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The weight-three Lambert sum at the golden ratio is irrational

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Abstract

For $|b| > 1$, define the weight-three Lambert sum by $Z(b) = \sum_{n\geq 1} n^2/(b^n - 1)$. For an algebraic integer b in this range, we prove that $Z(b) \notin \mathbb{Q}(b)$ if every conjugate outside the closed unit disk belongs to $\{b, \bar{b}\}$. This includes all Pisot and Salem bases. For a Salem base we bound the polynomial approximations even at conjugates on the unit circle, where the series diverges. At the golden ratio $\varphi = (1+\sqrt{5})/2$, the sum $Z(\varphi)$ lies outside $\mathbb{Q}(\sqrt{5})$ and is therefore irrational. A nonreal example is $Z(1+i) \notin \mathbb{Q}(i)$.

1 The golden-ratio value

For a complex number b with $|b| > 1$, define

$$Z(b) = \sum_{n=1}^{\infty} \frac{n^2}{b^n - 1} = \sum_{m=1}^{\infty} \frac{\sigma_2(m)}{b^m}, \qquad \sigma_2(m) = \sum_{d|m} d^2. \tag{1.1}$$

Expanding each denominator as a geometric series gives the second expression. Both series converge absolutely. At the golden ratio every partial sum belongs to the quadratic field $\mathbb{Q}(\sqrt{5})$.

Theorem 1.1. *For $\varphi = (1+\sqrt{5})/2$,*

$$\sum_{n=1}^{\infty} \frac{n^2}{\varphi^n - 1} \notin \mathbb{Q}(\sqrt{5}).$$

In particular this real number is irrational.

The second root of $x^2 - x - 1$ is $-1/\varphi$. Suppose the Lambert sum at φ were $\xi = u + v\sqrt{5}$ with $u, v \in \mathbb{Q}$. Its conjugate would be $Y = u - v\sqrt{5}$. Substituting $-1/\varphi$ in the Lambert series makes its nth term asymptotic to $-n^2$. To compare the two embeddings, we construct integer polynomials $P_K(t, X)$ of degree at most K in X. For some $c > 0$ we have $|P_K(\varphi, \xi)| \leq \exp(-cK^3)$ for all large K. At $(-1/\varphi, Y)$ we bound their growth by $\exp(o(K^3))$ *for every fixed* Y. Choose a positive integer d for which $d\xi$ is an algebraic integer. Then $d^{2K} P_K(\varphi, \xi) P_K(-1/\varphi, Y)$ is an integer. Its absolute value tends to zero, and we show it is nonzero for arbitrarily large K.

For a nonreal base b, absolute convergence gives $Z(\bar{b}) = \overline{Z(b)}$, so decay at b gives decay at $\bar{b}$ as well. It remains to control the other conjugates, which we can do whenever they all lie in the closed unit disk.

Theorem 1.2. *Let b be an algebraic integer with $|b| > 1$. If every conjugate distinct from b and its complex conjugate $\bar{b}$ has modulus at most one, then $Z(b) \notin \mathbb{Q}(b)$.*

A real algebraic integer $b > 1$ is a Pisot number when all its other conjugates have modulus less than one, and a Salem number when they lie in the closed unit disk with at least one on the unit circle. Theorem 1.2 covers both classes. At a unit-circle conjugate u of a Salem number, $|u^n - 1| \le 2$, so the terms of the series cannot tend to zero. Section 4 controls the polynomials there.

The Gaussian integer $1 + i$ has norm 2, the smallest norm above one in $\mathbb{Z}[i]$. Its only other conjugate is $1 - i$, so $Z(1 + i) \notin \mathbb{Q}(i)$. Every partial sum has rational real and imaginary parts. At least one of those coordinates is irrational in the limit.

In the Lambert-series convention for q-zeta values, $Z(b) = \zeta_{1/b}(3)$. Bundschuh and Zudilin singled out reciprocal Pisot and Salem parameters in their discussion of the arithmetic of these values [1, §6, after (41)]. For the weight-one sum $\sum_{n\ge1} 1/(b^n - 1)$, Tachiya's Theorems 1 and 2 give exclusion from $\mathbb{Q}(b)$ at $b = 1 + i$ and $b = \varphi$, respectively [6, p. 232]. Dixit, Kumar and Pathak prove this exclusion at every Pisot base [2, Corollary 1.2]. For the weight-three sum (1.1), uplift-universal-inc proved irrationality at each positive integer base $b \ge 2$ [7, Theorem 1.1].

The determinant and its cyclotomic factors come from the integer-base proof [7, §§1–5 and Appendix A]. Zudilin obtained cubic decay for Hankel determinants on geometric nodes in his work on the generalized q-logarithm [9, §4, (8)]. Krisman used a norm over a quadratic field and controlled the conjugate of a determinant for reciprocal Fibonacci cubes [4, §5]. Here a single estimate controls $P_K(t, Y)$ throughout the closed unit disk for every fixed Y. We estimate the determinant on a slightly larger circle, where successive powers of t are separated in absolute value. Since P_K is a polynomial in t, the maximum modulus principle carries that estimate to every remaining conjugate.

2 A single integral polynomial for every embedding

Put $q = b^{-1}$. To recover the summand $j^2/(b^j - 1)$ from x^j, apply the Euler derivative $\theta = x\, d/dx$ twice and sum over the points $q, q^2, \ldots$. Thus for a function f holomorphic on a neighborhood of $|x| \le |q|$, set

$$L_q(f) = \sum_{m\ge1} (\theta^2 f)(q^m).$$

Since $\theta^2 f(x) = xf'(x) + x^2 f''(x)$, the summands are $O_f(|q|^m)$. On monomials this gives

$$L_q(1) = 0, \qquad L_q(x^j) = \frac{j^2}{b^j - 1} \quad (j \ge 1).$$

The same functional recovers *tails* of $Z(b)$ on simple fractions. Indeed, with

$$g(y) = \sum_{j\ge1} j^2 y^j = \frac{y(1 + y)}{(1 - y)^3} \qquad (|y| < 1),$$

absolute summation gives $Z(b) = \sum_{m\ge1} g(q^m)$ and

$$L_q\big((1 - q^a x)^{-1}\big) = \sum_{m\ge1} g(q^{m+a}) = Z(b) - S_a(b), \qquad S_a(t) = \sum_{m=1}^{a} \frac{t^m(t^m + 1)}{(t^m - 1)^3}. \tag{2.1}$$

For example the first pole, at $x = b$, removes just $g(q)$ from $Z(b)$. Thus every pole introduces the same unknown value.

Now replace the base b and its value $Z(b)$ by independent variables t and X, respectively. The functional still acts in x. For $K \ge 1$ the first K poles give the denominator

$$D_{K,t}(x) = \prod_{a=1}^{K} (1 - t^{-a} x).$$

On functions $p(x)/D_{K,t}(x)$, define $\mathcal{L}_{t,X}$ by partial fractions over $\mathbb{Q}(t,X)$. On the polynomial part use $\mathcal{L}_{t,X}(1)=0$ and $\mathcal{L}_{t,X}(x^j)=\mu_j(t)=j^2/(t^j-1)$ for $j\ge 1$. On the simple fractions use $\mathcal{L}_{t,X}((1-t^{-a}x)^{-1})=X-S_a(t)$. By (2.1), at $t=b$, $X=Z(b)$ this agrees with L_q.

We arrange the moments in a determinant so that we can change the polynomial basis in both its rows and its columns. At b, monic polynomials with triple zeros at the first few nodes $q,q^2,\dots$ remove those nodes even after two derivatives. Successive rows and columns then supply the small factors that give cubic decay. In the monomial basis the determinant is

$$\Delta_{K,t}(X)=\det\left[\mathcal{L}_{t,X}\left(x^{i+j}/D_{K,t}(x)\right)\right]_{0\le i,j<K}. \tag{2.2}$$

The entries depend only on $i+j$, giving a Hankel matrix. Each is affine in X, so the determinant has degree at most K in X. Rational modifications of moment functionals appear in Uvarov's transformation of orthogonal polynomials [8, §1] and in Krattenthaler's determinant identity [3, Theorem 1]. The denominator $D_{K,t}$ and the functional in (2.2) are the specific weight-three construction of [7, §1].

Two features of this construction are essential for specialization. The poles of the coefficients in t, away from zero, occur at roots of unity where nodes coincide or moments have poles. Cyclotomic factors clear them, and t^{e_K} divides all the resulting coefficients, where $e_K=(K^3-K)/2$. Let Φ_d be the d-th cyclotomic polynomial. In units of $K^3\log|b|$, the determinant will save $1/9$ and the extracted power of t another $1/2$. The cost of clearing the denominators comes from the degree of

$$C_K(t)=\prod_{d=1}^{K}\Phi_d(t)^{c_{K,d}},\qquad c_{K,d}=\begin{cases}\lceil K^2/(4d)+2K+17d/16\rceil, & d\le K/2,\\ 5(K-d)+3, & K/2<d\le K.\end{cases} \tag{2.3}$$

Lemma 2.1. *The polynomial used at every embedding is*

$$P_K(t,X)=C_K(t)t^{-e_K}\Delta_{K,t}(X)\in\mathbb{Z}[t,X].$$

It has degree at most K in X. Moreover,

$$\deg C_K\le\gamma K^3+O(K^2\log(2K)),\qquad \gamma=\frac{321}{64\pi^2},\qquad \sum_{d=1}^{K}c_{K,d}=O(K^2\log(2K)). \tag{2.4}$$

Proof. At a primitive d-th root of unity, $t^a=t^c$ precisely when $a\equiv c\pmod d$. The first bound in (2.3) comes from [7, §4.2, (4.8)]. For $d>K/2$ each class contains at most two indices, and [7, Proposition A.1] gives the second bound. Combining these local bounds as in [7, §4.4] gives $C_K(t)\Delta_{K,t}(X)\in\mathbb{Z}[t,t^{-1}][X]$. Their cyclotomic degree sum gives the first bound in (2.4) [7, §4.5 and Proposition A.1]. The range $d\le K/2$ contributes $161/(64\pi^2)$ to its leading coefficient and the range $d>K/2$ contributes $5/(2\pi^2)$. The bound on the sum of exponents follows directly from (2.3).

It remains to justify using the extracted factor t^{e_K} as an identity in the base. We already know that the coefficients of P_K are integer Laurent polynomials. The integer-base divisibility result [7, Proposition 3.1 and Appendix A] gives $P_K(p,X)\in\mathbb{Z}[X]$ for every positive rational prime p. If a coefficient had a lowest term ct^{-h} with $h>0$ and $c\neq 0$, choose a prime $p\nmid c$. Multiplying its value at p by p^h gives an integer congruent to c mod p, which contradicts the integrality of the value itself. This proves the assertion over $\mathbb{Z}[t,X]$. The inverse-base polynomial normalization is also recorded in the rational-parameter companion to [7, §7.1]. The code in [10] checks this integrality coefficientwise for $1\le K\le 4$. □

For later use, if $t\neq 0$ is fixed and $|t|\neq 1$, Möbius inversion in the cyclotomic product gives

$$\log|\Phi_d(t)|=\phi(d)\log^+|t|+O_t(1),\qquad \log^+u=\log\max(1,u). \tag{2.5}$$

Here ϕ is Euler's totient. After factoring off the degree term, the absolute error is at most $\sum_{h\ge1}-\log(1-r^h)$ with $r=\min(|t|,|t|^{-1})<1$. The bound is independent of d.

3 Decay where the Lambert series converges

Fix any complex b of modulus greater than one and put $q = 1/b$, $r = |q|$. At $X = Z(b)$, the determinant in (2.2) is formed from the bilinear form $L_q(fg/D_{K,b})$. Its first evaluation node is $x = q$. A factor $(x-q)^3$ makes the value and the first two derivatives vanish at that node. Three more factors can remove the node q^2, and so on. This accounts for the multiplicity three in the following basis from [7, §2].

Proposition 3.1. *For every $|b| > 1$ and every $K \geq 1$,*

$$|P_K(b, Z(b))| \leq \exp\bigl(-\varepsilon K^3 \log|b| + O_b(K^2 \log(2K))\bigr), \qquad \varepsilon = \frac{11}{18} - \frac{321}{64\pi^2} > 0.$$

Proof. Define $B_0 = 1$ and, for $j \geq 1$,

$$B_j(x) = \prod_{\ell=1}^{j} (x - q^{\lceil \ell/3 \rceil}), \qquad s_j = \sum_{\ell=1}^{j} \lceil \ell/3 \rceil, \qquad R_j = \lceil j/3 \rceil.$$

Set $s_0 = R_0 = 0$. For example, $B_6 = (x-q)^3(x-q^2)^3$ and $s_6 = 3 \cdot 1 + 3 \cdot 2 = 9$. More generally, $s_{3h} = 3(1 + \cdots + h)$. Thus the exponent contributed by a row grows quadratically with its degree. Adding the exponents from all rows and columns will give the cubic power of K.

The change from monomials to the B_j is triangular with ones on its diagonal, so it preserves $\Delta_{K,b}(Z(b))$. In the entry indexed by i, j, put $R = \max(R_i, R_j)$ and $W = 1/D_{K,b}$. For $1 \leq m < R$ the node q^m contributes zero to $L_q(B_i B_j W)$ because it is a triple zero of $B_i B_j$.

For $m \geq \max(1, R)$, each factor $x - q^a$ at $x = q^m$ has modulus at most $2r^a$. This supplies $r^{s_i + s_j}$ in the product. To sum over the remaining nodes we also need decay in m. Both $\theta(x - q^a)$ and $\theta^2(x - q^a)$ equal x. At q^m their modulus is at most $r^a r^{m-R}$, since $a \leq R$. On the unit disk the function W is bounded by $\prod_{a \geq 1}(1 - r^a)^{-1}$. Cauchy's estimates then give, on the smaller disk $|x| \leq r$,

$$|W(x)| \leq C_r, \qquad |\theta W(x)| + |\theta^2 W(x)| \leq C_r |x|.$$

Consequently every term of $\theta^2(B_i B_j W)(q^m)$ has the extra summable factor r^{m-R}, including the term in which both derivatives fall on W. There are at most $(i + j + 1)^2$ ordered choices of the differentiated factors. Summing over m gives

$$|L_q(B_i B_j W)| \leq C_r'(i + j + 1)^2 \, 2^{i+j} r^{s_i + s_j}.$$

Take $2^i r^{s_i}$ from each row and $2^j r^{s_j}$ from each column. The remaining determinant has absolute value at most $K![C_r'(2K)^2]^K$. The extracted power of r has exponent $2\sum_{j<K} s_j = K^3/9 + O(K^2)$, since $s_j = j^2/6 + O(j)$. Thus

$$|\Delta_{K,b}(Z(b))| \leq \exp\bigl(-\tfrac{1}{9} K^3 \log|b| + O_b(K^2)\bigr). \tag{3.1}$$

This estimate uses only the moduli of the geometric nodes. Multiplying by b^{-e_K} saves another $(K^3 - K)\log|b|/2$ in the logarithm. Equations (2.4) and (2.5) bound the logarithm of the cost of $C_K(b)$ by $\gamma K^3 \log|b| + O_b(K^2 \log(2K))$. The claimed margin is $1/9 + 1/2 - \gamma > 0$, since $\pi > 3$. □

4 One bound on the closed unit disk

At a conjugate such as $-1/\varphi$, the norm argument needs only an upper bound $o(K^3)$ for $\log^+ |P_K(t, Y)|$ at each fixed Y. The individual moments can have poles in the closed unit disk. For example, $S_1(t)$ has a pole at $t = 1$, yet $P_1(t, Y) = (t-1)^3 Y - t(t+1)$ gives $P_1(1, Y) = -2$. Since $P_K(\cdot, Y)$ is a polynomial, we can estimate it on a circle just outside the disk.

Lemma 4.1. *For every fixed $Y \in \mathbb{C}$,*

$$\max_{|t|\le 1} \log^+ |P_K(t,Y)| = O_Y\left(K^2 \log^3(2K)\right).$$

Proof. Choose $|t| = 1 + 1/K$. The outward shift keeps the nodes $r_a = t^a$, $1 \le a \le K$, bounded by e. By (2.4), the degree term $(\deg C_K)\log|t|$ costs only $O(K^2)$ at this radius. At the same time, distinct nodes are separated uniformly in the argument of t. For $a \ne c$ the reverse triangle inequality gives

$$|1 - r_a/r_c| \ge |1 - |t|^{a-c}| \ge \frac{1}{K+1}.$$

Consequently the weights in the partial fractions of $1/D_{K,t}$ satisfy

$$\frac{1}{D_{K,t}(x)} = \sum_{a=1}^{K} \frac{w_a}{1 - x/r_a}, \qquad w_a = \prod_{c\ne a} (1 - r_a/r_c)^{-1}, \qquad |w_a| \le (K+1)^{K-1}.$$

To bound a moment with numerator x^k, put $U_{a,k} = \mathcal{L}_{t,Y}(x^k/(1 - x/r_a))$ and set $\mu_0 = 0$. The identity $x/(1 - x/r_a) = r_a((1 - x/r_a)^{-1} - 1)$ gives

$$U_{a,0} = Y - S_a(t), \qquad U_{a,k+1} = r_a(U_{a,k} - \mu_k(t)).$$

For $j \ge 1$ we have $|t^j - 1| \ge 1/K$. In (2.1) there are at most K terms, each of size $O(K^3)$, since $|t^m| \le e$ for $m \le K$. Hence $|S_a(t)| = O(K^4)$, and $|\mu_j(t)| \le 4K^3$ for $1 \le j \le 2K$. Iterating the recurrence yields $|U_{a,k}| \le e^{2K}(|Y| + O(K^4))$ throughout $0 \le k \le 2K - 2$. Each entry in (2.2) is $\sum_a w_a U_{a,k}$ with $k = i + j$, so its absolute value is at most $\exp(O_Y(K\log(2K)))$. Expanding the determinant gives

$$\log^+ |\Delta_{K,t}(Y)| = O_Y(K^2 \log(2K)).$$

It remains to bound the cyclotomic product on the same circle. For $h \ge 1$, we have $1/(K+1) \le |1 - t^{-h}| \le 2$. Möbius inversion for Φ_d therefore gives

$$\log|\Phi_d(t)| \le \phi(d)\log(1 + 1/K) + \tau(d)\log(2K+2),$$

where $\tau(d)$ is the number of positive divisors of d. After weighting by $c_{K,d}$, the first terms sum to the degree contribution above. For the divisor terms, (2.3) gives $c_{K,d} \le 8K^2/d$ and

$$\sum_{d\le K} \frac{\tau(d)}{d} = \sum_{uv\le K} \frac{1}{uv} \le (1 + \log K)^2.$$

Thus $\log^+ |C_K(t)| = O(K^2 \log^3(2K))$. Since $|t^{-e_K}| \le 1$, this also bounds $\log^+ |P_K(t,Y)|$ on the circle, with an implied constant depending on Y. Lemma 2.1 makes $t \mapsto P_K(t,Y)$ a polynomial. The maximum modulus principle gives the same bound on the closed unit disk, including the points where the original moment formulas have poles. □

5 Infinitely many nonzero determinants

To use the field norm we need arbitrarily large sizes at which the determinant is nonzero. For the functional L_q, which can take complex values, we detect nonvanishing through its Cauchy transform. For a fixed moment sequence, Kronecker related eventual vanishing of its Hankel determinants to rationality of its generating function [5, Arts. VII and IX]. Zudilin used this criterion for the generalized q-logarithm [9, §4]. Here the denominator changes with the size of the determinant.

The zeros $b, b^2, \dots, b^K$ of $D_{K,b}$ form nested interpolation sets. If all sufficiently large sizes were singular, one rational function would agree with the Cauchy transform at infinitely many of these nodes. The nodes tend to infinity, so the two functions would have the same Laurent expansion there. This is impossible because the transform has a pole at every b^{-m}.

Proposition 5.1. *For every complex b with $|b| > 1$, there are arbitrarily large K for which $P_K(b, Z(b)) \neq 0$.*

Proof. We adapt [7, Proposition 5.1] to complex nodes. Put $q = 1/b$ and let

$$\mathcal{C}(z) = L_q((z-x)^{-1}) = \sum_{m\geq 1} \frac{q^m(z+q^m)}{(z-q^m)^3} \qquad (|z| > |q|).$$

On compact sets disjoint from zero and the nodes, the tail is dominated by a constant times $\sum_m |q|^m$, so the series continues meromorphically to the punctured plane. The nodes have distinct moduli, and at each q^m the coefficient of $(z-q^m)^{-3}$ is $2q^{2m} \neq 0$. Thus the function analytic near infinity has infinitely many poles on continuation and cannot be rational.

Suppose the determinant of size k is zero. Write $D_k = D_{k,b}$. A nonzero null vector supplies a polynomial Q_k of degree less than k satisfying

$$L_q(x^j Q_k(x)/D_k(x)) = 0 \qquad (0 \leq j < k).$$

We seek a polynomial T_k such that the error $\mathcal{C}Q_k - T_k$ vanishes at $b, \dots, b^k$ and tends to zero at infinity. To impose the zeros, multiply the Cauchy transform of $Q_k(x)/D_k(x)$ by $D_k(z)$. Subtracting this error from $\mathcal{C}(z)Q_k(z)$ gives T_k. With L_q acting in x, the two expressions are

$$E_k(z) = D_k(z) L_q\left(\frac{Q_k(x)}{(z-x)D_k(x)}\right), \tag{5.1}$$

$$T_k(z) = \mathcal{C}(z)Q_k(z) - E_k(z) = L_q\left(\frac{Q_k(z)D_k(x) - D_k(z)Q_k(x)}{(z-x)D_k(x)}\right).$$

The numerator defining T_k is divisible by $z-x$, so T_k is a polynomial of degree less than k. The functional in (5.1) is holomorphic at every $z = b^a$, $1 \leq a \leq k$, giving the desired zeros. At infinity expand $1/(z-x)$ to k terms with remainder $x^k/(z^k(z-x))$. All k terms disappear by the moment equations. To bound the remainder, choose a disk about zero of radius strictly between $|q|$ and one. Throughout this disk the factor $x^k Q_k(x)/((z-x)D_k(x))$ is $O_k(|z|^{-1})$. Cauchy's inequalities bound its first two derivatives on $|x| \leq |q|$, and the Euler derivatives supply a summable factor $|q|^m$ at q^m. The remainder therefore contributes $O_k(|z|^{-k-1})$ to the functional, giving $E_k(z) = O_k(z^{-1})$ in (5.1). The same local bounds in z justify holomorphy at the exterior zeros.

If sizes k and $k+1$ both vanish, the polynomial

$$T_{k+1}Q_k - T_k Q_{k+1} = E_k Q_{k+1} - E_{k+1} Q_k$$

has degree at most $k-1$, by its growth at infinity, and vanishes at the k common exterior nodes. It is identically zero. Hence if all sufficiently large sizes vanished, the fractions T_k/Q_k would all represent one rational function T/Q in lowest terms, allowing $T = 0$, $Q = 1$ for the zero function. Then $T_k = V_k T$, $Q_k = V_k Q$ for a nonzero polynomial V_k. If $V_k(b^a) = 0$, the interpolation condition $E_k(b^a) = 0$ holds whatever value $\mathcal{C}$ takes there. We must bound how many interpolation nodes this common factor can account for.

Since $\mathcal{C}$ is nonrational, $E = \mathcal{C}Q - T$ is meromorphic at infinity and has a nonzero leading term there. The estimate $V_k E = O_k(1/z)$ forces $E(z) \sim cz^{-h}$ there for some $c \neq 0$ and integer $h \geq 1$. Thus $\deg V_k \leq h-1$ for every sufficiently large k, and $E(b^a) \neq 0$ for every sufficiently large a. Then (5.1) forces $V_k(b^a) = 0$ for all these $a \leq k$. The number of distinct zeros grows beyond $h-1$. This contradiction proves that $\Delta_{K,b}(Z(b)) \neq 0$ for arbitrarily large K. The factors relating P_K to $\Delta_{K,b}$ are nonzero at b. □

6 The field norm

Proof of Theorem 1.2. Suppose that $\xi = Z(b)$ belongs to $E = \mathbb{Q}(b)$. Choose a positive integer d such that $d\xi$ is in the ring of integers $\mathcal{O}_E$. Since $b \in \mathcal{O}_E$ and $P_K \in \mathbb{Z}[t, X]$ has X-degree at most K, we have

$$y_K = d^K P_K(b, \xi) \in \mathcal{O}_E.$$

By Proposition 5.1, $y_K \neq 0$ along an unbounded sequence of sizes. Its absolute norm at each such size is a positive integer, given by

$$|\,\mathrm{N}_{E/\mathbb{Q}}(y_K)| = d^{K[E:\mathbb{Q}]} \prod_{\sigma : E \hookrightarrow \mathbb{C}} |P_K(\sigma b, \sigma\xi)|.$$

At the chosen embedding, Proposition 3.1 supplies a negative term $-\varepsilon K^3 \log|b|$ in the logarithmic upper bound. When b is nonreal, the restriction of complex conjugation to E is the embedding into $\mathbb{C}$ that sends b to $\bar{b}$. It sends ξ to $\overline{Z(b)} = Z(\bar{b})$ and gives the same bound. For every other embedding, $|\sigma b| \leq 1$. Lemma 4.1, with the fixed argument $Y = \sigma\xi$, bounds the total logarithmic cost of these finitely many factors by $O_{b,\xi}(K^2 \log^3(2K)) = o(K^3)$.

Let $r_b = 1$ for real b and $r_b = 2$ otherwise. The fixed denominator contributes only $K[E : \mathbb{Q}] \log d$. On the nonzero subsequence we obtain

$$0 \leq \log|\,\mathrm{N}_{E/\mathbb{Q}}(y_K)| \leq -r_b\varepsilon(\log|b|)K^3 + o(K^3),$$

which is impossible for large K. This proves Theorem 1.2 and, at $b = \varphi$, Theorem 1.1. □

Verification

Another AI model independently audited the mathematics and exposition of the initial 18 November 2026 manuscript and found no gap in either theorem. The uniform disk estimate in Section 4 follows the proof supplied in that audit.

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