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GERSTEN EXACTNESS WITH INVERTIBLE COEFFICIENTS FOR EXCELLENT REGULAR LOCAL SURFACES

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ABSTRACT. Let R be an excellent regular local ring of dimension two, and let $N \geq 2$ be a natural number invertible in R. We prove exactness of the augmented Gersten complex for $K(R; \mathbf{Z}/N)$ in every degree. More generally, it suffices that the normalization of every reduced closed curve in $\operatorname{Spec} R$ be finite. The proof applies to ramified mixed-characteristic rings without a henselianity assumption. With coefficients of invertible order, we first prove that pushforward from a Cartier divisor on the spectrum of a noetherian semilocal ring annihilates every class coming from connective K-theory. For a curve in $\operatorname{Spec} R$, we then place its normalization and auxiliary regular curves as transverse Cartier divisors on a finite R-scheme. A third curve carries a prescribed residue class, and a quotient of the two divisor equations moves that residue to an auxiliary curve. The residues on the auxiliary curves cancel after transfer to the residue field of R. These constructions prove generic injectivity; vanishing of the closed-point pushforward then gives the full complex.

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1. INTRODUCTION

For a regular local ring R, Gersten's conjecture concerns the exactness of the complex obtained from the codimension filtration in algebraic K-theory. This paper treats regular local rings of dimension two with finite coefficients of invertible order. For a scheme X, let $K(X)$ denote nonconnective algebraic K-theory of perfect complexes and let $K^{\mathrm{cn}}(X)$ denote its connective cover. For a natural number $N \geq 2$ and $n \in \mathbf{Z}$, we use

$$K(X; \mathbf{Z}/N) = \operatorname{cofib}\big(N : K(X) \longrightarrow K(X)\big), \qquad K_n(X; \mathbf{Z}/N) = \pi_n K(X; \mathbf{Z}/N).$$

Thus $K_n(X; \mathbf{Z}/N)$ includes the contribution from the N-torsion in $K_{n-1}(X)$. For a noetherian scheme X, let $G(X)$ denote the connective K-theory spectrum of coherent modules. We use the same cofiber convention to define $G_q(X; \mathbf{Z}/N)$ for $q \in \mathbf{Z}$.

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Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a noetherian regular local ring of dimension two, with fraction field F. Suppose that the normalization of every reduced closed curve in* $\operatorname{Spec} R$ *is finite. In particular, this hypothesis holds if R is Nagata, and hence if R is excellent.*

For every natural number $N \geq 2$ invertible in R and every $n \in \mathbf{Z}$, the augmented Gersten complex

$$\begin{aligned} 0 \longrightarrow K_n(R; \mathbf{Z}/N) \longrightarrow K_n(F; \mathbf{Z}/N) \longrightarrow \bigoplus_{\operatorname{ht} \mathfrak{p}=1} K_{n-1}(\kappa(\mathfrak{p}); \mathbf{Z}/N) \\ \longrightarrow K_{n-2}(k; \mathbf{Z}/N) \longrightarrow 0 \end{aligned} \tag{1.1}$$

is exact. The differentials are the localization differentials of Quillen's codimension filtration.

Here a reduced closed curve is a nonempty reduced closed subscheme all of whose irreducible components have dimension one. A quotient of a Nagata ring is Nagata, a quasi-excellent ring is Nagata, and the normalization of a Nagata scheme is finite; this explains the Nagata and excellent cases [Sta26, Tag 032T, Tag 07QV, Tag 035S].

Relation with earlier results. Quillen proved the integral Gersten conjecture for regular local rings essentially of finite type over a field [Qui73, §7, Theorem 5.11]. Panin proved the integral conjecture for arbitrary equicharacteristic regular local rings [Pan03, Theorem A]. In mixed characteristic, Gillet proved the Gersten conjecture for discrete valuation rings with coefficients of order prime to the residue characteristic [Gil86]. Together with the relative theorem of Gillet and Levine [GL87, Corollary 6], this gives full exactness for regular local rings essentially smooth over a discrete valuation ring with such coefficients. Geisser and Levine supplied the residue-characteristic-primary case, and hence arbitrary finite coefficients in this smooth setting [GL00, Theorem 8.2]. Skalit obtained full finite-coefficient exactness for arbitrary unramified regular local rings [Ska17, Corollary 5.7]. With coefficients of order invertible in a discrete valuation ring V, applying the approximation and localization argument of [Ska17, Proposition 5.1, the proof of Theorem 5.4, and §5.2] over V gives full exactness for every regular local homomorphism $V \to R$. This applies whether or not V is ramified.

Gabber rigidity [Gab92] and Gillet's DVR theorem [Gil86] also imply generic injectivity for henselian regular local rings with invertible coefficients; a direct induction is proved in [Fel26b, Corollary 6.13]. In dimension two, this injectivity gives the full Gersten complex by the two localization sequences (6.1) and the DVR exact sequence of Corollary 3.6. The distinction between the henselian and general local cases is illustrated by Remark 6.3.

The scope of Theorem 1.1 includes nonhenselian ramified surfaces outside the case of a regular local homomorphism from a discrete valuation ring; Remark 6.4 gives an explicit example. Theorem 1.1 allows arbitrary residue fields and, in mixed characteristic $(0, p)$, arbitrary ramification $p \in \mathfrak{m}^2$. The invertibility of N cannot be omitted in general, even for complete regular local surfaces, by the counterexample in [Fel26a]; see Remark 6.5.

For étale cohomology, Sakagaito proved the corresponding full Gersten exact sequence over two-dimensional henselian regular local rings for Tate twists of the sheaf of N-th roots of unity, with N invertible in the ring [Sak20, p. 38, Theorem 9 of the published version].

The proof. The two principal constructions are Proposition 3.1 and Lemma 5.2. Both choose a unit on an open subscheme whose restriction to a specified closed subscheme is 1. Its integral K_1-class then acts by zero on the coherent theory of that closed subscheme, by (2.4). In Proposition 3.1, this is the complement of an étale neighborhood inside a finite compactification. In Lemma 5.2, it is the intersection of that open subscheme with a transverse curve carrying a prescribed residue class.

Vanishing for Cartier divisors. Gabber's rigidity theorem for henselian pairs gives the étale lifting of finite-coefficient classes in Lemma 3.4; see [Gab92] and [CMM21, Theorem 1.4]. For a noetherian semilocal ring B, a nonzerodivisor $g \in B$, and a natural number $N \geq 2$ invertible

in B, this lifting implies the vanishing of

$$K_q^{\mathrm{cn}}(B/gB;\mathbf{Z}/N) \xrightarrow{\text{Cartan}} G_q(B/gB;\mathbf{Z}/N) \longrightarrow G_q(B;\mathbf{Z}/N) \qquad (q\in\mathbf{N});$$

this is Proposition 3.1. The étale neighborhood is embedded in a finite B-scheme, where a defining equation of the divisor is chosen equal to 1 on the complement of that neighborhood; see Lemma 3.3 and (3.1). The corresponding integral K_1-class acts by zero on this complement by (2.4); the module-boundary formula (2.5) and Lemma 2.1 produce a class whose divisor boundary is the prescribed Cartan image. For regular B and B/gB, Proposition 3.1 and localization (2.3) give the short exact sequences of Corollary 3.5 in every dimension.

Two transverse Cartier divisors. Let $(R,\mathfrak{m},k)$ and N satisfy Theorem 1.1, let $D\subset \operatorname{Spec} R$ be a reduced closed curve, and let $\operatorname{Spec} A\to D$ be its finite normalization. Write the maximal ideals of A as $\mathfrak{a}_1,\dots,\mathfrak{a}_s$ and their residue fields as $L_i=A/\mathfrak{a}_i$. For $m\in\mathbf{N}^*$ and $\theta\in G_m(D;\mathbf{Z}/N)$, let $b_i\in K_{m-1}(L_i;\mathbf{Z}/N)$ be its image under the composite

$$\begin{array}{ccc} G_m(D;\mathbf{Z}/N) & \longrightarrow & G_m(\operatorname{Frac}(A_{\mathfrak{a}_i});\mathbf{Z}/N) \\ & & \downarrow{\scriptstyle \text{Cartan}^{-1}} \\ K_{m-1}(L_i;\mathbf{Z}/N) & \xleftarrow[\partial_{A_{\mathfrak{a}_i}}]{} & K_m(\operatorname{Frac}(A_{\mathfrak{a}_i});\mathbf{Z}/N) \end{array}$$

of generic restriction, inverse Cartan, and the DVR residue. These classes satisfy $\sum_i \operatorname{Tr}_{L_i/k}(b_i)=0$ by Lemma 5.1. Assume first that $|k|>s$. Lemmas 4.1 and 4.2 give regular parameters x,y of R with $xA=\mathfrak{m}A$ and finite free DVRs H_i over $R/(x)$, with uniformizer y and residue field L_i. Lemma 4.6 places these rings on a finite R-algebra B with global nonzerodivisors h,v and compatible isomorphisms

$$B/hB\simeq A, \qquad B/vB\simeq\prod_i H_i, \qquad B/(h,v)\simeq\prod_i L_i.$$

At $z_i=\ker(B\to L_i)$, the ring B_{z_i} is regular of dimension two with parameters h,v; this is the transversality required in Lemma 5.2.

Cancellation of one residue. By Lemma 4.9, there are a DVR B/e_iB and a class $b_i'\in K_{m-1}(B/e_iB;\mathbf{Z}/N)$ such that $b_i'|_{L_i}=b_i$ and $e_i\equiv h-v \pmod{z_i^2B_{z_i}}$. By Proposition 3.1 and the localization sequence (2.3), there is $\gamma_i\in G_m(B[1/e_i];\mathbf{Z}/N)$ satisfying

$$\partial^B_{V(e_i)}\gamma_i=\operatorname{Cartan}(b_i');$$

this is the first step of Lemma 5.2. The images of h and v are uniformizers of B/e_iB, so their quotient lifts to a unit $u_i\in B^\times$ by Lemma 3.2. The key equality is

$$\left.\frac{h}{u_iv}\right|_{(B/e_iB)[1/(hv)]}=1.$$

Consequently the e_i-boundary of $[h/(u_iv)]\cdot\gamma_i|_{B[1/(e_ihv)]}$ is zero, and this product extends to $G_{m+1}(B[1/(hv)];\mathbf{Z}/N)$ by (2.5) and Lemma 5.2. The valuations of $h/(u_iv)$ at the generic points of $V(h)$ and $V(v)$ are 1 and -1, respectively. Together with Lemma 2.2, these valuations give residue b_i at $\mathfrak{a}_i$, zero at the other $\mathfrak{a}_j$, and residue $-b_i$ on H_i; finite pushforward to $R/(x)$ turns the last residue into $-\operatorname{Tr}_{L_i/k}(b_i)$, as proved in Lemma 5.2.

From curves to the Gersten complex. Subtracting these localization boundaries from the image of θ in $G_m(D\cup V(x);\mathbf{Z}/N)$ eliminates every normalization residue; their residues on $R/(x)$ sum to zero by equations (5.1) and (5.2). The localization sequence (2.3), applied to A and $R/(x)$, then gives (5.3), and each remaining pushforward vanishes by Proposition 3.1 and Corollary 3.6. In Proposition 5.3, descent along a finite free étale extension of degree prime to N removes $|k|>s$, while rank and connectivity treat the nonpositive degrees. The localization sequence (2.3) gives generic injectivity in Corollary 5.4. Finally, the closed-point map into the connective K-theory of finitely generated torsion R-modules factors through $G(R/(x);\mathbf{Z}/N)$, as in (6.2), and vanishes on homotopy groups by Corollary 3.6. Corollary 5.4

and the two localization sequences (6.1) give equations (6.3) and (6.4); their splice is the augmented Gersten complex (1.1).

2. Coefficients, localization, and residues

All rings are commutative with identity. Schemes on which coherent G-theory is used are noetherian. We use $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$. All degree indices are integers, and every coefficient order N is a natural number with $N \geq 2$. A discrete valuation ring is abbreviated to DVR. A regular local surface means the spectrum of a regular local ring of dimension two. For a scheme X and a point $x \in X$, $\mathcal{O}_X$ denotes its structure sheaf and $\mathcal{O}_{X,x}$ its local ring at x. For points $x, y \in X$, we say that y is a specialization of x, or that x is a generalization of y, if $y \in \overline{\{x\}}$. On $X = \operatorname{Spec} B$, this means that the corresponding prime ideals satisfy $\mathfrak{p}_x \subset \mathfrak{p}_y$. For a prime $\mathfrak{p}$ of a ring R, $\kappa(\mathfrak{p})$ denotes the residue field of $R_\mathfrak{p}$. For a ring, K- and G-theory notation refers to its affine spectrum.

2.1. **The two K-theories.** For a scheme X, let $K(X)$ denote the nonconnective algebraic K-theory spectrum of perfect complexes, and let $K^{\mathrm{cn}}(X)$ denote its connective cover. We use the conventions of Thomason–Trobaugh [TT90]. For a noetherian scheme X, let $G(X)$ denote connective Quillen G-theory of coherent $\mathcal{O}_X$-modules. For $q \in \mathbf{Z}$, we write $K_q(X) = \pi_q K(X)$, $K_q^{\mathrm{cn}}(X) = \pi_q K^{\mathrm{cn}}(X)$, and $G_q(X) = \pi_q G(X)$. For an exact category $\mathcal{E}$, the notation $K^{\mathrm{cn}}(\mathcal{E})$ denotes its connective Quillen K-theory spectrum. For a noetherian scheme X, the inclusion of perfect complexes into bounded complexes with coherent cohomology induces the Cartan map

$$\mathrm{Cartan} : K^{\mathrm{cn}}(X) \longrightarrow G(X).$$

For an affine scheme, it is induced by regarding a finite projective module as a coherent module. The comparison with bounded complexes is the Gillet–Waldhausen theorem [TT90, Theorem 1.11.7, §§3.12–3.13]. If T is a regular noetherian ring of finite dimension, the resolution theorem and vanishing of negative K-groups give natural equivalences

$$K(T) \xleftarrow{\sim} K^{\mathrm{cn}}(T) \xrightarrow[\mathrm{Cartan}]{\sim} G(T) \tag{2.1}$$

by [Wei13, V.3.3] and [TT90, Proposition 6.8(b)]. We transport classes from $G(T)$ to $K(T)$ through the inverse of the right-hand equivalence followed by the left-hand map. The same convention applies after taking finite coefficients. In particular, for a finite morphism $f : Y \to \operatorname{Spec} T$, the pushforward with target $K_q(T; \mathbf{Z}/N)$ means the composite

$$G_q(Y; \mathbf{Z}/N) \xrightarrow{f_*} G_q(T; \mathbf{Z}/N) \xrightarrow{\mathrm{Cartan}^{-1}} K_q^{\mathrm{cn}}(T; \mathbf{Z}/N) \xrightarrow{\sim} K_q(T; \mathbf{Z}/N).$$

Write $\mathbb{S}/N$ for the cofiber of multiplication by N on the sphere spectrum. We use the same cofiber convention for $K^{\mathrm{cn}}(X; \mathbf{Z}/N)$ and $G(X; \mathbf{Z}/N)$, and smash the Cartan map with $\mathbb{S}/N$ to obtain its finite-coefficient version. We also use Cartan for the induced maps on integral and finite-coefficient homotopy groups. For a spectrum E, the cofiber sequence for $E \wedge \mathbb{S}/N$ gives the natural short exact sequence

$$0 \longrightarrow \pi_q(E)/N \longrightarrow \pi_q(E \wedge \mathbb{S}/N) \longrightarrow \pi_{q-1}(E)[N] \longrightarrow 0. \tag{2.2}$$

Here $\pi_{q-1}(E)[N]$ is the kernel of multiplication by N. In particular, an integer relatively prime to N acts invertibly on every group $\pi_q(E \wedge \mathbb{S}/N)$: it acts invertibly on both outer terms of (2.2), and hence on the middle term. We will use this observation for descent along a finite free extension.

A ring homomorphism $A \to B$ induces $K^{\mathrm{cn}}(A) \to K^{\mathrm{cn}}(B)$ by derived extension of scalars. This is pullback along $\operatorname{Spec} B \to \operatorname{Spec} A$. For $c \in K_q^{\mathrm{cn}}(A; \mathbf{Z}/N)$, its image is denoted $c|_B$; when $B = A/I$, it is the reduction of c modulo I. We use the same restriction notation for scheme pullbacks. Derived pullback on G-theory is used for morphisms of finite Tor dimension, so that bounded complexes with coherent cohomology remain bounded. The connecting homomorphisms in localization lower the degree by one.

Since $G(X)$ is connective, $G_q(X;\mathbf{Z}/N) = 0$ for $q < 0$. For a regular local ring R, its projective modules are free and its negative K-groups vanish. Consequently $K_0(R;\mathbf{Z}/N) = \mathbf{Z}/N$ by rank and $K_q(R;\mathbf{Z}/N) = 0$ for $q < 0$.

2.2. **Localization and its operations.** For a closed subscheme $Z \subset X$ with open complement U, coherent localization and dévissage give a cofiber sequence

$$G(Z) \longrightarrow G(X) \longrightarrow G(U) \xrightarrow{\partial_Z^X} \Sigma G(Z). \tag{2.3}$$

We use the same decoration ∂_Z^X for its connecting homomorphisms for $q \in \mathbf{Z}$, before and after taking finite coefficients:

$$\partial_Z^X : G_q(X \setminus Z;\mathbf{Z}/N) \longrightarrow G_{q-1}(Z;\mathbf{Z}/N).$$

For an affine ambient scheme $\operatorname{Spec} B$, the upper decoration is its coordinate ring B; thus $\partial^B_{V(I)}$ denotes the boundary for the closed subscheme $V(I) \subset \operatorname{Spec} B$. The lower decoration always specifies the closed subscheme or closed support in that ambient scheme. The G-theory with support depends only on the underlying closed set; thus Z may be replaced by its reduction. The localization sequence is functorial at the level of spectra [Wei13, V.4.1, V.6.11]. Smashing (2.3) with $\mathbb{S}/N$ gives the finite-coefficient sequence used throughout the proof.

We use Quillen's connecting homomorphisms with the index convention of [Wei13, V, Exercise 5.1 and Example 6.1.2]. If an endomorphism $f : M \to M$ of a coherent module becomes an automorphism on U, then

$$\partial_Z^X [f|_U] = [\operatorname{coker} f] - [\ker f]$$

in the Grothendieck group of coherent modules supported on Z, identified with $G_0(Z)$ by dévissage. For computations with complexes, this is the class of the two-term complex $[M \xrightarrow{f} M]$, with its target in cohomological degree zero. We use the natural Gillet–Waldhausen comparison with this Quillen construction [TT90, Theorem 1.11.7, §3.13].

A finite morphism $f : Y \to X$ gives a pushforward on G-theory by the exact restriction-of-scalars functor on coherent modules. It commutes with localization when the closed support in Y is the inverse image of the support in X. For a finite field extension L/k, the resulting map is denoted

$$\operatorname{Tr}_{L/k} : K_q(L;\mathbf{Z}/N) \longrightarrow K_q(k;\mathbf{Z}/N).$$

For nested closed supports, inclusion of the coherent support categories gives a map of localization sequences. We will use this as well as finite pushforward.

Tensor product with a perfect complex defines the integral action $K^{\mathrm{cn}}(X) \wedge G(X) \to G(X)$ and makes localization a sequence of $K^{\mathrm{cn}}(X)$-modules [Wei13, IV.6.6, V.3.12, V.6.11]. For the closed immersion $i : Z \hookrightarrow X$, a perfect complex $P^\bullet$ on X, and a bounded complex $M^\bullet$ of coherent $\mathcal{O}_Z$-modules, the projection formula is the natural equivalence

$$P^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} i_* M^\bullet \simeq i_*(Li^*P^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Z} M^\bullet). \tag{2.4}$$

Here i_* is exact, and $Li^*P^\bullet$ is perfect on Z; see [Sta26, Tag 0B54, Tag 0B55, Tag 09UA]. Under dévissage, this equivalence identifies the action on the supported term with the composite

$$K^{\mathrm{cn}}(X) \wedge G(Z) \xrightarrow{i^*\wedge 1} K^{\mathrm{cn}}(Z) \wedge G(Z) \longrightarrow G(Z).$$

Thus a perfect-complex class acts on $G(Z)$ through its restriction to Z. Smashing the integral action with $\mathbb{S}/N$ gives both pairings

$$K^{\mathrm{cn}}(X) \wedge G(X;\mathbf{Z}/N) \longrightarrow G(X;\mathbf{Z}/N),$$
$$K^{\mathrm{cn}}(X;\mathbf{Z}/N) \wedge G(X) \longrightarrow G(X;\mathbf{Z}/N).$$

For $r, q \in \mathbf{Z}$, take

$$(\alpha, \beta) \in K^{\mathrm{cn}}_r(X) \times G_q(U;\mathbf{Z}/N) \quad \text{or} \quad K^{\mathrm{cn}}_r(X;\mathbf{Z}/N) \times G_q(U),$$

respectively. Tensor product with a perfect complex on the right commutes with the connecting morphism: $\partial_Z^X(\beta \cdot \alpha|_U) = (\partial_Z^X \beta) \cdot \alpha|_Z$; this is the localization pairing of [Wei13, V,

Exercise 5.3]. The identity is natural at the level of spectra, so it also holds when either one of the two inputs carries $\mathbf{Z}/N$ coefficients, with the other input integral. Moving the degree-r perfect class to the left gives

$$\begin{aligned} \partial_Z^X(\alpha|_U \cdot \beta) &= (-1)^{rq}(\partial_Z^X \beta) \cdot \alpha|_Z \\ &= (-1)^{rq+r(q-1)} \alpha|_Z \cdot \partial_Z^X \beta \\ &= (-1)^r \alpha|_Z \cdot \partial_Z^X \beta \quad \text{in } G_{r+q-1}(Z; \mathbf{Z}/N). \end{aligned} \tag{2.5}$$

For a unit $a \in \Gamma(X, \mathcal{O}_X)^\times$, multiplication by $[a] \in K_1(X)$ uses its integral K-class.

Lemma 2.1. *Let B be a noetherian ring, let $g \in B$ be a nonzerodivisor, and let $N \geq 2$ be a natural number. For every $q \in \mathbf{N}$ and $c \in K_q^{\mathrm{cn}}(B; \mathbf{Z}/N)$, the localization boundary*

$$\partial_{V(g)}^B : G_{q+1}(B[1/g]; \mathbf{Z}/N) \longrightarrow G_q(B/gB; \mathbf{Z}/N)$$

satisfies

$$\partial_{V(g)}^B([g] \cdot \mathrm{Cartan}(c|_{B[1/g]})) = \mathrm{Cartan}(c|_{B/gB}).$$

Proof. Apply the index convention to multiplication by g on B. Its kernel is zero and its cokernel is B/gB, so

$$\partial_{V(g)}^B \mathrm{Cartan}([g]) = [B/gB] \quad \text{in } G_0(B/gB).$$

Equivalently, the supported two-term complex $[B \xrightarrow{g} B]$, with target in degree zero, is quasi-isomorphic to B/gB in degree zero. This is the arbitrary-ring calculation in [Wei13, V, Example 6.1.2]. Graded commutativity of the perfect-complex product, its compatibility with the Cartan map, and (2.5) give

$$\begin{aligned} \partial_{V(g)}^B([g] \cdot \mathrm{Cartan}(c|_{B[1/g]})) &= (-1)^q \partial_{V(g)}^B(c|_{B[1/g]} \cdot \mathrm{Cartan}([g])) \\ &= (-1)^{2q} c|_{B/gB} \cdot [B/gB] \\ &= \mathrm{Cartan}(c|_{B/gB}). \end{aligned}$$

The last equality is the definition of the Cartan map through tensor product with the structure module. □

For a discrete valuation ring V with residue field l, we use the single lower decoration ∂_V for its K-theory residue. For every $r \in \mathbf{Z}$, it is the homomorphism

$$\begin{aligned} &\partial_V : K_r(\mathrm{Frac}\, V; \mathbf{Z}/N) \longrightarrow K_{r-1}(l; \mathbf{Z}/N), \\ &\partial_V = \mathrm{Cartan}^{-1} \circ \partial_{\mathrm{Spec}\, l}^V \circ \mathrm{Cartan}\,. \end{aligned}$$

The Cartan maps use (2.1) for the two fields. The same notation and formula apply to integral groups. For $q \in \mathbf{Z}$, $a \in \mathrm{Frac}(V)^\times$, and $\alpha \in K_q(V; \mathbf{Z}/N)$, Lemma 2.1 gives

$$\partial_V([a] \cdot \alpha|_{\mathrm{Frac}(V)}) = \mathrm{ord}_V(a)\, \alpha|_l. \tag{2.6}$$

Here $\mathrm{ord}_V : \mathrm{Frac}(V)^\times \to \mathbf{Z}$ is the valuation normalized by $\mathrm{ord}_V(\pi) = 1$ for a generator π of the maximal ideal of V. For $q < 0$, both sides vanish by connectivity. For $q \in \mathbf{N}$, choose a uniformizer π and let $u = a\pi^{-\mathrm{ord}_V(a)} \in V^\times$. Then $[a] = \mathrm{ord}_V(a)[\pi] + [u]$ in $K_1(\mathrm{Frac}(V))$. Lemma 2.1 computes the first summand, and the second is the restriction of $[u] \cdot \alpha \in K_{q+1}(V; \mathbf{Z}/N)$, whose boundary is zero by exactness of (2.3).

2.3. **Transverse restrictions.** We record the compatibility of localization with Tor-independent pullback to a transverse divisor. The proof constructs the map of localization sequences and identifies its supported term; the latter identification is the G-theoretic base-change formula of [Wei13, V.3.7.2].

Lemma 2.2. *Let $(B, \mathfrak{n}, L)$ be a regular local ring of dimension two, let h, e be a regular system of parameters, and let $N \geq 2$ be a natural number. For $m \in \mathbf{Z}$ and $\gamma \in G_m(B[1/e]; \mathbf{Z}/N)$, one has*

$$\partial_{B/hB}(\mathrm{Cartan}^{-1}(\gamma|_{\mathrm{Frac}(B/hB)})) = \mathrm{Cartan}^{-1}((\partial^B_{V(e)}\gamma)|_L) \quad \text{in } K_{m-1}(L; \mathbf{Z}/N).$$

Here $\partial^B_{V(e)} : G_m(B[1/e]; \mathbf{Z}/N) \to G_{m-1}(B/eB; \mathbf{Z}/N)$ is the coherent localization boundary, whereas $\partial_{B/hB} : K_m(\mathrm{Frac}(B/hB); \mathbf{Z}/N) \to K_{m-1}(L; \mathbf{Z}/N)$ is the K-theory residue of the DVR B/hB. The restrictions inside the parentheses are derived pullbacks in G-theory. Each inverse Cartan map is followed by the connective-cover equivalence in (2.1), for $\mathrm{Frac}(B/hB)$ *on the left and L on the right.*

Proof. The rings B, B/hB, B/eB, their indicated localizations, and L are regular of finite dimension, so the comparisons (2.1) are equivalences. We first construct a diagram in integral G-theory from the Cartesian square

$$\begin{array}{ccc} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec}(B/eB) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(B/hB) & \longrightarrow & \operatorname{Spec} B \end{array}$$

using bounded complexes. For the regular ring B, every bounded complex of finite modules is represented by a bounded complex of finite projective modules. We use these complexes, with degreewise split monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences, to compute $G(B)$ by the resolution theorem [Wei13, V.3.3] and the Gillet–Waldhausen theorem [TT90, Theorem 1.11.7]. The same model computes $G(B/hB)$ over B/hB. Termwise tensor product with B/hB preserves these cofibrations and quasi-isomorphisms: the cone of a quasi-isomorphism between bounded projective complexes is bounded acyclic projective, hence contractible. For a bounded projective complex $P^\bullet$, there is a natural identification

$$(P^\bullet \otimes_B B/hB)[1/e] \simeq P^\bullet[1/e] \otimes_{B[1/e]} (B/hB)[1/e].$$

Consequently tensor product sends a complex acyclic after inverting e to one with the same property, and sends a map which is a quasi-isomorphism after inverting e to such a map. It therefore induces a map of Waldhausen localization sequences. The comparison with coherent localization and its supported terms is given by [TT90, Lemma 3.12 and Corollary 3.13] and [Wei13, V.6.1].

We identify the map on the closed terms. The sequence

$$0 \longrightarrow B \xrightarrow{h} B \longrightarrow B/hB \longrightarrow 0$$

is a free resolution. Tensoring it with B/eB gives $[B/eB \xrightarrow{h} B/eB]$. Since the image of h is a uniformizer of the DVR B/eB, this complex has cohomology L in degree zero and zero in every other degree. Thus

$$(B/eB) \otimes^{\mathbf{L}}_B B/hB \simeq L.$$

In particular, the square is Tor-independent. Its horizontal arrows have finite Tor dimension, as the displayed two-term resolutions show; see also [Sta26, Tag 068C]. For every bounded complex $M^\bullet$ of finite B/eB-modules, associativity of derived tensor product gives the natural identifications

$$\begin{aligned} M^\bullet \otimes^{\mathbf{L}}_B B/hB &\simeq M^\bullet \otimes^{\mathbf{L}}_{B/eB} ((B/eB) \otimes^{\mathbf{L}}_B B/hB) \\ &\simeq M^\bullet \otimes^{\mathbf{L}}_{B/eB} L. \end{aligned}$$

These identifications are compatible with regarding a B/eB-module as a coherent B-module supported on $V(e)$. Under dévissage, the induced map on the supported terms is therefore derived restriction $G(B/eB) \to G(L)$.

Since the image of e is a uniformizer of B/hB, one has $(B/hB)[1/e] = \mathrm{Frac}(B/hB)$. The resulting diagram is

$$\begin{array}{ccccc} G(B/eB) & \longrightarrow & G(B) & \longrightarrow & G(B[1/e]) \\ \downarrow & & \downarrow & & \downarrow \\ G(L) & \longrightarrow & G(B/hB) & \longrightarrow & G(\mathrm{Frac}(B/hB)). \end{array}$$

Its rows are instances of the coherent localization sequence (2.3). Smashing with $\mathbb{S}/N$ and taking connecting homomorphisms gives the equality of the two derived restrictions in $G_{m-1}(L;\mathbf{Z}/N)$. The Cartan equivalences for B/hB, $\mathrm{Frac}(B/hB)$, and L identify the lower boundary with the K-theory residue $\partial_{B/hB}$. Applying their inverses gives precisely the formula in the statement. □

3. Vanishing for Cartier divisors

Proposition 3.1. *Let B be a noetherian semilocal ring, let $g \in B$ be a nonzerodivisor, and let $N \geq 2$ be invertible in B. For every $q \in \mathbf{N}$, the composite*

$$K_q^{\mathrm{cn}}(B/gB;\mathbf{Z}/N) \longrightarrow G_q(B/gB;\mathbf{Z}/N) \longrightarrow G_q(B;\mathbf{Z}/N)$$

is zero. If B/gB is regular, the map $G_q(B/gB;\mathbf{Z}/N) \to G_q(B;\mathbf{Z}/N)$ is zero for every $q \in \mathbf{Z}$.

3.1. **Semilocal algebra and lifting coefficients.** We first record the algebraic facts used to prove Proposition 3.1.

Lemma 3.2. *Let B be a semilocal ring.*

(i) *For every ideal $I \subset B$, the homomorphism $B^\times \to (B/I)^\times$ is surjective.*

(ii) *Every invertible B-module is free of rank one. In particular, every invertible ideal of B is generated by a nonzerodivisor.*

Proof. For (i), choose a lift $a \in B$ of the prescribed unit. For every maximal ideal $\mathfrak{n}$ containing I, one has $a \notin \mathfrak{n}$. The ideal I and the finitely many maximal ideals not containing I are pairwise comaximal. The Chinese remainder theorem [Sta26, Tag 00DT] gives $b \in I$ such that $b \equiv 1 - a \pmod{\mathfrak{n}}$ at each of those maximal ideals. Then $a + b$ lifts the prescribed unit and lies in no maximal ideal of B, so $a + b \in B^\times$.

For (ii), let P be an invertible B-module. For each maximal ideal $\mathfrak{n}$, choose a basis of the one-dimensional $B/\mathfrak{n}$-vector space $P/\mathfrak{n}P$. Tensoring the Chinese-remainder isomorphism [Sta26, Tag 00DT] with P shows that $P \to \prod_{\mathfrak{n}} P/\mathfrak{n}P$ is surjective, where the product runs over the maximal ideals of B. Choose $p \in P$ having the prescribed images. The homomorphism $B \to P$, $b \mapsto bp$, becomes an isomorphism at every maximal ideal: $P_{\mathfrak{n}}$ is free of rank one, and the coordinate of p in a basis is a unit. Its kernel and cokernel therefore vanish, so $B \simeq P$; see also [Sta26, Tag 02M9]. If P is an ideal, the composite $B \simeq P \hookrightarrow B$ is multiplication by p and is injective. Thus p is a nonzerodivisor. □

Lemma 3.3 (Finite compactification)**.** *Let B be a ring and let C be a quasi-finite B-algebra of finite type. There are a finite B-algebra B' and an open immersion $j : \mathrm{Spec}\, C \hookrightarrow \mathrm{Spec}\, B'$ over $\mathrm{Spec}\, B$. Let A be a finite B-algebra equipped with a surjective B-algebra homomorphism $C \to A$. Then the composite $B' \to C \to A$ is surjective. If $\ker(C \to A)$ is an invertible C-module, then $\ker(B' \to A)$ is an invertible B'-module.*

Proof. The open immersion j is the affine form of Zariski's main theorem [Sta26, Tag 00QB]. The composite

$$\mathrm{Spec}\, A \hookrightarrow \mathrm{Spec}\, C \xrightarrow{j} \mathrm{Spec}\, B'$$

is an immersion. A finite set of B-module generators of A also generates A over B', because the B-action factors through B'. Hence $\mathrm{Spec}\, A \to \mathrm{Spec}\, B'$ is finite, and its image is closed [Sta26, Tag 01WM]. An immersion with closed image is a closed immersion [Sta26, Tag 01IQ]; equivalently, $B' \to A$ is surjective.

Let $J = \ker(B' \to A)$. Since $\operatorname{Spec} A \to \operatorname{Spec} B'$ factors through the open $\operatorname{Spec} C$, the canonical map $A \otimes_{B'} C \to A$ is an isomorphism. The B'-algebra C is flat, so tensoring $0 \to J \to B' \to A \to 0$ with C gives

$$0 \longrightarrow J \otimes_{B'} C \longrightarrow C \longrightarrow A \longrightarrow 0.$$

Consequently $JC = \ker(C \to A)$. The two opens $\operatorname{Spec} C$ and $\operatorname{Spec} B' \setminus \operatorname{Spec} A$ cover $\operatorname{Spec} B'$. On this cover the ideal sheaf of $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B'$ satisfies

$$\widetilde{J}|_{\operatorname{Spec} C} = \widetilde{\ker(C \to A)}, \qquad \widetilde{J}|_{\operatorname{Spec} B' \setminus \operatorname{Spec} A} = \mathcal{O}_{\operatorname{Spec} B' \setminus \operatorname{Spec} A}.$$

Both restrictions are invertible when $\ker(C \to A)$ is invertible. Thus $\widetilde{J}$ is invertible, as asserted. □

Lemma 3.4. *Let B be a commutative ring, let $I \subset B$ be an ideal, and let $N \geq 2$ be invertible in B. For every $q \in \mathbf{N}$ and every $c \in K_q^{\mathrm{cn}}(B/I; \mathbf{Z}/N)$, there are an étale B-algebra C and a class $c' \in K_q^{\mathrm{cn}}(C; \mathbf{Z}/N)$ such that the canonical map $B/I \to C/IC$ is an isomorphism and c' maps to c under*

$$K_q^{\mathrm{cn}}(C; \mathbf{Z}/N) \longrightarrow K_q^{\mathrm{cn}}(C/IC; \mathbf{Z}/N) \xrightarrow{\sim} K_q^{\mathrm{cn}}(B/I; \mathbf{Z}/N).$$

The last arrow is induced by the inverse of the canonical isomorphism $B/I \to C/IC$.

Proof. Let (B^h, IB^h) be the henselization of the pair (B, I). The canonical map $B/I \to B^h/IB^h$ is an isomorphism, and B^h is the filtered colimit of étale B-algebras C for which $B/I \to C/IC$ is an isomorphism; see the proof of [Sta26, Tag 0A02]. Gabber's rigidity theorem gives an equivalence

$$K^{\mathrm{cn}}(B^h; \mathbf{Z}/N) \xrightarrow{\sim} K^{\mathrm{cn}}(B/I; \mathbf{Z}/N)$$

by [CMM21, Theorem 1.4]; see also [Gab92]. Connective K-theory commutes with filtered colimits [Wei13, IV.6.4], as do smashing with $\mathbb{S}/N$ and homotopy groups. Consequently

$$\varinjlim_{C} K_q^{\mathrm{cn}}(C; \mathbf{Z}/N) \xrightarrow{\sim} K_q^{\mathrm{cn}}(B/I; \mathbf{Z}/N).$$

Represent c at one stage of this filtered colimit. That stage and its representative give C and c'. □

3.2. **Proof and localization consequences.**

Proof of Proposition 3.1. The case $B/gB = 0$ is immediate, so assume that the divisor $D = \operatorname{Spec}(B/gB)$ is nonempty, and write $i : D \hookrightarrow \operatorname{Spec} B$ for its closed immersion. Fix $c \in K_q^{\mathrm{cn}}(B/gB; \mathbf{Z}/N)$.

Apply Lemma 3.4 with $I = gB$. It gives an étale B-algebra C with canonical isomorphism $B/gB \xrightarrow{\sim} C/gC$ and a class $c' \in K_q^{\mathrm{cn}}(C; \mathbf{Z}/N)$ whose reduction corresponds to c.

A finite compactification. The étale B-algebra C is of finite type and quasi-finite [Sta26, Tag 00U5]. By Lemma 3.3, there are a finite B-algebra B' and an open immersion $\operatorname{Spec} C \hookrightarrow \operatorname{Spec} B'$ over $\operatorname{Spec} B$. Let $f : \operatorname{Spec} B' \to \operatorname{Spec} B$ be the finite structure morphism. The composite $B \to B' \to C \to C/gC \xrightarrow{\sim} B/gB$ is the quotient homomorphism; the last arrow is the inverse of the canonical isomorphism $B/gB \to C/gC$. Hence $B' \to B/gB$ is surjective. It defines a closed immersion $i' : D \hookrightarrow \operatorname{Spec} B'$ with $f \circ i' = i$. Let

$$J = \ker(B' \longrightarrow B/gB).$$

On $\operatorname{Spec} C$, the ideal sheaf $\widetilde{J}$ is the sheaf associated to $JC = gC$. Flatness of C over B preserves the exact sequence defining B/gB, giving

$$0 \longrightarrow C \xrightarrow{g} C \longrightarrow C/gC \longrightarrow 0.$$

Thus gC is free of rank one as a C-module; this is the flat-pullback criterion for an effective Cartier divisor [Sta26, Tag 02OO]. Applying the last assertion of Lemma 3.3 to the finite

B-algebra B/gB shows that J is invertible. The ring B' is semilocal because it is finite over B. By Lemma 3.2(ii), choose a nonzerodivisor $t \in B'$ with $J = tB'$.

Let $Z = V(I_Z)$ be the reduced closed complement of $\operatorname{Spec} C$ in $\operatorname{Spec} B'$, where $I_Z \subset B'$ is its radical ideal. Since $Z \cap D = \varnothing$, one has $J + I_Z = B'$ and hence $t \bmod I_Z \in (B'/I_Z)^\times$. By Lemma 3.2(i), choose $u \in (B')^\times$ with $u \bmod I_Z = (t \bmod I_Z)^{-1}$. Replacing t by ut preserves $J = tB'$ and gives

$$t \bmod I_Z = 1 \quad \text{in } B'/I_Z. \tag{3.1}$$

Extending a product across the closed complement. The closed subscheme Z lies in $\operatorname{Spec} B'[1/t]$, with open complement $\operatorname{Spec} C[1/t]$. The integral class $[t] \in K_1(B'[1/t])$ acts on the localization sequence for this closed immersion. Equations (2.4), (2.5), and (3.1) give

$$\begin{aligned}\partial_Z^{B'[1/t]}([t] \cdot \operatorname{Cartan}(c'|_{C[1/t]})) &= -[t \bmod I_Z] \cdot \partial_Z^{B'[1/t]} \operatorname{Cartan}(c'|_{C[1/t]}) \\ &= -[1] \cdot \partial_Z^{B'[1/t]} \operatorname{Cartan}(c'|_{C[1/t]}) = 0,\end{aligned}$$

since $[1] = 0$ in $K_1(Z)$. Exactness of (2.3) supplies

$$\delta \in G_{q+1}(B'[1/t]; \mathbf{Z}/N), \qquad \delta|_{C[1/t]} = [t] \cdot \operatorname{Cartan}(c'|_{C[1/t]}).$$

Computing its boundary at the divisor. The open subscheme $\operatorname{Spec} C \subset \operatorname{Spec} B'$ contains all of D. Denote the connecting homomorphisms for $D \hookrightarrow \operatorname{Spec} B'$ and $D \hookrightarrow \operatorname{Spec} C$ by $\partial_D^{B'}$ and ∂_D^C, respectively. Restriction along $\operatorname{Spec} C \hookrightarrow \operatorname{Spec} B'$ gives the commutative diagram

$$\begin{array}{ccccc} G_{q+1}(B'[1/t]; \mathbf{Z}/N) & \xrightarrow{\partial_D^{B'}} & G_q(D; \mathbf{Z}/N) & \xrightarrow{i'_*} & G_q(B'; \mathbf{Z}/N) \\ \downarrow & & \| & & \downarrow \\ G_{q+1}(C[1/t]; \mathbf{Z}/N) & \xrightarrow{\partial_D^C} & G_q(D; \mathbf{Z}/N) & \longrightarrow & G_q(C; \mathbf{Z}/N), \end{array}$$

whose rows are segments of (2.3). The outer vertical arrows are restriction, and the central vertical arrow is the identity. The image of t in C generates $JC = gC$. Applying Lemma 2.1 to C, t, and c' gives

$$\partial_D^{B'} \delta = \partial_D^C([t] \cdot \operatorname{Cartan}(c'|_{C[1/t]})) = \operatorname{Cartan}(c) \quad \text{in } G_q(D; \mathbf{Z}/N).$$

Here $C/tC = C/gC$ is identified with B/gB by the canonical isomorphism used to lift c. Functoriality of finite pushforward and $f \circ i' = i$ give

$$\begin{array}{ccc} G_q(D; \mathbf{Z}/N) & \xrightarrow{i'_*} & G_q(B'; \mathbf{Z}/N) \\ \| & & \downarrow f_* \\ G_q(D; \mathbf{Z}/N) & \xrightarrow{i_*} & G_q(B; \mathbf{Z}/N). \end{array}$$

Consequently, exactness of the top localization sequence yields

$$i_* \operatorname{Cartan}(c) = f_* i'_* \operatorname{Cartan}(c) = f_* i'_* \partial_D^{B'} \delta = 0.$$

If B/gB is regular, its connective perfect K-theory agrees with coherent G-theory in nonnegative degrees. Every class of $G_q(B/gB; \mathbf{Z}/N)$ is therefore covered by the argument for $q \in \mathbf{N}$. For $q < 0$ this group is zero by connectivity. □

Corollary 3.5. *Let B be a noetherian regular semilocal ring, let $g \in B$ be a nonzerodivisor such that B/gB is regular, and let $N \geq 2$ be a natural number invertible in B. For every $q \in \mathbf{Z}$, localization (2.3) gives a short exact sequence*

$$0 \longrightarrow K_q(B; \mathbf{Z}/N) \longrightarrow K_q(B[1/g]; \mathbf{Z}/N) \longrightarrow K_{q-1}(B/gB; \mathbf{Z}/N) \longrightarrow 0.$$

Proof. The noetherian regular semilocal ring B has finite dimension: its dimension is the maximum of the dimensions of its finitely many regular local rings at maximal ideals. Thus B, B/gB, and $B[1/g]$ are regular of finite dimension, so their coherent G-theories agree with their K-theories. By Proposition 3.1, the pushforwards

$$K_j(B/gB;\mathbf{Z}/N) \longrightarrow K_j(B;\mathbf{Z}/N)$$

are zero for $j = q$ and $j = q-1$. The long exact sequence of (2.3) for $\operatorname{Spec}(B/gB) \subset \operatorname{Spec} B$ gives the assertion. □

For discrete valuation rings, the following recovers Gillet's theorem [Gil86].

Corollary 3.6. *Let V be a noetherian discrete valuation ring with residue field l, and let $N \geq 2$ be invertible in V. Then $K_q(l;\mathbf{Z}/N) \to K_q(V;\mathbf{Z}/N)$ is zero for every $q \in \mathbf{Z}$. Consequently the localization sequence (2.3) gives*

$$0 \longrightarrow K_q(V;\mathbf{Z}/N) \longrightarrow K_q(\operatorname{Frac} V;\mathbf{Z}/N) \longrightarrow K_{q-1}(l;\mathbf{Z}/N) \longrightarrow 0$$

for every $q \in \mathbf{Z}$.

Proof. Apply Proposition 3.1 to a uniformizer of V. The residue field and V are regular, so their G-theories agree with their K-theories. The short exact sequences are also the DVR case of Corollary 3.5. □

4. Finite schemes and transverse curves

The construction of Lemma 5.2 requires both a finite scheme containing two transverse Cartier divisors and a third divisor carrying a prescribed finite-coefficient class. The first construction is completed in Lemma 4.6; the second, in Lemma 4.9. We give the algebraic preparations and geometric constructions in that order.

4.1. Parameters and auxiliary discrete valuation rings.

Lemma 4.1. *Let $(R,\mathfrak{m},k)$ be a regular local ring of dimension two, and let A be a finite regular R-algebra with s maximal ideals, whose maximal local rings all have dimension one. If $|k| > s$, there is a regular system of parameters x, y of R such that $xA = \mathfrak{m}A$. Moreover, x is a nonzerodivisor on A.*

Proof. Choose regular parameters a, b of R. For each maximal ideal $\mathfrak{a}_i$ of A, the ring $A_{\mathfrak{a}_i}$ is a DVR; write $L_i = A/\mathfrak{a}_i$, and choose a uniformizer π_i and its normalized valuation w_i, with $w_i(0) = +\infty$. For an element of R, its w_i-value means the value of its image under $R \to A_{\mathfrak{a}_i}$. Since A is finite over R, every $\mathfrak{a}_i$ lies over $\mathfrak{m}$. The algebra $A/\mathfrak{m}A$ is finite-dimensional over k, hence Artinian [Sta26, Tag 00J6]. Its localization

$$A_{\mathfrak{a}_i}/\mathfrak{m}A_{\mathfrak{a}_i} \simeq (A/\mathfrak{m}A)_{\mathfrak{a}_i}$$

is a local Artinian ring [Sta26, Tag 00JB]. Its maximal ideal is nilpotent [Sta26, Tag 00J8]; therefore every nonunit, and in particular every zero divisor, is nilpotent. This says that $\mathfrak{m}A_{\mathfrak{a}_i}$ is $\mathfrak{a}_iA_{\mathfrak{a}_i}$-primary. The ideal $\mathfrak{m}A_{\mathfrak{a}_i}$ is nonzero: if it were zero, the displayed local Artinian quotient would be $A_{\mathfrak{a}_i}$ itself, contradicting its dimension one. Since $\mathfrak{m} = (a,b)$, at least one of the images of a, b in $A_{\mathfrak{a}_i}$ is therefore nonzero. Both images lie in $\mathfrak{a}_iA_{\mathfrak{a}_i} = \pi_iA_{\mathfrak{a}_i}$. A nonzero element of $\pi_iA_{\mathfrak{a}_i}$ has positive integral valuation, while the zero element has valuation $+\infty$. Hence

$$d_i = \min\{w_i(a), w_i(b)\} \in \mathbf{N}^*, \qquad \mathfrak{m}A_{\mathfrak{a}_i} = (a,b)A_{\mathfrak{a}_i} = \pi_i^{d_i}A_{\mathfrak{a}_i}.$$

For the ideal equality, both generators are divisible by $\pi_i^{d_i}$, and at least one is a unit times $\pi_i^{d_i}$; see [Sta26, Tag 00PD].

The local homomorphism $R \to A_{\mathfrak{a}_i}$ induces the field embedding $k \hookrightarrow L_i$. Let $\bar{\lambda} \in k$ and let $\lambda, \lambda' \in R$ be two lifts. Since $\lambda - \lambda' \in \mathfrak{m}$ and $b \in \mathfrak{m}$, one has

$$(\lambda - \lambda')b \in \mathfrak{m}^2A_{\mathfrak{a}_i} = \pi_i^{2d_i}A_{\mathfrak{a}_i}, \qquad w_i((\lambda - \lambda')b) \geq 2d_i \geq d_i + 1.$$

Thus $(a+\lambda b)/\pi_i^{d_i}$ and $(a+\lambda' b)/\pi_i^{d_i}$ have the same image in L_i. For either lift, reduction modulo the maximal ideal gives

$$\overline{(a+\lambda b)/\pi_i^{d_i}} = \overline{a/\pi_i^{d_i}} + \bar{\lambda}\,\overline{b/\pi_i^{d_i}}.$$

The valuation of $a+\lambda b$ is d_i precisely when this element of L_i is nonzero.

At least one of the two coefficients on the right is nonzero. If $\overline{b/\pi_i^{d_i}} = 0$, the expression never vanishes. Otherwise vanishing is equivalent to

$$\bar{\lambda} = -\overline{a/\pi_i^{d_i}}\,\big(\overline{b/\pi_i^{d_i}}\big)^{-1} \quad \text{in } L_i.$$

The equation has a solution $\bar{\lambda} \in k$ only when its right-hand side belongs to the subfield $k \subset L_i$. Each of the s maximal ideals therefore excludes at most one element of k. In total at most s elements of k are excluded, and $s < |k|$ ensures that its complement in k is nonempty. Choose $\bar{\lambda}$ in that complement, choose a lift $\lambda \in R$, and let $x = a+\lambda b$, $y = b$. The change-of-generators matrix $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ is invertible over R. Hence x, y generate $\mathfrak{m}$ and form a regular system of parameters. For every i,

$$xA_{\mathfrak{a}_i} = \pi_i^{d_i} A_{\mathfrak{a}_i} = \mathfrak{m}A_{\mathfrak{a}_i}.$$

Equality after localization at every maximal ideal gives $xA = \mathfrak{m}A$. Multiplication by x is injective at each $A_{\mathfrak{a}_i}$, hence on A. □

Lemma 4.2. *Let V be a noetherian DVR with uniformizer π and residue field k. For every finite field extension L/k, there is a finite free local V-algebra H which is a DVR with uniformizer the image of π, together with a k-algebra isomorphism $H/\pi H \xrightarrow{\sim} L$.*

Proof. First suppose L/k is simple, and choose a k-algebra isomorphism $k[T]/(\bar{f}) \xrightarrow{\sim} L$ for a monic irreducible polynomial $\bar{f} \in k[T]$. Lift its coefficients to a monic polynomial $f \in V[T]$, and let $H = V[T]/(f)$. Division by a monic polynomial shows that H is free over V with basis $1, T, \dots, T^{\deg(f)-1}$, and reduction modulo π gives the required isomorphism $H/\pi H \simeq L$.

Every maximal ideal of H lies over (π), because H is finite over the local ring V. Since $H/\pi H$ is a field, H is local with maximal ideal πH. The injection $V \to H$ is integral, so $\dim H = \dim V = 1$ by [Sta26, Tag 00OK]. Its maximal ideal has one generator, hence H is regular local of dimension one and is a DVR [Sta26, Tag 00PD].

For a general finite extension, choose a tower of simple extensions from k to L and apply this construction at each step. Each resulting DVR is finite free over the preceding one, its uniformizer is still π, and its residue field is the next field in the tower. Composition gives the asserted finite free V-algebra and residue-field isomorphism. □

4.2. **Two transverse divisors on a finite scheme.**

Lemma 4.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $A \neq 0$ be a finite regular R-algebra whose maximal local rings have dimension one. Write its maximal ideals as $\mathfrak{a}_1, \dots, \mathfrak{a}_s$, and let $L_i = A/\mathfrak{a}_i$. Let $y \in \mathfrak{m}$, and let H_i be finite local R-algebras which are DVRs with uniformizer the image of y, equipped with k-algebra isomorphisms $H_i/yH_i \xrightarrow{\sim} L_i$. Let $H = \prod_i H_i$, and form the subring*

$$S = \{(a, (b_i)_i) \in A \times H : a \bmod \mathfrak{a}_i = b_i \bmod yH_i \text{ in } L_i \text{ for every } i\},$$

where the right-hand residues are viewed through the specified isomorphisms. Let $p_A : S \to A$ and $p_H : S \to H$ be the coordinate projections, and let

$$\rho : S \longrightarrow \prod_i L_i, \qquad (a, (b_i)_i) \longmapsto (a \bmod \mathfrak{a}_i)_i.$$

Then S is finite over R, both projections are surjective, and their kernels satisfy

$$I_A := \ker p_A = \{0\} \times \prod_i yH_i,$$
$$I_H := \ker p_H = \Big(\bigcap_i \mathfrak{a}_i\Big) \times \{0\},$$
$$I_A I_H = 0, \qquad I_A + I_H = \ker \rho.$$

The closed subschemes $\operatorname{Spec} A = V(I_A)$ *and* $\operatorname{Spec} H = V(I_H)$ *cover* $\operatorname{Spec} S$ *and have scheme-theoretic intersection* $\operatorname{Spec}(\prod_i L_i)$*. The maximal ideals of S are exactly*

$$\mathfrak{n}_i = \ker(S \xrightarrow{\rho} \prod_j L_j \longrightarrow L_i).$$

For each i, the projections induce an isomorphism $S_{\mathfrak{n}_i} \simeq A_{\mathfrak{a}_i} \times_{L_i} H_i$, and

$$\dim S_{\mathfrak{n}_i} = 1, \qquad \frac{\mathfrak{n}_i}{\mathfrak{n}_i^2} \simeq \frac{\mathfrak{a}_i A_{\mathfrak{a}_i}}{\mathfrak{a}_i^2 A_{\mathfrak{a}_i}} \oplus \frac{yH_i}{y^2 H_i}.$$

In particular, this cotangent space has dimension two over L_i.

Proof. To lift $a \in A$ along p_A, lift its ith residue to H_i for every i. To lift $(b_i)_i \in H$ along p_H, use the Chinese remainder theorem to choose $a \in A$ with the prescribed residues $b_i \bmod yH_i$ [Sta26, Tag 00DT]. This also proves that ρ is surjective. The kernel formulas follow from the coordinate projections. Multiplication is the multiplication of the subring $S \subset A \times H$; in particular, $(0,(b_i)_i)(a,0) = (0,0)$ for elements of I_A and I_H. Moreover, an element is in $\ker\rho$ precisely when its A-coordinate belongs to $\bigcap_i \mathfrak{a}_i$ and its H_i-coordinate belongs to yH_i for every i. Thus $I_A I_H = 0$ and $I_A + I_H = \ker\rho$.

The difference of the residue maps gives an exact sequence of R-modules

$$0 \longrightarrow S \longrightarrow A \oplus H \xrightarrow{(a,(b_i)_i)\mapsto(a \bmod \mathfrak{a}_i - b_i \bmod yH_i)_i} \prod_i L_i \longrightarrow 0.$$

Since R is noetherian and $A \oplus H$ is finite over R, S is finite over R. Every prime of S contains I_A or I_H, because their product is zero. This proves the asserted cover by closed subschemes. Their scheme-theoretic intersection has coordinate ring $S/(I_A + I_H) \simeq \prod_i L_i$. A maximal ideal containing I_A is the inverse image of some $\mathfrak{a}_i$; a maximal ideal containing I_H is the inverse image of $H_1 \times \cdots \times yH_i \times \cdots \times H_s$. Both inverse images are $\mathfrak{n}_i$, proving the list of maximal ideals.

Regard the displayed exact sequence as a sequence of S-modules through p_A, p_H, ρ, and localize at $\mathfrak{n}_i$. Localization is exact [Sta26, Tag 00CS]; the surjective projections identify the resulting sequence with

$$0 \longrightarrow S_{\mathfrak{n}_i} \longrightarrow A_{\mathfrak{a}_i} \oplus H_i \xrightarrow{(a,b)\mapsto \bar{a}-\bar{b}} L_i \longrightarrow 0.$$

Indeed, localizing the products H and $\prod_j L_j$ at the inverse image of the ith maximal ideal retains exactly their ith factors. The kernel of the last map is the ring $A_{\mathfrak{a}_i} \times_{L_i} H_i$, which gives the claimed ring isomorphism. Localization induces the canonical L_i-linear isomorphism

$$\frac{\mathfrak{n}_i}{\mathfrak{n}_i^2} \xrightarrow{\sim} \frac{\mathfrak{n}_i S_{\mathfrak{n}_i}}{\mathfrak{n}_i^2 S_{\mathfrak{n}_i}}, \qquad a + \mathfrak{n}_i^2 \longmapsto a/1 + \mathfrak{n}_i^2 S_{\mathfrak{n}_i}.$$

Indeed, $\mathfrak{n}_i/\mathfrak{n}_i^2$ is annihilated by $\mathfrak{n}_i$, so every element of $S \setminus \mathfrak{n}_i$ acts invertibly. Under the local fiber-product isomorphism, the maximal ideal of $S_{\mathfrak{n}_i}$ and its square are, respectively,

$$\mathfrak{a}_i A_{\mathfrak{a}_i} \oplus yH_i, \qquad \mathfrak{a}_i^2 A_{\mathfrak{a}_i} \oplus y^2 H_i.$$

The mixed products vanish, so quotienting gives the asserted cotangent-space isomorphism. Each summand is one-dimensional over L_i, since both local rings are DVRs.

Every chain of primes in $S_{\mathfrak{n}_i}$ lies in one of the two closed subsets corresponding to $A_{\mathfrak{a}_i}$ and H_i: its smallest prime contains one of the projection kernels, and hence so does every prime

in the chain. Both quotients have dimension one. They also provide chains of length one, so $\dim S_{\mathfrak{n}_i} = 1$. □

Lemma 4.4. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension two, and let $S \neq 0$ be a finite R-algebra with maximal ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_s$ and residue fields $L_i = S/\mathfrak{n}_i$. Suppose $\dim_{L_i}(\mathfrak{n}_i/\mathfrak{n}_i^2) = 2$ for every i. There are an affine scheme W of finite type over R and a closed immersion $j : \operatorname{Spec} S \hookrightarrow W$ such that $\mathcal{O}_{W,j(\mathfrak{n}_i)}$ is regular local of dimension two for every i.*

Proof. Choose $r \in \mathbf{N}$ and a surjective R-algebra homomorphism $\varphi : P = R[Z_1, \dots, Z_r] \to S$. Let $I = \ker \varphi$ and $\mathfrak{q}_i = \varphi^{-1}(\mathfrak{n}_i)$. Since S is finite over the local ring R, each $\mathfrak{n}_i$ lies over $\mathfrak{m}$ and L_i/k is finite. The quotient map $P \to P/\mathfrak{q}_i \simeq L_i$ therefore factors as

$$P \longrightarrow P/\mathfrak{m}P = k[Z_1, \dots, Z_r] \longrightarrow P/\mathfrak{q}_i \simeq L_i.$$

In particular, $\mathfrak{q}_i/\mathfrak{m}P$ is a maximal ideal, and

$$P_{\mathfrak{q}_i}/\mathfrak{m}P_{\mathfrak{q}_i} \simeq k[Z_1, \dots, Z_r]_{\mathfrak{q}_i/\mathfrak{m}P}$$

is regular local of dimension r [Sta26, Tag 00OP]. The map $R \to P_{\mathfrak{q}_i}$ is flat and local. The local dimension formula gives $\dim P_{\mathfrak{q}_i} = 2 + r$ [Sta26, Tag 00ON]. Localization identifies $\mathfrak{q}_i/\mathfrak{q}_i^2$ with $\mathfrak{q}_i P_{\mathfrak{q}_i}/\mathfrak{q}_i^2 P_{\mathfrak{q}_i}$ as L_i-vector spaces, since $\mathfrak{q}_i$ annihilates the former quotient. The right-exact sequence of cotangent spaces

$$(\mathfrak{m}/\mathfrak{m}^2) \otimes_k L_i \longrightarrow \mathfrak{q}_i/\mathfrak{q}_i^2 \longrightarrow \mathfrak{q}_i/(\mathfrak{q}_i^2 + \mathfrak{m}P) \longrightarrow 0$$

shows that the maximal ideal of $P_{\mathfrak{q}_i}$ has at most $2 + r$ generators: the first and last vector spaces have dimensions 2 and r. Consequently $P_{\mathfrak{q}_i}$ is regular local of dimension $2 + r$.

The surjection φ induces an exact sequence

$$0 \longrightarrow \frac{I + \mathfrak{q}_i^2}{\mathfrak{q}_i^2} \longrightarrow \frac{\mathfrak{q}_i}{\mathfrak{q}_i^2} \longrightarrow \frac{\mathfrak{n}_i}{\mathfrak{n}_i^2} \longrightarrow 0.$$

The last map sends the class of $a \in \mathfrak{q}_i$ to the class of $\varphi(a)$; the inverse image of $\mathfrak{n}_i^2$ is $I + \mathfrak{q}_i^2$. Its kernel thus has dimension r over L_i. Choose $f_{i,1}, \dots, f_{i,r} \in I$ whose classes form a basis of that kernel. The ideals $\mathfrak{q}_i^2$ are pairwise comaximal: a maximal ideal containing $\mathfrak{q}_i^2 + \mathfrak{q}_{i'}^2$ would contain both distinct maximal ideals $\mathfrak{q}_i$ and $\mathfrak{q}_{i'}$. The Chinese remainder theorem [Sta26, Tag 00DT] therefore gives elements $u_i \in P$ satisfying

$$u_i \equiv 1 \pmod{\mathfrak{q}_i^2}, \qquad u_i \equiv 0 \pmod{\mathfrak{q}_{i'}^2} \quad (i' \neq i).$$

For $1 \leq \nu \leq r$, let $f_\nu = \sum_{i=1}^s u_i f_{i,\nu} \in I$. Then $f_\nu \equiv f_{i,\nu} \pmod{\mathfrak{q}_i^2}$ for every i, ν. Thus the classes of $f_1, \dots, f_r$ form a basis of $(I + \mathfrak{q}_i^2)/\mathfrak{q}_i^2$ at each i.

Let $W = \operatorname{Spec}(P/(f_1, \dots, f_r))$. Since every f_ν belongs to I, φ factors through a surjection $P/(f_1, \dots, f_r) \to S$, which defines the closed immersion j. The point $z_i = j(\mathfrak{n}_i)$ corresponds to the maximal ideal $\mathfrak{q}_i/(f_1, \dots, f_r)$, and

$$\mathcal{O}_{W,z_i} \simeq P_{\mathfrak{q}_i}/(f_1, \dots, f_r)P_{\mathfrak{q}_i}.$$

The independent classes of $f_1, \dots, f_r$ extend to a basis of $\mathfrak{q}_i/\mathfrak{q}_i^2$. Nakayama's lemma [Sta26, Tag 00DV, (8)] therefore extends them to a minimal generating set of $\mathfrak{q}_i P_{\mathfrak{q}_i}$, which is a regular system of parameters. The displayed quotient is consequently regular local of dimension $(r + 2) - r = 2$ [Sta26, Tag 00NQ]. □

Lemma 4.5. *Use the notation and hypotheses of Lemma 4.3. Suppose in addition that R is regular local of dimension two, that x, y are regular parameters with $xA = \mathfrak{m}A$, and that $xH = 0$. Let $j : \operatorname{Spec} S \hookrightarrow W$ be a closed immersion into an affine scheme of finite type over R such that, for $z_i = j(\mathfrak{n}_i)$, each $\mathcal{O}_{W,z_i}$ is regular local of dimension two. Then the composite closed immersions $j_A : \operatorname{Spec} A \hookrightarrow W$ and $j_H : \operatorname{Spec} H \hookrightarrow W$ are effective Cartier divisors. Their scheme-theoretic intersection is $\operatorname{Spec}(\prod_i L_i)$, and their local equations at z_i form a regular system of parameters of $\mathcal{O}_{W,z_i}$. There is a principal open $W^\circ \subset W$ containing $j(\operatorname{Spec} S)$ for which $W^\circ \to \operatorname{Spec} R$ is quasi-finite.*

Proof. The two closed immersions have ideals

$$J_A = \ker(\Gamma(W, \mathcal{O}_W) \xrightarrow{j^\#} S \xrightarrow{p_A} A),$$
$$J_H = \ker(\Gamma(W, \mathcal{O}_W) \xrightarrow{j^\#} S \xrightarrow{p_H} H).$$

Localizing the corresponding quotient maps at z_i gives

$$0 \longrightarrow J_A\mathcal{O}_{W,z_i} \longrightarrow \mathcal{O}_{W,z_i} \longrightarrow A_{\mathfrak{a}_i} \longrightarrow 0,$$
$$0 \longrightarrow J_H\mathcal{O}_{W,z_i} \longrightarrow \mathcal{O}_{W,z_i} \longrightarrow H_i \longrightarrow 0.$$

Each quotient is regular local of dimension one. By [Sta26, Tag 00NR], there are nonzerodivisors $h_i, v_i \in \mathcal{O}_{W,z_i}$ such that

$$J_A\mathcal{O}_{W,z_i} = h_i\mathcal{O}_{W,z_i}, \qquad J_H\mathcal{O}_{W,z_i} = v_i\mathcal{O}_{W,z_i}.$$

The surjection $j^\#$ sends J_A, J_H onto I_A, I_H, respectively, and its kernel is contained in both ideals. Consequently

$$\Gamma(W, \mathcal{O}_W)/(J_A + J_H) \simeq S/(I_A + I_H) \simeq \prod_i L_i,$$
$$\mathcal{O}_{W,z_i}/(h_i, v_i) \simeq S_{\mathfrak{n}_i}/(I_A + I_H)S_{\mathfrak{n}_i} \simeq L_i.$$

The first line proves the intersection assertion. The second shows that h_i, v_i generate the maximal ideal of the regular local ring $\mathcal{O}_{W,z_i}$ of dimension two, and hence form a regular system of parameters.

For a point $w \in j_A(\operatorname{Spec} A)$, choose a maximal ideal $\mathfrak{a}_i$ containing the corresponding prime of A. Then w generalizes z_i, and the map $\mathcal{O}_{W,z_i} \to \mathcal{O}_{W,w}$ is a localization. Thus $J_A\mathcal{O}_{W,w} = h_i\mathcal{O}_{W,w}$, with h_i still a nonzerodivisor. Outside $j_A(\operatorname{Spec} A)$, the ideal J_A has unit stalks. Hence every stalk of its associated ideal sheaf is free of rank one. Since W is noetherian and J_A is finitely presented, this sheaf is invertible [Sta26, Tag 00NX]. The same argument applies to J_H, using the points of $\operatorname{Spec} H_i$ and their closed points z_i. This proves that the two closed subschemes are effective Cartier divisors [Sta26, Tag 01WR].

The equality $xA = \mathfrak{m}A$ identifies A/xA with the finite-dimensional k-algebra $A/\mathfrak{m}A$. The image of x in each DVR $A_{\mathfrak{a}_i}$ is nonzero: otherwise this DVR would be a localization of the Artinian ring A/xA. Therefore x is a nonzerodivisor on A. Since $y \in xA$, the equation $x(y/x) = y$ defines a unique element $y/x \in A$. The homomorphism $\Gamma(W, \mathcal{O}_W) \to A$ is surjective; choose $c \in \Gamma(W, \mathcal{O}_W)$ whose image is y/x. Then $y - cx \in J_A$, so for each i there is $u_i \in \mathcal{O}_{W,z_i}$ with

$$y - cx = h_iu_i.$$

Reduction under $\mathcal{O}_{W,z_i} \to H_i$ gives

$$y = (h_i \bmod v_i)(u_i \bmod v_i),$$

because $xH_i = 0$. The image $h_i \bmod v_i$ generates the maximal ideal of $H_i \simeq \mathcal{O}_{W,z_i}/v_i\mathcal{O}_{W,z_i}$, since h_i, v_i are regular parameters. Both that image and y are uniformizers of the DVR H_i. It follows that $u_i \bmod v_i \in H_i^\times$. The quotient map is local, so $u_i \in \mathcal{O}_{W,z_i}^\times$. Hence

$$\mathfrak{m}\mathcal{O}_{W,z_i} = (x, y)\mathcal{O}_{W,z_i} = (x, y - cx)\mathcal{O}_{W,z_i} = (x, h_i)\mathcal{O}_{W,z_i},$$

and the map to $A_{\mathfrak{a}_i}$ induces an isomorphism

$$\mathcal{O}_{W,z_i}/\mathfrak{m}\mathcal{O}_{W,z_i} \simeq (A/xA)_{\mathfrak{a}_i}.$$

This is a zero-dimensional local ring with residue field L_i, finite over k. The point z_i is closed in the closed fiber, and the displayed local ring has no other prime. Thus no other point of that fiber specializes to z_i, so $W \to \operatorname{Spec} R$ is quasi-finite at z_i [Sta26, Tag 01TH].

Let $Q \subset W$ be the locus where $W \to \operatorname{Spec} R$ is quasi-finite. It is open by [Sta26, Tag 01TI] and contains every z_i. Every prime of S lies in some maximal ideal $\mathfrak{n}_i$, so every point of $j(\operatorname{Spec} S)$ generalizes one of these z_i. Since Q is open, $j(\operatorname{Spec} S) \subset Q$. Let $J \subset \Gamma(W, \mathcal{O}_W)$ be the radical ideal defining $W \setminus Q$, and let $J_S = \ker j^\#$. The closed sets $V(J)$ and $V(J_S)$

are disjoint. If $J + J_S$ were proper, a maximal ideal containing it would give a point in their intersection. Therefore $J + J_S = \Gamma(W, \mathcal{O}_W)$. Choose $t \in J$ such that $1 - t \in J_S$ and let

$$W^\circ = \operatorname{Spec}(\Gamma(W, \mathcal{O}_W)[1/t]).$$

Since t restricts to 1 on $\operatorname{Spec} S$, this principal open contains $j(\operatorname{Spec} S)$; since $t \in J$, it is contained in Q. It is affine and of finite type over R, and its morphism to $\operatorname{Spec} R$ is quasi-finite at every point. Thus that morphism is quasi-finite. □

Lemmas 4.1 and 4.2 supply the parameters and auxiliary DVRs in Lemma 4.6 whenever the residue field has more elements than the finite algebra has maximal ideals.

Lemma 4.6. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension two, and let A be a finite regular R-algebra whose maximal local rings all have dimension one. Write its maximal ideals as $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ and their residue fields as $L_i = A/\mathfrak{a}_i$. Let x, y be regular parameters of R with $xA = \mathfrak{m}A$. For each i, let H_i be a finite local $R/(x)$-algebra which is a DVR with uniformizer the image of y, equipped with a k-algebra isomorphism $H_i/yH_i \xrightarrow{\sim} L_i$.*

There are a finite R-algebra B, nonzerodivisors $h, v \in B$, and R-algebra isomorphisms

$$B/hB \xrightarrow{\sim} A, \qquad B/vB \xrightarrow{\sim} \prod_{i=1}^{s} H_i, \qquad B/(h, v) \xrightarrow{\sim} \prod_{i=1}^{s} L_i,$$

compatible with the residue homomorphisms from A and $\prod_i H_i$ to $\prod_i L_i$. For $z_i = \ker(B \to L_i)$, the local ring B_{z_i} is regular of dimension two, with regular parameters h, v. At every other maximal ideal of B, both h and v are units.

Proof. If $A = 0$, take $B = R$ and $h = v = 1$. Suppose henceforth that $A \neq 0$. Form $H = \prod_i H_i$ and S as in Lemma 4.3, with their maps p_A, p_H, ρ and kernels I_A, I_H. Lemma 4.3 gives a finite R-algebra S whose cotangent spaces at all its maximal ideals $\mathfrak{n}_i$ have dimension two. Apply Lemma 4.4 to obtain $j : \operatorname{Spec} S \hookrightarrow W$ with $\mathcal{O}_{W, j(\mathfrak{n}_i)}$ regular local of dimension two. Since $xH = 0$, Lemma 4.5 gives an affine quasi-finite open $W^\circ \subset W$ containing $\operatorname{Spec} S$, on which $\operatorname{Spec} A$ and $\operatorname{Spec} H$ are effective Cartier divisors with the asserted local parameter systems.

By Lemma 3.3, there are a finite R-algebra B and an open immersion $W^\circ \hookrightarrow \operatorname{Spec} B$. Lemma 3.3 also shows that the composite immersion $\operatorname{Spec} S \hookrightarrow \operatorname{Spec} B$ is closed, since S is finite over R. Denote its surjective homomorphism by $\psi : B \to S$. The ideals of $\operatorname{Spec} A$ and $\operatorname{Spec} H$ in B are $\psi^{-1}(I_A)$ and $\psi^{-1}(I_H)$; by the Cartier assertion of Lemma 3.3, both ideals are invertible. Since B is finite over the local ring R, it is semilocal. By Lemma 3.2(ii), choose nonzerodivisors $h, v \in B$ such that

$$hB = \psi^{-1}(I_A), \qquad vB = \psi^{-1}(I_H).$$

The maps $p_A\psi, p_H\psi, \rho\psi$ then give compatible R-algebra isomorphisms

$$B/hB \simeq A, \qquad B/vB \simeq H, \qquad B/(h, v) \simeq S/(I_A + I_H) \simeq \prod_i L_i.$$

For $z_i = \psi^{-1}(\mathfrak{n}_i)$, the open immersion identifies B_{z_i} with $\mathcal{O}_{W, j(\mathfrak{n}_i)}$. It is therefore regular local of dimension two. The localizations of h, v generate the two Cartier ideals, whose local equations form a regular system of parameters by Lemma 4.5; hence so do h, v. A maximal ideal of B containing h corresponds under $B/hB \simeq A$ to some $\mathfrak{a}_i$, whose inverse image is z_i. Similarly, a maximal ideal containing v is one of the z_i. At every other maximal ideal both elements are therefore units. □

4.3. **A transverse curve carrying a prescribed class.**

Lemma 4.7 (Finite DVR quotients)**.** *Let B be a noetherian semilocal ring, and let z be a maximal ideal such that B_z is regular local of dimension two. Let A be a finite B-algebra which is a DVR with maximal ideal $\mathfrak{m}_A$. Assume that*

$$zA = \mathfrak{m}_A, \qquad B/z \xrightarrow{\sim} A/\mathfrak{m}_A$$

under the homomorphism induced by $B \to A$. Let $t \in z$ have nonzero image in zB_z/z^2B_z, and suppose that its image in A belongs to $\mathfrak{m}_A^2$. Then there is a nonzerodivisor $e \in B$ such that $B \to A$ induces an isomorphism $B/eB \xrightarrow{\sim} A$,

$$e \equiv t \pmod{z^2B_z},$$

and e is a unit in $B_\mathfrak{n}$ for every maximal ideal $\mathfrak{n} \neq z$ of B.

Proof. The residue-field hypothesis gives $(B \to A)^{-1}(\mathfrak{m}_A) = z$. Every element of $B \setminus z$ therefore maps to a unit of A, so the homomorphism $B \to A$ extends to $B_z \to A$. The canonical map $A \otimes_B B_z \to A$ is an isomorphism. Since A is finite over B_z and $B_z/zB_z \to A/zA$ is an isomorphism, Nakayama's lemma [Sta26, Tag 00DV, (6)] makes $B_z \to A$ surjective.

For a maximal ideal $\mathfrak{n} \neq z$ of B, choose $s \in \mathfrak{n} \setminus z$. Its image in A is a unit, so $\mathfrak{n}A = A$. The finite $B_\mathfrak{n}$-module $A \otimes_B B_\mathfrak{n}$ therefore vanishes by [Sta26, Tag 00DV, (2)]. The cokernel of $B \to A$ vanishes after localization at every maximal ideal of B, and hence $B \to A$ is surjective.

Let $J = \ker(B \to A)$. The quotient map induces an isomorphism

$$B_z/JB_z \xrightarrow{\sim} A. \tag{4.1}$$

By [Sta26, Tag 00NR], JB_z is generated by one regular parameter of B_z. For every maximal ideal $\mathfrak{n} \neq z$, the equality $A \otimes_B B_\mathfrak{n} = 0$ gives $JB_\mathfrak{n} = B_\mathfrak{n}$. The ideal J is finitely presented because B is noetherian. Its localization at every maximal ideal is free of rank one, so J is invertible by [Sta26, Tag 00NX, (5)]. Choose a nonzerodivisor $e_0 \in B$ such that $J = e_0B$, using Lemma 3.2(ii). The element $e_0/1$ is a regular parameter of B_z.

The surjection $B_z \to A$ inducing (4.1) maps zB_z onto $\mathfrak{m}_A$ and z^2B_z onto $\mathfrak{m}_A^2$. Its kernel is JB_z, so

$$(B_z \to A)^{-1}(\mathfrak{m}_A^2) = JB_z + z^2B_z.$$

It induces the exact sequence of B/z-vector spaces

$$0 \longrightarrow \frac{JB_z + z^2B_z}{z^2B_z} \longrightarrow \frac{zB_z}{z^2B_z} \longrightarrow \frac{\mathfrak{m}_A}{\mathfrak{m}_A^2} \longrightarrow 0,$$

where the last map sends the class of $a \in zB_z$ to the class of its image in A. The middle and right vector spaces have dimensions two and one, respectively. The left vector space therefore has dimension one. The nonzero class $t + z^2B_z$ belongs to this kernel by the assumption on the image of t in A. Both $t + z^2B_z$ and $e_0 + z^2B_z$ consequently generate the kernel. There is $\lambda \in (B/z)^\times$ such that

$$e_0 + z^2B_z = \lambda(t + z^2B_z).$$

By Lemma 3.2(i), choose $u \in B^\times$ with $u + z = \lambda^{-1}$, and define $e = ue_0$. Then

$$eB = J, \qquad e + z^2B_z = (u + z)\lambda(t + z^2B_z) = t + z^2B_z.$$

The element e is a nonzerodivisor. For every maximal ideal $\mathfrak{n} \neq z$, the equality $eB_\mathfrak{n} = B_\mathfrak{n}$ makes e a unit in $B_\mathfrak{n}$. □

Lemma 4.8 (Curves in an étale neighborhood)**.** *Let B be a noetherian semilocal ring, let z be a maximal ideal such that B_z is regular local of dimension two, and let $t \in z$ have nonzero image in zB_z/z^2B_z. Let C_0 be an étale B-algebra for which the canonical map $B/z \to C_0/zC_0$ is an isomorphism. There are a nonzerodivisor $e \in B$ and a B-algebra homomorphism $C_0 \to B/eB$ such that B/eB is a DVR with maximal ideal z/eB, $e \equiv t \pmod{z^2B_z}$, and*

$$\begin{array}{ccc} C_0 & \longrightarrow & B/eB \\ \downarrow & & \downarrow \\ C_0/zC_0 & \xrightarrow{\sim} & B/z \end{array}$$

commutes. The lower arrow is the inverse of the canonical map $B/z \to C_0/zC_0$. Moreover, e is a unit in $B_\mathfrak{n}$ for every maximal ideal $\mathfrak{n} \neq z$ of B.

Proof. Selecting and compactifying the neighborhood. For every maximal ideal $\mathfrak{n}$ of B, the fiber $C_0/\mathfrak{n}C_0$ is a finite product of finite separable extensions of $B/\mathfrak{n}$ [Sta26, Tag 00U3]. Since B is semilocal, there are finitely many maximal ideals of C_0 above maximal ideals of B, and the only one above z is zC_0. The Chinese remainder theorem [Sta26, Tag 00DT] gives $f \in C_0$ congruent to one modulo zC_0 and to zero modulo every maximal ideal of C_0 above a maximal ideal of B different from z. Let $C = C_0[1/f]$, with its localization homomorphism $C_0 \to C$. The induced maps

$$B/z \xrightarrow{\sim} C_0/zC_0 \xrightarrow{\sim} C/zC$$

are isomorphisms, since f has image one in C_0/zC_0. Localizing the other fibers at f gives

$$C/\mathfrak{n}C = 0 \quad \text{for every maximal ideal } \mathfrak{n} \neq z \text{ of } B. \tag{4.2}$$

The map $B \to C$ is étale [Sta26, Tag 00U2, (1) and (2)], hence quasi-finite and of finite type [Sta26, Tag 00U5]. By Lemma 3.3, there is a finite B-algebra B' and a B-morphism $\operatorname{Spec} C \hookrightarrow \operatorname{Spec} B'$ which is an open immersion.

Define

$$z' = \ker(B' \longrightarrow C \longrightarrow C/zC).$$

The composite is surjective because its restriction to B is the quotient onto B/z followed by $B/z \xrightarrow{\sim} C/zC$. Consequently z' is maximal, and

$$(B \to B')^{-1}(z') = z, \qquad (B' \to C)^{-1}(zC) = z'.$$

The induced residue homomorphisms are isomorphisms:

$$B/z \xrightarrow{\sim} B'/z' \xrightarrow{\sim} C/zC.$$

If a maximal ideal $\mathfrak{n}'$ of B' lies in $\operatorname{Spec} C$, let $\mathfrak{q} \subset C$ be the corresponding prime. Its contractions satisfy

$$(B' \to C)^{-1}(\mathfrak{q}) = \mathfrak{n}', \qquad (B \to C)^{-1}(\mathfrak{q}) = (B \to B')^{-1}(\mathfrak{n}').$$

The last ideal is maximal because B' is finite over B [Sta26, Tag 00GR]. By (4.2), this maximal ideal is z. Since $\operatorname{Spec}(C/zC)$ consists of the single point zC, one has $\mathfrak{q} = zC$ and $\mathfrak{n}' = z'$. Thus z' is the only maximal ideal of B' lying in $\operatorname{Spec} C$.

A regular divisor inside the neighborhood. The open immersion $\operatorname{Spec} C \hookrightarrow \operatorname{Spec} B'$ induces an isomorphism $B'_{z'} \xrightarrow{\sim} C_{zC}$. Hence $B_z \to B'_{z'}$ is essentially étale and flat [Sta26, Tag 00U2, (6)]. The ring $B'_{z'}$ is regular local of dimension two [Sta26, Tag 07QP, Tag 0AH0]. The open immersion and localization induce the isomorphisms

$$B'_{z'}/zB'_{z'} \xrightarrow{\sim} C_{zC}/zC_{zC} \xleftarrow{\sim} C/zC.$$

Thus $zB'_{z'} = z'B'_{z'}$, and the residue homomorphism

$$B_z/zB_z \xrightarrow{\sim} B'_{z'}/z'B'_{z'}$$

is induced by the isomorphism $B/z \to B'/z'$. Flatness of $B_z \to B'_{z'}$ gives

$$(zB_z/z^2B_z) \otimes_{B_z} B'_{z'} \xrightarrow{\sim} zB'_{z'}/z^2B'_{z'} = z'B'_{z'}/z'^2B'_{z'}.$$

Because zB_z annihilates zB_z/z^2B_z, the isomorphism $B_z/zB_z \xrightarrow{\sim} B'_{z'}/z'B'_{z'}$ identifies $(zB_z/z^2B_z) \otimes_{B_z} B'_{z'}$ with zB_z/z^2B_z. We obtain the canonical B/z-linear isomorphism

$$zB_z/z^2B_z \xrightarrow{\sim} z'B'_{z'}/z'^2B'_{z'}, \qquad a + z^2B_z \longmapsto a + z'^2B'_{z'}. \tag{4.3}$$

Here $a \in zB_z$, and the class on the right is formed after applying $B_z \to B'_{z'}$.

Let $I \subset B'$ be the radical ideal defining $\operatorname{Spec} B' \setminus \operatorname{Spec} C$. Since $I \not\subset z'$ and $\sqrt{z'^2} = z'$, the ideals I and z'^2 are comaximal. By the Chinese remainder theorem [Sta26, Tag 00DT], choose $e' \in B'$ so that

$$e' \equiv t \pmod{z'^2}, \qquad e' \equiv 1 \pmod{I}.$$

In particular, $e' \in z'$. The isomorphism (4.3) sends the nonzero class $t + z^2B_z$ to

$$t + z'^2B'_{z'} = e' + z'^2B'_{z'} \neq 0.$$

Choose $w \in z'B'_{z'}$ so that the classes of e' and w form a basis of $z'B'_{z'}/z'^2B'_{z'}$. Nakayama's lemma [Sta26, Tag 00DV, (8)] gives $z'B'_{z'} = (e', w)B'_{z'}$. By [Sta26, Tag 00NQ, Tag 00PD], $B'_{z'}/e'B'_{z'}$ is a DVR. Every maximal ideal of B' other than z' contains I, and hence does not contain e'. Therefore $B'/e'B'$ is local with maximal ideal $z'/e'B'$, and its localization homomorphism is an isomorphism

$$B'/e'B' \xrightarrow{\sim} B'_{z'}/e'B'_{z'}. \tag{4.4}$$

In particular, $B'/e'B'$ is a DVR. The relation $(e', I) = B'$ shows that $\operatorname{Spec}(B'/e'B') \subset \operatorname{Spec} C$. The closed immersion into this open subscheme gives a B'-algebra homomorphism $C \to B'/e'B'$.

Under (4.4), the equality $zB'_{z'} = z'B'_{z'}$ gives

$$z(B'/e'B') = z'/e'B', \qquad B/z \xrightarrow{\sim} (B'/e'B')/(z'/e'B') \xrightarrow{\sim} B'/z'.$$

The composite in this display is induced by $B \to B'$. The image of t in $B'/e'B'$ belongs to $(z'/e'B')^2$: under (4.4), this follows from $t - e' \in z'^2B'_{z'}$. Apply Lemma 4.7 to the finite B-algebra $B'/e'B'$ and the element t. It gives a nonzerodivisor e with the asserted congruence and unit properties, and a B-algebra isomorphism $B/eB \xrightarrow{\sim} B'/e'B'$. Define $C_0 \to B/eB$ as the composite

$$C_0 \longrightarrow C \longrightarrow B'/e'B' \xrightarrow{\sim} B/eB,$$

where the last arrow is the inverse of this isomorphism. Reduction to the residue fields gives the commutative diagram

$$\begin{array}{ccccc} C_0 & \longrightarrow & C & \longrightarrow & B'/e'B' \\ \downarrow & & \downarrow & & \downarrow \\ C_0/zC_0 & \xrightarrow{\sim} & C/zC & \xrightarrow{\sim} & B'/z'. \end{array}$$

The lower-right arrow is the inverse of the isomorphism $B'/z' \to C/zC$ induced by $B' \to C$: its composite with that isomorphism is the identity because $B' \to C \to B'/e'B'$ is the quotient homomorphism. The identification $B'/z' \simeq B/z$ therefore gives the commutative square in the statement. □

Lemma 4.9. *Let B be a noetherian semilocal ring, let z be a maximal ideal for which B_z is regular local of dimension two, and let $h, v \in B$ induce regular parameters of B_z. Let $L = B/z$, and let $N \geq 2$ be a natural number invertible in B. For every $q \in \mathbf{N}$ and $b \in K_q(L; \mathbf{Z}/N)$, there are a nonzerodivisor $e \in B$ and $b' \in K_q(B/eB; \mathbf{Z}/N)$ such that B/eB is a DVR with maximal ideal z/eB,*

$$e \equiv h - v \pmod{z^2B_z}, \qquad b' \longmapsto b \quad \text{under } K_q(B/eB; \mathbf{Z}/N) \longrightarrow K_q(B/z; \mathbf{Z}/N).$$

Moreover, e is a unit at every maximal ideal of B other than z.

Proof. Since L is a field, the comparison $K^{\mathrm{cn}}(L) \to K(L)$ is an equivalence. By Lemma 3.4, there are an étale B-algebra C_0 and a class $c_0 \in K_q^{\mathrm{cn}}(C_0; \mathbf{Z}/N)$ such that $B/z \to C_0/zC_0$ is an isomorphism and the composite

$$K_q^{\mathrm{cn}}(C_0; \mathbf{Z}/N) \longrightarrow K_q^{\mathrm{cn}}(C_0/zC_0; \mathbf{Z}/N) \xrightarrow{\sim} K_q(L; \mathbf{Z}/N)$$

sends c_0 to b. Since $(B \to B_z)^{-1}(zB_z) = z$, one has $h, v \in z$. The cosets $h + z^2B_z$ and $v + z^2B_z$ form a basis of zB_z/z^2B_z over L, so $h - v + z^2B_z \neq 0$. Apply Lemma 4.8 with $t = h - v$. It gives a nonzerodivisor e and a B-algebra homomorphism $C_0 \to B/eB$ with the required geometric properties. Let b' be the image of c_0 under

$$K_q^{\mathrm{cn}}(C_0; \mathbf{Z}/N) \longrightarrow K_q^{\mathrm{cn}}(B/eB; \mathbf{Z}/N) \xrightarrow{\sim} K_q(B/eB; \mathbf{Z}/N).$$

The last map is an isomorphism because B/eB is a DVR. The commutative square in Lemma 4.8 shows that the image of b' under $K_q(B/eB; \mathbf{Z}/N) \to K_q(B/z; \mathbf{Z}/N)$ is b. □

5. Cancellation of residues and vanishing for curves

5.1. **The transfer relation on the normalization.** Fix a regular local ring $(R, \mathfrak{m}, k)$ of dimension two and a natural number $N \geq 2$ invertible in R. Let $D \subset \operatorname{Spec} R$ be a reduced closed curve with finite normalization $\operatorname{Spec} A \to D$. The ring A is regular and semilocal: its local rings at closed points are one-dimensional noetherian normal local domains, hence DVRs [Sta26, Tag 00PD]. Write its maximal ideals as $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ and put $L_j = A/\mathfrak{a}_j$ for $1 \leq j \leq s$. All the closed points of $\operatorname{Spec} A$ lie over $\mathfrak{m}$.

Let $D^{(0)}$ denote the finite set of generic points of D. Every nonclosed point of the reduced local curve D is generic, and its local ring is its residue field. Normalization therefore identifies the open subschemes

$$D \setminus \{\mathfrak{m}\} \simeq \operatorname{Spec} A \setminus \{\mathfrak{a}_1, \dots, \mathfrak{a}_s\} \simeq \coprod_{\eta \in D^{(0)}} \operatorname{Spec} \kappa(\eta).$$

The generic restriction maps in coherent G-theory are

$$G_m(A; \mathbf{Z}/N) \longrightarrow \bigoplus_{\eta \in D^{(0)}} G_m(\kappa(\eta); \mathbf{Z}/N) \longleftarrow G_m(D; \mathbf{Z}/N).$$

Both arrows are induced by restriction to these open subschemes. We compose them with

$$\bigoplus_{\eta \in D^{(0)}} G_m(\kappa(\eta); \mathbf{Z}/N) \xrightarrow{\bigoplus_\eta \operatorname{Cartan}^{-1}} \bigoplus_{\eta \in D^{(0)}} K_m(\kappa(\eta); \mathbf{Z}/N),$$

using the identifications of (2.1).

The unique generic point of $\operatorname{Spec} A$ generalizing $\mathfrak{a}_i$ has residue field $\operatorname{Frac}(A_{\mathfrak{a}_i})$. For a tuple of generic classes, the restriction notation $|_{\operatorname{Frac}(A_{\mathfrak{a}_i})}$ means projection to this summand. The DVR residue

$$\partial_{A_{\mathfrak{a}_i}} : K_m(\operatorname{Frac}(A_{\mathfrak{a}_i}); \mathbf{Z}/N) \longrightarrow K_{m-1}(L_i; \mathbf{Z}/N)$$

is defined by the commutative square

$$\begin{array}{ccc} K_m(\operatorname{Frac}(A_{\mathfrak{a}_i}); \mathbf{Z}/N) & \xrightarrow{\text{Cartan}} & G_m(\operatorname{Frac}(A_{\mathfrak{a}_i}); \mathbf{Z}/N) \\ {\scriptstyle \partial_{A_{\mathfrak{a}_i}}}\downarrow & & \downarrow{\scriptstyle \partial^{A_{\mathfrak{a}_i}}_{\operatorname{Spec} L_i}} \\ K_{m-1}(L_i; \mathbf{Z}/N) & \xrightarrow{\text{Cartan}} & G_{m-1}(L_i; \mathbf{Z}/N). \end{array}$$

The right-hand map is the coherent connecting homomorphism for $\operatorname{Spec} L_i \subset \operatorname{Spec} A_{\mathfrak{a}_i}$, and both horizontal maps are isomorphisms.

Lemma 5.1 (Normalization residues)**.** *For every $m \in \mathbf{Z}$ and every $\theta \in G_m(D; \mathbf{Z}/N)$,*

$$\sum_{i=1}^{s} \operatorname{Tr}_{L_i/k}\left(\partial_{A_{\mathfrak{a}_i}}(\theta|_{\operatorname{Frac}(A_{\mathfrak{a}_i})})\right) = 0 \quad \text{in } K_{m-1}(k; \mathbf{Z}/N). \tag{5.1}$$

Proof. Apply (2.3) to D at its closed point and to $\operatorname{Spec} A$ at $\mathfrak{a}_1, \dots, \mathfrak{a}_s$. The latter set is the support of the full inverse image of $\mathfrak{m}$ under the finite normalization. Its reduced closed subscheme is $\operatorname{Spec}(\prod_i L_i)$. Dévissage identifies its G-theory with $\bigoplus_i G(L_i)$. Finite pushforward from this support to $\operatorname{Spec} k$ regards an L_i-vector space as a k-vector space. Consequently it commutes with Cartan as follows:

$$\operatorname{Cartan} \circ \operatorname{Tr}_{L_i/k} = (\operatorname{Spec} L_i \longrightarrow \operatorname{Spec} k)_* \circ \operatorname{Cartan}.$$

Finite pushforward and localization give the commutative diagram

$$\begin{array}{ccccc} G_m(A;\mathbf{Z}/N) & \longrightarrow & \bigoplus_{\eta\in D^{(0)}} G_m(\kappa(\eta);\mathbf{Z}/N) & \xrightarrow{\partial^A_{\mathrm{Spec}(\prod_i L_i)}} & \bigoplus_i G_{m-1}(L_i;\mathbf{Z}/N) \\ \downarrow & & \| & & \downarrow{\scriptstyle \sum_i (\mathrm{Spec}\, L_i\to \mathrm{Spec}\, k)_*} \\ G_m(D;\mathbf{Z}/N) & \longrightarrow & \bigoplus_{\eta\in D^{(0)}} G_m(\kappa(\eta);\mathbf{Z}/N) & \xrightarrow{\partial^D_{\mathrm{Spec}\, k}} & G_{m-1}(k;\mathbf{Z}/N). \end{array}$$

Both rows are segments of the coherent localization sequences [Wei13, V.4.1, V.6.11]. Localization at each $\mathfrak{a}_i$ identifies the ith component of $\partial^A_{\mathrm{Spec}(\prod_i L_i)}$ with $\partial^{A_{\mathfrak{a}_i}}_{\mathrm{Spec}\, L_i}$ after projection to $G_m(\mathrm{Frac}(A_{\mathfrak{a}_i});\mathbf{Z}/N)$. The preceding Cartan square for the DVR $A_{\mathfrak{a}_i}$ and compatibility of Cartan with the finite pushforwards therefore give

$$\begin{aligned} &\mathrm{Cartan}\left(\sum_i \mathrm{Tr}_{L_i/k}\left(\partial_{A_{\mathfrak{a}_i}}(\theta|_{\mathrm{Frac}(A_{\mathfrak{a}_i})})\right)\right) \\ &\quad = \partial^D_{\mathrm{Spec}\, k}((\mathrm{Cartan}(\theta|_{\kappa(\eta)}))_{\eta\in D^{(0)}}) \\ &\quad = 0. \end{aligned}$$

The last equality is exactness of the bottom row. The Cartan map for k is an isomorphism, which proves (5.1). □

5.2. **Cancellation of one normalization residue.**

Lemma 5.2. *Suppose that $|k| > s$. Choose regular parameters x, y of R with $xA = \mathfrak{m}A$ by Lemma 4.1. For each j, choose the finite free DVR H_j over $R/(x)$ with uniformizer y and residue field L_j supplied by Lemma 4.2. Choose the finite R-algebra B and global nonzerodivisors $h, v \in B$ of Lemma 4.6; thus*

$$B/hB \simeq A, \qquad B/vB \simeq \prod_{j=1}^s H_j, \qquad B/(h,v) \simeq \prod_{j=1}^s L_j.$$

These are R-algebra isomorphisms compatible with the residue maps. For $z_j = \ker(B \longrightarrow L_j)$, the local ring B_{z_j} is regular of dimension two with parameters h, v. At every other maximal ideal of B, both h and v are units. Let $U = \mathrm{Spec}\, R \setminus (D \cup V(x))$.

For $m \in \mathbf{N}^*$, $1 \le i \le s$, and $b \in K_{m-1}(L_i;\mathbf{Z}/N)$, there exists $\varepsilon_i \in K_{m+1}(U;\mathbf{Z}/N)$ with the following property. Let

$$\partial^R_{D\cup V(x)} : G_{m+1}(U;\mathbf{Z}/N) \longrightarrow G_m(D \cup V(x);\mathbf{Z}/N)$$

be the coherent localization boundary, and let

$$\alpha_i \in \bigoplus_{\eta\in D^{(0)}} K_m(\kappa(\eta);\mathbf{Z}/N), \qquad \beta_i \in K_m(\mathrm{Frac}(R/(x));\mathbf{Z}/N)$$

be the generic restrictions of $\partial^R_{D\cup V(x)}\,\mathrm{Cartan}(\varepsilon_i)$ on D and $V(x)$, respectively. For the residue

$$\partial_{R/(x)} : K_m(\mathrm{Frac}(R/(x));\mathbf{Z}/N) \longrightarrow K_{m-1}(k;\mathbf{Z}/N)$$

of the DVR $R/(x)$, one has

$$\partial_{A_{\mathfrak{a}_j}}(\alpha_i|_{\mathrm{Frac}(A_{\mathfrak{a}_j})}) = \begin{cases} b, & j = i, \\ 0, & j \ne i, \end{cases} \qquad \partial_{R/(x)}\beta_i = -\mathrm{Tr}_{L_i/k}(b).$$

Proof. The element x is a nonzerodivisor on A by Lemma 4.1. Thus $x \notin \mathfrak{p}$ for every height-one prime $\mathfrak{p}$ such that $V(\mathfrak{p})$ is an irreducible component of D. The regular local ring R is factorial [Sta26, Tag 0AG0]. The radical ideal of D is therefore generated by a product g of

generators of its distinct height-one primes, and $U = \operatorname{Spec} R[1/(gx)]$. Since $R/(x)$ is a DVR, x generates a prime ideal. The product gx has no repeated prime factor, so $(g) \cap (x) = (gx)$ and $D \cup V(x) = \operatorname{Spec}(R/(gx))$ is reduced.

A transverse curve with the prescribed coefficient. Apply Lemma 4.9 at z_i, with $q = m-1$. It gives a nonzerodivisor $e_i \in B$ and a class

$$b_i' \in K_{m-1}(B/e_iB; \mathbf{Z}/N), \qquad b_i'|_{L_i} = b,$$

such that B/e_iB is a DVR with maximal ideal z_i/e_iB,

$$e_i \equiv h - v \pmod{z_i^2 B_{z_i}},$$

and e_i is a unit at every maximal ideal of B other than z_i. By Proposition 3.1, the closed pushforward of $\operatorname{Cartan}(b_i')$ to $G_{m-1}(B; \mathbf{Z}/N)$ is zero. For the localization boundary

$$\partial^B_{V(e_i)} : G_m(B[1/e_i]; \mathbf{Z}/N) \longrightarrow G_{m-1}(B/e_iB; \mathbf{Z}/N),$$

exactness therefore supplies

$$\gamma_i \in G_m(B[1/e_i]; \mathbf{Z}/N), \qquad \partial^B_{V(e_i)}\gamma_i = \operatorname{Cartan}(b_i').$$

The local ring B/e_iB is canonically $B_{z_i}/e_iB_{z_i}$, since its only maximal ideal is z_i/e_iB. The vectors $h + z_i^2 B_{z_i}$ and $v + z_i^2 B_{z_i}$ form a basis of $z_iB_{z_i}/z_i^2B_{z_i}$. Its quotient by the line generated by the class of e_i is the cotangent space

$$\frac{z_iB_{z_i}}{e_iB_{z_i} + z_i^2B_{z_i}} \simeq \frac{z_iB_{z_i}/z_i^2B_{z_i}}{L_i \cdot (h - v + z_i^2B_{z_i})}.$$

The images of h and v in this quotient are equal and nonzero. Nakayama's lemma [Sta26, Tag 00DV, (8)] gives

$$h(B/e_iB) = v(B/e_iB) = z_i(B/e_iB).$$

Thus their images are uniformizers of B/e_iB, and $h/v \in (B/e_iB)^\times$. By Lemma 3.2(i), choose $u_i \in B^\times$ with image h/v in B/e_iB. Then

$$h/(u_iv) \in B[1/(hv)]^\times, \qquad (h/(u_iv))|_{(B/e_iB)[1/(hv)]} = 1.$$

Extension across the transverse curve. For every $q \in \mathbf{Z}$, the localization boundary is

$$\partial^{B[1/(hv)]}_{V(e_i)} : G_{q+1}(B[1/(e_ihv)]; \mathbf{Z}/N) \longrightarrow G_q((B/e_iB)[1/(hv)]; \mathbf{Z}/N).$$

Flat restriction of the preceding localization sequence identifies

$$\partial^{B[1/(hv)]}_{V(e_i)}(\gamma_i|_{B[1/(e_ihv)]}) = \operatorname{Cartan}(b_i'|_{(B/e_iB)[1/(hv)]}),$$

with $q = m-1$. The integral degree-one class $[h/(u_iv)]$, together with (2.4) and (2.5), gives

$$\begin{aligned}
&\partial^{B[1/(hv)]}_{V(e_i)}([h/(u_iv)] \cdot \gamma_i|_{B[1/(e_ihv)]}) \\
&\quad = -[(h/(u_iv))|_{(B/e_iB)[1/(hv)]}] \cdot \operatorname{Cartan}(b_i'|_{(B/e_iB)[1/(hv)]}) \\
&\quad = -[1] \cdot \operatorname{Cartan}(b_i'|_{(B/e_iB)[1/(hv)]}) = 0.
\end{aligned}$$

Since $[1] = 0$ in the integral K_1-group, exactness supplies

$$\widehat{\varepsilon}_i \in G_{m+1}(B[1/(hv)]; \mathbf{Z}/N), \qquad \widehat{\varepsilon}_i|_{B[1/(e_ihv)]} = [h/(u_iv)] \cdot \gamma_i|_{B[1/(e_ihv)]}.$$

Restrictions to the generic points. Every generic point ξ of $V(hv)$ belongs to exactly one of $V(h) \simeq \operatorname{Spec} A$ and $V(v) \simeq \coprod_j \operatorname{Spec} H_j$, since their intersection is the set of closed points $z_1, \dots, z_s$. The local ring B_ξ is a DVR: it is a localization of one of the regular local rings B_{z_j}, and its maximal ideal is generated by h or v, respectively. The element e_i is invertible in B_ξ. Choose j such that ξ generalizes z_j. If $j = i$, the pairs h, e_i and v, e_i are regular systems of parameters of B_{z_i}; hence e_i belongs to neither hB_{z_i} nor vB_{z_i}. If $j \neq i$, the element e_i is already a unit in B_{z_j}. In both cases it is a unit in B_ξ.

For such a point ξ, the notation $\gamma_i|_{\kappa(\xi)} \in K_m(\kappa(\xi);\mathbf{Z}/N)$ means the image of γ_i under the following composite:

$$\begin{array}{ccc} G_m(B[1/e_i];\mathbf{Z}/N) & \longrightarrow & G_m(B_\xi;\mathbf{Z}/N) \\ & & \big\downarrow{\scriptstyle -\otimes^{\mathbf{L}}_{B_\xi}\kappa(\xi)} \\ K_m(\kappa(\xi);\mathbf{Z}/N) & \xleftarrow[\text{Cartan}^{-1}]{} & G_m(\kappa(\xi);\mathbf{Z}/N) \end{array}$$

The first arrow is localization, and the right-hand arrow is defined because the residue field has a two-term free resolution over the DVR B_ξ. For a generic point of $V(h)$ we identify $\kappa(\xi)$ with its corresponding field $\kappa(\eta)$ for $\eta \in D^{(0)}$; for the generic point of $\operatorname{Spec} H_j$ it is $\operatorname{Frac} H_j$.

Let $\partial^B_{V(hv)}$ be the coherent boundary for $V(hv) \subset \operatorname{Spec} B$, and let ∂_{B_ξ} be the residue of the DVR B_ξ. Naturality of (2.3) for $\operatorname{Spec} B_\xi \to \operatorname{Spec} B$ gives the square

$$\begin{array}{ccc} G_{m+1}(B[1/(hv)];\mathbf{Z}/N) & \xrightarrow{\partial^B_{V(hv)}} & G_m(V(hv);\mathbf{Z}/N) \\ \big\downarrow & & \big\downarrow \\ K_{m+1}(\operatorname{Frac} B_\xi;\mathbf{Z}/N) & \xrightarrow{\partial_{B_\xi}} & K_m(\kappa(\xi);\mathbf{Z}/N). \end{array}$$

The vertical arrows are localization followed by inverse Cartan. On the lower open term, the restriction of $\widehat{\varepsilon}_i$ is the product of $[h/(u_iv)]$ with the generic restriction of $\text{Cartan}^{-1}(\gamma_i|_{B_\xi})$. Thus (2.6) gives

$$(\partial^B_{V(hv)}\widehat{\varepsilon}_i)|_{\kappa(\xi)} = \operatorname{ord}_{B_\xi}(h/(u_iv))\,\gamma_i|_{\kappa(\xi)}.$$

At a generic point of $V(h)$, the elements u_i, v are units and h is a uniformizer, so this valuation is one. At a generic point of $V(v)$, the elements u_i, h are units and v is a uniformizer, so the valuation is minus one. Consequently

$$\begin{aligned} (\partial^B_{V(hv)}\widehat{\varepsilon}_i)|_{\kappa(\eta)} &= \gamma_i|_{\kappa(\eta)} \qquad (\eta \in D^{(0)}), \\ (\partial^B_{V(hv)}\widehat{\varepsilon}_i)|_{\operatorname{Frac} H_j} &= -\gamma_i|_{\operatorname{Frac} H_j} \quad (1 \le j \le s). \end{aligned}$$

For each j, let

$$\partial_{H_j} : K_m(\operatorname{Frac} H_j;\mathbf{Z}/N) \longrightarrow K_{m-1}(L_j;\mathbf{Z}/N)$$

be the residue of the DVR H_j. Restrict $\partial^B_{V(e_i)}\gamma_i = \text{Cartan}(b_i')$ to B_{z_i}. Applying Lemma 2.2 first to h, e_i and then to v, e_i, and using $b_i'|_{L_i} = b$, gives

$$\partial_{A_{\mathfrak{a}_i}}(\gamma_i|_{\operatorname{Frac}(A_{\mathfrak{a}_i})}) = b, \qquad \partial_{H_i}(\gamma_i|_{\operatorname{Frac} H_i}) = b.$$

For $j \neq i$, the element e_i is a unit in B_{z_j}. The isomorphisms of the statement induce $B_{z_j}/hB_{z_j} \simeq A_{\mathfrak{a}_j}$ and $B_{z_j}/vB_{z_j} \simeq H_j$. Localization and the two derived pullbacks are the maps

$$\begin{aligned} G_m(B[1/e_i];\mathbf{Z}/N) &\longrightarrow G_m(B_{z_j};\mathbf{Z}/N), \\ G_m(B_{z_j};\mathbf{Z}/N) &\xrightarrow{-\otimes^{\mathbf{L}}_{B_{z_j}} A_{\mathfrak{a}_j}} G_m(A_{\mathfrak{a}_j};\mathbf{Z}/N), \\ G_m(B_{z_j};\mathbf{Z}/N) &\xrightarrow{-\otimes^{\mathbf{L}}_{B_{z_j}} H_j} G_m(H_j;\mathbf{Z}/N). \end{aligned}$$

The two quotient modules have two-term free resolutions because h, v are regular parameters of B_{z_j}. Associativity of derived tensor product identifies the generic restrictions of these two images of γ_i with $\gamma_i|_{\operatorname{Frac}(A_{\mathfrak{a}_j})}$ and $\gamma_i|_{\operatorname{Frac} H_j}$, respectively. Each of the following localization composites is zero:

$$\begin{aligned} G_m(A_{\mathfrak{a}_j};\mathbf{Z}/N) \longrightarrow K_m(\operatorname{Frac}(A_{\mathfrak{a}_j});\mathbf{Z}/N) &\xrightarrow{\partial_{A_{\mathfrak{a}_j}}} K_{m-1}(L_j;\mathbf{Z}/N), \\ G_m(H_j;\mathbf{Z}/N) \longrightarrow K_m(\operatorname{Frac} H_j;\mathbf{Z}/N) &\xrightarrow{\partial_{H_j}} K_{m-1}(L_j;\mathbf{Z}/N). \end{aligned}$$

The first arrows are generic restriction followed by inverse Cartan. It follows that

$$\partial_{A_{\mathfrak{a}_j}}(\gamma_i|_{\mathrm{Frac}(A_{\mathfrak{a}_j})}) = 0, \qquad \partial_{H_j}(\gamma_i|_{\mathrm{Frac}\,H_j}) = 0 \quad (j \neq i).$$

Finite pushforward. Let $f : \operatorname{Spec} B \to \operatorname{Spec} R$ be the finite morphism. The element g defining D maps to zero in $A = B/hB$, and x maps to zero in $B/vB = \prod_j H_j$. Hence

$$g \in hB, \qquad x \in vB, \qquad gx \in hvB.$$

This gives a closed immersion $V(hv) \hookrightarrow f^{-1}(D \cup V(x))$ and an open immersion $f^{-1}(U) \hookrightarrow \operatorname{Spec} B[1/(hv)]$. Inclusion of the closed supports, followed by finite pushforward, gives the commutative diagram of connecting homomorphisms

$$\begin{array}{ccc}
G_{m+1}(B[1/(hv)];\mathbf{Z}/N) & \xrightarrow{\partial^B_{V(hv)}} & G_m(V(hv);\mathbf{Z}/N) \\
\downarrow & & \downarrow \\
G_{m+1}(f^{-1}(U);\mathbf{Z}/N) & \xrightarrow{\partial^B_{f^{-1}(D\cup V(x))}} & G_m(f^{-1}(D\cup V(x));\mathbf{Z}/N) \\
{\scriptstyle (f|_{f^{-1}(U)})_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
G_{m+1}(U;\mathbf{Z}/N) & \xrightarrow{\partial^R_{D\cup V(x)}} & G_m(D\cup V(x);\mathbf{Z}/N).
\end{array}$$

The three horizontal arrows are the coherent boundaries for $V(hv) \subset \operatorname{Spec} B$, $f^{-1}(D \cup V(x)) \subset \operatorname{Spec} B$, and $D \cup V(x) \subset \operatorname{Spec} R$, respectively. The upper left arrow is restriction to the smaller open; the upper right arrow is pushforward along the displayed closed immersion. The lower vertical arrows are the exact restriction-of-scalars functors for the finite maps on the open and the full inverse-image support, respectively. In particular, the right-hand composite is pushforward along the finite map $V(hv) \to D \cup V(x)$. Since U is regular, define

$$\varepsilon_i = \mathrm{Cartan}^{-1}\left((f|_{f^{-1}(U)})_*(\widehat{\varepsilon}_i|_{f^{-1}(U)})\right) \quad \text{in } K_{m+1}(U;\mathbf{Z}/N).$$

The diagram proves

$$\partial^R_{D\cup V(x)}\,\mathrm{Cartan}(\varepsilon_i) = (V(hv) \longrightarrow D \cup V(x))_*(\partial^B_{V(hv)}\widehat{\varepsilon}_i).$$

The image in $G_m(f^{-1}(D \cup V(x));\mathbf{Z}/N)$ restricts to zero on $f^{-1}(D \cup V(x)) \setminus V(hv)$: the composite

$$\begin{aligned} G_m(V(hv);\mathbf{Z}/N) &\longrightarrow G_m(f^{-1}(D\cup V(x));\mathbf{Z}/N) \\ &\longrightarrow G_m(f^{-1}(D\cup V(x)) \setminus V(hv);\mathbf{Z}/N) \end{aligned}$$

is zero, because its composite functor on coherent modules is zero. In particular, the generic restriction is zero at every generic point of $f^{-1}(D \cup V(x))$ outside $V(hv)$.

We compute the generic restrictions of $\partial^R_{D\cup V(x)}\,\mathrm{Cartan}(\varepsilon_i)$. For a generic point λ of $D \cup V(x)$, the map $\operatorname{Spec}\kappa(\lambda) \to D \cup V(x)$ is flat: it is localization at a minimal prime of the reduced curve. Flat restriction commutes with finite pushforward, giving the commutative square of integral spectra

$$\begin{array}{ccc}
G(V(hv)) & \longrightarrow & G(D\cup V(x)) \\
\downarrow & & \downarrow \\
G(V(hv)\times_{D\cup V(x)}\operatorname{Spec}\kappa(\lambda)) & \longrightarrow & G(\kappa(\lambda)).
\end{array}$$

Its horizontal maps are finite pushforwards, and its vertical maps are flat restrictions. On a coherent module the two composites are restriction of scalars followed by localization, and localization followed by restriction of scalars; these are naturally isomorphic. Equivalently, this is the flat case of [TT90, Proposition 3.18]. Smashing with $\mathbb{S}/N$ and taking π_m gives the same square with finite coefficients.

If $\lambda \in D^{(0)}$, then x is nonzero in $\kappa(\lambda)$. Since $x \in vB$, the element v is invertible on the fiber. Its equation $hv = 0$ therefore reduces to $h = 0$, giving

$$\begin{aligned}(B/(hv)) \otimes_{R/(gx)} \kappa(\lambda) &\simeq A \otimes_{R/(g)} \kappa(\lambda)\\ &\simeq \kappa(\lambda).\end{aligned}$$

The last isomorphism is the generic-field identification for the normalization of D. If λ is the generic point of $V(x)$, then g is nonzero in $\kappa(\lambda)$, since $V(x)$ is not a component of D. Since $g \in hB$, the element h is invertible on the fiber. Its equation reduces to $v = 0$, and

$$\begin{aligned}(B/(hv)) \otimes_{R/(gx)} \operatorname{Frac}(R/(x)) &\simeq \Big(\prod_{j=1}^{s} H_j\Big) \otimes_{R/(x)} \operatorname{Frac}(R/(x))\\ &\simeq \prod_{j=1}^{s} \operatorname{Frac} H_j.\end{aligned}$$

The map $R/(x) \to H_j$ is injective: a nonzero kernel would contain a power of y, whereas y is a nonzero uniformizer in the domain H_j. Here $H_j \otimes_{R/(x)} \operatorname{Frac}(R/(x)) = H_j[1/y] = \operatorname{Frac} H_j$, because y is a uniformizer in both DVRs. Thus the generic restriction square and the previously computed boundary classes give

$$\alpha_i = \big(\gamma_i|_{\kappa(\eta)}\big)_{\eta \in D^{(0)}}, \qquad \beta_i = -\sum_{j=1}^{s} \operatorname{Tr}_{\operatorname{Frac} H_j / \operatorname{Frac}(R/(x))}\big(\gamma_i|_{\operatorname{Frac} H_j}\big).$$

All identifications of field G-groups with field K-groups here use inverse Cartan.

The inverse image of the closed point under $\coprod_j \operatorname{Spec} H_j \to \operatorname{Spec}(R/(x))$ is $\coprod_j \operatorname{Spec} L_j$, since y is a uniformizer in every H_j. Finite pushforward between the localization sequences therefore gives

$$\begin{array}{ccc}
\displaystyle\bigoplus_j K_m(\operatorname{Frac} H_j; \mathbf{Z}/N) & \xrightarrow{\sum_j \operatorname{Tr}_{\operatorname{Frac} H_j / \operatorname{Frac}(R/(x))}} & K_m(\operatorname{Frac}(R/(x)); \mathbf{Z}/N) \\
\Big\downarrow{\scriptstyle (\partial_{H_j})_j} & & \Big\downarrow{\scriptstyle \partial_{R/(x)}} \\
\displaystyle\bigoplus_j K_{m-1}(L_j; \mathbf{Z}/N) & \xrightarrow{\sum_j \operatorname{Tr}_{L_j/k}} & K_{m-1}(k; \mathbf{Z}/N).
\end{array}$$

It follows that

$$\partial_{R/(x)} \beta_i = -\sum_{j=1}^{s} \operatorname{Tr}_{L_j/k}\Big(\partial_{H_j}\big(\gamma_i|_{\operatorname{Frac} H_j}\big)\Big) = -\operatorname{Tr}_{L_i/k}(b).$$

The residues on A have the asserted values by the same generic identification and the computation above. □

5.3. **Vanishing for curves and generic injectivity.**

Proposition 5.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian regular local ring of dimension two. Let $N \geq 2$ be invertible in R, and let $D \subset \operatorname{Spec} R$ be a reduced closed curve with finite normalization. Then the pushforward*

$$G_m(D; \mathbf{Z}/N) \longrightarrow K_m(R; \mathbf{Z}/N)$$

is zero for every $m \in \mathbf{Z}$.

Proof. Let $\operatorname{Spec} A \to D$ be the finite normalization. The finite regular semilocal R-algebra A has maximal ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$; put $L_i = A/\mathfrak{a}_i$. Each $A_{\mathfrak{a}_i}$ is a DVR, and each L_i is a finite extension of k. We first assume $m \in \mathbf{N}^*$ and $|k| > s$. Choose x, y, the finite free DVRs

H_i over $R/(x)$, and the finite R-algebra B with global equations h, v as in Lemma 5.2; in particular,

$$B/hB \simeq A, \qquad B/vB \simeq \prod_i H_i, \qquad B/(h,v) \simeq \prod_i L_i.$$

Let $f : \operatorname{Spec} B \to \operatorname{Spec} R$ be the finite structure map, and let $U = \operatorname{Spec} R \setminus (D \cup V(x))$. Use the coherent boundary

$$\partial^R_{D\cup V(x)} : G_{m+1}(U; \mathbf{Z}/N) \longrightarrow G_m(D \cup V(x); \mathbf{Z}/N).$$

Cancellation of the residues. Fix $\theta \in G_m(D; \mathbf{Z}/N)$ and, for $1 \le i \le s$, let

$$b_i = \partial_{A_{\mathfrak{a}_i}}(\theta|_{\operatorname{Frac}(A_{\mathfrak{a}_i})}) \in K_{m-1}(L_i; \mathbf{Z}/N).$$

By Lemma 5.1,

$$\sum_{i=1}^{s} \operatorname{Tr}_{L_i/k}(b_i) = 0.$$

For each b_i, choose ε_i from Lemma 5.2, and define

$$\begin{aligned} \theta' &= (D \hookrightarrow D \cup V(x))_* \theta \\ &\quad - \sum_{i=1}^{s} \partial^R_{D\cup V(x)} \operatorname{Cartan}(\varepsilon_i) \quad \text{in } G_m(D \cup V(x); \mathbf{Z}/N). \end{aligned} \tag{5.2}$$

Exactness of (2.3) gives

$$(D \cup V(x) \hookrightarrow \operatorname{Spec} R)_* \partial^R_{D\cup V(x)} = 0.$$

Thus θ and θ' have the same image in $G_m(R; \mathbf{Z}/N)$, and hence in $K_m(R; \mathbf{Z}/N)$ under inverse Cartan. The generic restrictions of θ' satisfy

$$\partial_{A_{\mathfrak{a}_i}}(\theta'|_{\operatorname{Frac}(A_{\mathfrak{a}_i})}) = b_i - b_i = 0,$$

$$\partial_{R/(x)}(\theta'|_{\operatorname{Frac}(R/(x))}) = \sum_{i=1}^{s} \operatorname{Tr}_{L_i/k}(b_i) = 0.$$

For the second equality, the original class supported on D restricts to zero at the generic point of $V(x)$, and the ith correction has residue $-\operatorname{Tr}_{L_i/k}(b_i)$ by Lemma 5.2.

The sequence (2.3), applied to $\operatorname{Spec} A$ and the DVR $R/(x)$, therefore gives classes

$$a \in G_m(A; \mathbf{Z}/N), \qquad c \in K_m(R/(x); \mathbf{Z}/N)$$

satisfying

$$a|_{\kappa(\eta)} = \theta'|_{\kappa(\eta)} \quad (\eta \in D^{(0)}), \qquad c|_{\operatorname{Frac}(R/(x))} = \theta'|_{\operatorname{Frac}(R/(x))}.$$

The finite normalization induces the identity on the fields $\kappa(\eta)$, while the closed immersion of $V(x)$ induces the identity on $\operatorname{Frac}(R/(x))$. Their other generic contributions are zero, since the corresponding supports have disjoint generic points. Consequently, for every generic point λ of $D \cup V(x)$,

$$(\theta' - (\operatorname{Spec} A \longrightarrow D \cup V(x))_* a - (V(x) \hookrightarrow D \cup V(x))_* \operatorname{Cartan}(c))\,\big|_{\kappa(\lambda)} = 0.$$

The sequence (2.3), applied at the closed point of the reduced local curve $D \cup V(x)$, gives the exact sequence

$$G_m(k; \mathbf{Z}/N) \longrightarrow G_m(D \cup V(x); \mathbf{Z}/N) \longrightarrow \bigoplus_{\lambda \in (D\cup V(x))^{(0)}} K_m(\kappa(\lambda); \mathbf{Z}/N),$$

where $(D \cup V(x))^{(0)}$ is its set of generic points. Hence there is $\zeta \in K_m(k; \mathbf{Z}/N)$ such that

$$\begin{aligned} \theta' &- (\operatorname{Spec} A \longrightarrow D \cup V(x))_* a \\ &- (V(x) \hookrightarrow D \cup V(x))_* \operatorname{Cartan}(c) \\ &= (\operatorname{Spec} k \hookrightarrow D \cup V(x))_* \operatorname{Cartan}(\zeta). \end{aligned} \tag{5.3}$$

Vanishing of the remaining pushforwards. The factorization $\operatorname{Spec} A = V(h) \hookrightarrow \operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} R$ gives the commutative diagram

$$\begin{array}{ccccc} G_m(A;\mathbf{Z}/N) & \xrightarrow{0} & G_m(B;\mathbf{Z}/N) & \xrightarrow{f_*} & G_m(R;\mathbf{Z}/N) \\ & \searrow^{(\operatorname{Spec} A\to\operatorname{Spec} R)_*} & & & \downarrow{\scriptstyle \mathrm{Cartan}^{-1}} \\ & & & & K_m(R;\mathbf{Z}/N). \end{array}$$

The first horizontal arrow is closed pushforward from $V(h)$; it is zero by Proposition 3.1, since A is regular and B is semilocal. Proposition 3.1, applied to $x \in R$, gives

$$G_m(R/(x);\mathbf{Z}/N) \xrightarrow{(V(x)\hookrightarrow\operatorname{Spec} R)_*} G_m(R;\mathbf{Z}/N), \qquad (V(x) \hookrightarrow \operatorname{Spec} R)_* = 0.$$

The closed-point pushforward factors through this map in the commutative diagram

$$\begin{array}{ccccc} G_m(k;\mathbf{Z}/N) & \longrightarrow & G_m(R/(x);\mathbf{Z}/N) & & \\ & \searrow^{(\operatorname{Spec} k\hookrightarrow\operatorname{Spec} R)_*} & \downarrow{\scriptstyle 0} & & \\ & & G_m(R;\mathbf{Z}/N) & \xrightarrow{\mathrm{Cartan}^{-1}} & K_m(R;\mathbf{Z}/N). \end{array}$$

The upper horizontal arrow regards a finite-dimensional k-vector space as an $R/(x)$-module. Thus the images of a, $\mathrm{Cartan}(c)$, and $\mathrm{Cartan}(\zeta)$ in $G_m(R;\mathbf{Z}/N)$ vanish. Pushing (5.3) to R gives that the image of θ' is zero, and the equality of its image with that of θ proves the assertion in this case.

Finite residue fields. Suppose now that k is finite. Choose $d \in \mathbf{N}^*$ such that

$$d \equiv 1 \pmod{N \prod_i [L_i : k]}, \qquad |k|^d > s,$$

and let k'/k be the extension of degree d. Lift a monic irreducible polynomial defining k'/k to a monic polynomial over R, and let R' be the quotient of $R[T]$ by that polynomial. The ring R' is finite free of rank d over R. Its closed fiber is the field k', so R' is local. The derivative of the defining polynomial is invertible modulo its maximal ideal, because finite field extensions are separable; it is therefore a unit in R'. Thus R'/R is étale. In particular, R' is a regular local ring of dimension two.

Let $D' = D \times_R R'$ and $A' = A \otimes_R R'$. The curve D' is reduced because it is étale over D. The ring A' is finite over R' and regular because it is étale over A. Over the generic locus it agrees with D'. It is therefore the normalization of D'. For each i, the algebra $L_i \otimes_k k'$ is a field: finite fields of relatively prime extension degrees are linearly disjoint, and $\gcd(d, [L_i : k]) = 1$. It follows that A' has exactly s closed points. Since $|k'| > s$, the case already proved applies to $D' \subset \operatorname{Spec} R'$.

Let $\rho : \operatorname{Spec} R' \to \operatorname{Spec} R$ be the finite étale morphism, and let $\alpha \in K_m(R;\mathbf{Z}/N)$ be the image of θ. Flat base change gives the commutative square

$$\begin{array}{ccc} G_m(D;\mathbf{Z}/N) & \longrightarrow & K_m(R;\mathbf{Z}/N) \\ \downarrow & & \downarrow{\scriptstyle \rho^*} \\ G_m(D';\mathbf{Z}/N) & \xrightarrow{0} & K_m(R';\mathbf{Z}/N). \end{array}$$

The horizontal arrows are closed pushforwards followed by inverse Cartan, and the left vertical arrow is flat pullback. The bottom arrow is zero by the case already proved; thus $\rho^*\alpha = 0$. The projection formula [TT90, Proposition 3.17] gives

$$0 = \rho_*\rho^*\alpha = [R'] \cdot \alpha = d\alpha,$$

since the finite free R-module R' has class d in $K_0(R)$. As $\gcd(d, N) = 1$, multiplication by d is invertible on $K_m(R;\mathbf{Z}/N)$ by (2.2). Hence $\alpha = 0$.

Degree zero and negative degrees. The target $K_0(R;\mathbf{Z}/N)$ is $\mathbf{Z}/N$ by rank. The image of every coherent module supported on D has rank zero. As $G_0(D;\mathbf{Z}/N) = G_0(D)/N$, the pushforward in degree zero is zero. For $m < 0$ the source is zero by connectivity. This completes the proof. □

Corollary 5.4. *Let R be a noetherian regular local ring of dimension two whose reduced closed curves have finite normalization. Then $K_n(R;\mathbf{Z}/N) \to K_n(\operatorname{Frac} R;\mathbf{Z}/N)$ is injective for every natural number $N \geq 2$ invertible in R and every $n \in \mathbf{Z}$.*

Proof. For $n \in \mathbf{N}^*$, let α have zero image in the fraction field. The field is the filtered colimit of the rings $R[1/f]$ with $0 \neq f \in R$. Continuity [TT90, Theorem 7.2], followed by smashing with $\mathbb{S}/N$, gives one such f on which α vanishes. If f is a unit, α is already zero. Otherwise the reduced scheme $D = V(f)_{\mathrm{red}}$ is a closed curve: the minimal primes of a nonzero principal ideal in the regular local domain R have height one. The sequence (2.3), with dévissage for the reduction of $V(f)$, expresses $\mathrm{Cartan}(\alpha)$ as the pushforward of a class of $G_n(D;\mathbf{Z}/N)$. Its normalization is finite by hypothesis. Proposition 5.3 gives $\alpha = 0$. In degree zero the map is the identity on ranks modulo N, and in negative degrees both groups vanish. □

6. The full Gersten complex

Proof of Theorem 1.1. Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension two whose reduced closed curves have finite normalization, let $F = \operatorname{Frac} R$, and let $N \geq 2$ be a natural number invertible in R. We prove exactness of (1.1) for every $n \in \mathbf{Z}$ using the codimension filtration of coherent R-modules.

Let $\mathcal{M}^1(R)$ be the exact category of finitely generated torsion R-modules, with its usual exact sequences. We use its connective K-theory spectrum $K^{\mathrm{cn}}(\mathcal{M}^1(R))$ and its cofiber of multiplication by N, denoted $K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N)$.

Dévissage [Qui73, §5, Theorem 4 and Corollary 1] identifies the K-theory of the subcategory of finite-length modules with $K(k)$. The exact functor $M \mapsto (M_{\mathfrak{p}})_{\operatorname{ht}\mathfrak{p}=1}$ induces an equivalence from the Serre quotient of $\mathcal{M}^1(R)$ by its finite-length subcategory to the finite-support direct sum of the categories of finite-length $R_{\mathfrak{p}}$-modules, for $\operatorname{ht}\mathfrak{p} = 1$ [Qui73, §7]. Each such local category has K-theory $K(\kappa(\mathfrak{p}))$ by the same dévissage result [Qui73, §5, Theorem 4 and Corollary 1]. The direct sum has finite support: a finitely generated torsion module has only finitely many height-one primes in its support. Quillen localization [Qui73, §5, Theorem 5] identifies the homotopy fibers in nonnegative degrees. The relevant maps on integral degree-zero groups are surjective. For the quotient by finite-length modules, the map is

$$K_0^{\mathrm{cn}}(\mathcal{M}^1(R)) \longrightarrow \bigoplus_{\operatorname{ht}\mathfrak{q}=1} K_0(\kappa(\mathfrak{q})) = \bigoplus_{\operatorname{ht}\mathfrak{q}=1} \mathbf{Z}.$$

For each height-one prime $\mathfrak{p}$, the class of $R/\mathfrak{p}$ maps to its corresponding basis vector, since

$$\operatorname{length}_{R_{\mathfrak{q}}}((R/\mathfrak{p})_{\mathfrak{q}}) = \begin{cases} 1, & \mathfrak{q} = \mathfrak{p}, \\ 0, & \mathfrak{q} \neq \mathfrak{p} \end{cases} \qquad (\operatorname{ht}\mathfrak{q} = 1).$$

For the quotient by torsion modules, the rank maps identify $K_0(R) \to K_0(F)$ with the identity of $\mathbf{Z}$. The fibers of the corresponding maps of connective spectra are consequently connective: their possible degree-minus-one group is the cokernel of the displayed degree-zero map, and all lower groups vanish. Quillen localization [Qui73, §5, Theorem 5] therefore identifies the fibers as spectra, so the integral sequences are also cofiber sequences. All field spectra and $K(R)$ are connective by regularity. Smashing with $\mathbb{S}/N$ gives

$$\begin{aligned} K(k;\mathbf{Z}/N) \longrightarrow K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N) &\longrightarrow \bigoplus_{\operatorname{ht}\mathfrak{p}=1} K(\kappa(\mathfrak{p});\mathbf{Z}/N), \\ K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N) \longrightarrow K(R;\mathbf{Z}/N) &\longrightarrow K(F;\mathbf{Z}/N). \end{aligned} \tag{6.1}$$

These are precisely the two steps of the codimension filtration; see also [Wei13, V.9]. The second uses regularity to identify the coherent G-theory of R with $K(R)$.

Choose any regular system of parameters x, y of R. Restriction of scalars gives the factorization

$$K(k;\mathbf{Z}/N) \longrightarrow G(R/(x);\mathbf{Z}/N) \longrightarrow K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N). \tag{6.2}$$

The first functor regards a finite-dimensional k-vector space as a finite-length $R/(x)$-module; the second regards a finitely generated $R/(x)$-module as a torsion R-module. The first map is zero on every homotopy group by Corollary 3.6, applied to the DVR $R/(x)$ and its uniformizer y. Thus the first map in the first sequence of (6.1) is zero on all homotopy groups. The long exact sequence gives, for every n,

$$\begin{aligned} 0 \longrightarrow \pi_{n-1}K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N) \longrightarrow \bigoplus_{\operatorname{ht}\mathfrak{p}=1} K_{n-1}(\kappa(\mathfrak{p});\mathbf{Z}/N) \\ \longrightarrow K_{n-2}(k;\mathbf{Z}/N) \longrightarrow 0. \end{aligned} \tag{6.3}$$

By Corollary 5.4, generic restriction is injective in both degrees n and $n-1$. The second sequence of (6.1) therefore gives

$$0 \longrightarrow K_n(R;\mathbf{Z}/N) \longrightarrow K_n(F;\mathbf{Z}/N) \longrightarrow \pi_{n-1}K^{\mathrm{cn}}(\mathcal{M}^1(R);\mathbf{Z}/N) \longrightarrow 0. \tag{6.4}$$

Splice (6.4) with (6.3). The composite from $K_n(F;\mathbf{Z}/N)$ to the direct sum of the height-one field groups is the localization boundary followed by the projection to the first layer of the codimension filtration, hence the first Gersten differential. The last map in (6.3) is its second differential. The resulting exact sequence is (1.1). This argument applies to all integer degrees. In degree zero it reads $0 \to \mathbf{Z}/N \to \mathbf{Z}/N \to 0$; in negative degrees all its terms vanish. □

Remark 6.1. The first Gersten differential is the sum of the localization residues for the DVRs $R_{\mathfrak{p}}$. The second can be described using normalization. For a height-one prime $\mathfrak{p}$, normalize $R/\mathfrak{p}$ in $\kappa(\mathfrak{p})$. Its normalization is finite by hypothesis, and its closed local rings are DVRs. The component $K_{n-1}(\kappa(\mathfrak{p});\mathbf{Z}/N) \to K_{n-2}(k;\mathbf{Z}/N)$ is the sum of their residues followed by the finite residue-field transfers to k. Indeed, finite normalization induces the identity on $\kappa(\mathfrak{p})$. Finite pushforward and the functoriality of (2.3) identify the boundary for $R/\mathfrak{p}$ with the sum of the normalization residues followed by their residue-field transfers.

Corollary 6.2. *Let X be a noetherian regular scheme of dimension at most two. Assume that, for every $y \in X$, the normalization of every reduced closed curve in* $\operatorname{Spec}\mathcal{O}_{X,y}$ *is finite. In particular, this condition holds if X is Nagata. Let $N \geq 2$ be a natural number invertible on X. For every $n \in \mathbf{Z}$, let $\mathcal{K}_n(-;\mathbf{Z}/N)$ be the Zariski sheaf associated to $U \mapsto K_n(U;\mathbf{Z}/N)$. For a point $x \in X$, write $i_x : \operatorname{Spec}\kappa(x) \to X$ for the canonical morphism, and let $X^{(p)}$ denote the points of codimension p. In the direct images below, each group $K_j(\kappa(x);\mathbf{Z}/N)$ is regarded as the constant sheaf on* $\operatorname{Spec}\kappa(x)$*. The Gersten sheaf complex*

$$\begin{aligned} 0 \longrightarrow \mathcal{K}_n(-;\mathbf{Z}/N) \longrightarrow \bigoplus_{x\in X^{(0)}} (i_x)_*K_n(\kappa(x);\mathbf{Z}/N) \longrightarrow \bigoplus_{x\in X^{(1)}} (i_x)_*K_{n-1}(\kappa(x);\mathbf{Z}/N) \\ \longrightarrow \bigoplus_{x\in X^{(2)}} (i_x)_*K_{n-2}(\kappa(x);\mathbf{Z}/N) \longrightarrow 0 \end{aligned} \tag{6.5}$$

is a flasque resolution. Its augmentation is generic restriction, and its other arrows are the localization differentials of the codimension filtration.

Proof. If X is Nagata, its local rings and their quotient rings are Nagata, so the stated normalization condition holds [Sta26, Tag 032U, Tag 032T, Tag 035S]. Fix $y \in X$. Continuity [TT90, Theorem 7.2] identifies the stalk at y of $\mathcal{K}_n(-;\mathbf{Z}/N)$ with $K_n(\mathcal{O}_{X,y};\mathbf{Z}/N)$. The stalk of each of the remaining terms is the direct sum over its indicated points generalizing y. These are precisely the points of the corresponding codimension in $\operatorname{Spec}\mathcal{O}_{X,y}$. Compatibility of localization with restriction identifies the stalk complex with the Gersten complex of this

regular local ring. For dimension two it is exact by Theorem 1.1; for dimension one it is exact by Corollary 3.6; in dimension zero it is the identity on the K-theory of a field. The sheaf complex is therefore exact at every stalk.

Each direct image $(i_x)_*K_j(\kappa(x);\mathbf{Z}/N)$ has the given group on an open containing x and zero otherwise, and is flasque. Their displayed direct sums are also flasque on the noetherian scheme X. More explicitly, a section of such a sum on an open has finite support: sections are locally finite sums, and the open is quasi-compact. Each finite list of components on a smaller open extends to a larger open by the same values and zero in the remaining components. Thus every restriction map is surjective. This proves the assertion. □

Remark 6.3 (Passage to a henselization). The map to a henselization can kill finite-coefficient classes. For example, let $V = \mathbf{Q}[t]_{(t)}$ and let V^h be its henselization. Since $K_0(V) = \mathbf{Z}$ and $K_1(V) = V^\times$, (2.2) gives a canonical isomorphism

$$K_1(V;\mathbf{Z}/2) \simeq V^\times/(V^\times)^2.$$

The element $1+t$ is a unit of V, since its image modulo t is 1. In the DVR $\mathbf{Q}[t]_{(t+1)}$, however, $t+1$ generates the maximal ideal. Thus

$$\begin{aligned}\operatorname{ord}_{\mathbf{Q}[t]_{(t+1)}}(1+t) &= 1,\\ \operatorname{ord}_{\mathbf{Q}[t]_{(t+1)}}(r^2) &= 2\operatorname{ord}_{\mathbf{Q}[t]_{(t+1)}}(r) \in 2\mathbf{Z} \qquad (r \in \mathbf{Q}(t)^\times).\end{aligned}$$

Hence $1+t$ is not a square in $\mathbf{Q}(t)$, and its class in $K_1(V;\mathbf{Z}/2)$ is nonzero. On the other hand, $T^2-(1+t)$ reduces modulo t to T^2-1, whose derivative at the root 1 is $2 \in \mathbf{Q}^\times$. The defining lifting property of the henselian ring V^h gives

$$u \in (V^h)^\times, \qquad u \equiv 1 \pmod{tV^h}, \qquad u^2 = 1+t;$$

see [Sta26, Tag 04GF]. The class of $1+t$ consequently maps to zero in $K_1(V^h;\mathbf{Z}/2)$. Thus $K_1(V;\mathbf{Z}/2) \to K_1(V^h;\mathbf{Z}/2)$ has a nonzero kernel.

Remark 6.4 (A nonhenselian ramified surface). The ring

$$R = (\mathbf{Z}[x,y]/(3+x^2-y^3))_{(3,x,y)}$$

is an excellent regular local ring of dimension two, of mixed characteristic $(0,3)$, with maximal ideal $\mathfrak{m} = (x,y)R$. It is nonhenselian and receives no regular local homomorphism from a DVR. In particular, Theorem 1.1 applies for every natural number $N \geq 2$ prime to 3.

Proof. The ring $\mathbf{Z}[x,y]_{(3,x,y)}$ is regular local of dimension three. Its maximal ideal has the minimal generating set $3+x^2-y^3, x, y$, because $3 = (3+x^2-y^3) - x^2 + y^3$. Quotienting by the first regular parameter gives a regular local ring of dimension two with maximal ideal $(x,y)R$ and residue field $\mathbf{F}_3$ [Sta26, Tag 00NQ]. Excellence follows from excellence of $\mathbf{Z}$ and its stability under finite-type extensions and localization [Sta26, Tag 07QW, Tag 07QU].

The polynomial x^2-y^3 is irreducible in $\mathbf{F}_3(y)[x]$: the y-adic valuation of y^3 is three, whereas that of a square is even. Since x^2-y^3 is monic in x, division by it shows that $\mathbf{F}_3[x,y]/(x^2-y^3)$ is free over $\mathbf{F}_3[y]$ with basis $1, x$. Thus the natural map

$$\mathbf{F}_3[x,y]/(x^2-y^3) \longrightarrow \mathbf{F}_3(y)[x]/(x^2-y^3)$$

is injective. Its target is a field by the irreducibility just proved. Hence the following quotient is a one-dimensional local domain:

$$R/3R \simeq (\mathbf{F}_3[x,y]/(x^2-y^3))_{(x,y)}.$$

Its cotangent space has the basis given by the images of x, y, since $x^2-y^3 \in (x,y)^2$. Its dimension is two, so $R/3R$ is singular. The equality $\dim(R/3R) = 1 < \dim R$ shows that $3 \neq 0$ in R. Since R is a domain with residue field of characteristic 3, its characteristic is 0 or 3; the inequality $3 \neq 0$ excludes the latter. Thus R has mixed characteristic $(0,3)$, and $3 = y^3 - x^2 \in \mathfrak{m}^2$.

We prove nonhenselianity using $T^2-(1+x)\in R[T]$. Modulo $\mathfrak{m}$, this polynomial has the root 1, where its derivative is $2\in\mathbf{F}_3^\times$. Consider the one-dimensional $\mathbf{Q}$-algebra

$$C=\mathbf{Q}[x,y]/(3+x^2-y^3).$$

Division by $3+x^2-y^3$, which is monic in x, shows that $\mathbf{Z}[x,y]/(3+x^2-y^3)$ and C are free with basis $1,x$ over $\mathbf{Z}[y]$ and $\mathbf{Q}[y]$, respectively. Hence the natural localization maps give injections

$$\mathbf{Z}[x,y]/(3+x^2-y^3)\hookrightarrow C\hookrightarrow\mathbf{Q}(y)[x]/(3+x^2-y^3).$$

The rightmost ring is a field: y^3-3 cannot be a square in $\mathbf{Q}(y)$, since for nonzero $a,b\in\mathbf{Q}[y]$ the equation $a^2=(y^3-3)b^2$ would give $2\deg a=3+2\deg b$. Thus $\mathbf{Z}[x,y]/(3+x^2-y^3)$ is a domain. Both R and C are localizations of this domain, so $\operatorname{Frac}C=\operatorname{Frac}R$. Moreover,

$$C/(x+1)C\simeq\mathbf{Q}[y]/(y^3-4)$$

is a field. Indeed, a root $r\in\mathbf{Q}$ of this cubic would give $3\operatorname{ord}_2(r)=2$, where ord_2 is the normalized 2-adic valuation on $\mathbf{Q}^\times$, which takes integer values. Consequently $C_{(x+1)}$ is a one-dimensional noetherian local domain with maximal ideal generated by $1+x$. It is a DVR with uniformizer $1+x$ [Sta26, Tag 00PD], so

$$\operatorname{ord}_{C_{(x+1)}}(1+x)=1,\qquad \operatorname{ord}_{C_{(x+1)}}(a^2)=2\operatorname{ord}_{C_{(x+1)}}(a)\in 2\mathbf{Z}\quad(a\in(\operatorname{Frac}R)^\times).$$

Thus $1+x$ is not a square in $\operatorname{Frac}R$. The simple root 1 does not lift to R, proving that R is nonhenselian [Sta26, Tag 04GF].

Finally, suppose that $\varphi:V\to R$ is a regular local homomorphism from a DVR with uniformizer π. By definition, φ is flat with geometrically regular fibres [Sta26, Tag 07BZ]. Since it is local, it is faithfully flat, hence injective, and it induces an embedding

$$V/\pi V\hookrightarrow R/\mathfrak{m}=\mathbf{F}_3;$$

see [Sta26, Tag 00HR, Tag 05CK]. Therefore V has mixed characteristic $(0,3)$, and $3=u\pi^e$ for $u\in V^\times$ and $e\in\mathbf{N}^*$. If $e\geq 2$, then

$$0\neq\varphi(\pi)+3R\in R/3R,\qquad(\varphi(\pi)+3R)^e=0.$$

To verify the inequality, membership $\varphi(\pi)\in 3R=\varphi(\pi)^eR$ would give $1\in\varphi(\pi)^{e-1}R\subseteq\mathfrak{m}$ after cancellation in the domain R. This contradicts reducedness of $R/3R$. If $e=1$, then

$$R\otimes_V(V/\pi V)=R/\varphi(\pi)R=R/3R$$

is a regular fibre of φ, contradicting the singularity of $R/3R$ already proved. □

Remark 6.5 (The coefficient hypothesis). The invertibility hypothesis in Theorem 1.1 cannot be omitted in general. Let $\mathbf{Z}_3$ denote the ring of 3-adic integers. The ring

$$A=\mathbf{Z}_3[\![x,y]\!]/(3+x^2-y^3)$$

is a complete regular local domain of dimension two, hence excellent [Sta26, Tag 07QW], and

$$\ker\bigl(K_2(A;\mathbf{Z}/3)\longrightarrow K_2(\operatorname{Frac}A;\mathbf{Z}/3)\bigr)\neq 0$$

by [Fel26a, Theorem 1.1].

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AI use and exposition. AI tools assisted with mathematical discussion and with drafting and revising the manuscript. I have carefully audited the paper line by line and vouch for its mathematical correctness.

At my request, the exposition gives explicit hypotheses, maps, and intermediate calculations, and separates auxiliary arguments into lemmas. Proofs of standard facts are included where they make the dependencies easier to check. This level of detail is intended to facilitate independent verification and future formalization in Lean.

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