Irrational values of Dirichlet series with growing period

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Abstract

Let f be a nonzero rational-valued function of period D. We count the exponents $2 \le j \le s$ of prescribed parity for which $L_f(j) = \sum_{n\ge1} f(n)n^{-j}$ is irrational. For fixed $0 < \eta < 1$ our lower bound of order $\sqrt{s \log\log s/(D \log(sD))}$ has explicit constants and holds simultaneously for all $D \le s^{1-\eta}$ and all such f. For the real and imaginary parts of the quartic character modulo five we improve the previous lower-bound coefficient at even exponents by a factor of $2/\sqrt{3}$.

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1 The count when the period can change

Fix a positive integer D. Assign rational numbers to its residue classes, at least one nonzero, and extend them periodically to a function f. Its Dirichlet series is $L_f(x) = \sum_{n\ge1} f(n)n^{-x}$ for $x > 1$. For $\delta \in \{0, 1\}$ write

$$N_{f,\delta}(s) = \#\{j \in \mathbb{Z} : 2 \le j \le s,\ j \equiv \delta \pmod 2,\ L_f(j) \notin \mathbb{Q}\}.$$

The fixed-period bound [13, Theorem 1.1] holds for all coefficients once s is sufficiently large in terms of D. When D varies with s, we need one starting cutoff for a whole range of periods. Here D can be any period of f, and all logarithms are natural.

Writing φ for Euler's totient function, set

$$C_D = \prod_{p\mid D}\left(1 + \frac{1}{p(p-1)}\right), \qquad K_D = \frac{C_D}{\varphi(D)}, \qquad R_D = \prod_{p\mid D}\left(1 + \frac{1}{p}\right). \tag{1.1}$$

Theorem 1.1. *For every $0 < \eta, \varepsilon < 1$ there is $s_0(\eta, \varepsilon)$ such that, for every integer $s \ge s_0$, every $D \le s^{1-\eta}$, every nonzero rational-valued function f of period D, and $\delta = 0, 1$,*

$$N_{f,\delta}(s) \ \ge\ (1-\varepsilon)\, 2\sqrt{\frac{K_D}{R_D}}\,\sqrt{\frac{s \log\log s}{\log(sD)}}. \tag{1.2}$$

If f vanishes on the nonunits modulo D, one may replace $2\sqrt{K_D/R_D}$ by $2\sqrt{K_D}$.

There is also a bound independent of the factorization of D. Let $\zeta(x) = \sum_{n\ge1} n^{-x}$ for $x > 1$ be the Riemann zeta function and set $\kappa = C_1 = \zeta(2)\zeta(3)/\zeta(6)$. Since $(p^2 - p + 1)(p + 1) = p^3 + 1$ and $\prod_p(1 + p^{-3}) = \zeta(3)/\zeta(6)$,

$$DK_D/R_D = \kappa \prod_{p\mid D}(1 + p^{-3})^{-1} \ \ge\ \zeta(2).$$

Consequently every profile in Theorem 1.1 satisfies

$$N_{f,\delta}(s) \ \ge\ (1-\varepsilon)\,\pi\sqrt{\frac{2}{3}}\,\sqrt{\frac{s \log\log s}{D \log(sD)}}. \tag{1.3}$$

If $\log D = o(\log s)$, the square root in (1.2) has the fixed-period scale $\sqrt{s \log\log s/\log s}$. For D of order s^θ the logarithm costs a factor tending to $1/\sqrt{1+\theta}$. Even at $D = \lfloor s^{1-\eta} \rfloor$, (1.3) with $\varepsilon = \frac12$ gives $\gg_\eta s^{\eta/2}\sqrt{\log\log s/\log s}$ irrational exponents of each parity. The coefficients and their denominators may change with s.

Why two counts enter. We use grids of fractions j/q with $D \mid q$ and $\varphi(q) \le B$. Too few irrational exponents would let a linear combination of the grids eliminate every irrational value. Clearing denominators gives integer determinants, at least one nonzero. To show its absolute value is less than one, we count the different fractions sampled. Take $D = 4$. The 8- and 12-grids both have totient 4, so each supplies one elimination weight at $B = 4$. Assign each fraction to its smallest supporting grid. The 8-grid receives four fractions, all of reduced denominator 8. The 12-grid receives eight, of reduced denominators $3, 6, 12$. We carry the fraction count into the determinant estimate. A coarse bound of order B^2 would lose the range of large periods. For sparse coefficients even the numerator matters, as the quartic character illustrates in Section 6.

Earlier work. Fischler's theorem for periodic coefficients [4, Theorem 2] allows one to count values outside a fixed finite-dimensional rational vector space, with a cutoff depending on the period. For real characters of conductor $T \ge 2$, Kumar and Tosi [7, Theorem 1] state a lower bound of order $\sqrt{s/\log s}$ at exponents of opposite parity.[1] Mistiaen [9, Théorème 1.0.5] gives a dimension bound in that parity for a character of modulus $N \ge 3$ with $4 \nmid N$. In the determinant approach, Fischler, Lai, Sprang and Zudilin [5] combined the elimination method of Fischler, Sprang and Zudilin [6], the inverse-totient grids of Lai and Yu [8], and the positive Hankel determinant used by Fauzan [3]. Their nonvanishing theorem permits signed measures. The reflected-tail construction for even beta values [12] and its periodic extension [13] use this additional freedom. We use the arithmetic normalization and the largest-grid witness of [13]. The new steps are the uniform weighted grid count and a size estimate in terms of the actual number of fractions, including a separate cost for each retained numerator.

2 Counting grids and fractions uniformly

Writing $q = Dr$ isolates the effect of an increasing period. The ratio $\varphi(pD)/\varphi(D)$ is p if $p \mid D$ and $p-1$ if $p \nmid D$. Thus each prime divisor of r new to D contributes one factor $1-1/p$ to $\varphi(Dr)/(r\varphi(D))$. Accordingly define

$$h_D(r) = \prod_{\substack{p\mid r\\ p\nmid D}} (1-1/p)^{-1}, \qquad \lambda_D(r) = \frac{r}{h_D(r)} = \frac{\varphi(Dr)}{\varphi(D)}, \qquad v_D(r) = \prod_{\substack{p\mid r\\ p\mid D}} (1-1/p). \tag{2.1}$$

The grid cutoff is $\lambda_D(r) \le x := B/\varphi(D)$. A reduced denominator b first appears on the grid $\operatorname{lcm}(b, D)$. Summing $\varphi(b)$ over the denominators first appearing at Dr gives $D\lambda_D(r)v_D(r)$ fractions, including the endpoint for $b = 1$ (Section 2.1). We therefore need a weighted first moment as well as the grid count.

Lemma 2.1. *For $w_D = 1$ or $w_D = v_D$, put $V_D(x; w) = \sum_{\lambda_D(r)\le x} w_D(r)$, where $r \ge 1$. As $x \to \infty$, uniformly over all positive integers D,*

$$V_D(x; w) = x(c_D(w) + o(1)), \tag{2.2}$$

$$\sum_{\lambda_D(r)\le x} \lambda_D(r) w_D(r) = \frac{x^2}{2}(c_D(w) + o(1)), \tag{2.3}$$

$$c_D(1) = C_D, \qquad c_D(v) = C_D \prod_{p\mid D} (1-p^{-2}).$$

Both constants lie between $1/\zeta(2)$ and κ, so the errors are also uniform in relative terms.

[1] The proof of [7, Theorem 1] in arXiv:2610.41458v1 has a gap in its nonvanishing argument. In Lemma 18 (pp. 12–13), take $a = b = 1$ and $\ell = h = y = 0$. For every sufficiently large admissible n, the factor at $j = rn - 1$ is t, so $K_0(0) = 0$. The empty product gives $g_{0,1,1}(0) = 1$, so the lemma would assign a finite p_n-adic valuation to zero. Our proof uses [5, Proposition 5.4].

Proof. For $D = 1$, Erdős credits the linear asymptotic to joint work with Turán [2, p. 540] and controls the tail by a second moment of $r/\varphi(r)$ [2, pp. 543–544]. Dressler [1, Theorem 1.1] computes κ. For varying D and both weights, we truncate at a fixed prime cutoff y. Both products then have one common period, regardless of D. We first bound the truncation error.

Write $h(r) = r/\varphi(r)$. Expanding $h(r)^2$ over its squarefree divisors and using $\lfloor T/d \rfloor \le T/d$ gives

$$\sum_{r \le T} h(r)^2 \le JT, \qquad J = \prod_p \left(1 + \frac{(1-1/p)^{-2} - 1}{p}\right) < \infty.$$

The coefficients are nonnegative and the factors of J are $1 + O(p^{-2})$. For $R \ge 2$ and $x \ge 1$ this implies

$$\#\{r > Rx : r \le x h(r)\} \ll x/R. \tag{2.4}$$

Indeed on $2^k Rx < r \le 2^{k+1} Rx$ we need $h(r) > 2^k R$. Summing the second-moment bounds $2Jx/(2^k R)$ proves (2.4).

A subscript y will mean that only primes $p \le y$ are kept. Writing μ for the Möbius function, the expansion $h(r) = \sum_{d \mid r} \mu(d)^2/\varphi(d)$ gives

$$\frac{1}{T} \sum_{r \le T} (h_D(r) - h_{D,y}(r)) \le e_y, \qquad e_y = \sum_{\substack{d \ge 1 \\ \exists p \mid d,\ p > y}} \frac{\mu(d)^2}{d\varphi(d)} \longrightarrow 0.$$

The full series converges to κ. Also $0 \le v_{D,y}(r) - v_D(r) \le \sum_{p \mid r,\, p > y} 1/p$. Its mean over $r \le T$ is at most $u_y = \sum_{p > y} p^{-2}$. These bounds hold for all D.

Let $P_y = \prod_{p \le y} p$. Both truncated products are periodic modulo P_y. In a residue class a the cutoff is now the ordinary interval $r \le x h_{D,y}(a)$. Summing over classes gives

$$\sum_{r \le x h_{D,y}(r)} w_{D,y}(r) = x M_{D,y} + O(P_y), \qquad M_{D,y} = \frac{1}{P_y} \sum_{a \bmod P_y} h_{D,y}(a) w_{D,y}(a).$$

By the Chinese remainder theorem the mean factors over primes. For $p \nmid D$ its factor is $1 + 1/[p(p-1)]$. For $p \mid D$ it is 1 at weight 1, and $1 - p^{-2}$ at weight v_D. Thus $|M_{D,y} - c_D(w)| \ll u_y$ uniformly.

To restore the full cutoff, discard $r > Rx$ from both counts by (2.4). Restoring the weights below Rx costs at most Rxu_y. Given $\tau > 0$, at most Rxe_y/τ integers there have $h_D - h_{D,y} > \tau$. Every other change of membership lies in

$$x h_{D,y}(r) < r \le x(h_{D,y}(r) + \tau).$$

In each class modulo P_y this interval has length τx, so the total number of changes is at most $\tau x + P_y$. After division by x the combined errors are

$$\ll R^{-1} + R(u_y + e_y/\tau) + \tau + P_y/x + u_y.$$

Take R large, then τ small, then y large, and finally x large. This proves (2.2) with choices independent of D.

For the weighted sum, partial summation gives the exact expression $xV_D(x; w) - \int_0^x V_D(t; w)\,\mathrm{d}t$, including the terms at the endpoint. Since $\lambda_D(r) \ge \varphi(r) \ge 1$, the count vanishes below 1. It is $O(t)$ for $t \ge 1$ by (2.4). For any fixed t_0 the part of the integral below t_0 costs $O(t_0^2)$. Above t_0 the relative error in (2.2) can be made uniformly small by taking t_0 large. This proves (2.3). □

2.1 The fractions that the grids sample

Let $\mathcal{Q} = \{q \ge 1 : D \mid q,\ \varphi(q) \le B\}$. For $x = B/\varphi(D) \to \infty$, Lemma 2.1 gives

$$|\mathcal{Q}| = (1 + o(1)) K_D B \tag{2.5}$$

uniformly in D. Keep the endpoint 1 separate. For a general profile let Θ comprise the reduced fractions $a/b \in (0,1)$ with $\varphi(\mathrm{lcm}(b,D)) \le B$. These are exactly the proper fractions on our grids. Indeed $\mathrm{lcm}(b,D)$ divides every supporting grid denominator and is itself admissible if its totient is at most B. We used that divisibility implies the corresponding inequality of totients. For f supported on units, an active fraction j/q has $\gcd(j,D)=1$, so its reduced denominator remains a multiple of D. Here take all proper reduced fractions with denominator in $\mathcal{Q}$. Both choices are closed under $\theta \mapsto 1-\theta$.

Write $h = |\Theta|$, $H = h+1$, and set

$$G = \begin{cases} K_D, & \text{for the choice on units,} \\ K_D R_D, & \text{for the general choice.} \end{cases}$$

We claim that, uniformly as $B/\varphi(D) \to \infty$,

$$H = (1+o(1))GB^2/2. \tag{2.6}$$

On units we have $h = \varphi(D)\sum_{\lambda_D(r)\le x} \lambda_D(r) - \mathbf{1}_{D=1}$. Apply (2.3) with weight one. The endpoint does not affect the asymptotic since $K_D B^2 = C_D \varphi(D) x^2 \ge x^2$.

For general coefficients, a point $j/(Dr)$, $1 \le j \le Dr$, lies on the Du-grid precisely when $r \mid uj$. Its smallest supporting grid therefore has denominator $Dr/\gcd(j,r)$. It first appears at Dr precisely when $\gcd(j,r)=1$, giving $D\varphi(r)$ points, including the endpoint when $r=1$. Grouping by reduced denominator b and using the Euler product for $\varphi(r) = \lambda_D(r) v_D(r)$ gives

$$\sum_{\mathrm{lcm}(b,D)=Dr} \varphi(b) = \varphi(Dr) \prod_{\substack{p\mid D \\ p\nmid r}} (1-1/p)^{-1} = D\lambda_D(r) v_D(r). \tag{2.7}$$

In the example $D=4$, the 8-grid contributes $\varphi(8)=4$ and the 12-grid contributes $\varphi(3)+\varphi(6)+\varphi(12) = 8$. Sum (2.7) and apply (2.3) with weight v_D. Since $Dc_D(v)/\varphi(D)^2 = K_D R_D$, this proves (2.6).

2.2 The cost of a selected fraction

A reduced fraction $\theta = a/b$ enters the numerator of our moment function as a shift in a product of n linear factors. Section 3.1 will clear its denominators at a cost of at most the n-th power of $b\prod_{p\mid b} p^{1/(p-1)}$. Put

$$A = \prod_{a/b \in \Theta} \left(b \prod_{p\mid b} p^{1/(p-1)} \right). \tag{2.8}$$

Even when just some reduced numerators at a denominator are retained,

$$\log A \le h(\log B + O((\log\log(3B))^2)) \qquad (B \ge 2). \tag{2.9}$$

Indeed $\varphi(b) \le B$. If b has k distinct prime divisors, the j-th smallest is at least $j+1$. Hence $b/\varphi(b) \le \prod_{j=1}^{k} (j+1)/j = k+1 \le 1 + \log b/\log 2$. This first gives $\log b \le 2\log B + O(1)$, then by substitution $\log b \le \log B + O(\log\log(3B))$. For $t = \max(2, \log(2b))$, the part of $\sum_{p\mid b} \log p/(p-1)$ from $p \le t$ is at most $\sum_{2\le j\le t} \log j/(j-1) \ll (\log(2t))^2$. The part from $p > t$ is at most $\log b/(t-1)$. Summing over the h fractions gives (2.9).

3 An integer determinant on the selected fractions

Fix a cutoff $s \equiv \delta \bmod 2$ and put $e = (-1)^s$. If $f(j) = ef(-j)$ on every residue class, Section 5 uses Fourier series to give the count directly. Here a class with $f(j) \ne ef(-j)$ will give nonvanishing.

Multiplication by a nonzero integer preserves irrationality, so assume f is integral. Fix B with $\mathcal{Q} \neq \varnothing$. Here we allow any finite reflection-closed $\Theta \subset \mathbb{Q} \cap (0,1)$ containing every proper fraction j/q with $f(j) \neq 0$ on the grids. This permits the smaller choice of numerators in Section 6.

If $N_{f,\delta}(s) < |\mathcal{Q}|$, there are integers w_q, not all zero, satisfying

$$\sum_{q \in \mathcal{Q}} w_q q^j = 0 \quad \text{for every irrational } L_f(j) \text{ counted by } N_{f,\delta}(s). \tag{3.1}$$

These are fewer homogeneous rational equations than unknowns. Let A_0 be a positive integer clearing all rational $L_f(j)$ in the same range and parity. The quantities s, D, f, B, w_q, A_0 stay fixed as the auxiliary integer n grows.

Take even n divisible by each $p-1$ for the primes dividing the denominators of Θ. Then A^n is integral. For the rising factorial $(t)_r = t(t+1)\cdots(t+r-1)$ put

$$R_n(t) = A^n (n!)^{s-h} \frac{\prod_{\theta \in \Theta} (t+\theta)_n}{(t)_{n+1}^s}.$$

Reflection gives $R_n(-n-t) = eR_n(t)$. For $\deg U \leq (n+1)s - nh - 2$ define the convergent linear functional

$$\mathcal{L}_n(U) = A_0 \sum_{q \in \mathcal{Q}} w_q \sum_{j=1}^{q} f(j) \sum_{v \geq 0} R_n(t) \big(U(t+n/2) + U(-t-n/2)\big)\big|_{t=v+j/q}. \tag{3.2}$$

Let $d_n = \operatorname{lcm}(1, \ldots, n)$. For each i put $E_{n,i} = (n!)^{\lfloor i/n \rfloor} (i \bmod n)!$ and $U_i(z) = (z)_i / E_{n,i}$. Write $\Delta_{n,m} = \det_{0 \leq i,j < m} \mathcal{L}_n(z^{i+j})$ for the monomial determinant. The integer we must make small is

$$\boxed{J_{n,m} = \det_{0 \leq i,j < m} \big(d_n^s \mathcal{L}_n(U_i U_j)\big) = \frac{d_n^{sm}}{\prod_{0 \leq i < m} E_{n,i}^2} \Delta_{n,m}.} \tag{3.3}$$

The two changes of basis, in the rows and in the columns, each have diagonal $1/E_{n,i}$. On the left the rising factorials let us prove integrality. On the right we can estimate $\Delta_{n,m}$ using monic polynomials that vanish at the first points on both tails (Section 4).

3.1 Arithmetic for a partial set of numerators

For $2m \leq (n+1)s - nh$, each entry $d_n^s \mathcal{L}_n(U_i U_j)$ in (3.3) belongs to $\mathbb{Z}$. We give the argument because removing individual numerators changes the cost A of the original construction [13, Section 2]. Its arithmetic basis is from [5, Proposition 3.2].

For $U = U_i U_j$, write the proper rational function $G_U(t) = R_n(t)[U(t+n/2) + U(-t-n/2)]$ in (3.2) as $\sum_{k=0}^{n} \sum_{\ell=1}^{s} a_{\ell,k}/(t+k)^\ell$. First, $d_n^{s-\ell} a_{\ell,k} \in \mathbb{Z}$. Following Sprang's progression count and partial-fraction argument [11, Lemmas 1.2–1.3], we check the Taylor coefficients at $-k$ for one shift $\theta = a/b$. Set

$$M_b = b^n \prod_{p \mid b} p^{\lfloor n/(p-1) \rfloor}.$$

In $M_b (t + a/b)_n / n!$, the b^n clears all denominators from a/b, and for $p \mid b$ the extra power covers $v_p(n!)$. For $p \nmid b$, at an integer center the progression of step b has at least $\lfloor n/p^j \rfloor$ multiples of p^j. Each summand of an order-u Taylor coefficient deletes u factors, losing at most u such multiples at each level $p^j \leq n$. Hence d_n^u clears that coefficient. The product of the M_b, one per selected fraction, divides A^n. The integer-shifted rising factorials in U_i, U_j split into blocks $(z+c)_r / r!$ of lengths $r \leq n$ and obey the same bound. Finally, $n!(t+k)/(t)_{n+1}$ has integral constant at $-k$, and each inverse gap in its expansion has denominator dividing d_n. Multiplication gives the claim at order $u = s - \ell$.

Write $\rho_\ell = \sum_{k=0}^{n} a_{\ell,k}$. The rational function is $O(t^{-2})$, so $\rho_1 = 0$. Reflection exchanges $-k$ and $-(n-k)$ with coefficient sign $(-1)^{s+\ell}$, so only orders of parity δ survive in ρ_ℓ. At an active shift $\theta = j/q$ the finite correction is

$$E_U(\theta) = -\sum_{k=1}^{n}\sum_{v=0}^{k-1}\sum_{\ell=1}^{s}\frac{a_{\ell,k}}{(v+\theta)^\ell}.$$

For $\theta = 1$ the multiplier d_n^s clears each term. For a proper shift, fix one prime p and one k, v. If $v_p(v+\theta) \le v_p(d_n)$, each term is p-integral after scaling. Otherwise put $t_0 = -k+v+\theta$. This is a regular point at which the reflected numerator factor $(t+1-\theta)_n$ vanishes. The sum at pole k therefore equals the negative sum at the other poles. For $k' \neq k$, $v_p(t_0+k') = v_p(k'-k) \le v_p(d_n)$. This proves the integrality of the grouped correction by the numerator-zero argument of [11, Lemma 1.4] and [6, Lemma 2], in the reflected form of [13, Section 2].

For $\ell = 1$ truncate at a common upper limit and then pass to infinity using $\rho_1 = 0$. With $\zeta(\ell,\theta) = \sum_{v\ge 0}(v+\theta)^{-\ell}$ for $\ell > 1$, reindexing each partial fraction gives

$$\sum_{v\ge 0} G_U(v+\theta) = E_U(\theta) + \sum_{\substack{2\le\ell\le s\\ \ell\equiv s \bmod 2}} \rho_\ell\,\zeta(\ell,\theta).$$

The coefficients ρ_ℓ are the same at every shift. Summing on the q-grid gives $\sum_{j=1}^{q} f(j)\zeta(\ell, j/q) = q^\ell L_f(\ell)$ by periodicity. Equation (3.1) therefore removes the irrational terms, and the factors A_0, d_n^s clear the rational ones. This proves the entrywise assertion in (3.3).

3.2 A nonzero determinant

The weights w_q in (3.1) may have either sign. Let $j_0 \le D$ be the first positive integer with $f(j_0) \neq ef(-j_0)$, and q_* the largest denominator with $w_{q_*} \neq 0$. Its grid detects this mismatch at $\theta_* = j_0/q_*$. Every smaller grid that samples θ_* sees a positive integer $q\theta_* < j_0$, where the reflected values still agree. The nonvanishing argument of [13, Section 3] uses this surviving mismatch.

More precisely, put $d = s+h+2$ and $T = (n+1)s - nh - 1$. If $d \le T$, then for every integer $M \ge 1$ such that $2(M+d-1) - 2 < T$, there exists

$$M \le m \le M+d-1 \qquad \text{with } \Delta_{n,m} \neq 0. \tag{3.4}$$

We verify the hypotheses of the signed-moment criterion [5, Lemma 5.1 and Proposition 5.4]. Put $X = \mathbb{Z} + (\Theta \cup \{1\})$ and define the one-periodic weight $W(x) = \sum_{q\in\mathcal{Q},\, qx\in\mathbb{Z}} w_q f(qx)$. Writing $S(z) = R_n(z - n/2)$, the two tails of (3.2) give a signed measure on X with masses

$$a_x = \begin{cases} A_0 W(x) S(x), & x > n/2,\\ eA_0 W(-x) S(x), & x < -n/2,\\ 0, & |x| \le n/2. \end{cases}$$

Extend a_x by zero off X. Since $S(x) = O(|x|^{-T-1})$, the absolute moments through degree $T-1$ exist. The quotient of successive values of S gives the recurrence $P(x)a_{x+1} = Q(x)a_x$ with

$$P(z) = (z+1+n/2)^{s+1}(z-n/2)\prod_{\theta\in\Theta}(z+\theta-n/2),$$
$$Q(z) = (z-n/2)^{s+1}(z+1+n/2)\prod_{\theta\in\Theta}(z+\theta+n/2).$$

Indeed $S(z+1)/S(z) = Q(z)/P(z)$. At the fractional boundaries $x = n/2 - 1 + \theta$ and $x = -n/2 - \theta$, the required zero occurs in P and Q, respectively, by reflection of Θ. The extra linear factors handle $x = n/2$ and $x = -n/2 - 1$. Both polynomials have degree d.

By [5, Lemma 5.1], the Cauchy transform $F(z) = \sum_{x\in X} a_x/(z-x)$ satisfies $P(z)F(z+1) - Q(z)F(z) = C(z)$ for a polynomial C. A rational solution r of the same equation would make $(F-r)/S$ one-periodic. At the fraction θ_* chosen above, every smaller grid contributes zero to the difference. Hence

$$W(\theta_*) - eW(-\theta_*) = w_{q_*}(f(j_0) - ef(-j_0)) \neq 0.$$

The class of θ_* is in X, even if only its reflection has nonzero weight or if $\theta_* = 1$. At $k+\theta_*$ and $-k+\theta_*$ for large integers k, the residues of $(F-r)/S$ are $A_0W(\theta_*)$ and $eA_0W(-\theta_*)$. Here a missing mass contributes residue zero. The two points differ by $2k$, so these unequal residues contradict one-periodicity. This excludes a rational solution. Proposition 5.4 of [5] now gives (3.4).

4 Bounding the determinant by the number of shifts

The grids supply $|\mathcal{Q}|$ elimination variables, but their active fractions at a fixed level lie at just H possible positions on each tail. A polynomial of degree $2H$ can vanish at all these positions. Repeating this for successive levels forces high-degree moments to start farther out where R_n is small. We keep this degree cost $2H$ in the analytic basis of [5, Section 4], as adapted in [13, Section 5].

Proposition 4.1. *Fix the data of Section 3. There is a constant $C_0 > 0$ independent of n, m such that, for sufficiently large admissible n and $2m \le (s-H-10)n$,*

$$|J_{n,m}| \le (C_0 m)^m A^{mn} \exp\!\big(m^2 + (2s+10H)mn\big) \left(\frac{10Hn}{m}\right)^{(s-H)mn-2m^2}. \tag{4.1}$$

Proof. Let $\nu_i = \lfloor i/(2H) \rfloor$ and choose the monic polynomial

$$V_i(z) = z^{i-2H\nu_i} \prod_{\theta\in\Theta\cup\{1\}} \prod_{0\le v<\nu_i} (z^2 - (n/2+v+\theta)^2).$$

On each later level $t = v + j/q$, every positive root in the product is at most $t + n/2$. Each quadratic factor at either sign of $t+n/2$ thus has absolute value at most $(t+n/2)^2$, giving $|V_i(\pm(t+n/2))| \le (t+n)^i$. Let S_i be the sum of these squared polynomial values with the absolute masses of the individual summands in (3.2). Cauchy–Schwarz gives $|\mathcal{L}_n(V_iV_j)| \le (S_iS_j)^{1/2}$. The monic change of basis preserves the moment determinant. Its expansion therefore gives $|\Delta_{n,m}| \le m! \prod_{i<m} S_i$.

Write $B_0 = 1 + \max \mathcal{Q}$. For $1 \le k \le n$ we have $t + k \ge k(1 + t/n)$. Also $B_0 t \ge 1 + t/n$ because $t \ge 1/(B_0 - 1)$. Hence

$$(t)_{n+1} \ge (n!/B_0)(1+t/n)^{n+1}, \qquad 0 < R_n(t) \le B_0^s A^n \exp(hn)(1+t/n)^{-((n+1)s-nh)}.$$

For the second estimate use $(t+\theta)_n \le (t+n)^n$ and $n! \ge (n/\mathrm{e})^n$. Set $e_i = (n+1)s - nh - 2i$ and $E = (s-H)n - 2m$. For $i < m$ we have $e_i - 1 \ge E \ge 10n$. Comparing the decreasing summand with its integral gives, uniformly in $0 < \theta \le 1$,

$$\sum_{v\ge\nu_i} (1 + (v+\theta)/n)^{-e_i} \le (1+\nu_i/n)^{-e_i} + \frac{n}{e_i - 1}(1+\nu_i/n)^{1-e_i} \le 2(1+\nu_i/n)^{1-e_i}.$$

Since $1 + \nu_i/n \ge \max(1, (i+1)/(2Hn))$, the last power is at most $(2Hn/(i+1))^E$. Absorbing the fixed weights into $C_0 = 4A_0B_0^s \max_j |f(j)| \sum_{q\in\mathcal{Q}} q|w_q|$ gives

$$S_i \le C_0 A^n \exp(hn) n^{2i} \left(\frac{2Hn}{i+1}\right)^E.$$

The product over $i < m$ contributes $n^{m(m-1)}\big((2Hn)^m/m!\big)^E$. The first factor is canceled by the arithmetic basis in (3.3). Indeed $d_n \le \exp(2n)$ for sufficiently large n, by the prime number theorem,

and $\log E_{n,i} \geq i \log n - i - n$. For the latter write $i = an + r$ and use $\log r! \geq r \log r - r$ and $r \log(r/n) \geq -n$, with $0 \log 0 = 0$. Thus the multiplier of $\Delta_{n,m}$ in (3.3) is at most $n^{-m(m-1)} \exp(m^2 + 2(s+1)mn)$. Using $m! \geq (m/\mathrm{e})^m$ inside the power E, $m! \leq m^m$ outside it, $2\mathrm{e} \leq 10$, and $h + 2 \leq 10H$ proves (4.1). □

For one fixed cutoff choose $C > 0$ with $2C < s - H - 10$. Use $M = \lceil Cn \rceil$ in (3.4). The window length d is fixed as $n \to \infty$, so the moment, degree and size hypotheses hold throughout the window for large admissible n. Choose m there with $\Delta_{n,m} \neq 0$. By (3.3), $|J_{n,m}| \geq 1$. Divide the logarithm of (4.1) by mn and let n grow. Since $m/n \to C$, this gives

$$0 \leq \log A + C + 2s + 10H + (s - H - 2C) \log(10H/C). \tag{4.2}$$

The term $\log(C_0 m)/n$ has vanished for these fixed data. This is where the coefficient heights of f, the denominator A_0 and the size of the eliminating weights leave the proof. We may now vary the entire profile and its period.

5 Proof of the growing-period bound

Proof of Theorem 1.1. Assume first that $s \equiv \delta \bmod 2$ and $f(j) \neq (-1)^s f(-j)$ for some j. In (4.2) the term $\log A$ pays for the selected fractions. The negative term comes from the choice of moment size $m \sim Cn$. We want C large relative to H, so that $10H/C$ is small, and $C = o(s)$ to preserve the coefficient $s - H - 2C$. At the scale verified below, the negative term has size $s \log\log s$. There are about $GB^2/2$ shifts, each costing at most $(\frac{1}{2} + o(1)) \log(sD)$ in $\log A$. Balancing $(GB^2/4) \log(sD)$ against $s \log\log s$ gives the factor of 2 in our choice of B. For a fixed $0 < \rho < 1$ take

$$B = 2(1-\rho)\sqrt{\frac{s \log\log s}{G \log(sD)}}, \qquad C = \frac{s}{\log\log s}. \tag{5.1}$$

The factor $1 - \rho$ leaves the strict margin in the final inequality.

For use of Lemma 2.1, observe first that

$$\varphi(D)^2 G = \begin{cases} C_D \varphi(D), & \text{on units,} \\ C_D D \prod_{p \mid D} (1 - p^{-2}), & \text{in general.} \end{cases}$$

Both are at most κD. Since $D \leq s^{1-\eta}$ and $\log(sD) \leq 2 \log s$, we have

$$x = \frac{B}{\varphi(D)} \gg_\rho s^{\eta/2} \sqrt{\frac{\log\log s}{\log s}} \longrightarrow \infty$$

uniformly. Also $G \geq 1/D$, so $\log B \leq (\frac{1}{2} + o(1)) \log(sD)$. Inserting (5.1) in (2.6) and then in (2.9) gives

$$H = \big(2(1-\rho)^2 + o(1)\big) \frac{s \log\log s}{\log(sD)}, \tag{5.2}$$

$$\log A \leq \big((1-\rho)^2 + o(1)\big) s \log\log s. \tag{5.3}$$

Here $(\log\log(3B))^2 = o(\log(sD))$. All errors are uniform in D and in the two choices of support, for fixed ρ, η. In particular $H = o(s)$ and $2C < s - H - 10$ for large s. Equation (5.2) also gives

$$\log(10H/C) = -\log\log(sD) + 2 \log\log\log s + O_\rho(1) = -(1 + o(1)) \log\log s.$$

If $N_{f,\delta}(s) < |\mathcal{Q}|$, inequality (4.2) now implies $0 \leq ((1-\rho)^2 - 1 + o(1)) s \log\log s$, a contradiction. By (2.5) the number of irrational exponents is therefore at least

$$|\mathcal{Q}| = (1 - \rho + o(1)) \frac{2K_D}{\sqrt{G}} \sqrt{\frac{s \log\log s}{\log(sD)}}. \tag{5.4}$$

The order of the two limits matters. We derived (4.2) at each fixed s, D, f before taking $s \to \infty$ here.

For matching parity, $f(-j) = (-1)^s f(j)$ for every j. The finite Fourier expansion of f and the Bernoulli Fourier formula [10, §24.8(i)] show that $L_f(j)$ is an algebraic multiple of π^j at each eligible j [13, Section 6]. Each nonzero value is therefore transcendental. To see the dependence of the zero count on D, write $\Gamma(x)L_f(x)$ for $x > 1$ as the Mellin transform of

$$t \longmapsto \frac{\sum_{r=1}^{D} f(r)e^{-rt}}{1 - e^{-Dt}}.$$

The numerator divided by e^{-t} is a nonzero polynomial in e^{-t} of degree at most $D - 1$. Its number of changes of sign bounds the number of distinct real zeros of the transform. Indeed, if t_0 is a sign-change point, multiplication of the integrand by $\log(t/t_0)$ removes that sign change. With $F(x)$ denoting the original transform, the new transform is $t_0^x(t_0^{-x}F(x))'$. Rolle's theorem and induction give the bound, starting with an integrand of constant sign. Differentiation is valid locally for $x > 1$ after any finite number of logarithmic factors. It follows that at most $D - 1$ eligible exponents give zero. For $D \le s^{1-\eta}$ this leaves at least $s/3$ for all sufficiently large s. This exceeds the required bound, since $K_D/\sqrt{G} \le \sqrt{\kappa}$.

Finally let σ be the largest integer at most s of parity δ. Since $s-1 \le \sigma \le s$, we have $s^{1-\eta} \le \sigma^{1-\eta/2}$ for large s. Apply the parity-aligned argument with $\eta/2$ at σ. The ratio of the square-root factors at σ and s tends to one uniformly because $\log(sD) \ge \log s$. Take $\rho = \varepsilon/4$ and enlarge s_0 to obtain (1.2) for either parity. □

6 Two sparse profiles modulo five

The real and imaginary parts of the quartic character χ with $\chi(2) = i$ are supported on $\{1, 4\}$ and $\{2, 3\}$ modulo five, respectively. Their Dirichlet values at real arguments are the real and imaginary parts of L_χ, and $\chi(-1) = i^2 = -1$ gives both profiles odd parity. Both series are positive for real $x > 1$, since their successive blocks are $(5k+1)^{-x} - (5k+4)^{-x}$ and $(5k+2)^{-x} - (5k+3)^{-x}$, respectively, for $k \ge 0$. At every odd integer $j \ge 3$, the two values are therefore nonzero algebraic multiples of π^j by Section 5, hence transcendental.

For the even-exponent count, consider the first support pair and reduced denominator 10. On the 10-grid, $1/10$ is available but $3/10$ samples the other pair. The 20-grid supplies $3/10 = 6/20$, since $6 \equiv 1 \bmod 5$, at a totient cost of 8 instead of 4. On an odd denominator the same doubling leaves its totient unchanged. Keeping track of this difference reduces the number of shifts while leaving the number of eliminating weights unchanged.

Corollary 6.1. *Let f be a nonzero rational-valued function of period five whose support is contained in either $\{1, 4\}$ or $\{2, 3\}$ modulo five. For every $\varepsilon > 0$, for all sufficiently large s independent of f, and for either parity,*

$$N_{f,\delta}(s) \ge (1 - \varepsilon)\, 4\sqrt{K_5/3}\, \sqrt{\frac{s \log\log s}{\log s}}, \qquad K_5 = \frac{5\kappa}{21}.$$

For $\operatorname{Re}\chi$ and $\operatorname{Im}\chi$ this improves the previous fixed-period lower-bound coefficient $2\sqrt{K_5}$ at even exponents by the factor $2/\sqrt{3}$.

Proof. Use the same grids $5 \mid q,\ \varphi(q) \le B$. For Θ retain the fractions sampled by the chosen residue pair. This set is reflection-closed even if f is nonzero at just one member of the pair. A proper reduced a/b occurs if $q = kb$ is admissible and ka belongs to the pair modulo five. This forces $5 \nmid k$ and $5 \mid b$. For a numerator in the chosen pair, $k = 1$ works as soon as $\varphi(b) \le B$. For the complementary pair,

$k = 2$ works at a cost ratio $\varphi(2b)/\varphi(b)$ of one on odd b and two on even b. Every $k \geq 3$ has

$$\frac{\varphi(kb)}{\varphi(b)} = k \prod_{\substack{p \mid k \\ p \nmid b}} (1 - 1/p) \geq \varphi(k) \geq 2.$$

Thus odd denominators use all numerators up to the full cutoff. For even denominators exactly the complementary half of numerators has the stricter cutoff $\varphi(b) \leq B/2$. It is half because reduction of units modulo b onto units modulo five is surjective with equal fibers.

Write $U_D(y) = \sum_{D \mid b,\, \varphi(b) \leq y} \varphi(b)$. Equation (2.3) at fixed D gives $U_D(y) = (K_D/2 + o(1))y^2$. Since the even denominators are precisely the multiples of ten among the multiples of five, the number of retained proper fractions is

$$h = U_5(B) - \tfrac{1}{2}(U_{10}(B) - U_{10}(B/2)) = \frac{B^2}{2}(K_5 - \tfrac{3}{8}K_{10} + o(1)).$$

By (1.1), $K_{10}/K_5 = 2/3$, so this gives $G_* = 3K_5/4$ in (2.6). Restricting to one residue pair keeps every grid and uses asymptotically three quarters as many shifts as retaining all units. The cost bound (2.9) and the integrality argument of Section 3.1 both apply to individual selected numerators. Use G_* in (5.1). The proof of (5.4) now yields $2K_5/\sqrt{G_*} = 4\sqrt{K_5/3}$, and $\log(5s)/\log s \to 1$. At matching parity the same zero bound applies. □

Verification. Independent AI agents reviewed the 18 November 2026 research proof and first manuscript draft for correctness and exposition. A final independent audit of the exposition revision verified Theorem 1.1 and Corollary 6.1. The accompanying program `checks/check_grids.py` in the paper project reconstructs the fractions directly and verifies the finite grid identities in Sections 2 and 6 for the documented test cases.

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