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MODULARITY OF THE FACTORS OF AN AUTOMORPHIC TENSOR PRODUCT

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Abstract. Let F be a totally real field and let ρ_1, ρ_2 be two-dimensional p-adic representations of G_F which are Lie-irreducible and potentially modular, each over its own extension. We prove the following: if $\rho_1 \otimes \rho_2$ is attached to a regular L-algebraic cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_F)$, then ρ_1 and ρ_2 are modular and $\Pi \simeq \pi_1 \boxtimes \pi_2$. This is a converse to Ramakrishnan's automorphy of the Rankin–Selberg product. The proof combines the descent of Hundley–Sayag from GL_4 to GSpin_4 with L-function arguments based on potential modularity. To pin down the weights of the factors we use the Galois representations attached to the adjoint lifts $\mathrm{ad}^0\pi_i'$ on GL_3; this rules out factors of mixed parity.

1. Introduction

Let F be a totally real field, p a prime and $\iota : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$.

Conventions. We write Frob_v for a geometric Frobenius element at a finite place v, q_v for the cardinality of the residue field, and $\varepsilon_{\mathrm{cyc}}$ for the p-adic cyclotomic character, so that $\varepsilon_{\mathrm{cyc}}(\mathrm{Frob}_v) = q_v^{-1}$; it has Hodge–Tate weight -1. Class field theory is normalized so that uniformizers correspond to geometric Frobenius elements, and we use the same letter for a Hecke character of finite order and the corresponding Galois character. If π_v is an unramified representation of $\mathrm{GL}_n(F_v)$, its Satake parameter $t(\pi_v)$ is the semisimple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ with $L(s, \pi_v) = \det(1 - t(\pi_v) q_v^{-s})^{-1}$; thus $t(\pi_v \otimes |\det|^s) = q_v^{-s} t(\pi_v)$. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ *corresponds to* a Galois representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ if $t(\pi_v) = \iota(r(\mathrm{Frob}_v))$ for almost all v. No normalization of automorphic representations is imposed: a cuspidal representation is a twist of a unitary one by a real power of $|\det|$. For a Galois representation r and a finite set S of places containing the places where r is ramified and those above p,

$$L^S(s, r) = \prod_{v \notin S} \det(1 - \iota r(\mathrm{Frob}_v) q_v^{-s})^{-1}.$$

A cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is *regular L-algebraic* [BG14] if at every real place the restriction to $\mathbb{C}^\times$ of its Langlands parameter is a sum $\bigoplus_{j=1}^n z^{p_j} \bar{z}^{q_j}$ with $p_j, q_j \in \mathbb{Z}$ and with the p_j pairwise distinct. For n even, this means that $\Pi \otimes |\det|^{1/2}$ is regular algebraic in the sense of Clozel [Clo90]; for n odd the two notions agree.

Let D_k, $k \geq 2$, be the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k with unitary central character sgn^k. By a *Hilbert cusp form* we mean a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with $\pi_\tau \simeq D_{k_\tau}$ at every real place τ, with $k_\tau \geq 2$ all of the same parity; it has a Galois representation ρ_π, see (R7). A representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *modular* if $\rho \simeq \rho_\pi \otimes \psi$ for a Hilbert cusp form π and a character ψ of the form (finite order)$\cdot \varepsilon_{\mathrm{cyc}}^n$, $n \in \mathbb{Z}$. Equivalently, ρ corresponds to $\pi(\rho) = \pi \otimes \eta |\det|^{n-m/2}$, where π is a Hilbert cusp form, η has finite order, $n \in \mathbb{Z}$

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and $m = \max_\tau k_\tau - 1$. A modular representation is irreducible, odd, and de Rham with distinct Hodge–Tate weights at every embedding.

A representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *Lie-irreducible* if it is irreducible on every open subgroup. (For an irreducible representation with distinct Hodge–Tate weights this means that it is not induced from a character of a quadratic extension, which is then necessarily CM.) It is *potentially modular* if there is a finite Galois totally real extension F'/F such that $\rho|_{G_{F'}}$ is modular.

For a Lie-irreducible representation, $\mathrm{ad}^0\rho$ is irreducible. Indeed, let G be the Zariski closure of its image. The inverse image of G° is open, so G° acts irreducibly. A proper connected algebraic subgroup of PGL_2 is solvable and its inverse image in GL_2 preserves a line. Thus the projective image of G° is PGL_2, whose adjoint representation is irreducible.

Remark 1.1 (Solvable subextensions). Let ρ be potentially modular over F', and assume that $\rho|_{G_{F'}}$ is irreducible (this holds if ρ is Lie-irreducible). Let $F'' \subseteq F'$ be a subfield with $\mathrm{Gal}(F'/F'')$ solvable. Then $\rho|_{G_{F''}}$ is modular. Indeed, filter $\mathrm{Gal}(F'/F'')$ by normal subgroups with cyclic quotients of prime order; at each step, the cuspidal representation attached to the restriction of ρ to the larger field is invariant under the Galois group of the cyclic extension, so it is the base change of a cuspidal representation of the smaller field, unique up to twist by a character of that Galois group [AC89]; a twist matches the Galois representation, by Chebotarev and the irreducibility of $\rho|_{G_{F'}}$. See [Tay02, §4]. This is how potential modularity enters Lemma 2.2.

Theorem 1.2. *Let $\rho_1, \rho_2 : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be continuous Lie-irreducible representations. Assume:*

- (H1) *ρ_1 and ρ_2 are potentially modular, over fields F_1' and F_2' respectively.*
- (H2) *There is a regular L-algebraic cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_F)$ which corresponds to $\rho_1 \otimes \rho_2$: for almost all v, $t(\Pi_v)$ is the conjugacy class of $\iota(\rho_1 \otimes \rho_2)(\mathrm{Frob}_v)$.*

Then ρ_1 and ρ_2 are modular, and $\Pi \simeq \pi(\rho_1) \boxtimes \pi(\rho_2)$.

We do not require ρ_1 and ρ_2 to become modular over a common field. Nor do we require anything about the relative parity of their weights. By Lemma 2.1, hypothesis (H1) implies that ρ_1 and ρ_2 are de Rham at the places above p; this is used in Steps 6–8.

Remark 1.3 (On the potential modularity hypothesis). Hypothesis (H1) is known in many cases. One must always assume that ρ is odd, i.e. $\det \rho(c) = -1$ for every complex conjugation c, and in general that the residual image is large. The typical situations are the following.

(1) *Large residual image and local hypotheses at p.* Suppose that ρ is unramified outside finitely many places, totally odd, regular and potentially diagonalizable at every place above p. Examples of the last condition include ordinary crystalline representations with distinct labelled Hodge–Tate weights, and crystalline representations over unramified extensions of $\mathbb{Q}_p$ whose labelled weights lie in an interval of length at most $p-2$; see [BLGGT14, Lemma 1.4.3]. If $p \geq 7$ and $\bar\rho|_{G_{F(\zeta_p)}}$ is absolutely irreducible, then [BLGGT14, Corollary 4.5.2], with polarization given by $\det \rho$, gives a finite Galois totally real extension over which ρ is modular.

(2) *p totally split in F.* If p splits completely in F, the local conditions in (1) can be relaxed, because modularity lifting theorems are available for $\mathrm{GL}_2(\mathbb{Q}_p)$ without potential diagonalizability [Kis09, Paš15, HT15]. One needs a Galois totally real extension F'/F in which p still splits completely and over which $\bar\rho$ is modular; such extensions are constructed in [Tay02, Tay06, Sno09] under hypotheses on $\bar\rho$ at the places above p. The lifting theorems also carry hypotheses on $\bar\rho$: irreducibility over $F'(\zeta_p)$, and the exclusion of some small primes and of some shapes of $\bar\rho|_{G_{\mathbb{Q}_p}}$. We refer to these papers for the precise statements.

(3) *Compatible systems.* If ρ belongs to a regular, totally odd weakly compatible system of representations of dimension two, every member of which is irreducible, then the system

is potentially modular [BLGGT14, Theorem 5.4.1]. The determinant supplies the polarization. Purity follows from this theorem and need not be imposed separately; see also [PT15].

By Remark 1.1, in all these cases $\rho|_{G_{F''}}$ is also modular for every $F'' \subseteq F'$ with F'/F'' solvable; this is the form of (H1) used in Lemma 2.2. In Theorem 1.2 it is applied to ρ_1 and ρ_2 separately.

Remark 1.4 (Relation with the literature). We compare Theorem 1.2 with related results.

(1) *Patrikis.* Assuming the Fontaine–Mazur–Langlands conjecture, [Pat19, Proposition 3.3.8] proves an automorphic tensor factorization when a cuspidal L-algebraic representation has a Galois representation of the form $\rho_1 \otimes \rho_2$. The given factors in that statement need not themselves be geometric. In the setting of Theorem 1.2, hypothesis (H1) makes both factors geometric, so the conjecture would imply their modularity. Patrikis also explains that potential automorphy gives some cases after a finite base change. Theorem 1.2 gives the conclusion over F under the stated potential modularity hypotheses.

(2) *Mixed parity forms.* Consider two cuspidal representations on $\mathrm{GL}_2(\mathbb{A}_F)$ with discrete series at every real place, neither induced from a quadratic extension. Suppose that each has both even and odd weights, that their weight parities agree at each real place, and that their tensor product is cuspidal. With the normalization of [Pat19, Proposition 2.6.7], this tensor product is L-algebraic and its Galois representation has tensor factors no twist of which is de Rham; see also [HL24, Remark 1.5]. These factors are not potentially modular in our sense, since potential modularity implies the de Rham property by Lemma 2.1. Thus replacing (H1) by a condition allowing arbitrary twists that are not de Rham would admit counterexamples. The de Rham property is used in Steps 6–8 below.

(3) *Direct sums.* The analogous case $\sigma \oplus \sigma'$ is treated in [Ram13, Lemma 5.1 and Theorem C]; see Remark 4.2.

(4) *The opposite direction.* [LY16] and [Ram02] prove automorphy of representations of GO_4-type by splitting them into factors and applying [Ram00]. [Raj04] and [Pat19, Cor. 2.6.3] treat the uniqueness of the factorization, on the Galois and automorphic sides respectively.

(5) *Over* $\mathbb{Q}$. The statement is essentially known. Under mild hypotheses on the residual representation, a two-dimensional odd, irreducible, de Rham and regular ρ over $\mathbb{Q}$, unramified outside finitely many primes, is modular [Kis09, Paš15, HT15], without passing through Π. Moreover, under its hypotheses on connected monodromy and the tautological representation, [HL24, Theorem 1.4] gives tensor decompositions into symmetric powers of representations attached to modular forms, up to a character; in the case of two factors of dimension two, the symmetric powers have degree one. The interest of Theorem 1.2 lies in $F \neq \mathbb{Q}$.

(6) *The automorphic factorization.* Steps 1–3 of the proof, which show that $\Pi \simeq \pi_1' \boxtimes \pi_2'$ for cuspidal π_1', π_2' on $\mathrm{GL}_2(\mathbb{A}_F)$, are not new. Asgari and Shahidi [AS14] prove, using the descent of Hundley–Sayag [HS16], that a cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ whose twisted symmetric square L-function $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \chi^{-1})$ has a pole at $s = 1$ is a transfer from a split or quasi-split even general spin group; and in the classification of Asgari and Raghuram [AR11, Thm. 1.1] of the cuspidal representations Π of $\mathrm{GL}_4(\mathbb{A}_F)$ with non-cuspidal exterior square, the transfers from split GSpin_4 are the automorphic tensor products $\pi_1 \boxtimes \pi_2$, and those from quasi-split non-split GSpin_4 are Asai transfers. Since $\omega_\Pi \chi^{-2} = 1$ in our situation, the descent is to the split group, and $\Pi \simeq \pi_1' \boxtimes \pi_2'$. We include the argument, with the identification of split GSpin_4 with a subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_2$ (Remark 4.3), to fix the normalizations used in Steps 4–8. The additional argument in Steps 4–8 passes from this automorphic factorization, whose factors need not initially be algebraic, to the modularity of the given ρ_i over F. The use of adjoint representations for forms of mixed parity also appears in [Pat19, Proposition 2.6.7(iii)].

(7) *Shavali.* In [Sha26, Proposition 5.1], a regular algebraic cuspidal representation $\Pi = \pi_1 \boxtimes \pi_2$, with neither factor induced from a quadratic extension, has associated Galois representations whose monodromy Lie algebra has semisimple part $\mathfrak{so}_4$. Its proof uses a decomposition into two factors of dimension two, which need not themselves be geometric over F. Theorem 1.2 addresses the modularity over F of factors given in advance and assumed to be potentially modular.

(8) *Calegari–Gee.* In [CG13, §6.4] the tensor case $r \simeq a \otimes b$ appears in the study of irreducibility for $n = 4$, without any conclusion on the modularity of the factors. Their corrigendum [CG17] explains that the proof of their Theorem 4.1 for $n = 4, 5$ is not valid as written.

2. Preliminaries

2.1. **Known results.** We shall use the following results.

(R1) *Tensor product* [Ram00]. If π, π' are cuspidal on $\mathrm{GL}_2(\mathbb{A}_F)$, there is an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}_4(\mathbb{A}_F)$ which is functorial at all places, the archimedean ones included: the Langlands parameter of $(\pi \boxtimes \pi')_v$ is the tensor product of those of π_v and π'_v.

(R2) *Exterior square identity.* For representations V_1, V_2 of dimension two in characteristic zero,

$$\wedge^2(V_1 \otimes V_2) \otimes (\det V_1 \det V_2)^{-1} \simeq \mathrm{ad}^0 V_1 \oplus \mathrm{ad}^0 V_2.$$

This follows from $\wedge^2(V_1 \otimes V_2) \simeq (\mathrm{Sym}^2 V_1 \otimes \det V_2) \oplus (\det V_1 \otimes \mathrm{Sym}^2 V_2)$. We use it for Galois representations and for Satake parameters.

(R3) *Poles, nonvanishing and multiplicity one* [JS76, JS81]. Let c be unitary cuspidal and let $X = \boxplus_j c_j^{\boxplus e_j}$ be an isobaric sum of distinct unitary cuspidal representations.

- $L^S(s, c \times c^\vee)$ has a simple pole at $s = 1$.
- $L^S(s, c)$ is holomorphic and nonzero at $s = 1$, except when c is the trivial character of GL_1, in which case it has a simple pole. Hence $-\mathrm{ord}_{s=1} L^S(s, X)$ is the multiplicity of the trivial character in X.
- Strong multiplicity one holds for isobaric representations.

Here S is any finite set containing the ramified places. Removing the local factors preserves these assertions at $s = 1$, since the removed factors are holomorphic and nonzero there.

(R4) *Gelbart–Jacquet* [GJ78]. If π is cuspidal on $\mathrm{GL}_2(\mathbb{A}_F)$, then $\mathrm{ad}^0\pi$ is an isobaric automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, self-dual with trivial central character, functorial at all places, and it is cuspidal if and only if π is not dihedral. Its isobaric constituents are unitary. Indeed, a real norm twist of π disappears in $\mathrm{ad}^0\pi$. In the cuspidal case the trivial central character gives unitarity. If $\pi = \mathrm{AI}_{K/F}(\theta)$, then $\mathrm{ad}^0\pi = \eta_K \boxplus \mathrm{AI}_{K/F}(\theta/\theta^c)$, where c generates $\mathrm{Gal}(K/F)$ and η_K is its quadratic character; the quotient θ/θ^c is unitary.

(R5) *Descent* [HS16, Thm. 16.3.1]. Let Π be a unitary cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$ and χ a Hecke character. Suppose that $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \chi^{-1})$ has a pole at $s = 1$, i.e. Π is χ^{-1}-orthogonal, and that $\omega_\Pi \chi^{-2} = 1$. Then the descent $DC^a_\chi(\Pi)$ to the split group GSpin_4 is a nonzero cuspidal representation of $\mathrm{GSpin}_4(\mathbb{A}_F)$, generic and with central character χ, and every irreducible automorphic representation σ contained in it lifts weakly to Π with respect to the inclusion $\mathrm{GSO}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$. This is the case $n = 2$, $r = 1$, $\tau = \Pi$, $\omega = \chi$ of the cited theorem, with a the class of squares: the quadratic character ω_Π/χ^2 is trivial, hence compatible with a in the sense of [HS16, Def. 16.1.13], and G_4^a is the split group [HS16, Rem. 16.1.6]. See Remark 4.3.

(R6) *Purity* [Clo90, Lemme 4.9]. Let Π be cuspidal on $\mathrm{GL}_n(\mathbb{A}_F)$ such that some twist $\Pi \otimes |\det|^s$, $s \in \mathbb{R}$, is algebraic in the sense of Clozel (for instance Π L-algebraic, or a unitary twist of such a representation). If $\phi_{\Pi_\tau}|_{\mathbb{C}^\times} = \bigoplus_j z^{p_j} \bar{z}^{q_j}$, then $p_j + q_j$ is independent of j.

(R7) *Galois representations of Hilbert modular forms* [Car86, Tay89, BR93, Sai09, Ski09]. Let π be a Hilbert cusp form, with weights $k_\tau \geq 2$ of the same parity, and let $m = \max_\tau k_\tau - 1$.

Then there is a continuous irreducible representation $\rho_\pi : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which corresponds to $\pi \otimes |\det|^{-m/2}$, that is, $\iota\rho_\pi(\mathrm{Frob}_v)$ has eigenvalues $q_v^{m/2}$ times the Satake parameters of π_v for almost all v. It is de Rham at the places above p, with Hodge–Tate weights differing by $k_\tau - 1$ at τ. The central character ω_π has finite order, and $\det \rho_\pi = \omega_\pi \varepsilon_{\mathrm{cyc}}^{-m}$; in particular, the sum of these weights is m at every embedding.

(R8) *Galois representations for* GL_3 [BLGHT11, Theorem 1.1], using [Shi11, CH13]. Let Σ be a cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_F)$ with $\Sigma^\vee \simeq \Sigma$ and trivial central character. Suppose that its infinitesimal character at τ is $(d_\tau, 0, -d_\tau)$, with $d_\tau \geq 1$ an integer. Then there is a continuous semisimple Galois representation R_Σ which corresponds to Σ, is de Rham above p, has determinant 1, and has labelled Hodge–Tate weights $\{d_\tau, 0, -d_\tau\}$.

To specify the normalization, the algebraic weight in the cited theorem is $a_\tau = (d_\tau - 1, 0, 1 - d_\tau)$, which is dominant and equal to its dual. The representation $r_{p,\iota}(\Sigma)$ of that theorem corresponds to $\Sigma \otimes |\det|^{-1}$ and has weights $\{d_\tau + 1, 1, 1 - d_\tau\}$. Thus

$$R_\Sigma = r_{p,\iota}(\Sigma) \otimes \varepsilon_{\mathrm{cyc}}$$

has the stated Frobenius normalization and weights. Its determinant is 1 by the unramified Satake comparison and Chebotarev. The condition on the archimedean values of the character in the cited theorem holds because that character is trivial.

2.2. **Lemmas.**

Lemma 2.1. *Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be potentially modular. Then ρ is unramified outside finitely many places, totally odd, and de Rham at the places above p, with distinct Hodge–Tate weights at every embedding. Moreover, $\det \rho = \chi_\rho \varepsilon_{\mathrm{cyc}}^{-w_\rho}$ with χ_ρ of finite order and $w_\rho \in \mathbb{Z}$. The sum of the Hodge–Tate weights of ρ at every embedding is w_ρ.*

Proof. The modular restriction to $G_{F'}$ is unramified outside finitely many places. Since F'/F is finite, the same holds for ρ. Total oddness descends because F' is totally real. The restriction is de Rham with distinct labelled Hodge–Tate weights by (R7). The de Rham property descends through a finite extension of local fields, and the labelled weights are preserved under extension of embeddings.

By (R7) and the definition of modularity, there is an integer w_ρ such that $(\det \rho)\varepsilon_{\mathrm{cyc}}^{w_\rho}$ has finite image on $G_{F'}$. A character with finite image on an open subgroup has finite image on the whole group. This proves the determinant formula, and its Hodge–Tate weight gives the asserted sum. □

Lemma 2.2 (Non-vanishing at $s = 1$ via Brauer)**.** *Let ρ be two-dimensional, Lie-irreducible and potentially modular over F', and let η be a finite order Hecke character of F. Then $L^S(s, \mathrm{ad}^0\rho \otimes \eta)$ is meromorphic on $\mathbb{C}$, and holomorphic and nonzero at $s = 1$.*

Proof. Enlarge S so that it contains the places which ramify in F' and those where η is ramified. By Brauer's theorem, $1_G = \sum_i n_i \mathrm{Ind}_{H_i}^G \psi_i$ in $G = \mathrm{Gal}(F'/F)$, with $n_i \in \mathbb{Z}$, H_i elementary (hence solvable) and ψ_i characters of H_i. Let $F_i = F'^{H_i}$, let S_i be the set of places of F_i above S, and let $\eta_{F_i} = \eta \circ N_{F_i/F}$. Since $r \simeq \sum_i n_i \mathrm{Ind}_{G_{F_i}}^{G_F}(r|_{G_{F_i}} \otimes \psi_i)$ as virtual representations for $r = \mathrm{ad}^0\rho \otimes \eta$, and all the places outside S are unramified,

$$L^S(s, \mathrm{ad}^0\rho \otimes \eta) = \prod_i L^{S_i}(s, \mathrm{ad}^0\rho|_{G_{F_i}} \otimes \psi_i\eta_{F_i})^{n_i}.$$

By Remark 1.1, applied to the solvable extension F'/F_i, $\rho|_{G_{F_i}}$ is modular, so it corresponds to a cuspidal representation σ_i of $\mathrm{GL}_2(\mathbb{A}_{F_i})$, and σ_i is not dihedral because ρ is Lie-irreducible. Hence $\mathrm{ad}^0\sigma_i$ is cuspidal by (R4); it is unitary, and $\mathrm{ad}^0\sigma_i \otimes \psi_i\eta_{F_i}$ is a unitary cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_{F_i})$ which corresponds to $\mathrm{ad}^0\rho|_{G_{F_i}} \otimes \psi_i\eta_{F_i}$. By (R3), each factor $L^{S_i}(s, \mathrm{ad}^0\sigma_i \otimes \psi_i\eta_{F_i})$ is meromorphic on $\mathbb{C}$, and holomorphic and nonzero at $s = 1$. The same holds for the product of their powers. □

Lemma 2.3. *Let σ, σ' be two-dimensional with $\mathrm{ad}^0\sigma \simeq \mathrm{ad}^0\sigma'$ irreducible. Then $\sigma' \simeq \sigma \otimes \psi$ for a continuous character ψ. If σ and σ' are de Rham, so is ψ.*

Proof. An isomorphism $\mathrm{ad}^0\sigma \simeq \mathrm{ad}^0\sigma'$ carries the Killing form to a multiple of the Killing form, because $\mathrm{ad}^0\sigma$ is irreducible; after scaling and changing its sign it lies in $\mathrm{SO}_3 \simeq \mathrm{PGL}_2$. Hence the projectivizations of σ and σ' are conjugate, and after conjugation $\sigma' = \sigma \otimes \psi$ for a continuous character ψ; compare [Ram02, Thm. 3.10]. For the second assertion, ψ is a summand of $\sigma^\vee \otimes \sigma' \simeq \sigma^\vee \otimes \sigma \otimes \psi$, which is de Rham. □

Lemma 2.4 (A twist normalizing the archimedean components)**.** *Let ω be a Hecke character of F with $\omega_\tau(x) = \mathrm{sgn}(x)^{e_\tau}|x|^{s_\tau}$ at each real place τ. Then there is a Hecke character μ with $\mu_\tau(x) = \mathrm{sgn}(x)^{f_\tau}|x|^{s_\tau/2}$ for some $f_\tau \in \{0, 1\}$.*

Proof. Let $U \subset \widehat{\mathcal{O}}_F^\times$ be an open compact subgroup with $\omega|_U = 1$, and let E^+ be the group of totally positive units in U. For $\varepsilon \in E^+$, the condition $\omega(\varepsilon) = 1$ gives $\prod_\tau |\varepsilon|_\tau^{s_\tau} = 1$. Hence $\prod_\tau |\eta|_\tau^{s_\tau/2} = 1$ for all η in $(E^+)^2$, a subgroup of finite index.

By Chevalley's theorem [Che51], every finite index subgroup of $\mathcal{O}_F^\times$ contains a congruence subgroup $\mathcal{O}_F^\times(\mathfrak{a}) = \{\varepsilon \equiv 1 \bmod \mathfrak{a}\}$. So there is an open compact $U' \subseteq U$ such that the totally positive units in U' lie in $(E^+)^2$. Then the character $x_\infty u \mapsto \prod_\tau |x_\tau|^{s_\tau/2}$ of $H = F_{\infty,+}^\times U'$ is trivial on $F^\times \cap H$. It therefore defines a character of $F^\times H/F^\times$, an open subgroup of finite index of $\mathbb{A}_F^\times/F^\times$, which extends to $\mathbb{A}_F^\times/F^\times$. Such an extension exists because $\mathbb{C}^\times$ is divisible, and it is continuous because the subgroup is open. On the component groups at the real places it has values in $\{1, -1\}$, giving the exponents f_τ. □

3. Proof of Theorem 1.2

Let $\rho = \rho_1 \otimes \rho_2$. By Lemma 2.1, $\det \rho_i = \chi_i \varepsilon_{\mathrm{cyc}}^{-w_i}$ with χ_i of finite order and $w_i \in \mathbb{Z}$. Put

$$\chi = \chi_1\chi_2, \qquad w = w_1 + w_2, \qquad \Pi^u = \Pi \otimes |\det|^{w/2}.$$

Then $t(\Pi_v^u) = q_v^{-w/2}\, \iota\rho(\mathrm{Frob}_v)$ for almost all v. Since $\det \rho = \chi^2 \varepsilon_{\mathrm{cyc}}^{-2w}$, the central character of Π^u agrees with χ^2 at almost all places, hence $\omega_{\Pi^u} = \chi^2$. In particular Π^u is a cuspidal representation with unitary central character, so it is unitary. It is a twist of Π, so (R6) applies to it and its archimedean parameters are regular. In Steps 1–7 we write Π for Π^u.

Step 1: orthogonal type. On the Galois side, $\rho^\vee \otimes \det \rho_1 \det \rho_2 \simeq \rho$. Hence $t(\Pi_v)^{-1}\chi(\varpi_v) = t(\Pi_v)$ as conjugacy classes for almost all v, and by strong multiplicity one,

$$\Pi \simeq \Pi^\vee \otimes \chi, \qquad \omega_\Pi = \chi^2.$$

Step 2: pole of the symmetric square. Since $\Pi^\vee \simeq \Pi \otimes \chi^{-1}$,

$$L^S(s, \Pi \times \Pi^\vee) = L^S(s, \Pi, \mathrm{Sym}^2 \otimes \chi^{-1})\, L^S(s, \Pi, \wedge^2 \otimes \chi^{-1}).$$

The left hand side has a simple pole at $s = 1$. On the Galois side, $\wedge^2\rho \otimes (\det \rho_1 \det \rho_2)^{-1} = \mathrm{ad}^0\rho_1 \oplus \mathrm{ad}^0\rho_2$, and $\wedge^2 t(\Pi_v) \cdot \chi(\varpi_v)^{-1} = \iota(\wedge^2\rho \otimes (\chi\varepsilon_{\mathrm{cyc}}^{-w})^{-1})(\mathrm{Frob}_v)$. Hence

$$L^S(s, \Pi, \wedge^2 \otimes \chi^{-1}) = L^S(s, \mathrm{ad}^0\rho_1)\, L^S(s, \mathrm{ad}^0\rho_2),$$

which is meromorphic on $\mathbb{C}$, and holomorphic and nonzero at $s = 1$ by Lemma 2.2, applied to each ρ_i with its own field F_i' and $\eta = 1$. Therefore the quotient $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \chi^{-1})$ is meromorphic near $s = 1$ and has a simple pole at $s = 1$.

Step 3: descent. Steps 1–3 recover a known result (Remark 1.4(6)); see [AS14] and [AR11, Thm. 1.1]. By Step 2 and $\omega_\Pi = \chi^2$, (R5) applies: there is an irreducible cuspidal σ on $\mathrm{GSpin}_4(\mathbb{A}_F)$ lifting weakly to Π. The dual group of GSpin_4 is $\mathrm{GSO}_4(\mathbb{C}) = \beta(\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C}))$, with $\beta(g_1,g_2) = g_1\otimes g_2$ [AR11, §2.2]. Hence

$$\mathrm{GSpin}_4 \simeq H := \{(g_1,g_2)\in \mathrm{GL}_2\times\mathrm{GL}_2 : \det g_1 = \det g_2\},$$

and the lift to GL_4 corresponds to the tensor product.

We apply [LS19, Thm. 1.1.1 and Thm. 5.2.2] to the inclusion $f : H \hookrightarrow \mathrm{GL}_2\times\mathrm{GL}_2$. It is injective, and on derived groups it is the identity of $\mathrm{SL}_2\times\mathrm{SL}_2$, in particular a central isogeny. The case $\mathrm{SL}_2\subset\mathrm{GL}_2$ is [LL79]. Hence, by the second assertion of [LS19, Thm. 1.1.1], σ occurs in the restriction to $H(\mathbb{A}_F)$ of the space of a cuspidal representation $\pi_1'\otimes\pi_2'$ of $(\mathrm{GL}_2\times\mathrm{GL}_2)(\mathbb{A}_F)$; the π_i' are cuspidal, not necessarily unitary. For almost all v, both σ_v and $\pi_{1,v}'\otimes\pi_{2,v}'$ are unramified, and σ_v is the constituent of the restriction of $\pi_{1,v}'\otimes\pi_{2,v}'$ to $H(F_v)$ which is unramified for $H(\mathcal{O}_{F_v})$. Its Satake parameter is therefore the image of that of $(\pi_{1,v}',\pi_{2,v}')$ under the dual map $\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})\to\mathrm{GSO}_4(\mathbb{C})$, and its image in $\mathrm{GL}_4(\mathbb{C})$ is $t(\pi_{1,v}')\otimes t(\pi_{2,v}')$. So $\pi_1'\boxtimes\pi_2'$, which is automorphic by (R1), and Π have the same Satake parameters at almost all places. By strong multiplicity one (R3), $\Pi\simeq\pi_1'\boxtimes\pi_2'$, and by (R1), $\Pi_v\simeq\pi_{1,v}'\boxtimes\pi_{2,v}'$ at all places, including the real ones.

Let $\varepsilon = \omega_{\pi_1'}\omega_{\pi_2'}\chi^{-1}$; then $\varepsilon^2 = \omega_\Pi\chi^{-2} = 1$. By (R4), the representation

$$X := (\mathrm{ad}^0\pi_1'\otimes\varepsilon)\boxplus(\mathrm{ad}^0\pi_2'\otimes\varepsilon) \quad\longleftrightarrow\quad \mathrm{ad}^0\rho_1\oplus\mathrm{ad}^0\rho_2 \tag{1}$$

is isobaric automorphic with unitary constituents. The displayed correspondence follows from (R2): at almost all places, both sides have Satake or Frobenius parameter $\wedge^2 t(\Pi_v)\chi(\varpi_v)^{-1}$.

Step 4: the π_i' are not dihedral. Suppose that π_1' is automorphically induced from a quadratic extension K/F, with quadratic character η_K. Then $\mathrm{ad}^0\pi_1'\simeq\eta_K\boxplus(\cdots)$, so the isobaric representation $X\otimes\eta_K\varepsilon\simeq(\mathrm{ad}^0\pi_1'\otimes\eta_K)\boxplus(\mathrm{ad}^0\pi_2'\otimes\eta_K)$, a sum of unitary cuspidal representations, contains the trivial character, and $L^S(s,X\otimes\eta_K\varepsilon)$ has a pole at $s=1$ by (R3). But by (1) this function is

$$L^S(s,\mathrm{ad}^0\rho_1\otimes\eta_K\varepsilon)\,L^S(s,\mathrm{ad}^0\rho_2\otimes\eta_K\varepsilon),$$

which is holomorphic and nonzero at $s=1$ by Lemma 2.2, applied with $\eta=\eta_K\varepsilon$, a contradiction. The same argument applies to π_2'. By (R4), $\mathrm{ad}^0\pi_1'$ and $\mathrm{ad}^0\pi_2'$ are cuspidal.

Step 5: archimedean components. Let τ be real and let ϕ_i be the parameter of $\pi_{i,\tau}'$. By Step 3 and (R1), the parameter of Π_τ is $\phi_1\otimes\phi_2$. Write $\phi_\Pi|_{\mathbb{C}^\times} = (\phi_1\otimes\phi_2)|_{\mathbb{C}^\times} = \bigoplus_j z^{p_j}\bar z^{q_j}$. By (R6) and the finite order central character $\omega_\Pi=\chi^2$, $p_j+q_j=0$, so $p_j=\frac12(p_j-q_j)$. Regularity of Π requires the differences p_j-q_j to be distinct.

The representation $\pi_{i,\tau}'$ is irreducible and generic, so it is either a principal series or a twist of a discrete series. If ϕ_1 were a principal series parameter, a sum of two characters of $\mathbb{R}^\times$, then $\phi_1|_{\mathbb{C}^\times} = |z|^{2c}\oplus|z|^{2c'}$ and each difference p_j-q_j of ϕ_2 would occur twice in $\phi_1\otimes\phi_2$, contradicting regularity. The same holds for ϕ_2. Therefore

$$\pi_{i,\tau}'\simeq D_{k_{i,\tau}}\otimes|\det|^{t_{i,\tau}},\qquad k_{i,\tau}\geq 2,\quad t_{i,\tau}\in\mathbb{C}.$$

Consequently $\mathrm{ad}^0\pi_{i,\tau}'$ has exponents $\{k_{i,\tau}-1,0,1-k_{i,\tau}\}$, since the twists $|\det|^{t_{i,\tau}}$ cancel in ad^0. So $\mathrm{ad}^0\pi_i'$ is cuspidal (Step 4), self-dual and regular algebraic, even if π_i' is not algebraic.

Step 6: projective matching and weights. Let $R_i := R_{\mathrm{ad}^0\pi_i'}$, given by (R8) with $d_\tau = k_{i,\tau}-1$. Step 5 verifies its archimedean hypothesis, and (R4) gives the trivial central character. By (1), Chebotarev and the semisimplicity of both sides,

$$(R_1\otimes\varepsilon)\oplus(R_2\otimes\varepsilon)\simeq\mathrm{ad}^0\rho_1\oplus\mathrm{ad}^0\rho_2.$$

Since the $\mathrm{ad}^0\rho_i$ are irreducible of dimension 3, each $R_i \otimes \varepsilon$ is isomorphic to one of them; after renumbering π_1', π_2', we have $R_i \otimes \varepsilon \simeq \mathrm{ad}^0\rho_i$. The central character of $\mathrm{ad}^0\pi_i'$ is trivial, so $\det R_i = 1$ by (R8), and $\det(R_i \otimes \varepsilon) = \varepsilon^3 = \varepsilon$. As $\det \mathrm{ad}^0\rho_i = 1$, we get $\varepsilon = 1$, that is,

$$R_{\mathrm{ad}^0\pi_i'} \simeq \mathrm{ad}^0\rho_i \qquad (i = 1, 2).$$

By (R8), the labelled Hodge–Tate weights of R_i are exactly $\{k_{i,\tau}-1, 0, 1-k_{i,\tau}\}$. The representations ρ_i are de Rham by Lemma 2.1, and the Hodge–Tate weights of $\mathrm{ad}^0\rho_i$ at τ are $\{A_{i,\tau} - B_{i,\tau}, 0, B_{i,\tau} - A_{i,\tau}\}$, where $\{A_{i,\tau}, B_{i,\tau}\}$ are those of ρ_i. Hence

$$k_{i,\tau} - 1 = |A_{i,\tau} - B_{i,\tau}| \qquad \text{for all } \tau.$$

In other words, at every real place each π_i' carries the weights of its own ρ_i, and the weights cannot be mixed between the two factors.

Step 7: parity and algebraicity. By Lemma 2.1, $A_{i,\tau} + B_{i,\tau} = w_i$ for all τ. By Step 6, $k_{i,\tau} - 1 = |A_{i,\tau} - B_{i,\tau}| \equiv w_i \pmod 2$ for all τ, so each π_i' has paritious weights.

The central character of π_i' satisfies $\omega_{\pi_i',\tau} = \mathrm{sgn}^{k_{i,\tau}} |\cdot|^{2t_{i,\tau}}$. By Lemma 2.4, applied to $\omega = \omega_{\pi_i'}^{-1}$, there is μ_i with $\mu_{i,\tau} = \mathrm{sgn}^{f_\tau} |\cdot|^{-t_{i,\tau}}$. Since $D_k \otimes \mathrm{sgn} \circ \det \simeq D_k$, we get

$$(\pi_i' \otimes \mu_i)_\tau \simeq D_{k_{i,\tau}} \qquad \text{for all } \tau.$$

Thus $\pi_i' \otimes \mu_i$ is a Hilbert cusp form, and by (R7) it has a Galois representation $r_i = \rho_{\pi_i' \otimes \mu_i}$, which is irreducible and de Rham. Since $\mathrm{ad}^0(\pi_i' \otimes \mu_i) = \mathrm{ad}^0\pi_i'$ and the scalar $q_v^{m/2}$ of (R7) cancels in ad^0, the semisimple representations $\mathrm{ad}^0 r_i$ and $R_{\mathrm{ad}^0\pi_i'}$ have the same characteristic polynomials of Frobenius at almost all places, and Chebotarev gives $\mathrm{ad}^0 r_i \simeq R_{\mathrm{ad}^0\pi_i'} \simeq \mathrm{ad}^0\rho_i$.

Step 8: conclusion. By Lemma 2.3, applied to the de Rham representations r_i and ρ_i, we have $\rho_i \simeq r_i \otimes \psi_i$ with ψ_i a de Rham character of G_F. Put $m_i = \max_\tau k_{i,\tau} - 1$. By Lemma 2.1 and (R7),

$$\psi_i^2 = \frac{\det \rho_i}{\det r_i} = \xi_i \varepsilon_{\mathrm{cyc}}^{m_i - w_i},$$

where ξ_i has finite order. Since ψ_i is de Rham, its Hodge–Tate weight at every embedding is the integer $h_i = (w_i - m_i)/2$. Thus $(\psi_i \varepsilon_{\mathrm{cyc}}^{h_i})^2 = \xi_i$ has finite image, so $\psi_i \varepsilon_{\mathrm{cyc}}^{h_i}$ itself has finite image. It follows that ψ_i is a finite order character times an integral power of $\varepsilon_{\mathrm{cyc}}$. Therefore ρ_i is modular, and it corresponds to a cuspidal representation $\pi(\rho_i)$, a twist of π_i' by a Hecke character. Finally we return to the original Π of (H2). By (R1), the isobaric representation $\pi(\rho_1) \boxtimes \pi(\rho_2)$ corresponds to $\rho_1 \otimes \rho_2$, as does Π, so $\Pi \simeq \pi(\rho_1) \boxtimes \pi(\rho_2)$ by strong multiplicity one. □

4. Remarks

Remark 4.1 (The role of each ingredient). We summarize how the hypotheses enter the proof.

- Potential modularity is used through Lemma 2.2, always for a single ρ_i: for $\mathrm{ad}^0\rho_i$ and its quadratic twists. No common field of potential modularity is needed. It is also used through Lemma 2.1, which gives the de Rham property needed in Steps 6–8.
- The deep input in Steps 1–3 is the descent (R5), which recognizes the image of $\mathrm{GL}_2 \times \mathrm{GL}_2$; this part is known [AS14, AR11].
- The role of (R8) is to match each automorphic adjoint with the adjoint of one of the given Galois factors, using one global ordering. This gives the labelled weights of that factor at every real place and proves that its weights have constant parity. If the weights of the automorphic factors are already known to be paritious, (R7) can instead be applied directly to $\pi_i' \otimes \mu_i$, and the same comparison of adjoints applies. This is automatic when $F = \mathbb{Q}$.

Remark 4.2 (Comparison with the direct sum case). In [Ram13, Lemma 5.1 and Theorem C], under his crystallinity and weight hypotheses, Ramakrishnan treats $\rho = \sigma \oplus \sigma'$ using Taylor's potential modularity, Brauer's theorem, Kim's exterior square transfer [Kim03], and results of Jacquet and Shalika. Here the elementary identity (R2), together with the adjoint lifts after descent, suffices. There are two differences.

(1) In the direct sum case, the decomposition $\pi \simeq \eta \boxplus \eta'$ is read off from the order of the pole of $L(s, \pi \times \pi^\vee)$, and it is unique. Here we use descent to GSpin_4, and the factorization has the ambiguity of interchanging the factors and replacing them by $(\pi_1' \otimes \mu, \pi_2' \otimes \mu^{-1})$. Steps 5–8 identify automorphic representations corresponding to the given Galois factors.
(2) In the direct sum case, the archimedean parameters of η and η' are summands of those of π, so there is no ambiguity in the weights. Here there is, and Step 6 resolves it.

Remark 4.3 (On the descent (R5)). We add four comments on the use of [HS16].

(1) The *Main Theorem (even case)* of [HS16, §12] assumes $n \geq 2$ and that ω is not the square of another Hecke character, whereas in our situation $\omega = \chi$ may be a square. The refined statement [HS16, Thm. 16.3.1], which is the one we use, is formulated for an arbitrary Hecke character ω, and so are the results on Eisenstein series [HS16, Thm. 15.0.12] and on the vanishing of incompatible descents [HS16, Thm. 16.2.1] on which it rests. By [HS16, Rem. 18.0.5], the hypothesis that ω is not a square only serves to force the rank m to be even under ω^{-1}-orthogonality, and the proof is blind to this parity. Here $m = 4$ is already even.
(2) By [HS16, Rem. 2.2.2(1)], orthogonality may be defined using either complete or partial L-functions.
(3) The passage from GSpin_4 to $\mathrm{GL}_2 \times \mathrm{GL}_2$ is not written out in [AR11, §5.2]. There it is only stated that Π is as in case (ii)(a) when $\mu = \omega_\Pi \chi^{-2}$ is trivial, and GSpin_4 is described in §2.2. Step 3 supplies this passage via [LS19]. That a pole of the twisted symmetric square L-function implies a transfer from an even general spin group is [AS14], based on [HS16].
(4) By [HS16, Thm. 16.3.1(2)], the central character of σ is χ. With the identification $\mathrm{GSpin}_4 \simeq H$ this yields $\omega_{\pi_1'} \omega_{\pi_2'} = \chi$, that is, $\varepsilon = 1$ already in Step 3. We do not use this: the determinant argument of Step 6 gives $\varepsilon = 1$ directly.

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