$C_h(t) = W[t, t+2h] + W[-t, 2h-t], \qquad \rho_{\text{win}}(V; h, \lambda) = \frac{1}{4\lambda} \sup_{|t| \leq 2h} C_h(t).$$

Exact dictionary with Anthropic. In [4, Appendix B, equations (54)–(56)], $V = \mathbb{Q}[t_1, t_2, \ldots]$ has integer weights, $\dim V_E = p(E)$, and the 2^λ inputs are encoded by distinct monomials in weight h (chosen as the least integer with $p(h) \geq 2^\lambda$). The pieces are indexed by $I = \{-2h, -2h+1, \ldots, 2h\}$, including $s = 0$. In Definition 2.1 we denote the shift by t and allow real t, while the source indexes pieces by integer s; we use logarithms of window dimensions, whereas the source takes the ceilings of the binary logarithms. We also allow any λ satisfying the encoding condition, whereas the source chooses the *least integer* h for which $p(h) \geq 2^\lambda$. These differences must be maintained when comparing costs; all arguments below use the full real window range and the stated grading. The row and column factorization dimensions of piece s are, respectively,

$$R_s^{\text{row}} = \sum_{i=0}^{2h} p(s+i), \qquad R_s^{\text{col}} = \sum_{i=0}^{2h} p(2h-s-i),$$

where $p(j) = 0$ for $j < 0$, and their bit costs are the ceilings of the binary logarithms. In [4, Definition 7], the parameters satisfy $1/2 \leq \rho < 1$ and $c \geq 1$ is an integer. The *two-direction bit-budget condition* is specifically Definition 7(b):

$$c_s^{\text{row}} + c_s^{\text{col}} \leq 4\rho\lambda + c \log(2\lambda). \tag{2.1}$$

Separately, Definition 7(c) requires

$$|I| \leq (2\lambda)^c,$$

and imposes magnitude bounds on the indices, coefficients and factorization data, together with effective-computability requirements. Definition 7(a) is the crossing-identity requirement. Thus (2.1) is only the bit-budget portion of the source definition, not a claim that a new graded space provides all admissible pieces. The $4\rho\lambda + 8$ estimate in the source Lemma 2(c) concerns its main-text $\rho = 11/12$ pieces and should *not* replace Definition 7(b) in the Appendix-B comparison. The bound on ρ_{win} transfers to the specified window-cost form of (2.1), with the logarithmic tolerance, in Corollary 6.13. Our construction of graded spaces alone supplies no operators satisfying the source identities.

A family of matrices $\{A_s\}_{s \in I}$ summing to dX_λ ($d \neq 0$) is a window factorization over V when each row flattening factors through $V_{[s,s+2h]}$ and each column flattening through $V_{[-s,2h-s]}$.

Lemma 2.2 (Ranks versus windows)**.** *For such a factorization,*

$$\rho_{\text{rank}} := \frac{1}{4\lambda} \max_s \left(\log_2 \max\{1, \operatorname{rank}_{\text{row}} A_s\} + \log_2 \max\{1, \operatorname{rank}_{\text{col}} A_s\}\right) \leq \rho_{\text{win}}.$$

The reverse inequality need not hold. The convention $\log_2 \max\{1, \text{rank}\}$ *allows zero matrices.*

Proof. The rank of a factorization through an N-dimensional space is at most N. □

The source encoding gives the constraint

$$2^\lambda \leq \dim V_h, \qquad \lambda \leq D(h) \qquad \text{(E)}$$

Lemma 2.3 (Rank floor)**.** *If dX_λ is the sum of $|I|$ such pieces, then*

$$2\lambda \leq D(4h) + \log_2 |I| \qquad \text{(R)}$$

Proof. Put $N = 2^\lambda$. The row-wise flattening separates (x, y) from (u, v), and the column-wise flattening separates (x, u) from (y, v). Since $X_\lambda(x, y; u, v) = [x = v][y = u]$, both are $N^2 \times N^2$ permutation matrices and therefore have rank $N^2 = 2^{2\lambda}$. Rank subadditivity in the row-wise flattening gives $2^{2\lambda} \le \sum_s \operatorname{rank}_{\mathrm{row}} A_s$. For $s \in [-2h, 2h]$, the stated row and column windows $[s, s+2h]$ and $[-s, 2h-s]$ lie inside $[-2h, 4h]$. Since the grading is nonnegative, each intermediate window has dimension at most $\dim V_{\le 4h}$. Thus $2^{2\lambda} \le |I| \dim V_{\le 4h}$. The column-wise flattening yields the analogous estimate. □

Lemma 2.4 (Central window)**.** *The definition over the* full *range* $|t| \le 2h$ *gives*

$$\rho_{\mathrm{win}}(V; h, \lambda) \ge \frac{D(2h)}{2\lambda}. \tag{2.2}$$

Under (E) this implies $\rho_{\mathrm{win}} \ge D(2h)/(2D(h))$*; under (R) it implies* $\rho_{\mathrm{win}} \ge D(2h)/(D(4h) + \log_2 |I|)$.

Proof. At $t = 0$ both windows are $V_{\le 2h}$, so $C_h(0) = 2D(2h)$. □

Role of the central piece. Theorems 4.6 and 6.5 bound the prescribed window cost by estimating this one term of the supremum. Their arguments require the encoding inequality (E), but not the separate rank floor (R). In particular, (2.2) gives no lower bound on the intrinsic ranks of the actual factorization matrices: by Lemma 2.2, those ranks can be smaller than the intermediate-window dimensions.

3 Dimension-only bounds and their limits

Set $\ell = \log_2 |I|/\lambda$. In the original construction $|I| = 4h + 1$ and $\ell \to 0$.

Proposition 3.1 (Profile bounds)**.** *Assume (E) and (R).*

(i) *If* $D(2h) \ge \frac{2}{3}D(h) + \frac{1}{3}D(4h)$ *(in particular for a concave extension on* $[h, 4h]$*), then* $\rho_{\mathrm{win}} \ge \frac{2}{3} - \ell/6$.

(ii) *If* $\chi_h = D(2h)^2/(D(h)D(4h))$*, then* $\rho_{\mathrm{win}} \ge \sqrt{\chi_h/2}\sqrt{1 - \ell/2}$.

(iii) *If* $D(E) = cE^\alpha(1 + \delta_E)$ *for* $E = h, 2h, 4h$*,* $|\delta_E| \le \delta_0 < 1$*, then*

$$\rho_{\mathrm{win}} \ge \frac{1 - \delta_0}{1 + \delta_0} \max\{2^{\alpha-1},\ 2^{-\alpha}(1 - \ell/2)\} \ge \frac{1 - \delta_0}{1 + \delta_0}\sqrt{\frac{1 - \ell/2}{2}} \qquad (0 \le \ell < 2).$$

Proof. By (E) and (R), $D(h) \ge \lambda$ and $D(4h) \ge (2 - \ell)\lambda$. Insert each hypothesis into Lemma 2.4. For (ii), $D(2h)^2 = \chi_h D(h)D(4h) \ge \chi_h(2 - \ell)\lambda^2$. For (iii), compare separately $D(2h)/D(h)$ and $D(2h)/D(4h)$. For $0 \le \ell < 2$, the product of the two branches $2^{\alpha-1}$ and $2^{-\alpha}(1 - \ell/2)$ is $(1 - \ell/2)/2$, so their maximum is at least $\sqrt{(1 - \ell/2)/2}$. The exponent $\alpha = 1/2$ minimizes the uncorrected ($\ell = 0$) maximum. When $\ell \ge 2$, the asserted positive square-root refinement is not used. □

Attribution. The concave-profile obstruction in part (i) and the exact-power-profile obstruction underlying part (iii) are already present in Majdoddin [6, Section 4.7, Proposition 4.24]; see also Brito [5] for power-profile rank bounds. The explicit $\ell = \log_2 |I|/\lambda$ and δ_0 terms are recorded here as elementary quantitative variants, without claiming priority. Part (ii) is likewise included as an elementary comparison rather than a novelty claim. The next proposition addresses dimension-only examples, not realizable operator identities.

Proposition 3.2 (Dimension-only countermodels)**.** (i) *Let* $\dim V_0 = 1$*,* $\dim V_h = 2^\lambda$*,* $\dim V_{4h} = 2^{2\lambda}$*, and all other components vanish. Then (E) and the numerical rank inequality (R) hold, but*

$$\rho_{\mathrm{win}}(V; h, \lambda) = \frac{\log_2(1 + 2^\lambda)}{2\lambda} = \frac{1}{2} + O(2^{-\lambda}/\lambda).$$

(ii) *For a relaxed profile* $D(E) = a + cE^p$ *on* $[h, 4h]$ *(*$p > 0$*),* $\tau = a/(ch^p) > -1$*, the two constraints alone imply*

$$\rho_{\text{win}} \geq (1 - \ell/2)F_p(\tau), \qquad F_p(\tau) = \max\left\{\frac{\tau + 2^p}{2(\tau+1)}, \frac{\tau+2^p}{\tau+4^p}\right\}.$$

The minimum over τ *occurs at* $\tau = 4^p - 2$*, with value* $\frac{1}{2} + \frac{1}{2(2^p+1)}$*, which tends to* $1/2$ *as* $p \to \infty$*.*

Proof. For (i), windows with $|t| \leq h$ contain at most $V_0 \oplus V_h$ in each direction; at $t = 0$ both attain that dimension, whereas at an endpoint $t = \pm 2h$ one of the windows contains V_{4h} and the other is the vacuum, giving cost 2λ in the numerator. The stated central cost dominates. For (ii) the first displayed branch decreases with τ and the second increases; equality gives $\tau = 4^p - 2$. □

These are numerical Hilbert-space witnesses: they do not supply the operators or computational certificates required by the source algorithm.

Why impose thermodynamic structure? The countermodels in Proposition 3.2 separate a dimension-profile possibility from an actual algebraic construction: the numerical constraints (E) and (R) alone leave room for a sharp change of growth. The next sections investigate more constrained families using partition functions and graded character identities. In particular, Kleinian glue codes allow weight distributions that need not be concave or convergent across the family; the sectorwise argument in Section 6 still yields the fixed-density window barrier. This is a statement about counted intermediate dimensions, not about the existence or optimality of operator factorization identities.

4 A thermodynamic saturation criterion

4.1 Legendre envelopes and scaling

Let M be graded, with character $Z_M(\beta) = \sum_E (\dim M_E) e^{-\beta E}$ finite for $\beta > 0$. For an arbitrary normalization $m \geq 1$, define

$$\varphi_M(\beta) = \frac{1}{m}\log Z_M(\beta), \qquad \mathcal{L}_M(y) = \inf_{\beta>0}[\varphi_M(\beta) + \beta y] \in [-\infty, \infty). \tag{4.1}$$

The exponential generating-function (Chernoff) bound gives

$$\log \dim M_{\leq my} \leq m\mathcal{L}_M(y). \tag{4.2}$$

Remark 4.1 (Gibbs entropy and logarithmic slope). For a differentiable positive character $Z_M(\beta)$ define its mean energy and Gibbs entropy (in natural-log units) by

$$U_M(\beta) = -\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z_M(\beta), \qquad \mathsf{S}_M(\beta) = \log Z_M(\beta) + \beta U_M(\beta).$$

A direct differentiation gives the identity

$$\frac{\mathrm{d}}{\mathrm{d}\beta}[\beta \log Z_M(\beta)] = \mathsf{S}_M(\beta) - 2\beta U_M(\beta). \tag{4.3}$$

Consequently, for $\mathsf{S}_M(\beta) > 0$, the monotonicity condition $(\beta \log Z_M)' \leq 0$ is *equivalent* to $\beta U_M/\mathsf{S}_M \geq 1/2$. If the Legendre infimum in (4.1) is attained smoothly at β with $U_M(\beta) = my$ and $\mathcal{L}_M(y) > 0$, then

$$m\mathcal{L}_M(y) = \mathsf{S}_M(\beta), \qquad \frac{\mathrm{d}\log \mathcal{L}_M(y)}{\mathrm{d}\log y} = \frac{y\mathcal{L}_M'(y)}{\mathcal{L}_M(y)} = \frac{\beta U_M(\beta)}{\mathsf{S}_M(\beta)} \geq \frac{1}{2}.$$

Thus the criterion bounds the logarithmic entropy-growth slope from below. Since $\varphi_M = m^{-1}\log Z_M$, its sign condition $(\beta\varphi_M)' \leq 0$ is the same. This is an interpretation under the stated differentiability and positivity assumptions, *not* an extra hypothesis for the next scaling lemma, which requires neither differentiability nor a minimizing inverse temperature.

Lemma 4.2 (Scaling lemma)**.** *If $\psi(\beta) := \beta\varphi(\beta)$ is nonincreasing and $\mathcal{L}(y) = \inf_{\beta>0}[\varphi(\beta) + \beta y]$, then for $c \geq 1$ and $y \in \mathbb{R}$,*

$$\mathcal{L}(cy) \geq \sqrt{c}\,\mathcal{L}(y). \tag{4.4}$$

This includes extended-real infima.

Proof. Set $\beta = \gamma/\sqrt{c}$. Then

$$\mathcal{L}(cy) = \sqrt{c}\inf_{\gamma>0}\left[\frac{\psi(\gamma/\sqrt{c})}{\gamma} + \gamma y\right] \geq \sqrt{c}\inf_{\gamma>0}\left[\frac{\psi(\gamma)}{\gamma} + \gamma y\right] = \sqrt{c}\,\mathcal{L}(y).$$

□

4.2 Product sectors and a finite-length estimate

Definition 4.3 (Block alphabet and sector)**.** *A block alphabet is a finite family $\mathcal{A} = (Z^{(1)}, \ldots, Z^{(r)})$ of characters having nonnegative integral coefficients, finite values for $\beta > 0$, infinitely many nonzero coefficients, and lowest energies $g_j \geq 0$. A product sector M of length m has character*

$$Z_M(\beta) = a_M \prod_{i=1}^{m} Z^{(j_i)}(\beta), \qquad a_M \in \mathbb{Z}_{\geq 1}, \qquad g_M = \sum_i g_{j_i}.$$

Only graded dimensions are used: M need not be a literal tensor product.

Lemma 4.4 (Finite-length Chernoff–Chebyshev)**.** *Let $\mathcal{A}$ be a block alphabet and let $L \geq 0$, $0 < y_1 \leq y_2 < \infty$ and $\delta > 0$. There is a constant $C = C(\mathcal{A}, L, y_2, \delta)$ such that for every length-m product sector M with $a_M \leq e^{Lm}$ and every $y \in [y_1, y_2]$ satisfying $y - g_M/m \geq \delta$,*

$$m\mathcal{L}_M(y) - C\sqrt{m} \leq \log\dim M_{\leq my} \leq m\mathcal{L}_M(y). \tag{4.5}$$

Proof. The upper bound is the classical Chernoff inequality in the large-deviation method (see, e.g., [16]). Put $\mu_j(\beta) = -(\log Z^{(j)})'$ and $v_j(\beta) = (\log Z^{(j)})''$. These are the mean and variance of the canonical energy; μ_j is continuous decreasing, $\mu_j(0+) = \infty$ and $\mu_j(\infty) = g_j$. The first limit follows from the infinitely many nonzero coefficients, with $Z^{(j)}(\beta) \to \infty$ as $\beta \downarrow 0$.

Choose $0 < \beta_{\mathrm{lo}} < \beta_{\mathrm{hi}} < \infty$ such that $\mu_j(\beta_{\mathrm{lo}}) > y_2$ and $\mu_j(\beta_{\mathrm{hi}}) < g_j + \delta/4$ for all j. Put $V_* := \max_{j,\beta\in[\beta_{\mathrm{lo}},\beta_{\mathrm{hi}}]} v_j(\beta) < \infty$. The sector Gibbs distribution is a sum of m independent block energies; its mean is $\mu_m(\beta) = \sum_i \mu_{j_i}(\beta)$ and its standard deviation $\sigma_m(\beta) \leq \sqrt{mV_*}$.

Assume $m > 64V_*/(9\delta^2)$. The continuous function $f(\beta) = \mu_m(\beta) + 2\sigma_m(\beta)$ lies above my at β_{lo} and below my at β_{hi}. Indeed $f(\beta_{\mathrm{lo}}) > my_2 \geq my$, whereas $f(\beta_{\mathrm{hi}}) < g_M + m\delta/4 + 2\sqrt{mV_*} < my$. Hence there is $\beta_1 \in [\beta_{\mathrm{lo}}, \beta_{\mathrm{hi}}]$ with $f(\beta_1) = my$. Chebyshev puts at least $3/4$ of Gibbs mass in $[my - 4\sigma_m(\beta_1), my]$. On this interval the weights $e^{-\beta_1 E}$ are at most $e^{-\beta_1(my - 4\sigma_m(\beta_1))}$, so

$$\dim M_{\leq my} \geq \frac{3}{4} e^{\beta_1(my - 4\sigma_m(\beta_1))} Z_M(\beta_1).$$

In particular, the explicit intermediate estimate is

$$\log\dim M_{\leq my} \geq m\mathcal{L}_M(y) - 4\beta_1\sigma_m(\beta_1) - \log(4/3) \geq m\mathcal{L}_M(y) - 4\beta_{\mathrm{hi}}\sqrt{mV_*} - \log(4/3). \tag{4.6}$$

Finitely many smaller m are handled by the sector ground state and $a_M \leq e^{Lm}$, absorbing the bounded range into C. The upper endpoint β_{hi} can be chosen using only the gap δ and the finite alphabet; the constant does not require a separate dependence on y_1 once $y - g_M/m \geq \delta$ is imposed. □

4.3 Assembly

Proposition 4.5 (Sector assembly)**.** *Let $\mathcal{A}$ contain a vacuum character $Z^{(j_0)}$ with $g_{j_0} = 0$. Fix $L, c \geq 0$ and $0 < x_1 \leq x_2 < \infty$. There is C' depending only on these data such that if $V = \bigoplus_{\sigma\in\Sigma} M_\sigma$ is a sum of length-m product sectors satisfying*
1. $|\Sigma| \leq (m+1)^c$ and each multiplicity $a_{M_\sigma} \leq e^{Lm}$;
2. for every sector, $\beta \mapsto \beta\varphi_{M_\sigma}(\beta)$ is nonincreasing;
3. some sector has character $(Z^{(j_0)})^m$;
then for every $h/m \in [x_1, x_2]$ and every $\lambda \geq 1$ with $2^\lambda \leq \dim V_h$,

$$\rho_{\mathrm{win}}(V; h, \lambda) \geq \frac{1}{\sqrt{2}} - C' m^{-1/2}. \tag{4.7}$$

Proof. Write $x = h/m$, $\mathcal{L}_\sigma = \mathcal{L}_{M_\sigma}$, and $b = \max_\sigma \mathcal{L}_\sigma(x)$. The vacuum sector gives $b \geq b_0(x_1) > 0$, where $b_0(y) = \inf_{\beta>0}[\log Z^{(j_0)}(\beta) + \beta y]$: the Legendre envelope is nondecreasing in y because each affine function in its infimum has slope $\beta > 0$. Its positivity at $x_1 > 0$ follows from the positive-energy states of the vacuum character and the ground term. Chernoff and the number-of-sectors bound give the following estimate in *natural logarithms*; the factor $\log 2$ converts the binary encoding parameter λ to natural-log units:

$$\lambda \log 2 \leq \log \dim V_{\leq h} \leq mb + c \log(m+1). \tag{4.8}$$

Choose a maximizing sector σ^*. It cannot have ground energy $g_{\sigma^*} > mx$, since then $\mathcal{L}_{\sigma^*}(x) = -\infty$. Hence $2x - g_{\sigma^*}/m \geq x \geq x_1$, so Lemma 4.4 applied at energy $2h$ gives

$$\log \dim V_{\leq 2h} \geq m\mathcal{L}_{\sigma^*}(2x) - C\sqrt{m} \geq \sqrt{2}\, mb - C\sqrt{m},$$

where the last step is Lemma 4.2. Now Lemma 2.4 and (4.8) yield

$$\rho_{\mathrm{win}} \geq \frac{\sqrt{2}\, mb - C\sqrt{m}}{2(mb + c\log(m+1))} \geq \frac{1}{\sqrt{2}} - O_{\mathcal{A},L,c,[x_1,x_2]}(m^{-1/2}),$$

using $b \geq b_0(x_1) > 0$. If the displayed numerator is negative, the inequality is trivial because $\rho_{\mathrm{win}} \geq 0$. □

4.4 Tensor powers

Theorem 4.6 (Tensor powers)**.** *If B is graded, $B_0 \neq 0$, its character is finite for $\beta > 0$ with infinitely many states, and $\beta \mapsto \beta \log Z_B(\beta)$ is nonincreasing, then for h/m in any fixed compact of $(0, \infty)$ and $2^\lambda \leq \dim(B^{\otimes m})_h$,*

$$\rho_{\mathrm{win}}(B^{\otimes m}; h, \lambda) \geq 1/\sqrt{2} - O(m^{-1/2}).$$

Proof. Apply Proposition 4.5 to the single-letter alphabet (Z_B), with one sector. □

4.5 Local refinement

Lemma 4.7 (Local scaling)**.** *For any $\varphi : (0, \infty) \to \mathbb{R}$, $\psi(\beta) = \beta\varphi(\beta)$, $\mathcal{L}(y) = \inf_{\beta>0}[\varphi(\beta) + \beta y]$, and $\beta > 0$,*

$$\varphi(\beta) + 2y\beta \geq \sqrt{2}\, \mathcal{L}(y) + \frac{\psi(\beta) - \psi(\sqrt{2}\beta)}{\beta}. \tag{4.9}$$

Proof. Subtract $\sqrt{2}[\varphi(\sqrt{2}\beta) + \sqrt{2}\beta y]$ from the left side. The difference is $\varphi(\beta) - \sqrt{2}\varphi(\sqrt{2}\beta) = [\psi(\beta) - \psi(\sqrt{2}\beta)]/\beta$; the bracketed quantity is at least $\sqrt{2}\, \mathcal{L}(y)$. □

5 Free fields

Proposition 5.1 (Bosonic Fock spaces)**.** *Let $c_n \geq 0$ be nondecreasing nonnegative integers, not all zero, with $\sum_{n\geq 1} n c_n e^{-\beta n} < \infty$ for $\beta > 0$. For $Z(\beta) = \prod_{n\geq 1}(1-e^{-\beta n})^{-c_n}$, the function $\beta \log Z(\beta)$ is strictly decreasing.*

Proof. Writing $G_c(q) = \sum_{n\geq 1} c_n q^n$ for the mode-multiplicity series and expanding the logarithm, we have $\beta \log Z(\beta) = \sum_{k\geq 1} k^{-2}\Phi(k\beta)$ with $\Phi(u) = uG_c(e^{-u})$. If $d_j = c_j - c_{j-1} \geq 0$ ($c_0 = 0$), then

$$\Phi(u) = \sum_{j\geq 1} d_j e^{-(j-1)u}\frac{u}{e^u - 1}.$$

Every summand decreases, and at least one strictly decreases. Hence so do Φ and the original series. □

This includes a single free boson ($c_n = 1$) and plane-partition-type multiplicities ($c_n = n$).

Proposition 5.2 (Neveu–Schwarz fermion)**.** *For $Z_F(\beta) = \prod_{n\geq 1}(1+e^{-\beta(n-1/2)})$, the function $\beta \log Z_F(\beta)$ is strictly decreasing.*

Proof. Expanding $\log(1+x)$ yields

$$\beta \log Z_F(\beta) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^2} g(k\beta), \quad g(u) = \frac{u}{2\sinh(u/2)} = H(u^2/4), \quad H(s) = \prod_{n\geq 1}\left(1 + \frac{s}{n^2\pi^2}\right)^{-1}.$$

The product implies $b(s) := -H'(s) > 0$ is strictly decreasing to 0. Thus differentiating termwise on compact positive β-intervals gives

$$(\beta \log Z_F)'(\beta) = -\frac{\beta}{2}\sum_{k\geq 1}(-1)^{k+1} b(k^2\beta^2/4) < 0$$

by the alternating-series test. □

Corollary 5.3. *Finite tensor products of the Fock spaces in Propositions 5.1–5.2 meet the hypotheses of Theorem 4.6.*

Remark 5.4 (Finite blocks). If $1 < \dim B < \infty$ and $B_0 \neq 0$, then $(\beta \log Z_B)' \to \log \dim B > 0$ as $\beta \downarrow 0$. This violates the criterion but does not create admissible pieces with improved ρ.

6 Kleinian code lattices

6.1 Codes, lattices and characters

Put $\mathbb{K} = \mathbb{F}_2^2$, the Klein four-group, and endow it with the alternating form

$$\omega((a,b),(a',b')) = ab' + a'b.$$

The product symplectic form is $\omega^{\oplus m}$. An isotropic code means $\omega^{\oplus m}|_S = 0$; the 2-rank is $\log_2|S| = \dim_{\mathbb{F}_2} S$. For $S \subset \mathbb{K}^m$ an additive isotropic ($\mathbb{F}_2$-linear and $\omega|_S = 0$) code, define $\mathrm{wt}(s) = |\{i : s_i \neq 0\}|$. We *define* the graded vector space V_S, with degrees in $\frac{1}{2}\mathbb{Z}_{\geq 0}$, by prescribing the Hilbert series of its codeword sectors: each $s \in S$ contributes $A^{m-\mathrm{wt}(s)}B^{\mathrm{wt}(s)}$, with the series A, B specified in (6.1). Their coefficients are nonnegative integers (Lemma 6.1), so this character prescribes a graded vector space without requiring a vertex-algebra construction. For background on GF(4)-based stabilizer codes, see [12]. The discriminant group $D_4^*/D_4 \cong \mathbb{K}$ carries the bilinear form $\omega/2 \pmod{\mathbb{Z}}$; its three nonzero cosets have identical theta series.

Optional lattice interpretation. The preimage Λ_S of S between D_4^m and $(D_4^*)^m$ is an integral (not necessarily even) lattice. Standard lattice-character formulas realize the same Hilbert series through a lattice vertex (super)algebra [18, 14], but no such algebraic structure is needed for the dimension estimates below. In particular, no Appendix-B-compatible operators are claimed on V_S.

Set $q = e^{-\beta}$, $P(q) = \prod_{n\geq 1}(1-q^n)$ and

$$\theta_3 = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \quad \theta_4 = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}, \quad \theta_2 = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}.$$

The classical Jacobi quartic identity $\theta_3^4 = \theta_2^4 + \theta_4^4$ gives the equivalent forms of B below. In our normalization, the D_4 theta series begins

$$\Theta_{D_4}(q) = \frac{\theta_3^4 + \theta_4^4}{2} = 1 + 24q + 24q^2 + 96q^3 + \cdots.$$

The vacuum and the three equivalent nonzero-coset characters are

$$A(\beta) = \frac{\theta_3^4 + \theta_4^4}{2P^4}, \qquad B(\beta) = \frac{\theta_2^4}{2P^4} = \frac{\theta_3^4 - \theta_4^4}{2P^4}, \quad Z_{V_S} = \sum_{s\in S} A^{m-\mathrm{wt}(s)} B^{\mathrm{wt}(s)}. \tag{6.1}$$

Lemma 6.1 (Positive coefficients)**.** *$A = 1 + 28q + 134q^2 + 568q^3 + \cdots$, and $B = q^{1/2}\mathcal{C}(q)$ with $\mathcal{C}(q) = 8 + 64q + 288q^2 + 1024q^3 + \cdots$. Both $A - 1$ and $\mathcal{C}$ have nonnegative integral coefficients. In particular A, $-A'$ and $\mathcal{C}(e^{-\beta})$ are nonincreasing in β.*

Proof. The numerator is the theta series of a D_4 coset and P^{-4} is a product of geometric series. Coefficient positivity and the first coefficients follow by expansion. □

Lemma 6.2 (Decreasing ratio)**.** *$u := B/A \in (0,1)$ is strictly decreasing in β, and $v := (\beta \log u)' < 0$.*

Proof. The Jacobi triple product gives

$$r = \left(\frac{\theta_4}{\theta_3}\right)^4 = \prod_{n\geq 1}\left(\frac{1-q^{n-1/2}}{1+q^{n-1/2}}\right)^8, \qquad u = \frac{1-r}{1+r}.$$

Each factor of r increases with β, whence $u' < 0$ and $v = \log u + \beta u'/u < 0$. □

6.2 Sectors and weight bounds

Write $a_{m,w} = |\{s \in S : \mathrm{wt}(s) = w\}|$. Grouping characters gives

$$V_S = \bigoplus_{w: a_{m,w} > 0} M_w, \qquad Z_{M_w} = a_{m,w} A^{m-w} B^w, \tag{6.2}$$

and the ground energy of the sector is $w/2$. The vacuum sector M_0 has character A^m with multiplicity exactly $a_{m,0} = 1$, because the only codeword of weight zero is 0.

Lemma 6.3 (Weight bounds)**.** *For every isotropic additive $S \subset \mathbb{K}^m$,*

$$a_{m,w} \leq 2^m, \qquad a_{m,w} \leq \binom{m}{w} 2^w; \qquad \frac{1}{m}\log a_{m,w} \leq \min\{\log 2,\ H(\vartheta) + \vartheta \log 2\}, \tag{6.3}$$

where $\vartheta = w/m$ and $H(\vartheta) = -\vartheta\log\vartheta - (1-\vartheta)\log(1-\vartheta)$.

Proof. S is isotropic in a symplectic space of dimension $2m$, so $\dim_{\mathbb{F}_2} S \leq m$. For any fixed w-subset of coordinates T, words supported in T form an isotropic subspace of $\mathbb{K}^T$, of size at most 2^w. Finally $\binom{m}{w} \leq e^{mH(w/m)}$. The additivity of S is essential. □

6.3 Sectorwise criterion

Define $\varphi_{m,w} = m^{-1} \log Z_{M_w}$ and

$$g(\beta) = (\beta \log A)', \quad F_1 = g + \log 2, \quad F_2 = g + \log(1 + 2e^v), \quad F = \min\{F_1, F_2\}. \tag{6.4}$$

Theorem 6.4 (Sectorwise criterion)**.** *For every $m \geq 1$, additive isotropic code $S \subset \mathbb{K}^m$, and present sector w, $\beta \mapsto \beta\varphi_{m,w}(\beta)$ is strictly decreasing on $(0, \infty)$.*

Proof. By (6.2) and $\log B = \log A + \log u$,

$$(\beta\varphi_{m,w})' = \frac{1}{m} \log a_{m,w} + g + \vartheta v.$$

The first inequality in (6.3), together with $v < 0$, bounds this by F_1; the second gives

$$(\beta\varphi_{m,w})' \leq g + \sup_{0 \leq \vartheta \leq 1} \{H(\vartheta) + \vartheta(\log 2 + v)\} = g + \log(1 + 2e^v) = F_2.$$

Thus it is at most $F(\beta) < 0$ by Proposition A.1. No weight-spectrum limit or concavity is used. □

6.4 Main theorem and transfer

Here an *admissible level* for V_S is a positive degree $h \in \frac{1}{2}\mathbb{Z}_{>0}$ of its grading with $\dim(V_S)_h > 0$; for the source's integer-degree comparison we additionally require $h \in \mathbb{Z}_{>0}$. The constants in the next theorem depend on the density endpoints x_1, x_2 and the fixed alphabet (A, B), not on m, h, λ or the code S.

Theorem 6.5 (Strict fixed-density window barrier)**.** *Fix $\mathcal{J} = [x_1, x_2] \Subset (0, \infty)$. There are constants $\eta_{\mathcal{J}} > 0$ and $C_{\mathcal{J}} > 0$, independent of m, the additive isotropic code $S \subset \mathbb{K}^m$, the admissible level h, and $\lambda \geq 1$, such that, whenever $h/m \in \mathcal{J}$ and $2^\lambda \leq \dim(V_S)_h$,*

$$\rho_{\mathrm{win}}(V_S; h, \lambda) \geq \frac{1}{\sqrt{2}} + \eta_{\mathcal{J}} - \frac{C_{\mathcal{J}}}{\sqrt{m}}. \tag{6.5}$$

In particular, for each fixed $\mathcal{J}$, the prescribed window cost is strictly greater than $1/\sqrt{2}$ for all sufficiently large m. The bound does not claim uniformity as $h/m \to \infty$.

Proof. Write $x = h/m$ and let M_w denote the sectors in (6.2). Set $b = \max_{w : a_{m,w} > 0} \mathcal{L}_{M_w}(x)$ and choose a maximizing sector w^*. By the encoding assumption and the sector Chernoff bound,

$$\lambda \log 2 \leq \log \dim(V_S)_h \leq mb + \log(m + 1). \tag{6.6}$$

For both single-letter characters A, B, choose positive $\beta_{\mathrm{lo}}(\mathcal{J}) < \beta_{\mathrm{hi}}(\mathcal{J})$ such that

$$\mu_j(\beta_{\mathrm{lo}}) > 2x_2, \qquad \mu_j(\beta_{\mathrm{hi}}) < g_j + x_1/4, \quad (j = A, B), \qquad g_A = 0,\ g_B = 1/2.$$

Existence follows from their positive excitation coefficients and the limits of their mean energies. These choices work uniformly for the maximizing sector, since its ground energy is at most x whenever its Legendre envelope at x is finite. Let $V_*(\mathcal{J})$ be a uniform bound on the relevant single-letter variances in this compact inverse-temperature interval. For all sufficiently large m, the constructive Chebyshev argument in Lemma 4.4 (with target $y = 2x$ and gap $2x - w^*/(2m) \geq x_1$) selects an inverse temperature $\beta_1 \in [\beta_{\mathrm{lo}}, \beta_{\mathrm{hi}}]$ for which

$$\log \dim(V_S)_{\leq 2h} \geq m[\varphi_{m,w^*}(\beta_1) + 2x\beta_1] - C'_{\mathcal{J}}\sqrt{m}. \tag{6.7}$$

Here one uses the *explicit* bound (4.6), not merely the lower bound involving the infimum $\mathcal{L}_{M_{w^*}}(2x)$: this distinction is needed in the next step. Indeed, applying Lemma 4.7 at that same β_1 (and recalling $\mathcal{L}_{M_{w^*}}(x) = b$) gives

$$\varphi_{m,w^*}(\beta_1) + 2x\beta_1 \geq \sqrt{2}\, b + \frac{\psi_{m,w^*}(\beta_1) - \psi_{m,w^*}(\sqrt{2}\beta_1)}{\beta_1}.$$

The sectorwise sign criterion (Theorem 6.4) supplies a continuous function $F(\beta) < 0$, independent of m, w, S, such that $\psi'_{m,w}(\beta) \le F(\beta)$. Hence the last summand is at least

$$\delta_{\mathcal{J}} := (\sqrt{2}-1) \min_{\beta \in [\beta_{\mathrm{lo}}(\mathcal{J}), \sqrt{2}\beta_{\mathrm{hi}}(\mathcal{J})]} (-F(\beta)) > 0.$$

For all sectors, the Chernoff envelope also has the uniform upper bound

$$b \le b^{\max}_{\mathcal{J}} := \log 2 + \max\{\log A(1), \log B(1)\} + x_2.$$

Combining the central-window inequality $\rho_{\mathrm{win}} \ge D(2h)/(2\lambda)$ with (6.6) and (6.7) gives

$$\rho_{\mathrm{win}} \ge \frac{m(\sqrt{2}\, b + \delta_{\mathcal{J}}) - C'_{\mathcal{J}}\sqrt{m}}{2mb + 2\log(m+1)} \ge \frac{1}{\sqrt{2}} + \frac{\delta_{\mathcal{J}}}{2b^{\max}_{\mathcal{J}}} - \frac{C_{\mathcal{J}}}{\sqrt{m}}.$$

In the last step, the positive vacuum-sector lower bound on b for $x \in \mathcal{J}$ controls the denominators, and $\log(m+1)/m = O(m^{-1/2})$; these bounds are independent of S. Set $\eta_{\mathcal{J}} = \delta_{\mathcal{J}}/(2b^{\max}_{\mathcal{J}}) > 0$. Increasing $C_{\mathcal{J}}$ absorbs the finitely many shorter lengths. This establishes the claimed fixed-compact estimate without an assumption on concavity or convergence of code weight spectra. □

Underlying cumulative-dimension growth. For every $m \ge 1$, every additive isotropic code $S \subset \mathbb{K}^m$, and every admissible h with $h/m \in \mathcal{J}$, the same proof establishes the following inequality in natural logarithms, with $C''_{\mathcal{J}}$ depending only on $\mathcal{J}$ and the fixed character alphabet:

$$\log \dim(V_S)_{\le 2h} \ge \sqrt{2}\, \log \dim(V_S)_h + \delta_{\mathcal{J}} m - C''_{\mathcal{J}}\sqrt{m}.$$

Indeed, the local scaling lemma (Lemma 4.7) and the strict sign margin $\delta_{\mathcal{J}}$ give the $m(\sqrt{2}\, b + \delta_{\mathcal{J}})$ term in (6.7). Subtract $\sqrt{2}\log \dim(V_S)_h$ from that estimate and use (6.6); absorb $\sqrt{2}\log(m+1)$ into $C''_{\mathcal{J}}\sqrt{m}$. Finitely many small m are absorbed by enlarging $C''_{\mathcal{J}}$. The window theorem follows from the central-window inequality. This formulation highlights the actual entropy-growth statement, rather than attributing an operator-rank consequence to the result.

Remark 6.6 (Scope of isotropy). Isotropy is used through the local weight bound $a_{m,w} \le \binom{m}{w} 2^w$ in Lemma 6.3. It is sufficient for the present proof; this does not show that isotropy is necessary for every individual additive code. Unrestricted codes are not covered: for $S = \mathbb{K}^m$ the character is Ψ^m with $\Psi(q) = A(q) + 3B(q)$, and the local bound fails. In fact, the thermodynamic monotonicity condition fails for this unrestricted family. Put $\widehat{A}(\gamma) = A(e^{-\gamma})$, $\Delta_A(\gamma) = -\widehat{A}'(\gamma) \ge 0$, $a = 2\pi^2/3$ and $\beta^* = 4\pi^2/\beta$. The modular transformation (C.1) at β^* gives the exact identity

$$\Psi(e^{-\beta}) = 2e^{a/\beta - \beta/6}\widehat{A}(\beta^*).$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\big[\beta \log \Psi(e^{-\beta})\big] = \log 2 - \frac{\beta}{3} + \log \widehat{A}(\beta^*) + \frac{\beta^* \Delta_A(\beta^*)}{\widehat{A}(\beta^*)} > 0 \qquad (0 < \beta < 3\log 2).$$

Thus the sign criterion used in Theorem 6.4 does not extend indiscriminately to $S = \mathbb{K}^m$ (a *proved* fact). Numerical window costs for that code at some fixed densities fall below $1/\sqrt{2}$; that separate observation is *not* promoted to a rigorous asymptotic countertheorem.

Corollary 6.7 (Uniform fixed-density window barrier)**.** *Under the hypotheses of Theorem 6.5,*

$$\rho_{\mathrm{win}}(V_S; h, \lambda) \ge \frac{1}{\sqrt{2}} - \frac{C_{\mathcal{J}}}{\sqrt{m}}$$

for a constant independent of the code S and m.

Proof. Immediate from Theorem 6.5, since $\eta_{\mathcal{J}} > 0$. □

Corollary 6.8 (Uniform strict separation)**.** *Fix $\mathcal{J} \Subset (0,\infty)$. For all sufficiently large m (depending only on $\mathcal{J}$), every family as in Theorem 6.5 satisfies $\rho_{\mathrm{win}} \geq 1/\sqrt{2} + \eta_{\mathcal{J}}/2$, regardless of whether its code-weight spectra are concave or converge.*

Proof. Take $m \geq (2C_{\mathcal{J}}/\eta_{\mathcal{J}})^2$ in Theorem 6.5. □

Remark 6.9 (Whole-character estimate). For $0 \leq z \leq 1$, the support estimate gives $W_S(1,z) := \sum_{s\in S} z^{\mathrm{wt}(s)} \leq (1+z)^m$. Set $\ell = \log(B/A)$ and $L(u) = \log W_S(1,e^u)$. The function L is convex as a log-sum-exponential function. Because $v = (\beta\ell)' = \ell + \beta\ell'$, its tangent inequality gives $L(\ell) + (v-\ell)L'(\ell) \leq L(v)$. Thus

$$\frac{1}{m}(\beta \log Z_{V_S})' = g + \frac{L(\ell) + (v-\ell)L'(\ell)}{m} \leq g + \frac{L(v)}{m} \leq g + \log(1+e^v),$$

using $v < 0$ and the support estimate at $e^v \leq 1$. This does not alone control microcanonical windows without extra hypotheses; the proof above proceeds sector by sector.

Lemma 6.10 (Even subcode and integer-degree part)**.** *For isotropic S, the map $s \mapsto \mathrm{wt}(s) \bmod 2$ is $\mathbb{F}_2$-linear. Its kernel S_0 is isotropic, all its weights are even, $[S : S_0] \leq 2$, and*

$$\bigoplus_{E\in\mathbb{Z}} (V_S)_E \cong V_{S_0}$$

as graded vector spaces.

Proof. For $Q : \mathbb{K} \to \mathbb{F}_2$, $Q(0) = 0$ and $Q(x) = 1$ for $x \neq 0$, direct inspection gives $Q(x+y) + Q(x) + Q(y) = \omega(x,y)$. Thus $\mathrm{wt}(s) \bmod 2 = Q^{\oplus m}(s)$ is linear on isotropic S. A sector of weight w has character $a_{m,w} q^{w/2} A^{m-w} \mathcal{C}^w$ with $A, \mathcal{C} \in \mathbb{Z}[[q]]$, so it has integer degrees precisely for even w. Their sum is the character and the sector sum of V_{S_0}. □

Remark 6.11 (Integer degrees and admissibility). For integer-energy windows as in the source Appendix B, apply Corollary 6.7 to S_0; it remains a code in the same class, with the same universal constants. This does not by itself provide the requisite multiplication operators or factorization identities.

Lemma 6.12 (Growth at the exact integer level)**.** *Let $\mathcal{J} = [x_1, x_2] \Subset (0,\infty)$. There exist $c_{\mathcal{J}}, C_{\mathcal{J}} > 0$ and an integer $m_{\mathcal{J}}$ such that, for every $m \geq m_{\mathcal{J}}$ and every* integer *h with $h/m \in \mathcal{J}$, every isotropic S satisfies*

$$c_{\mathcal{J}} m \leq \log_2 \dim(V_S)_h \leq C_{\mathcal{J}} m. \tag{6.8}$$

Proof. The vacuum sector gives $\dim(V_S)_h \geq [q^h]A^m$. Write $A = N(q)P^{-4}$ with $N = (\theta_3^4 + \theta_4^4)/2 \in \mathbb{Z}_{\geq 0}[[q]]$ and constant term 1. Because $P^{-4m} = (1-q)^{-1}G(q)$ for some positive-coefficient G, its coefficients are nondecreasing; multiplication by N^m preserves this property. Therefore

$$[q^h]A^m \geq \frac{1}{h+1} \dim(A^m)_{\leq h}.$$

Apply Lemma 4.4 to the single-letter alphabet (A) with $y = h/m \in \mathcal{J}$ and gap $\delta = x_1$. It yields $\log \dim(A^m)_{\leq h} \geq m b_A(x_1) - C\sqrt{m}$ with $b_A(x_1) > 0$, hence $\log \dim(V_S)_h \geq m b_A(x_1) - C\sqrt{m} - \log(h+1)$. For example, set $b_* = b_A(x_1) > 0$ and choose $m_{\mathcal{J}}$ sufficiently large that the finite-length lemma applies and, for every $m \geq m_{\mathcal{J}}$,

$$C\sqrt{m} \leq \tfrac{1}{4} m b_*, \qquad \log(x_2 m + 1) \leq \tfrac{1}{4} m b_*.$$

Then one may take $c_{\mathcal{J}} = b_*/(2\log 2)$; this specifies a finite threshold $m_{\mathcal{J}}$ without claiming its numerical optimality. For the other direction, use (4.2) for each of the at most $m+1$ sectors at $\beta = 1$: $\log \dim(V_S)_h \leq \log(m+1) + m[\log 2 + \max\{\log A(1), \log B(1)\} + x_2]$. □

Corollary 6.13 (Transfer to Definition 7(b))**.** *Fix a compact density interval $\mathcal{J} \Subset (0,\infty)$, $\theta > 0$ and an integer $c \geq 1$. Assume:*

(i) $S \subset \mathbb{K}^m$ *is additive isotropic and* $h/m \in \mathcal{J}$*;*

(ii) $\lambda \geq \theta m$ *and* $2^\lambda \leq \dim(V_S)_h$*;*

(iii) *the piece index set* I *contains* 0*;*

(iv) *the two bit costs at each* s *are the* ceilings *of the binary logarithms of the prescribed window dimensions and obey* (2.1)*.*

Then

$$\rho \geq \frac{1}{\sqrt{2}} + \eta_{\mathcal{J}} - C(\mathcal{J}, \theta, c) m^{-1/2}. \tag{6.9}$$

For integer h *one may choose* $\lambda = \lfloor \log_2 \dim(V_S)_h \rfloor$*, which obeys* $\lambda = \Theta_{\mathcal{J}}(m)$ *by Lemma 6.12. The conclusion applies equally to the integer-degree even subcode* S_0*.*

Proof. At $s = 0$ the two costs are each at least $D(2h)$, and (2.1) gives

$$\rho \geq \frac{D(2h)}{2\lambda} - \frac{c \log(2\lambda)}{4\lambda}.$$

Theorem 6.5 gives the first term at least $1/\sqrt{2} + \eta_{\mathcal{J}} - O_{\mathcal{J}}(m^{-1/2})$. For all sufficiently large m, the lower bound $\lambda \geq \theta m \geq 2$ and monotonic decrease of $u \mapsto \log(2u)/u$ for $u \geq 2$ give $\log(2\lambda)/\lambda \leq \log(2\theta m)/(\theta m) = O_\theta(\log m/m)$. Any remaining small m are absorbed into the theorem's constant, without requiring an upper bound on λ. This is a statement only for the specified cost model and does not rule out cheaper intrinsic factorizations or assert the existence of the required operators. □

7 Uniform high-density saturation

7.1 The high-density problem and admissible levels

The fixed-density estimate of Theorem 6.5 applies when h/m remains in a compact subset of $(0, \infty)$; it does not by itself supply an error bound that is uniform as $h/m \to \infty$. We now address that regime for the *prescribed graded-window cost*. The conclusions below rest on the analytic input proved in Appendix C.

Throughout this section, $S \subset \mathbb{K}^m$ is an additive subgroup of 2-rank $r_S = \log_2 |S| \leq m$; this includes every isotropic code of Section 6, but isotropy is used here only through $r_S \leq m$. For non-isotropic S, we extend the notation V_S by *defining* the formal graded space with character $\sum_{s \in S} A^{m - \mathrm{wt}(s)} B^{\mathrm{wt}(s)}$ as the direct sum of its codeword sectors. Coefficient positivity makes this definition valid without assuming a lattice or vertex-algebra realization. For this V_S, put

$$N_s(E) = \dim(V_s)_{\leq E}, \quad N_S(E) = \dim(V_S)_{\leq E}, \quad D_S(E) = \log_2 N_S(E), \quad \kappa = \frac{\pi}{\ln 2}\sqrt{\frac{8}{3}}.$$

A positive level $h \in \frac{1}{2}\mathbb{Z}$ is *admissible* when $\dim(V_S)_h > 0$. For $h \geq m/2$, admissibility holds if and only if h is an integer or h is a non-integral half-integer and S contains an odd-weight word: each sector of weight w has support in $w/2 + \mathbb{Z}_{\geq 0}$, with strictly positive coefficients there (Lemma C.6). Indeed, every positive admissible level has exact-level dimension at least 2: an integer $h \geq 1$ contains the vacuum-sector coefficient $[q^h]A^m \geq 28m$; a non-integral half-integer admissible level is supported by an odd-weight sector with coefficient at least 8 (Lemma C.6). Thus, for every admissible h, the *maximal encoding*

$$\lambda_h := \lfloor \log_2 \dim(V_S)_h \rfloor \geq 1$$

is defined. An *admissible encoding* is an integer $1 \leq \lambda \leq \lambda_h$, equivalently $2^\lambda \leq \dim(V_S)_h$. The full supremum in Definition 2.1 runs over real shifts; the reduction to the half-lattice by Lemma C.8 is needed only to enumerate numerical checks; the analytic upper bound holds for all real shifts.

7.2 Main theorem

Theorem 7.1 (Code-uniform high-density saturation)**.** *Let $m \geq 1$, $S \subset \mathbb{K}^m$ be additive of 2-rank $r_S \leq m$, and $h \in \frac{1}{2}\mathbb{Z}_{>0}$ be admissible with $h \geq 15m$. Set*

$$X = \kappa\sqrt{mh}, \qquad e_1 = \frac{4}{3}m + 24(mh)^{1/4} + 1, \qquad e_2 = e_1 + 1.$$

Then the following statements hold.

(i) Universal lower bound. *For every admissible integer encoding $1 \leq \lambda \leq \lambda_h$,*

$$\begin{aligned}\rho_{\mathrm{win}}(V_S; h, \lambda) &\geq \frac{D_S(2h)}{2D_S(h)} \geq \frac{1}{\sqrt{2}}\frac{X - e_1/\sqrt{2}}{X + r_S} \\ &\geq \frac{1}{\sqrt{2}} - 0.19\sqrt{\frac{m}{h}} - 1.7(mh)^{-1/4}.\end{aligned} \tag{7.1}$$

(ii) Maximal-encoding upper bound. *If $X > e_2$, then*

$$\rho_{\mathrm{win}}(V_S; h, \lambda_h) \leq \frac{1}{\sqrt{2}}\frac{X + r_S/\sqrt{2}}{X - e_2}. \tag{7.2}$$

(iii) Uniform quantitative saturation. *If additionally $mh \geq 4000$,*

$$\begin{aligned}\frac{1}{\sqrt{2}} - 0.19\sqrt{\frac{m}{h}} - 1.7(mh)^{-1/4} &\leq \rho_{\mathrm{win}}(V_S; h, \lambda_h) \\ &\leq \frac{1}{\sqrt{2}} + 0.4\sqrt{\frac{m}{h}} + 5(mh)^{-1/4}.\end{aligned} \tag{7.3}$$

For every real $x \geq 4000$, let $\mathcal{H}_x$ denote the set of triples (m, S, h) with $m \geq 1$, $S \subset \mathbb{K}^m$ additive of 2-rank at most m, and $h \in \frac{1}{2}\mathbb{Z}_{>0}$ admissible with $h \geq xm$. Then the uniform modulus is

$$\sup_{(m,S,h)\in\mathcal{H}_x} \left|\rho_{\mathrm{win}}(V_S; h, \lambda_h) - \frac{1}{\sqrt{2}}\right| \leq 0.4x^{-1/2} + 5x^{-1/4}. \tag{7.4}$$

Thus for every sequence with $m_n \geq 1$, $r_{S_n} \leq m_n$, admissible h_n and $h_n/m_n \to \infty$,

$$\rho_{\mathrm{win}}(V_{S_n}; h_n, \lambda_{h_n}) \longrightarrow \frac{1}{\sqrt{2}}$$

uniformly over the codes with the stated rank restriction.

(iv) Other encodings. *For every admissible λ,*

$$\rho_{\mathrm{win}}(V_S; h, \lambda) = \frac{\lambda_h}{\lambda}\rho_{\mathrm{win}}(V_S; h, \lambda_h). \tag{7.5}$$

Along sequences in the above high-density regime, $\rho_{\mathrm{win}}(V_{S_n}; h_n, \lambda_n) \to 1/\sqrt{2}$ if and only if $\lambda_n/\lambda_{h_n} \to 1$.

Proof. Appendix C proves the all-energy bound

$$D_S(E) \leq \kappa\sqrt{mE} + r_S \qquad (E \geq 0) \tag{7.6}$$

(Proposition C.2), the sectorwise lower bound

$$\log_2 N_s(E) \geq \kappa\sqrt{mE} - \frac{4}{3}m - 20(mE)^{1/4} - 1 \quad (E \geq 15m,\ s \in S) \tag{7.7}$$

(Proposition C.3), and hence the same lower bound for $D_S(E)$ (Proposition C.4). For exact levels, choose the vacuum sector when h is integral and an odd-weight sector when h is a non-integral

half-integer. Lemma C.6 supplies monotonicity *inside the chosen sector*, not across different degree parities. Thus $\dim(V_S)_h \geq N_s(h)/(h+1)$. Applying (7.7) and $\log_2(h+1) \leq 4(mh)^{1/4}$ gives

$$X - e_2 \leq \lambda_h \leq D_S(h) \leq X + r_S. \tag{7.8}$$

This is precisely Corollary C.7 of the appendix and is valid for both integer and admissible half-integer levels.

For (i), the choice of shift $t = 0$ in the central window gives

$$\rho_{\mathrm{win}}(V_S; h, \lambda) \geq \frac{D_S(2h)}{2\lambda} \geq \frac{D_S(2h)}{2D_S(h)}.$$

At $E = 2h$, the cumulative lower bound and $20 \cdot 2^{1/4} < 24$ give $D_S(2h) \geq \sqrt{2}X - e_1$; at $E = h$, the upper bound gives $D_S(h) \leq X + r_S$. If $\sqrt{2}X - e_1 < 0$, the claimed middle inequality is trivial, and otherwise division gives it. The last inequality follows using $r_S \leq m$, $(1-u)/(1+v) \geq 1-u-v$ for $u, v \geq 0$, and $(2/3 + 1/\sqrt{2})/\kappa < 0.19$ and $12.5/\kappa < 1.7$.

For (ii), with $u = |t| \leq 2h$, the two windows are contained in cumulative spaces of energies $2h + u$ and $2h - u$. By (7.6) and concavity of the square root,

$$C_h(t) \leq \kappa\sqrt{m}(\sqrt{2h+u} + \sqrt{2h-u}) + 2r_S \leq 2\kappa\sqrt{2mh} + 2r_S.$$

Divide by $4\lambda_h$ and apply the lower bound in (7.8), for $X > e_2$.

For (iii), put $b_{\mathrm{hd}} = e_2/X$ and $y = r_S/(\sqrt{2}X)$. If $h \geq 15m$ and $mh \geq 4000$, the certified bounds give $b_{\mathrm{hd}} < 1/2$, so

$$\frac{1+y}{1-b_{\mathrm{hd}}} \leq (1+y)(1+2b_{\mathrm{hd}}) \leq 1 + 2y + 2b_{\mathrm{hd}}.$$

Use $y \leq \sqrt{m/h}/(\sqrt{2}\kappa)$ and $b_{\mathrm{hd}} \leq [(4/3)\sqrt{m/h} + 26(mh)^{-1/4}]/\kappa$. The constants 0.4 and 5 follow from $(1 + 4\sqrt{2}/3)/\kappa < 0.4$ and $26\sqrt{2}/\kappa < 5$. Together with (i) this proves (7.3). For $h \geq xm$ and $m \geq 1$, one has $mh \geq h/m \geq x$; therefore both error terms are at most $0.4x^{-1/2} + 5x^{-1/4}$, establishing (7.4).

Finally, (iv) holds because the numerator defining ρ_{win} is independent of λ. Since the maximal-encoding cost tends to the nonzero number $1/\sqrt{2}$, the exact scaling gives the stated necessary and sufficient condition on λ/λ_h. □

Corollary 7.2 (No fixed positive high-density gap)**.** *There is no absolute $\eta > 0$ for which every high-density maximal encoding has $\rho_{\mathrm{win}}(V_S; h, \lambda_h) \geq 1/\sqrt{2} + \eta$. Indeed, even the vacuum code $S = \{0\}$ has maximal-encoding cost converging to $1/\sqrt{2}$ along every admissible sequence with $h/m \to \infty$.*

Remark 7.3 (Compatibility of the two regimes). Fix $0 < x_1 < x_2$ with $x_2 \geq 15$, and let $\eta_{[x_1,x_2]} > 0$ be a valid margin in Theorem 6.5. For the vacuum code $S = \{0\}$, take $h_m = \lfloor x_2 m \rfloor$, so $h_m/m \to x_2$ and the levels lie in $[x_1, x_2]$ for all sufficiently large m. Comparing the fixed-density lower bound with Theorem 7.1(iii) and letting $m \to \infty$ gives

$$\eta_{[x_1,x_2]} \leq 0.4\, x_2^{-1/2}.$$

Thus the strict-gap constants cannot remain uniformly positive as the upper density endpoint diverges.

Corollary 7.4 (Combining fixed and high density for isotropic codes)**.** *For each $x_0 > 0$ and $\varepsilon > 0$ there exists m_0 such that for all $m \geq m_0$, all* isotropic *codes $S \subset \mathbb{K}^m$, all admissible levels h with $h/m \geq x_0$, and every $1 \leq \lambda \leq \lambda_h$,*

$$\rho_{\mathrm{win}}(V_S; h, \lambda) \geq \frac{1}{\sqrt{2}} - \varepsilon.$$

Proof. Choose $R > \max\{15, x_0\}$ large enough that the error in Theorem 7.1(i) is at most $\varepsilon/2$ for $h/m \geq R$ and all sufficiently large m. For $h/m \in [x_0, R]$ apply the fixed-compact result Theorem 6.5, whose hypotheses include isotropy; the finite-size error is eventually at most ε. Combine the two regions. The restriction to isotropic codes is necessary here because the fixed-density theorem is only stated for that class, whereas the high-density theorem uses merely $r_S \leq m$. □

Remark 7.5 (Scope and quantitative limitations). The numerical constants in (7.3) are conservative: they do not imply that each finite cost exceeds $1/\sqrt{2}$, and they become informative only at large parameter values. For example, at $m = 10$, $h = 400$, the threshold $mh = 4000$ is met, but the resulting two-sided interval is much wider than the limiting value. The $O((mh)^{-1/4})$ term comes from a Chebyshev estimate; a finer local-limit theorem would be needed to replace it by a smaller error. The square-root leading term is consistent with classical partition/Cardy asymptotics [8, 13, 9], and the novelty claim here is limited to the explicitly quantified code-uniform formulation. A finer fixed-density formula depending on r_S/m has been suggested numerically; it remains unproved and is not used here. All results concern prescribed graded-window dimensions, not intrinsic matrix ranks or a new operator identity.

8 Limitations and open problems

Scope of the partial answer. The bound of Theorem 6.5 excludes a limiting fixed-density improvement below $1/\sqrt{2}$ *within the prescribed window-cost model* for the indicated Kleinian families. Since $\rho_{\text{rank}} \leq \rho_{\text{win}}$ and no Appendix-B-compatible operators have been constructed for them, it does not exclude a lower intrinsic cost, a different window geometry, or a distinct four-leg identity that could address [6, Section 5, Problem 1].

Question 8.1 (The remaining low-density regime). The high-density part of the variable-density question is resolved by Theorem 7.1: the lower estimate $1/\sqrt{2} - o(1)$ persists as $h/m \to \infty$, and maximal encodings saturate $1/\sqrt{2}$. What are the sharp uniform estimates when $h/m \to 0$? The compact inverse-temperature bounds used in the fixed-density proof degenerate in this regime.

Remark 8.2 (Finer corrections remain open). At high density, the explicit $O((mh)^{-1/4})$ finite-size error may be nonoptimal. An expression depending on the ratio r_S/m was suggested at fixed positive density, but remains heuristic and is *not* asserted or used as a theorem here.

Question 8.3 (Other glue groups). Can the sector criterion be established for other lattices and discriminant groups, with complete weight enumerators when the nonzero cosets are inequivalent? Proposition 4.5 would then apply without change.

Question 8.4 (Admissible graded operators). Do the operator relations required by [4, Appendix B] force a corresponding thermodynamic saturation condition, or can exact efficiently computable pieces be built on a family with a genuinely smaller window cost? Such operators are not constructed in this paper.

Remark 8.5 (Two intrinsic-rank questions). Two logically different questions arise for the central piece A_0 of the Anthropic construction: *(1) fixed encoding*, determine the intrinsic row and column flattening ranks for the particular partition encoding prescribed in [4, Appendix B], including whether they are bounded below by $\dim V_{\leq 2h}/\operatorname{poly}(h)$; and *(2) generalized admissible encodings*, ask for an analogous lower bound *uniformly across all admissible encodings*. The quantifiers differ. Neither question follows from the prescribed window-dimension theorems of this paper.

Question 8.6 (Universal floor). For a genuinely broader class of exact algebraic pieces, is there any $\varepsilon > 0$ that forces $\rho \geq 1/2 + \varepsilon$? This is well beyond the results of this note.

Question 8.7 (Relaxing isotropy). Is there a useful larger class of additive codes for which the local weight-enumerator bound of Lemma 6.3 or a suitable replacement suffices for the fixed-density criterion? Isotropy is sufficient in our theorem, but is not proved necessary for each code.

Question 8.8 (Unrestricted formal characters). For $S = \mathbb{K}^m$, the character is $(A + 3B)^m$ and numerical fixed-density window costs may drop below $1/\sqrt{2}$. Can these formal dimensions be analyzed asymptotically with rigorous error bounds? No admissible 3-SAT factorization operators are claimed.

Computational verification and AI use

The analytical proofs are accompanied by reproducible computational checks available at `https://github.com/MFRADJ/graded-window-barrier-certificates/releases/tag/v1.1.0`. The recorded reference run uses Windows, CPython 3.14.3, `python-flint==0.9.0` (FLINT 3.6.0) and `mpmath==1.3.0`. It reports 14/14 successful suites, zero skipped or failed, and 37/37 additional consistency checks. The archive includes execution logs, SHA-256 source digests, and a runner using isolated workspaces.

The Arb interval-arithmetic checks [19] address the sector-sign inequalities, high-density constants (31 Arb checks and three separate non-certifying numerical tests), further constant bounds, and the modular chain-rule estimates. Other checks use `mpmath.iv`, exact integer coefficients, or floating-point comparisons after exact coefficient computation. In particular, the 6,118-case finite audit and 26 near-threshold tests are not universal interval certificates. These computations support specified numerical premises subject to the analytical reductions in the text; they do not certify the full proof or constitute formal proof-assistant verification. They do not establish an intrinsic matrix-rank bound or a proof or refutation of the Exponential Time Hypothesis.

This research was conducted in an author-directed process with substantial assistance from ChatGPT (OpenAI) and Claude (Anthropic). The text of the manuscript is primarily AI-generated. AI systems contributed extensively to mathematical exploration, proposed arguments and proofs, drafting and revision, software development, numerical checks, and critical examination of the reasoning. The human author initiated and directed the project, selected the research questions and scope, organized the results and computational evidence, and made the publication decisions. No exclusive individual discovery attribution to either AI system is asserted, and no AI system is listed as an author.

Human understanding and verification of the mathematical contribution are partial. The author is studying the underlying mathematics and has not yet personally mastered or independently verified all of the technical arguments, including the complete proofs of Theorems 6.5 and 7.1. The computational checks support specified numerical premises but do not establish the correctness of the complete mathematical proofs. Independent qualified human mathematical peer review and formal proof checking of the complete argument remain outstanding. The author takes responsibility for the submission, its mathematical claims, software, citations, provenance disclosures, and any necessary corrections, without claiming independent human verification of every proof.

A Proof of the sector sign

Proposition A.1 (Global sign). *For F in* (6.4), $F(\beta) < 0$ *for every* $\beta > 0$. *Specifically* $F_1 < 0$ *on* $(0, 4]$ *and* $F_2 < 0$ *on* $[4, \infty)$.

The sign proof uses the three regimes $(0, 4]$, $[4, 5]$, and $[5, \infty)$. The lemmas below establish the required inequalities on these intervals; their numerical enclosures can be checked with the companion Arb script.

Lemma A.2 (Modular transformation). *Put* $\beta^* = 4\pi^2/\beta$, $\Psi(q) = (\theta_3(q)^4 + \theta_2(q)^4)/P(q)^4$, *and* $\psi_\Psi(y) = \log \Psi(e^{-y})$. *Then*

$$\log A(\beta) = \frac{\beta^* - \beta}{6} - \log 2 + \psi_\Psi(\beta^*), \quad F_1(\beta) = -\frac{\beta}{3} + \psi_\Psi(\beta^*) - \beta^* \psi'_\Psi(\beta^*). \tag{A.1}$$

Moreover, by the Jacobi identity, $\Psi = A + 3B$, with the explicit initial expansion

$$\Psi(q) = 1 + 24q^{1/2} + 28q + 192q^{3/2} + 134q^2 + O(q^{5/2}).$$

Thus $\Psi(e^{-y}) = 1 + \sum_{\nu \in \frac{1}{2}\mathbb{Z}, \nu \geq 1/2} c_\nu e^{-\nu y}$ with $c_\nu \geq 0$. These coefficients permit an independent low-order check of the point bound $R(\pi^2) \approx 10.2925020821$, while its rigorous enclosure is supplied by the certificate.

Proof. The standard theta and eta transformation formulas (see, e.g., [11]) and Poisson summation supply $\theta_3(e^{-\beta}) = (2\pi/\beta)^{1/2}\theta_3(e^{-\beta^*})$ and $\theta_4(e^{-\beta}) = (2\pi/\beta)^{1/2}\theta_2(e^{-\beta^*})$. The eta transformation yields $P(e^{-\beta}) = (2\pi/\beta)^{1/2} e^{(\beta-\beta^*)/24} P(e^{-\beta^*})$. Substitution and differentiation using $\beta\beta^* = 4\pi^2$ give (A.1). Positivity follows from theta and P^{-4} series. □

Lemma A.3 (Small inverse temperature)**.** *$F_1(\beta) < 0$ for $0 < \beta \leq 4$.*

Proof. Let $T_\Psi(y) := -\mathrm{d}\Psi(e^{-y})/\mathrm{d}y = \sum_\nu \nu c_\nu e^{-\nu y} > 0$. Since all $\nu \geq 1/2$ and $\Psi \geq 1$, $\psi_\Psi \leq \Psi - 1 \leq 2T_\Psi$ and $-\psi'_\Psi \leq T_\Psi$. Thus

$$F_1(\beta) \leq -\frac{\beta}{3} + (2 + \beta^*)T_\Psi(\beta^*) = \frac{R(\beta^*) - 4\pi^2/3}{\beta^*}, \quad R(y) = y(2+y)T_\Psi(y).$$

Each term of R decreases for $y > 1 + \sqrt{5}$; for $\beta \leq 4$ we have $\beta^* \geq \pi^2$. The certified point bound is $R(\pi^2) < 10.29251 < 4\pi^2/3 \approx 13.15947$. Hence the right-hand side is negative. □

Lemma A.4 (Intermediate boxes)**.** *$F_2(\beta) < 0$ for $4 \leq \beta \leq 5$.*

Proof. Let $\Delta_A(\beta) = -A'(\beta) \geq 0$ and write $B = q^{1/2}\mathcal{C}$. Then

$$g = \log A - \frac{\beta\Delta_A}{A}, \quad v \leq -\beta + \log \mathcal{C} - \log A + \frac{\beta\Delta_A}{A}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{C}(e^{-\beta}) \leq 0$ and $A \geq 1$, the preceding estimate also gives the bound required in the final regime:

$$e^{v(\beta)} \leq e^{-\beta}\mathcal{C}(e^{-\beta}) \exp\bigl(\beta\Delta_A(\beta)\bigr). \tag{A.2}$$

For $4 \leq b' \leq \beta \leq b'' \leq 5$, coefficient positivity and the decrease of $A, \Delta_A, \mathcal{C}$ imply

$$F_2(\beta) \leq U(b', b'') := \log A(b') - \frac{b'\Delta_A(b'')}{A(b')} + \log\left(1 + \frac{2e^{-b'}\mathcal{C}(e^{-b'})}{A(b'')} e^{b''\Delta_A(b')/A(b'')}\right). \tag{A.3}$$

Ten intervals $[4 + i/10, 4 + (i+1)/10]$ give strictly negative upper bounds U, with the weakest no larger than -0.1618. The interval table was regenerated under `mpmath.iv`; the local Arb verification independently confirmed the aggregate statement that all ten box upper bounds are negative, with the largest upper endpoint enclosed near -0.161844090586. The generated table appears in Table 2; the corresponding scripts and execution records are described in Appendix B. □

Lemma A.5 (Large inverse temperature)**.** *$F_2(\beta) < 0$ for $\beta \geq 5$.*

Proof. Write $\varepsilon = A - 1 \geq 28q$, $\Delta_A = -A' \geq \varepsilon$. Since $\log(1 + \varepsilon) \leq \varepsilon$,

$$g \leq \frac{\varepsilon(1 + \varepsilon - \beta)}{1 + \varepsilon}.$$

Furthermore (A.2) gives $e^v \leq q\mathcal{C}(e^{-\beta}) \exp(\beta\Delta_A)$; also $\log(1 + 2e^v) \leq 2e^v$. The coefficients of $A - 1$ occur in degrees $n \geq 1$, so

$$\frac{\mathrm{d}}{\mathrm{d}\beta}(\beta\Delta_A(\beta)) = \sum_{n \geq 1} n a_n (1 - \beta n) e^{-\beta n} \leq 0 \qquad (\beta \geq 5).$$

β	$A(\beta)$	$\Delta_A(\beta)$	$\mathcal{C}(e^{-\beta})$	$U[\beta, \beta+.1]$ upper bound
4.0	1.5614948801	0.6140840464	9.2754793271	-0.1618
4.1	1.5035672897	0.5459791406	9.1446615704	-0.2186
4.2	1.4520193458	0.4862735504	9.0280978931	-0.2654
4.3	1.4060725724	0.4337747160	8.9240792320	-0.3029
4.4	1.3650575244	0.3874875235	8.8311300163	-0.3318
4.5	1.3283959991	0.3465767090	8.7479711845	-0.3528
4.6	1.2955865877	0.3103372854	8.6734899373	-0.3669
4.7	1.2661928643	0.2781711162	8.6067148458	-0.3750
4.8	1.2398336691	0.2495682349	8.5467952494	-0.3778
4.9	1.2161750683	0.2240918518	8.4929841075	-0.3763
5.0	1.1949236629	0.2013662464	8.4446236514	–

Table 2: Values for the sign check on $[4, 5]$. The first three data columns are 10-decimal approximations to narrow interval enclosures; the last column gives outward-rounded upper bounds (four decimals) for U on ten subintervals. The table is computed with `mpmath.iv`; Arb independently verifies the ten-box maximum, not each printed row.

Therefore ε, $\mathcal{C}$, and $\beta\Delta_A$ are all nonincreasing on $[5, \infty)$, and

$$F_2(\beta) \le -e^{-\beta}\left[\frac{28(\beta - 1 - \varepsilon(5))}{1 + \varepsilon(5)} - 2\mathcal{C}(e^{-5})e^{5\Delta_A(5)}\right].$$

The bracket increases with β; at $\beta = 5$ certified enclosures give $89.162296\ldots - 46.224428\ldots > 42.9378 > 0$. □

Proof of Proposition A.1. Lemma A.3 gives $F(\beta) \le F_1(\beta) < 0$ up to 4; Lemmas A.4 and A.5 give $F(\beta) \le F_2(\beta) < 0$ above 4. □

B Reproducibility

The source code, dependency specifications, recorded outputs, source hashes, and verification logs for the results discussed in this paper are available in the public companion archive: `https://github.com/MFRADJ/graded-window-barrier-certificates/releases/tag/v1.1.0`. The archive provides the interval certificates used in Appendix A, the modular and high-density constant checks, exact-coefficient tests, and a driver for fourteen principal verification suites.

The reference environment is Windows 64-bit, CPython 3.14.3, `python-flint==0.9.0` (FLINT 3.6.0 with Arb), and `mpmath==1.3.0`. From the root of the companion archive, the commands are:

```
python -m pip install -r requirements.txt
python -m pip check
python reproduce.py --mode verify
python reproduce.py --mode all --timeout 3600
python reproduce.py --mode f3
```

The timeout applies to each suite. Python must be invoked without `-O`, because certain checks use assertions. The driver verifies the input source hashes, executes independent copies in `results/local_runs/`, and writes per-suite logs and a machine-readable `CERTIFICATION_RESULT.json`. Failed checks, missing dependencies, altered inputs, and timeouts prevent a successful full-run result.

For direct inspection of the sector-sign computations, the individual scripts can also be invoked as follows:

```
python certify_arb_independent.py
python supplement/series_exact.py
```

```
python supplement/certify_mpmath_iv_crosscheck.py
python supplement/regenerate_interval_table.py
```

The interval-arithmetic computations certify the specified numerical inequalities, subject to the mathematical reductions provided in the text. They do not constitute formal verification of the complete paper.

C Analytic and sectorwise high-density estimates

This appendix provides the quantitative analytic input and sectorwise proof needed for Theorem 7.1. Put $q = e^{-\beta}$ and

$$a = \frac{2\pi^2}{3}, \qquad \kappa = \frac{2\sqrt{a}}{\ln 2} = \frac{\pi}{\ln 2}\sqrt{\frac{8}{3}}.$$

The characters A and B are exactly those of (6.1). For every $s \in S$, the sector character is $A^{m-w}B^w$, $w = \mathrm{wt}(s)$. Let $N_s(E) = \dim(V_s)_{\le E}$, $N_S(E) = \dim(V_S)_{\le E}$ and $D_S(E) = \log_2 N_S(E)$. Throughout this appendix $S \subset \mathbb{K}^m$ is additive of 2-rank $r_S \le m$, including the isotropic codes of the main paper.

C.1 Modular input

The functions F, G, g, r defined in this appendix are local modular remainder functions, distinct from the thermodynamic functions F and g defined in Section 6. The triangular-number series below is denoted by $\mathcal{T}$ to distinguish it from the modular series Ψ of Appendix A. Write $\tilde{q} = e^{-4\pi^2/\beta}$ and $y = \tilde{q}^{1/2} = e^{-c/\beta}$, $c = 2\pi^2$. With the conventions above, Poisson summation gives $\theta_3(e^{-\beta}) = \sqrt{2\pi/\beta}\,\theta_3(\tilde{q})$ and $\theta_4(e^{-\beta}) = \sqrt{2\pi/\beta}\,\theta_2(\tilde{q})$, while $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$ with $\tau = i\beta/(2\pi)$ gives $P(e^{-\beta}) = \sqrt{2\pi/\beta}\,e^{\beta/24-\pi^2/(6\beta)}P(\tilde{q})$. The factors $(2\pi/\beta)^2$ cancel in A and B, and Jacobi's identity $\theta_3^4 = \theta_2^4 + \theta_4^4$ turns $(\theta_3^4 + \theta_2^4)/(2P^4)$ into $(A + 3B)/2$ and $\theta_4^4/(2P^4)$ into $(A - B)/2$. Hence

$$A(e^{-\beta}) = \tfrac{1}{2}e^{a/\beta-\beta/6}(A + 3B)(\tilde{q}), \qquad B(e^{-\beta}) = \tfrac{1}{2}e^{a/\beta-\beta/6}(A - B)(\tilde{q}). \tag{C.1}$$

Define $F(y) = A(y^2) + 3B(y^2)$, $G(y) = A(y^2) - B(y^2)$; since $B(y^2) = 8y\mathcal{T}(y^2)^4P(y^2)^{-4}$ with $\mathcal{T}(q) = \sum_{n\ge0} q^{n(n+1)/2}$, both are power series in y with radius of convergence 1. Then

$$\log A(e^{-\beta}) = \frac{a}{\beta} - \frac{\beta}{6} - \log 2 + g(\beta), \quad \frac{B(e^{-\beta})}{A(e^{-\beta})} = \frac{G(y)}{F(y)}, \quad g = \log F(y), \quad r = \log(G(y)/F(y)).$$

The first terms of the nonnegative series A and B are $A = 1 + 28q + 134q^2 + \cdots$ and $B = 8q^{1/2}(1 + 8q + 36q^2 + \cdots)$; hence $F(y) = 1 + 24y + O(y^2)$ and $G(y) = 1 - 8y + O(y^2)$.

Lemma C.1 (Analytic modular bounds)**.** *For all $\beta > 0$, $0 < B(e^{-\beta})/A(e^{-\beta}) < 1$ and*

$$A(e^{-\beta}) < e^{a/\beta}. \tag{C.2}$$

For $0 < \beta \le 1$ and $\varepsilon_0 = 10^{-4}$,

$$0 \le g, g' \le \varepsilon_0, \qquad -\varepsilon_0 \le r, r' \le 0, \qquad |g''|, |r''| \le \varepsilon_0.$$

Proof. Jacobi's identity gives $A(q) - B(q) = \theta_4(q)^4/P(q)^4 > 0$ for $0 < q < 1$ ($\theta_4(q) > 0$ by the triple product), while $B(q) > 0$. Hence $0 < B/A < 1$. All coefficients of A and B are nonnegative and $A(0) = 1$.

Cauchy bounds. For $|z| \le R = 1/16$ put $v = R^2 = 1/256$. Since $|\theta_{3,4}(z^2)| \le \sum_n R^{|n|}$, $|1 - w| \ge 1 - |w|$ and $-\log(1 - v^n) \le v^n/(1 - v)$,

$$|\theta_3(z^2)|, |\theta_4(z^2)| \le \frac{1 + R}{1 - R}, \quad |P(z^2)|^{-4} \le \exp\Big(\frac{4v}{(1 - v)^2}\Big) \le \frac{1}{1 - 4v/(1 - v)^2}, \quad |\mathcal{T}(z^2)| \le \frac{1}{1 - v},$$

so that

$$|A(z^2)| \le \Big(\frac{1+R}{1-R}\Big)^4 \frac{1}{1-4v/(1-v)^2} < 1.68, \quad |B(z^2)| \le \frac{8R}{(1-v)^4}\frac{1}{1-4v/(1-v)^2} < 0.52.$$

Thus $|F(z)|, |G(z)| < 4$ on $|z| = 1/16$ and $|[z^n]F|, |[z^n]G| \le 4 \cdot 16^n$. For $0 \le y \le 3 \cdot 10^{-9}$ and $u = 16y$, keeping the exact linear coefficients 24 and -8 and bounding the remaining coefficients by Cauchy,

$$\begin{aligned} |F(y)-1|, |G(y)-1| &\le 24y + \frac{4u^2}{1-u} < 10^{-7}, \\ |F_y(y)| &\le 24 + 64((1-u)^{-2}-1) < 25, \qquad |G_y(y)| \le 8 + 64((1-u)^{-2}-1) < 9, \\ |F_{yy}(y)|, |G_{yy}(y)| &\le 2048(1-u)^{-3} < 2049, \end{aligned}$$

and $F_y > 0 > G_y$, since the derivative tails are below 10^{-5}.

Derivatives of y. From $3.14 < \pi < 22/7$, $19.7 < c < 20$ and $e^{-c} < 3 \cdot 10^{-9}$. For $0 < \beta \le 1$ and $j \le 4$, the function $\beta^{-j}e^{-c/\beta}$ increases on $(0, c/j] \supset (0, 1]$, so it is at most e^{-c}. Since $y' = cy/\beta^2$ and $y'' = (c^2/\beta^4 - 2c/\beta^3)y$ with $c > 2\beta$,

$$0 < y < 3 \cdot 10^{-9}, \quad 0 < y' \le ce^{-c} < 6 \cdot 10^{-8}, \quad |y''| \le c^2 e^{-c} < 1.32 \cdot 10^{-6}.$$

Chain rule. Since $|F|, |G| > 1 - 10^{-7}$,

$$\begin{aligned} |g''| &\le \frac{25|y''|}{1-10^{-7}} + \left(\frac{2049}{1-10^{-7}} + \frac{25^2}{(1-10^{-7})^2}\right)|y'|^2 < 10^{-4}, \\ |r''| &\le \frac{34|y''|}{1-10^{-7}} + \left(\frac{4098}{1-10^{-7}} + \frac{25^2+9^2}{(1-10^{-7})^2}\right)|y'|^2 < 10^{-4}. \end{aligned}$$

The bounds on $|g|, |r|, |g'|, |r'|$ are smaller. The signs $g \ge 0$, $g' \ge 0$, $r < 0$, $r' < 0$ follow from $F(y) \ge 1$, $F_y > 0$, $y' > 0$, $0 < G(y) < F(y)$ and $G_y < 0$.

Global bound (C.2). Put $q_* = e^{-2\pi} < 1/500$ and $R_* = e^{-\pi} < 1/23$. The theta numerator of $A(y^2)$ is the even part in y of $T(y)^4$, $T(y) = \sum_n y^{n^2}$, and T is coefficientwise dominated by $(1+y)/(1-y)$. Since the resulting bounds increase in R and q,

$$A(q_*) \le \frac{1}{2}\left[\left(\frac{1+R_*}{1-R_*}\right)^4 + \left(\frac{1-R_*}{1+R_*}\right)^4\right]\frac{1}{1-4q_*/(1-q_*)^2} < 1.08, \qquad B(q_*) < 0.36,$$

so $F(R_*) < 2.2$. If $0 < \beta \le 1$, then $F(y) < 2$. If $1 \le \beta \le 2\pi$, then $y \le R_*$ and, F being increasing on $[0, 1)$, $F(y) \le F(R_*) < 2.2 < 2e^{1/6} \le 2e^{\beta/6}$. In both cases (C.1) gives $A(e^{-\beta}) < e^{a/\beta}$. Finally, for $\beta \ge 2\pi$, $(e^{-\beta})^n \le e^{2\pi-\beta}q_*^n$ for $n \ge 1$, so

$$\log A(e^{-\beta}) \le A(e^{-\beta}) - 1 \le e^{2\pi-\beta}(A(q_*) - 1) < 0.08e^{2\pi-\beta} < \frac{a}{\beta},$$

because $\beta e^{2\pi-\beta} \le 2\pi$ for $\beta \ge 2\pi$ and $0.16\pi < a$. □

C.2 Uniform cumulative estimates

Proposition C.2 (All-energy upper bound)**.** *For every $E \ge 0$,*

$$N_S(E) \le 2^{r_S}e^{2\sqrt{amE}}, \qquad D_S(E) \le \kappa\sqrt{mE} + r_S. \tag{C.3}$$

Proof. For every $\beta > 0$, coefficient positivity, $B < A$ and Lemma C.1 give

$$N_S(E) \le e^{\beta E}\sum_{s\in S} A^{m-w}B^w \le 2^{r_S}A^m e^{\beta E} < 2^{r_S}e^{\beta E + am/\beta}.$$

For $E > 0$ take $\beta = \sqrt{am/E}$; at $E = 0$, $N_S(0) = 1$. □

Proposition C.3 (Uniform lower bound for every sector)**.** *Let $m \geq 1$, $s \in S$, and $E \geq 15m$. Then*

$$\log_2 N_s(E) \geq \kappa\sqrt{mE} - m(1.001 + 1.24\sqrt{m/E}) - 20(mE)^{1/4} - 1 \geq \kappa\sqrt{mE} - \frac{4}{3}m - 20(mE)^{1/4} - 1. \tag{C.4}$$

Proof. Put $w = \mathrm{wt}(s) \leq m$ and $\Lambda = \log(A^{m-w}B^w) = m\log A + wr$, and endow the energies of V_s with Gibbs probabilities proportional to $\dim(V_s)_e e^{-\beta e}$. Their mean and variance are $\mu = -\Lambda'$ and $\sigma^2 = \Lambda''$, so

$$\mu = m\Big(\frac{a}{\beta^2} + \frac{1}{6}\Big) - mg' - wr', \qquad \sigma^2 = \frac{2am}{\beta^3} + mg'' + wr''.$$

For $0 < \beta \leq 1$, Lemma C.1 gives $-mg' - wr' \in [-m\varepsilon_0, w\varepsilon_0]$ (this uses the signs of g' and r'), hence

$$\left|\mu - m\left(\frac{a}{\beta^2} + \frac{1}{6}\right)\right| \leq m\varepsilon_0, \qquad \left|\sigma^2 - \frac{2am}{\beta^3}\right| \leq 2m\varepsilon_0, \qquad \Lambda + \beta\mu \geq \frac{2am}{\beta} - m\ln 2 - 2m\varepsilon_0, \tag{C.5}$$

the last because $\Lambda + \beta\mu = m(2a/\beta - \ln 2 + g - \beta g') + w(r - \beta r')$. The continuous function $\mu + 2\sigma$ tends to $+\infty$ as $\beta \downarrow 0$, while, using $\sqrt{m} \leq m$,

$$\mu(1) + 2\sigma(1) \leq m\left(a + \frac{1}{6} + \varepsilon_0 + 2\sqrt{2a + 2\varepsilon_0}\right) < 14.01m < E.$$

Thus there is a $\beta \in (0,1)$ with $E = \mu + 2\sigma$. By Chebyshev, at least $3/4$ of the Gibbs mass lies in $[E - 4\sigma, E]$, where $e^{-\beta e} \leq e^{-\beta(E-4\sigma)}$; therefore

$$\log N_s(E) \geq \Lambda + \beta(E - 4\sigma) - \ln\tfrac{4}{3} \geq \frac{2am}{\beta} - m\ln 2 - 2m\varepsilon_0 - 2\beta\sigma - \ln\tfrac{4}{3}.$$

Set $\delta = m(1/6 + \varepsilon_0) + 2\sigma$. By (C.5), $E - \delta \leq am/\beta^2 \leq E$. Since $\sqrt{\max(0, E-\delta)} \geq \sqrt{E} - \delta/\sqrt{E}$,

$$\frac{2am}{\beta} \geq 2\sqrt{amE} - 2\delta\sqrt{am/E}.$$

Using $\beta \leq 1$ to absorb $2m\varepsilon_0$ in $\sigma^2 \leq m(2a + 2\varepsilon_0)\beta^{-3}$, together with $1/\beta \leq \sqrt{E/(am)}$, one gets, with $c_1 = \sqrt{2a + 2\varepsilon_0}\, a^{-1/4} < 2.27$,

$$\beta\sigma \leq c_1(mE)^{1/4}, \qquad \sigma\sqrt{am/E} \leq c_1(mE)^{1/4}.$$

Consequently

$$\log N_s(E) \geq 2\sqrt{amE} - m\left[\ln 2 + 2\varepsilon_0 + (\tfrac{1}{3} + 2\varepsilon_0)\sqrt{am/E}\right] - 6c_1(mE)^{1/4} - \ln\tfrac{4}{3}.$$

Dividing by $\ln 2$ yields (C.4), since $1 + 2\varepsilon_0/\ln 2 < 1.001$, $(1/3 + 2\varepsilon_0)\sqrt{a}/\ln 2 < 1.24$, $6c_1/\ln 2 < 20$, $\log_2(4/3) < 1$, and $1.001 + 1.24/\sqrt{15} < 4/3$. □

Proposition C.4 (Uniform high-density growth)**.** *For every $m \geq 1$, every S with $r_S \leq m$ and every $E \geq 15m$,*

$$-\frac{4}{3}m - 20(mE)^{1/4} - 1 \leq D_S(E) - \kappa\sqrt{mE} \leq r_S \leq m. \tag{C.6}$$

For $0 \leq E < 15m$, $-29m \leq D_S(E) - \kappa\sqrt{mE} \leq m$. All constants are absolute.

Proof. The zero codeword supplies the sector A^m, so $N_S(E) \geq N_0(E)$ and Proposition C.3 applies. Proposition C.2 supplies the upper bound. For $E < 15m$, $D_S(E) \geq 0$ and $\kappa\sqrt{15} < 29$. □

Remark C.5 (Robustness). In Proposition C.3, if only $|g'|, |r'| \leq \varepsilon_0$ is used, without the signs, the constants become $1 + 4\varepsilon_0/\ln 2 < 1.001$ and $(1/3 + 4\varepsilon_0)\sqrt{a}/\ln 2 < 1.24$, so (C.4) is unchanged.

C.3 Exact levels and real windows

Lemma C.6 (Exact-level monotonicity)**.** *In a sector of weight w, the coefficients at the energies $w/2 + j$, $j \in \mathbb{Z}_{\geq 0}$, are nondecreasing in j and at least 8^w; all other energies have zero coefficient. In particular, for every $h \geq w/2$ in $w/2 + \mathbb{Z}$,*

$$\dim(V_s)_h \geq \frac{N_s(h)}{h+1}.$$

Proof. With $N = \frac{1}{2}(\theta_3^4 + \theta_4^4)$ and $\theta_2^4/2 = 8q^{1/2}\mathcal{T}^4$,

$$A^{m-w}B^w = q^{w/2}(1-q)^{-1}H(q), \qquad H = 8^w N^{m-w}\mathcal{T}^{4w}(1-q)^{-(4m-1)}\prod_{n\geq 2}(1-q^n)^{-4m},$$

where H has nonnegative integer coefficients and $H(0) = 8^w$. Multiplication by $(1-q)^{-1}$ forms partial sums. The cumulative count up to h involves at most $h+1$ levels of the sector, and the last coefficient dominates their average. □

Corollary C.7 (Largest admissible encoding)**.** *For every admissible $h \geq 15m$, $\dim(V_S)_h \geq 2$, so $\lambda_h \geq 1$ is defined, and*

$$\kappa\sqrt{mh} - \frac{4}{3}m - 20(mh)^{1/4} - \log_2(h+1) - 1 \leq \log_2 \dim(V_S)_h \leq \kappa\sqrt{mh} + r_S.$$

Thus, with $e_2 = \frac{4}{3}m + 24(mh)^{1/4} + 2$, one has $\lambda_h \geq \kappa\sqrt{mh} - e_2$.

Proof. For integer h choose $s = 0$; for half-integer h choose an odd-weight $s \in S$, which exists by admissibility. As $w \leq m \leq h$, the level h occurs in that sector, and Lemma C.6 gives $\dim(V_S)_h \geq \dim(V_s)_h \geq 8$ if w is odd, and $\geq [q^1]A^m = 28m$ if $s = 0$. Apply Lemma C.6, Proposition C.3, and $\log_2(1+x) \leq 4x^{1/4}$ ($x \geq 0$). For completeness, let $f(x) = 4x^{1/4} - \log_2(1+x)$. For $x > 0$, $f'(x) > 0$ since $\ln 2\,(1+x)x^{-3/4} \geq 4 \cdot 3^{-3/4}\ln 2 > 1$; the expression $(1+x)x^{-3/4}$ is minimized at $x = 3$, and $f(0) = 0$. This yields $\log_2(h+1) \leq 4(mh)^{1/4}$ because $m \geq 1$; one unit accounts for the floor. □

Lemma C.8 (Continuous shifts reduce to the half-lattice)**.** *For $h \in \frac{1}{2}\mathbb{Z}_{>0}$, the supremum in ρ_{win} over all real $t \in [-2h, 2h]$ is attained at some $t \in \frac{1}{2}\mathbb{Z}$.*

Proof. Let $t \in (j/2, (j+1)/2)$. In half-unit coordinates, the graded levels in the closed window $[t, t+2h]$ have indices $j+1, \ldots, j+4h$, a subset of $j, \ldots, j+4h$ at $t = j/2$; those in $[-t, 2h-t]$ have indices $-j, \ldots, 4h-j-1$, a subset of $-j, \ldots, 4h-j$. Both window dimensions are therefore no larger than at $t = j/2 \in [-2h, 2h)$. □

The analytic proof of Theorem 7.1 bounds all real shifts without using this lemma. It justifies evaluating ρ_{win} on the half-lattice in numerical checks.

C.4 Numerical checks and reproducibility status

The companion archive includes 31 Arb checks of the high-density constants, three separate non-certifying numerical consistency checks, a further constant checker, and a twelve-check Arb chain-rule program. It also includes a 6,118-case finite audit, near-threshold tests, source-specific logs, and the recorded execution results described in Appendix B. Several exact-coefficient window audits use floating-point logarithms in their final comparisons. Those finite checks are not rigorous universal interval bounds.

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