hexagon:2611.00621v1

COMPLETE MODULUS LOCKING IN THE EXCEPTIONAL SET OF THE ABC CONJECTURE

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Abstract. Let $N_\lambda(X)$ count primitive positive triples $a+b=c\le X$ with $\mathrm{rad}(abc)<c^\lambda$. We prove, using inherited anatomic estimates and a new finite exact-rational implication, that

$$N_\lambda(X)\ll_{\lambda,\varepsilon} X^{U+\varepsilon}\qquad (0<\lambda\le 1),\quad U=\frac{11680007}{20000012}<\frac{73}{125}=0.584.$$

The new arithmetic-computational step exhausts all third-letter whole-value modulus sets. Index 1 is eliminated by a one-Lipschitz argument, leaving 32 subsets of $\{2,3,4,5,6\}$. Grouped power-difference bounds are used in every fixed degree $2\le e\le 6$. A branch-and-bound tree with 65,576 leaves is verified by an independent standard-library checker which reconstructs every inequality and checks every rational multiplier; floating-point linear programming is used only to discover the tree.

We also prove that the enlargement is mathematically essential, not merely a faster solver. An explicit rational point feasible for the old modulus family has $D=403/690$, whereas FullOC gives the collision bound $53/92$ at the same point. Thus the old and new exponent-program optima satisfy $D^*_{\mathrm{FullOC}}\le U<403/690\le D^*_{\mathrm{old}}$. A second explicit point gives $7/12\le D^*_{\mathrm{FullOC}}$. These are statements about exponent programs, not lower bounds for abc triples. No assertion of abc, Szpiro, finiteness of the exceptional set, or an Arakelov degeneration functor is made.

Contents

1. Introduction and status of the results 2
1.1. Main arithmetic theorem 2
1.2. What is new 3
1.3. Provenance and logical status 3
2. Anatomic boxes and the inherited exponent system 3
2.1. Powerful-coordinate boxes 3
2.2. The four inherited estimates 4
3. Uniform power differences and twisted energy 5
3.1. Weighted differences of arbitrary fixed degree 5
3.2. Energy with an arbitrary unit twist 6
4. The collision inequality with a third-letter modulus 7
5. Complete modulus locking and elimination of index 1 9
5.1. The deletion inequality 9
5.2. The finite collision families 10
6. Piecewise-linear encoding and exact rational certificates 10
6.1. The 25-variable root system 10
6.2. Branch semantics 11
6.3. Rational leaf proofs 11
6.4. The FullOC upper certificate 11
6.5. Independent verification 12
7. Passage from the exponent program to $N_\lambda(X)$ 12
8. Strict separation and exact program obstructions 13
8.1. An exact feasible point for the old family 13
8.2. The new family cuts this point 14

Date: 19 November 2026.

2020 *Mathematics Subject Classification.* Primary 11D45; Secondary 11N36, 11D75, 11Y16.

Key words and phrases. abc conjecture, exceptional set, radical, collision inequality, exact rational certificate, branch and bound.

8.3. A lower obstruction for the FullOC program 15
9. Scope, limitations and relation to abc–Szpiro 15
9.1. What the theorem does and does not say 15
9.2. Limitations of the present computation 16
9.3. Conclusion 16
Appendix A. Certificate data and reproduction 16
A.1. Data schema 16
A.2. Verification commands 16
A.3. Principal SHA-256 hashes 17
A.4. Audit boundary 17
Data availability 17
References 17

1. Introduction and status of the results

For $\lambda > 0$, define

$$N_\lambda(X) = \#\left\{(a,b,c) \in \mathbb{N}^3 : \ a+b=c \le X, \ \gcd(a,b,c)=1, \ \operatorname{rad}(abc) < c^\lambda\right\}.$$

The abc conjecture predicts that $N_\lambda(X)$ is bounded for each fixed $\lambda < 1$. This paper concerns unconditional upper bounds for this counting function; it does not assert the pointwise radical inequality required by abc.

Bernert, Browning, Lichtman and Teräväinen introduced the anatomic exponent program used below and obtained the exponent $3/5$ for $0 < \lambda < 1$, together with a budget-dependent bound for $0 < \lambda \le 2$ [1]. Li subsequently introduced collision inequalities and radical-budget refinements [2], and then grouped power differences, released coordinates and a whole-value residue bound [3]. The present argument starts from that framework. Its new ingredients are the complete whole-value modulus family, its exact linearization in degrees through 6, and the rational separation and obstruction analysis.

1.1. **Main arithmetic theorem.**

Theorem 1.1 (Computer-assisted theorem with exact rational proof data)**.** *Put*

$$U = \frac{11680007}{20000012} = 0.583999999600000240\ldots.$$

For every $0 < \lambda \le 1$ and every $\varepsilon > 0$,

$$N_\lambda(X) \ll_{\lambda,\varepsilon} X^{U+\varepsilon}. \tag{1.1}$$

In particular,

$$N_\lambda(X) \ll_{\lambda,\varepsilon} X^{73/125+\varepsilon} = X^{0.584+\varepsilon}, \qquad \frac{73}{125} - U = \frac{1}{2500001500}.$$

The finite optimization step is not inferred from a floating-point optimum. It is proved by a complete disjunctive tree and positive rational multipliers, independently reconstructed and checked in exact arithmetic. The passage from the exponent program to (1.1), including all logarithmic, divisor and decomposition losses, is given in Section 7.

For orientation, Table 1 lists the stated exponents in the principal sources used here. Decimal values are rounded only for display. No claim of a comprehensive literature optimum is made.

Relative to the exact exponent of [3], the gain is

$$\frac{589903}{1000008} - U = \frac{29491659695}{5000043000024} = 0.0058982812137\ldots.$$

TABLE 1. Selected general bounds for $N_\lambda(X)$ at the budget endpoint $\lambda = 1$; the BBLT 3/5 bound is stated for $\lambda < 1$.

Source	Certified exponent	Approximation
BBLT [1]	3/5	0.600000000
Li, collision program [2]	54/91	0.593406593
Li, squares and twisted energy [3]	589903/1000008	0.589898281
Theorem 1.1	**11680007/20000012**	**0.5839999996**

1.2. **What is new.** The argument has five connected outputs.

(1) **All fixed degrees through 6.** The grouped difference estimate is used with $e = 2, 3, 4, 5, 6$. The underlying uniform count for $aU^e - bV^e = m$ in arbitrary fixed degree is already proved in [3, Appendix B]; we reproduce its proof and do not claim a new general-degree counting theorem.

(2) **Complete whole-value modulus locking.** For fixed remaining collision data, all third-letter sets $O_C \subseteq \{1, \dots, 6\}$ are considered. A one-Lipschitz deletion argument proves index 1 redundant, leaving the 32 subsets of $\{2, 3, 4, 5, 6\}$. This completeness statement is specific to the whole-value lock and the savings $\delta_0, \delta_2, \dots, \delta_6$; it is not asserted for an arbitrary coordinate-lock program.

(3) **A new exact upper certificate.** The resulting finite implication has 65,576 leaves, 4,591 geometric nodes and 7,623 collision nodes. Its maximum leaf bound is exactly U.

(4) **Strict separation from the old family.** The old modulus family has an exact feasible point with $D = 403/690$. At that point a new FullOC instance gives $53/92 < D$. Consequently the stronger arithmetic family produces an independent exponent improvement.

(5) **A precise remaining program obstruction.** A symmetric rational point satisfies all 15 exponent pairs and all 32 FullOC sets with $D = 7/12$. Thus the present program cannot prove an exponent below 7/12, and its remaining possible gain below U is at most $U - 7/12$, without any assertion that this gain is attainable.

The two feasible points are lower obstructions for linear programs. They do not construct integer triples and give no lower bound for $N_\lambda(X)$.

1.3. **Provenance and logical status.** Propositions taken from the literature are labelled as inherited inputs. Section 2 records the BBLT anatomic estimates in the exponent form used by Li. Section 3 reproduces the proofs of the fixed-degree weighted-difference estimate and the arbitrary-unit energy estimate from [3], so that the later collision proof is self-contained. The new work begins with the exhaustive degree-and-modulus implementation of the collision inequality, the index-1 elimination, the finite certificates and the exact program separation.

No floating-point quantity is used as a proof hypothesis. The numerical optimizer discovers branches and candidate multipliers only. The proof consists of the arithmetic inequalities, complete child coverage, and exact rational row and multiplier identities. The main result is therefore a computer-assisted theorem, not a numerical conjecture; it has not been represented here as peer reviewed.

Section 9 explains the boundary with abc and Szpiro. In particular, neither a finiteness assertion for $N_\lambda(X)$ when $\lambda < 1$ nor a degeneration functor on a cyclotomic tower is constructed.

2. Anatomic boxes and the inherited exponent system

This section records the external arithmetic framework. Nothing in it is claimed as new. We use the formulation of [3, Section 2], which in turn cites [1, Propositions 2.1, 2.3, 3.1 and 4.1].

2.1. **Powerful-coordinate boxes.** Write $x \sim P$ for $P < x \le 2P$, with the lowest dyadic block enlarged to include 1. After dyadic decomposition in height and radical, and after fixing a decomposition error $\varepsilon_0 > 0$, there is a fixed $d = d(\varepsilon_0) \ge 6$ such that the restricted count is at

most X^{ε_0} times a constant times the largest relevant box count. In a box, small coefficients are fixed and the three letters have the form

$$\mathsf{a}_0\prod_{i=1}^{d}x_{a,i}^{i},\qquad \mathsf{b}_0\prod_{i=1}^{d}x_{b,i}^{i},\qquad \mathsf{c}_0\prod_{i=1}^{d}x_{c,i}^{i},\qquad x_{s,i}\sim P_{s,i}.$$

The equation is

$$\mathsf{a}_0\prod_{i}x_{a,i}^{i}+\mathsf{b}_0\prod_{i}x_{b,i}^{i}=\mathsf{c}_0\prod_{i}x_{c,i}^{i}. \tag{2.1}$$

The small coefficients are pairwise coprime and at most X^{ε_0}. We may restrict to solutions whose three displayed letter values are pairwise coprime: in a primitive triple $a+b=c$, a common divisor of any two letters divides the third. Discarding further within-letter restrictions can only enlarge an upper bound.

The dyadic parameters obey the radical and height budgets

$$2^{-d-1}X^{\mu}\le\prod_{s,i}P_{s,i}\le X^{\mu+3\varepsilon_0/4},\qquad \prod_{i}P_{s,i}^{i}\le X, \tag{2.2}$$

and the original third letter satisfies $\prod_i P_{c,i}^{i}\ge X^{1-\varepsilon_0/2}$. Here $0<\mu\le\lambda$ is the radical budget of the restricted count. All later estimates are uniform over boxes and frozen coefficients once d is fixed.

Set

$$P_{s,i}=X^{\alpha_{s,i}},\qquad \alpha_s=\sum_{i=1}^{d}\alpha_{s,i},\qquad L=\alpha_a+\alpha_b+\alpha_c, \tag{2.3}$$

and write the nonzero box count as X^{D}. The quantities $\alpha_{s,i}$, α_s and D depend on the box, but we suppress this in the notation. Errors below are $O(\varepsilon_0)+o(1)$ as $X\to\infty$, with d fixed.

2.2. **The four inherited estimates.** The limiting exponent system uses only coordinates $1\le i\le 6$ explicitly; the remaining tail is retained through $\alpha_s-\sum_{i\le 6}\alpha_{s,i}$. All variables are nonnegative.

Proposition 2.1 (Inherited height and radical rows)**.** *Every anatomic box satisfies, up to the stated limiting errors,*

$$L\le\lambda,\qquad \sum_{i=1}^{6}\alpha_{s,i}\le\alpha_s,\qquad 7\alpha_s-\sum_{i=1}^{6}(7-i)\alpha_{s,i}\le 1\quad(s\in\{a,b,c\}). \tag{2.4}$$

The last row follows from $\sum_{i\le d}i\alpha_{s,i}\le 1$ by using $i\ge 7$ in the tail.

Proposition 2.2 (Inherited pair count)**.** *One has*

$$D\le\alpha_a+\alpha_b,\qquad D\le\alpha_a+\alpha_c,\qquad D\le\alpha_b+\alpha_c. \tag{2.5}$$

Proposition 2.3 (Inherited Fourier estimate)**.** *Define*

$$F_s=\max\{\alpha_{s,2}+\alpha_{s,4}+\alpha_{s,6},\ \alpha_{s,3}+\alpha_{s,6},\ \alpha_{s,5}\}. \tag{2.6}$$

Then

$$F_a+F_b+F_c\le 4L-6D. \tag{2.7}$$

Proposition 2.4 (Inherited geometric estimate)**.** *For arbitrary $I_a,I_b,I_c\subseteq\{1,\dots,6\}$, put*

$$\Sigma_0=\sum_{s\in\{a,b,c\}}\sum_{i\in I_s}\alpha_{s,i},\qquad \Sigma_1=\sum_{s\in\{a,b,c\}}\sum_{i\in I_s}(i-1)\alpha_{s,i}.$$

Then

$$D\le\max\{L-\Sigma_0,\ L-1+\Sigma_1\}. \tag{2.8}$$

For proofs of (2.5)–(2.8), see [1, Propositions 2.3, 3.1 and 4.1]. In the last two estimates the numerator initially contains the actual product $\prod_{s,i} P_{s,i}$, before any replacement by the radical budget. This gives L, rather than merely λ, in (2.7) and (2.8). The factor $X^{-1+O(\varepsilon_0)}$ in the second geometric alternative comes from the height of the original third letter and is independent of the radical budget. Thus λ enters these limiting rows only through $L \le \lambda$.

For computation, each F_s is a nonnegative auxiliary variable bounded below by the three expressions in (2.6). The complete inequality system is invariant under permutations of the letters. We may therefore impose the symmetry-breaking order

$$\alpha_{a,4} + 2\alpha_{a,5} \ge \alpha_{b,4} + 2\alpha_{b,5} \ge \alpha_{c,4} + 2\alpha_{c,5}. \tag{2.9}$$

After relabelling, every feasible point has a representative satisfying (2.9). The geometric estimate remains valid after such a permutation: its final displayed expression is symmetric, although its proof uses the height of the original third letter.

3. Uniform power differences and twisted energy

The two arithmetic lemmas in this section are inherited from [3, Sections 3.1–3.2 and Appendix B]. Their proofs are included because the new certificate uses them in degrees and modulus configurations not exhausted there.

3.1. **Weighted differences of arbitrary fixed degree.**

Lemma 3.1 (Inherited uniform count; proof reproduced)**.** *Fix an integer $e \ge 2$ and positive real numbers C, η. For $T \ge 2$, positive integers a, b and a nonzero integer m, the number of positive pairs (U, V) satisfying*

$$\mathsf{a}U^e - \mathsf{b}V^e = m, \qquad \mathsf{a}, \mathsf{b}, U, V, |m| \le T^C \tag{3.1}$$

is $O_{e,C,\eta}(T^\eta)$, uniformly in $\mathsf{a}, \mathsf{b}, m$. No coprimality hypothesis is required.

Proof. Multiply by a^{e-1} and put

$$t = \mathsf{a}^{e-1}\mathsf{b}, \qquad H = \mathsf{a}^{e-1}m, \qquad \theta = t^{1/e} > 0, \qquad K = \mathbb{Q}(\theta), \qquad n = [K : \mathbb{Q}] \le e.$$

Then θ is an algebraic integer and

$$(\mathsf{a}U)^e - tV^e = H \ne 0.$$

Let f be the monic minimal polynomial of θ. Since $f \in \mathbb{Z}[Z]$, monic division in $\mathbb{Z}[Z]$ gives

$$Z^e - t = f(Z)g(Z), \qquad g \in \mathbb{Z}[Z], \qquad \deg g = e - n.$$

For the algebraic integer $\alpha = \mathsf{a}U - \theta V$, define

$$z := \mathrm{N}_{K/\mathbb{Q}}(\alpha) = V^n f(\mathsf{a}U/V).$$

Evaluating the factorization of $Z^e - t$ yields

$$H = z\, V^{e-n} g(\mathsf{a}U/V). \tag{3.2}$$

Both factors on the right are integers. Since $H \ne 0$, the norm z is a nonzero divisor of H.

All quantities in (3.1) have polynomial size in T. Hence there is $C_1 = C_1(e, C)$ such that every embedding $\sigma : K \hookrightarrow \mathbb{C}$ satisfies $|\sigma(\alpha)| \le T^{C_1}$. The product of these absolute values is the nonzero integer $|z|$, so

$$T^{-C_1(n-1)} \le |\sigma(\alpha)| \le T^{C_1}. \tag{3.3}$$

Moreover, the integral ideal (α) has norm $|z|$ dividing $|H|$.

We count possible ideals uniformly in K. Above a rational prime p there are at most e prime ideals. If $p^h \| H$, an ideal whose norm divides H has exponent at most h at each of these primes. Unique factorization of ideals therefore gives at most

$$\prod_{p^h \| H} (h+1)^e = \tau(|H|)^e \tag{3.4}$$

integral ideals whose norms divide $|H|$. This bound has no discriminant-dependent constant. Since $|H| \le T^{Ce}$, the divisor bound makes (3.4) at most $O_{e,C,\eta}(T^{\eta/2})$.

Fix one generator α_0 arising from a counted solution for each principal ideal that occurs. Every other counted generator of that ideal is $\alpha = \epsilon\alpha_0$ for a unit ϵ. By (3.3), the vector $(\log|\sigma(\epsilon)|)_\sigma$ lies in a cube of side $O_{e,C}(\log T)$.

We need a separation constant uniform over fields of degree at most e. Algebraic integers of degree at most e whose conjugates all have modulus at most $\exp(1)$ form a finite set: their monic minimal polynomials have integer coefficients bounded in terms of e. If all conjugates of an algebraic integer have modulus 1, all its positive powers belong to this same finite set, so it is a root of unity. Consequently there is $c_e > 0$ such that every non-torsion unit of degree at most e has

$$\max_\sigma |\log|\sigma(\epsilon)|| \ge c_e.$$

Outside the finite set a conjugate already has modulus greater than $\exp(1)$; inside it, take the least positive distance. The number of roots of unity of degree at most e is bounded by the same finiteness argument.

Apply this separation to the quotient of two units occurring in K. Distinct unit logarithmic vectors are at distance at least c_e in the supremum norm. Thus cubes of side $c_e/2$ contain at most one vector, and each vector has a bounded number of unit preimages. One ideal therefore contributes $O_{e,C}((1+\log T)^e)$ generators.

If $n \ge 2$, then θ is irrational and $(U,V) \mapsto \mathsf{a}U - \theta V$ is injective. The ideal and generator counts now prove the lemma after absorbing the logarithm into $T^{\eta/2}$.

Finally, if $n = 1$, then θ is a positive integer. The nonzero integer $s = \mathsf{a}U - \theta V$ divides H. For each signed divisor s, substitution gives

$$(\theta V + s)^e - (\theta V)^e = H.$$

This is a nonzero polynomial equation of degree $e-1$, with leading coefficient $e\theta^{e-1}s$. It has at most $e-1$ roots, each determining U. The divisor bound for H completes the proof. □

Corollary 3.2 (Grouped values)**.** *Let $\varnothing \ne I \subseteq \{1,\dots,6\}$ and $x_i, x_i' \sim X^{\alpha_i}$ for $i \in I$. For fixed nonzero m of polynomial size in X, the number of pairs of values*

$$v = \prod_{i\in I} x_i^i, \qquad v' = \prod_{i\in I}(x_i')^i, \qquad v - v' = m$$

is at most $X^{\delta+o(1)}$, where δ may be the minimum of

$$\delta_0 = \sum_{i\in I}\alpha_i, \qquad \delta_e = 2\sum_{\substack{i\in I\\ e\nmid i}}\alpha_i \quad (e \ge 2 \text{ fixed}). \tag{3.5}$$

Proof. The bound δ_0 counts v, after which v' is determined. For δ_e, freeze both copies of every coordinate with $e \nmid i$. The remaining products have the form U^e and V^e, so the equation becomes $\mathsf{a}U^e - \mathsf{b}V^e = m$. Lemma 3.1 applies uniformly, and the frozen choices cost δ_e. A value of U has at most $\tau(U)^6 = X^{o(1)}$ coordinate representations, since every coordinate divides U; the same holds for V. If no index in I is divisible by e, then $\delta_e = 2\delta_0$ and this alternative is redundant. □

The numerical certificate below uses δ_e precisely for $e = 2,3,4,5,6$. The new feature is this systematic use, not Lemma 3.1 itself.

3.2. **Energy with an arbitrary unit twist.**

Lemma 3.3 (Inherited energy estimate; proof reproduced)**.** *Fix positive integers j,k. Let $M \ge 1$, let γ be a unit modulo M, and let $P,Q \ge 1$. The number $\mathcal{K}_\gamma$ of unit-coordinate solutions to*

$$x^j y^k \equiv \gamma (x')^j (y')^k \pmod M, \qquad x,x' \sim P, \quad y,y' \sim Q, \tag{3.6}$$

satisfies, for every $\eta > 0$,

$$\mathcal{K}_\gamma \ll_{j,k,\eta} (PQM)^\eta PQ\sqrt{(1+P^2/M)(1+Q^2/M)}. \tag{3.7}$$

In particular, no coprimality condition on j, k is needed.

Proof. Let $G = (\mathbb{Z}/M\mathbb{Z})^\times$, interpreted as the one-element group when $M = 1$. For $g \in G$, let $R_P(g)$ count pairs (x, x') with $x/x' = g$, and define

$$S_j(t) = \sum_{g^j = t} R_P(g), \qquad S_k(t) = \sum_{h^k = t} R_Q(h).$$

Writing $g = x/x'$ and $h = y/y'$, the congruence becomes $g^j h^k = \gamma$. Therefore

$$\mathcal{K}_\gamma = \sum_{t \in G} S_j(t) S_k(\gamma/t).$$

Since $t \mapsto \gamma/t$ permutes G, Cauchy–Schwarz gives

$$\mathcal{K}_\gamma^2 \le \|S_j\|_2^2 \|S_k\|_2^2 \le \rho_j(M)\rho_k(M)\mathcal{M}_P\mathcal{M}_Q, \tag{3.8}$$

where $\mathcal{M}_P = \sum_g R_P(g)^2$ and $\rho_j(M)$ is the kernel size of $g \mapsto g^j$. The second inequality applies Cauchy–Schwarz on each fibre of the power map.

For an odd prime power the unit group is cyclic, and for a power of 2 it is a product of at most two cyclic groups. Hence

$$\rho_j(M) \le j^{2\omega(M)} \ll_{j,\eta} M^\eta,$$

including $M = 1$. Next, $\mathcal{M}_P$ counts $xt' \equiv x't \pmod M$ with all four coordinates in the interval of size P. Fixing (x, t') gives $O(P^2)$ choices. The product $x't$ then lies in one residue class in $[1, 4P^2]$, so it takes $O(1 + P^2/M)$ integer values. Each value has $P^{o(1)}$ factorizations. Thus

$$\mathcal{M}_P \ll_\eta P^{2+\eta}(1 + P^2/M).$$

The analogous estimate for $\mathcal{M}_Q$, inserted into (3.8), proves (3.7) after reducing the intermediate error exponents. The argument is uniform in γ. □

If $P = X^p$, $Q = X^q$, $M \ge X^w$ and $M \le X^{O(1)}$, then (3.7) has exponent

$$\kappa(p, q; w) = p + q + \tfrac{1}{2}\big((2p - w)_+ + (2q - w)_+\big). \tag{3.9}$$

Only the lower bound for M is used in the two main factors; the polynomial upper bound absorbs M^η.

4. The collision inequality with a third-letter modulus

We now reproduce the collision formula of [3, Section 4] in the generality needed later. The formula is an inherited arithmetic input; its exhaustive degree-and-modulus deployment and the resulting certificates are new.

Fix distinct letters A, B, C. Choose nonempty $I_A, I_B \subseteq \{1, \dots, 6\}$, a pair $j < k$ in I_A, and a possibly empty set $O_C \subseteq \{1, \dots, 6\}$. Define

$$s_A = \sum_{i \in I_A} \alpha_{A,i}, \qquad s_B = \sum_{i \in I_B} \alpha_{B,i}, \qquad p = \alpha_{A,j}, \quad q = \alpha_{A,k}, \quad r = s_A - p - q, \tag{4.1}$$

$$f_A = \alpha_A - s_A, \qquad f_B = \alpha_B - s_B, \qquad f_C = \sum_{i \in O_C} \alpha_{C,i}, \qquad w_C = \sum_{i \in O_C} i\alpha_{C,i}, \tag{4.2}$$

$$w_B = \sum_{\substack{i \le 6 \\ i \notin I_B}} i\alpha_{B,i} + 7\Big(\alpha_B - \sum_{i \le 6} \alpha_{B,i}\Big), \qquad h_B = \sum_{i \in I_B} i\alpha_{B,i}. \tag{4.3}$$

The energy exponent is

$$K = 2r + \kappa(p, q; w_B), \tag{4.4}$$

with κ as in (3.9). In addition to $\delta_0 = s_B$ and the grouped exponents δ_e on letter B, define

$$\delta_{\mathrm{coord}} = \min_{t \in I_B} \{s_B - \alpha_{B,t} + (\alpha_{B,t} - w_C)_+\}, \qquad \delta_{\mathrm{value}} = (h_B - w_C)_+. \tag{4.5}$$

Proposition 4.1 (Inherited collision inequality; proof reproduced)**.** *Let δ be the minimum of any nonempty subcollection of δ_0, δ_e $(2 \le e \le 6)$, δ_{coord} and δ_{value}. Then every anatomic box satisfies*

$$D \le f_A + f_B + f_C + \max\left\{\alpha_C - f_C,\ \frac{\alpha_C - f_C + K + \delta}{2}\right\} + o(1). \tag{4.6}$$

The estimate is uniform over frozen coefficients for fixed d. Separate savings are combined by taking their minimum, never by adding them.

Proof. Freeze all coordinates of A outside I_A, all coordinates of B outside I_B, and the coordinates of C in O_C. The first two frozen sets include every index above 6. There are at most $X^{f_A+f_B+f_C+o(1)}$ choices. In one fixed fibre, write the equation as

$$\epsilon_A M_A u + \epsilon_B M_B v = \epsilon_C M_C w, \qquad \epsilon_A, \epsilon_B, \epsilon_C \in \{1, -1\}, \tag{4.7}$$

where u, v are the free monomials and w is the product of the remaining coordinates of C. The fixed small coefficient of each letter is included in its M, and all three M are positive. By (4.3) and the weight at least 7 of every tail coordinate,

$$M_B \ge X^{w_B}, \qquad M_C \ge X^{w_C},$$

and all quantities are at most $X^{O_d(1)}$.

Each positive monomial value has $X^{o(1)}$ coordinate representations: each of the bounded number of coordinates divides the value. It therefore suffices to count pairs of values (u, v), restoring this factor at the end. Let $\mathcal{W}$ be the set of possible w values, so

$$|\mathcal{W}| \le X^{\alpha_C - f_C + o(1)}.$$

For $w \in \mathcal{W}$, let n_w count permitted pairs (u, v), and put $N = \sum_w n_w$. If E_{off} counts ordered collisions between distinct pairs with the same w, Cauchy–Schwarz on the fibres gives

$$N^2 \le |\mathcal{W}| \sum_w n_w^2 = |\mathcal{W}|(N + E_{\mathrm{off}}), \qquad N \ll |\mathcal{W}| + \sqrt{|\mathcal{W}| E_{\mathrm{off}}}. \tag{4.8}$$

A nonempty fibre has $\gcd(M_A, M_B) = 1$, and every coordinate of u is a unit modulo M_B by pairwise coprimality of the original letters. Subtracting two equations (4.7) with the same w gives

$$M_A(u - u') = \pm M_B(v - v').$$

For distinct pairs, both differences are nonzero: if either vanished, the equation would force the other to vanish. In particular $u \equiv u' \pmod{M_B}$.

Fix independently the two copies of the coordinates in $I_A \setminus \{j, k\}$, at a cost $X^{2r+o(1)}$. Since these coordinates are units modulo M_B, the remaining congruence is exactly (3.6) for some unit γ. Lemma 3.3 is uniform in that unit. Thus the number of possible pairs (u, u') is at most $X^{K+o(1)}$, with K as in (4.4).

Now fix such a pair (u, u'). The nonzero integer

$$m = v - v' = \pm M_A(u - u')/M_B$$

is fixed. Counting all possible v gives $X^{s_B+o(1)}$, since v' is then determined. Corollary 3.2 gives each grouped bound $X^{\delta_e+o(1)}$.

There are two residue-class alternatives. In a nonempty fibre, M_B is invertible modulo M_C. Reducing (4.7) modulo M_C determines the residue class of v from u. Also

$$1 \le v \le 2^{|I_B|} X^{h_B}.$$

A single residue class in this interval has $O_d(1 + X^{h_B}/M_C)$ elements. Restoring divisor-sized coordinate multiplicity gives $X^{\delta_{\mathrm{value}}+o(1)}$. This is a whole-value interval count; it remains valid when an individual coordinate range is larger than the modulus.

Alternatively, fix every coordinate of v except the one indexed by $t \in I_B$. The congruence becomes $x_t^t \equiv \xi \pmod{M_C}$ for a unit ξ. Its solutions occupy at most $\rho_t(M_C) = X^{o(1)}$ residue classes, by the kernel estimate in Lemma 3.3. Each class contributes $O(1 + X^{\alpha_{B,t}}/M_C)$ integers

in the dyadic interval. The other fixed coordinates cost $X^{s_B-\alpha_{B,t}+o(1)}$. Taking the minimum over t proves the coordinate bound. The case $M_C = 1$ is included.

All these estimates count the same conditional set of pairs (v, v'), so their minimum is valid. Consequently

$$E_{\text{off}} \le X^{K+\delta+o(1)}.$$

Insert this into (4.8), restore frozen choices and coordinate multiplicities, and take logarithms base X. The two alternatives in (4.8) give respectively the diagonal and off-diagonal expressions in (4.6). No coprimality assumption between different solutions is used. □

Lemma 4.2 (Release of all first-letter coordinates)**.** *For fixed j, k, I_B, O_C and a fixed choice of admissible savings, the strongest member of (4.6) is obtained with $I_A = \{1, \dots, 6\}$.*

Proof. Adding a coordinate of exponent α to I_A subtracts α from f_A and adds α to r, hence adds 2α to K. The off-diagonal expression is unchanged, while the diagonal expression decreases by α. □

The older pairwise difference refinements are repeated instances of Corollary 3.2. For example, $I_B = \{2, 3\}$ gives $2\alpha_{B,3}$ and $2\alpha_{B,2}$; $I_B = \{3, 4\}$ gives $2\alpha_{B,3}$ and $2\alpha_{B,4}$; and each of $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$ gives $2\alpha_{B,5}$. Retaining these labels in the data does not require another arithmetic theorem.

Lemma 3.3 permits all 15 pairs $1 \le j < k \le 6$. The upper tree in this paper uses only the 11 coprime pairs, which are sufficient for the stated bound. Exact lower-obstruction checks use all 15 pairs. Thus noncoprime pairs are a legal extension, not a hidden hypothesis and not a claimed source of the upper certificate.

5. Complete modulus locking and elimination of index 1

Throughout the finite programs, Lemma 4.2 is used to set $I_A = \{1, \dots, 6\}$. The remaining collision data are an ordered pair of distinct letters (A, B), an exponent pair (j, k), a nonempty $I_B \subseteq \{1, \dots, 6\}$, and a third-letter modulus set O_C.

The old modulus family is

$$\mathcal{O}_{\text{old}} = \big\{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{2, 5\}, \{3, 5\}\big\}. \tag{5.1}$$

FullOC starts with all 64 subsets of $\{1, \dots, 6\}$. We prove that, for the whole-value lock and the grouped savings used here, it suffices to retain

$$\mathcal{O}_{\text{FullOC}} = \{O_C : O_C \subseteq \{2, 3, 4, 5, 6\}\}, \qquad |\mathcal{O}_{\text{FullOC}}| = 32. \tag{5.2}$$

5.1. **The deletion inequality.** Fix A, B, C, j, k, I_B and put

$$\delta_* = \min\{\delta_0, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}.$$

This quantity is independent of O_C. For any $O_C \subseteq \{1, \dots, 6\}$, define

$$H(O_C) = f_C + \min\{\delta_*, (h_B - w_C)_+\}. \tag{5.3}$$

The diagonal expression in (4.6) is

$$f_A + f_B + f_C + \alpha_C - f_C = f_A + f_B + \alpha_C,$$

so it is independent of O_C. With $\delta = \min\{\delta_*, \delta_{\text{value}}\}$, the off-diagonal expression is

$$f_A + f_B + \frac{\alpha_C + K}{2} + \frac{H(O_C)}{2}.$$

Thus only $H(O_C)$ matters when comparing modulus sets.

Proposition 5.1 (Index-1 elimination)**.** *If $1 \in O_C$ and $O'_C = O_C \setminus \{1\}$, then*

$$H(O'_C) \le H(O_C). \tag{5.4}$$

Consequently, the collision row associated with O'_C is at least as strong as that associated with O_C. In the whole-value lock subfamily with savings $\delta_0, \delta_2, \dots, \delta_6$, every set containing 1 may be discarded.

Proof. Write $\alpha = \alpha_{C,1} \geq 0$. Deleting index 1 gives

$$f'_C = f_C - \alpha, \qquad w'_C = w_C - \alpha, \qquad h_B - w'_C = h_B - w_C + \alpha.$$

The function

$$t \longmapsto \phi(t) = \min\{\delta_*, (t)_+\}$$

is nondecreasing and 1-Lipschitz: it is 0 for $t \leq 0$, then t up to δ_*, and then constant. Therefore

$$\phi(h_B - w'_C) \leq \phi(h_B - w_C) + \alpha,$$

and hence

$$H(O'_C) = f_C - \alpha + \phi(h_B - w'_C) \leq f_C + \phi(h_B - w_C) = H(O_C).$$

The diagonal row is unchanged and the off-diagonal row is monotone in H, proving the final assertion. □

Applying Proposition 5.1 once to every set containing 1 leaves exactly the 32 sets in (5.2). This is an exhaustive treatment of the whole-value modulus family, not a heuristic list of promising sets.

Remark 5.2 (Scope of the elimination)**.** The proof uses the independence of δ_* from O_C and the one-Lipschitz whole-value expression. It does not automatically eliminate index 1 in a different program whose selected savings include arbitrary coordinate locks or additional modulus-dependent terms. The word "complete" below always refers to the subfamily specified in Proposition 5.1.

5.2. **The finite collision families.** For nonempty O_C, the tree imposes simultaneously the affine rows coming from

$$\delta_{\text{value}}, \qquad \delta_0, \qquad \delta_e \quad (2 \leq e \leq 6).$$

On a fixed sign octant, each is a valid upper bound for the same minimum, so imposing all resulting rows is legitimate. For $O_C = \varnothing$, one has $w_C = 0$ and

$$\delta_{\text{coord}} = \min_{t \in I_B}\{s_B - \alpha_{B,t} + \alpha_{B,t}\} = \delta_0.$$

The tree uses a uniform coordinate-sign encoding of this fact and additionally retains the older pair bounds and $\delta_2, \ldots, \delta_6$.

For exact program comparisons, we use all 15 exponent pairs. The upper certificate uses only the 11 coprime pairs. Table 2 counts the collision instances before their diagonal and sign-octant disjunctions; there are six ordered letter pairs and 63 nonempty choices of I_B.

Table 2. Collision-instance counts in the exact programs.

Family	Exponent pairs	Modulus sets	Instances
Old-family control	15	9	$6 \cdot 15 \cdot 63 \cdot 9 = 51{,}030$
FullOC, all legal pairs	15	32	$6 \cdot 15 \cdot 63 \cdot 32 = 181{,}440$
FullOC, upper tree	11 coprime pairs	32	$6 \cdot 11 \cdot 63 \cdot 32 = 133{,}056$

The old and FullOC control programs below share the base rows, all 64^3 geometric choices, released first-letter coordinates, grouped savings and all 15 exponent pairs. They differ only in the modulus family (5.1) versus (5.2). Since the upper tree uses a subfamily of FullOC, its upper bound also applies to the all-pair FullOC program.

6. Piecewise-linear encoding and exact rational certificates

6.1. **The 25-variable root system.** The vector $x \geq 0$ consists of

$$(\alpha_{s,i})_{s \in \{a,b,c\},\, 1 \leq i \leq 6}, \qquad (\alpha_a, \alpha_b, \alpha_c), \qquad (F_a, F_b, F_c), \qquad D,$$

for $18 + 3 + 3 + 1 = 25$ variables. The root has the 22 affine rows encoding (2.4)–(2.7) and (2.9). The three expressions in each Fourier maximum are imposed separately as lower bounds on F_s. The geometric and collision inequalities are disjunctive and are introduced by the tree.

6.2. **Branch semantics.** A *geometric node* selects (I_a, I_b, I_c) in (2.8) and has exactly two children, carrying respectively

$$D \le L - \Sigma_0, \qquad D \le L - 1 + \Sigma_1.$$

Thus every point satisfying the selected geometric inequality belongs to at least one child.

A *collision node* selects $(A, B, C, j, k, I_B, O_C)$ and a distinguished $k_B \in I_B$ used only for the uniform empty-modulus encoding. It has one diagonal child and eight off-diagonal children. The diagonal child carries

$$D \le f_A + f_B + \alpha_C.$$

The eight children cover the closed sign octants of

$$2p - w_B, \qquad 2q - w_B, \qquad \begin{cases} h_B - w_C, & O_C \neq \varnothing, \\ \alpha_{B,k_B}, & O_C = \varnothing. \end{cases}$$

On each octant, κ, the whole-value lock and the coordinate lock are affine. The corresponding off-diagonal rows for every admissible saving are imposed together. This is valid because Proposition 4.1 permits the minimum of those savings; it is not a choice of a favorable saving after seeing the point. In the empty-modulus case the nonpositive branch of α_{B,k_B} is redundant except on its zero boundary, which is harmless.

Every geometric or collision node therefore has complete child coverage. A path from the root collects affine rows $A_\ell x \le b_\ell$ at its leaf ℓ.

6.3. **Rational leaf proofs.**

Lemma 6.1 (Leaf multiplier criterion)**.** *Suppose nonnegative rational multipliers y_ℓ satisfy*

$$(y_\ell^T A_\ell)_D = 1, \qquad (y_\ell^T A_\ell)_v \ge 0 \quad (v \neq D), \qquad y_\ell^T b_\ell = \beta_\ell. \tag{6.1}$$

Then every nonnegative x satisfying the leaf rows obeys $D \le \beta_\ell$.

Proof. Multiplying the rows by y_ℓ gives

$$D + \sum_{v \neq D} (y_\ell^T A_\ell)_v x_v = y_\ell^T A_\ell x \le y_\ell^T b_\ell = \beta_\ell.$$

All summands on the left are nonnegative by (6.1) and $x \ge 0$. □

Floating-point linear programming is used only to select violated branches and discover candidate duals. Candidate multipliers are converted to rational numbers and repaired, if necessary, before storage. Neither an LP status nor a displayed numerical optimum is a proof step. The proof is (6.1), complete child coverage and Lemma 6.1.

6.4. **The FullOC upper certificate.**

Theorem 6.2 (Exact finite implication)**.** *Every nonnegative point satisfying the root system, all geometric disjunctions, and the FullOC collision disjunctions with $I_A = \{1, \dots, 6\}$, the* 11 *coprime exponent pairs, all* 32 *modulus sets and $\delta_0, \delta_2, \dots, \delta_6$ satisfies*

$$D \le U = \frac{11680007}{20000012}.$$

The same conclusion holds for the program enlarged to all 15 *exponent pairs.*

Computer-assisted proof. The certified tree has the statistics in Table 3. Each leaf stores positive rational multipliers satisfying Lemma 6.1. The exact maximum of its leaf bounds is U. Since every feasible point belongs to a leaf, this proves the first assertion. Enlarging the exponent-pair family only restricts the feasible set, so the second assertion follows. □

The exact target margin is

$$\frac{73}{125} - U = \frac{1}{2500001500} > 0.$$

The certificate uses δ_e for all $e = 2, \dots, 6$; its machine-data mode is `all_deg`.

TABLE 3. Metadata of the main FullOC certificate.

Radical budget	$\lambda = 1$
Requested target	73/125
Exact maximum leaf bound	11680007/20000012
Leaves	65,576
Geometric nodes	4,591
Collision nodes	7,623
Maximum depth	36
LP calls during discovery	466,244
Discovery time in the recorded run	1965.523944 s

6.5. **Independent verification.** A separate checker, using only the standard library, reconstructs the root, geometric, collision, sign-octant, whole-value lock and higher-degree rows directly from the formulas. It does not import the search program or its engine. For every node it checks the schema and complete child list; for every leaf it recomputes the dot products in (6.1) with rational arithmetic; and at the end it recomputes the leaf count and the exact maximum bound.

The recorded run returned

$$\texttt{upper} = 11680007/20000012, \qquad \texttt{leaf} = 65576, \qquad \texttt{geo} = 4591, \qquad \texttt{ss} = 7623,$$

followed by `VERIFIED`. Three deliberate faults—a changed multiplier, a deleted child and an altered declared maximum—were all rejected. Appendix A gives the command, file hashes and obstruction-checking routines.

7. Passage from the exponent program to $N_\lambda(X)$

We now prove Theorem 1.1. The purpose of this limiting argument is to account explicitly for decomposition errors, divisor losses and the finite but box-dependent nature of the estimates.

Let $N_\mu^*(Y)$ count primitive triples with

$$Y/2 \le c \le Y, \qquad c^\mu/2 \le \mathrm{rad}(abc) < c^\mu.$$

Dyadic decomposition in height and radical gives [1, Section 2]

$$N_1(X) \ll (\log X)^2 \max_{1\le Y\le X} \max_{0<\mu\le 1} N_\mu^*(Y). \tag{7.1}$$

Bounded small-height terms can be added without affecting the asserted bound.

Suppose that the conclusion at $\lambda = 1$ fails for some $\varepsilon > 0$. Then (7.1) supplies arbitrarily large Y and some $\mu \le 1$ for which

$$N_\mu^*(Y) \ge Y^{U+\varepsilon/2}.$$

Choose a real D_0 with

$$U < D_0 < U + \varepsilon/2.$$

For each sufficiently large integer n, apply the anatomic box decomposition with $\varepsilon_0 = 1/n$. The fixed integer $d = d(1/n)$ may change with n, but it is fixed during each individual box estimate. Choosing $Y = Y_n$ sufficiently large, we obtain a box whose count has exponent $D_n \ge D_0$ and for which the errors in all inherited inequalities and in the finite collision family used by the tree are at most $O(1/n)$. This uses uniformity over boxes and frozen coefficients, together with finiteness of the collision family.

Record the 18 first-six coordinates and the three totals. By the height inequalities, they lie in a fixed compact set independently of d. The pair bounds bound D_n. Pass to a convergent subsequence and relabel the letters to satisfy (2.9). Define the limiting Fourier variables by the maxima in (2.6).

The limiting point satisfies (2.4)–(2.8), all required instances of (4.6), and $D \geq D_0$. Indeed, these are finitely many closed inequalities involving continuous maxima, minima and positive parts, and their errors tend to zero. The tail enters only through

$$\alpha_s - \sum_{i \leq 6} \alpha_{s,i}.$$

In particular, $L \leq 1$ in the limit even though each finite box has a small budget error, and the actual radical sub-budget $\mu \leq 1$ is absorbed by the root row $L \leq 1$.

Theorem 6.2 now gives $D \leq U$, contradicting $D_0 > U$. This proves

$$N_1(X) \ll_\varepsilon X^{U+\varepsilon}.$$

The logarithmic and divisor losses have been absorbed into the arbitrary ε. Finally, for $0 < \lambda \leq 1$ and $c \geq 2$, the condition $\mathrm{rad}(abc) < c^\lambda$ implies $\mathrm{rad}(abc) < c$. Thus the set counted by $N_\lambda(X)$ is contained in the set counted by $N_1(X)$. This proves Theorem 1.1 in its stated uniform range.

The argument proves no corresponding lower bound. A feasible point of the limiting exponent system is only a possible optimizer of a collection of upper estimates; it need not arise from an anatomic box or from any integer triple.

8. Strict separation and exact program obstructions

Let D^*_{old} and D^*_{FullOC} denote the maxima of D over the old-family and FullOC limiting programs defined in Section 5, both with all 15 exponent pairs. These numbers are optima of finite closed exponent programs. They are not counting exponents attained by actual triples.

8.1. **An exact feasible point for the old family.** Consider the rational point

$$\begin{aligned} \alpha_{a,\bullet} = \alpha_{b,\bullet} &= \left(\frac{1}{46}, \frac{3}{46}, \frac{3}{23}, \frac{3}{46}, \frac{9}{230}, 0\right), & \alpha_a = \alpha_b &= \frac{37}{115}, \\ \alpha_{c,\bullet} &= \left(\frac{5}{69}, \frac{9}{230}, \frac{3}{23}, \frac{79}{690}, 0, 0\right), & \alpha_c &= \frac{41}{115}, \\ D &= \frac{403}{690}, & F_a = \frac{73}{345}, \quad F_b &= \frac{3}{23}, \quad F_c = \frac{53}{345}. \end{aligned} \tag{8.1}$$

Here $\alpha_{s,\bullet}$ denotes $(\alpha_{s,1}, \ldots, \alpha_{s,6})$. All three tails are zero, $L = 1$, and F_a, F_b, F_c are auxiliary values satisfying the Fourier lower rows; they need not all be minimal.

Proposition 8.1 (Exact old-family obstruction)**.** *The point* (8.1) *satisfies every base row, all* 64^3 *geometric inequalities and all* 51,030 *old-family collision inequalities. Hence*

$$D^*_{\mathrm{old}} \geq \frac{403}{690} = 0.5840579710144927\ldots. \tag{8.2}$$

In particular, the old-family program cannot certify the exponent 0.584*.*

Exact rational computation. Direct substitution gives zero maximum violation of the base rows. The minimum geometric right-hand side is 403/690, attained for example by

$$I_a = \{1\}, \qquad I_b = \{1, 3\}, \qquad I_c = \{1, 2, 3\}.$$

Enumeration of the 51,030 old-family collision instances in rational arithmetic gives minimum right-hand side 403/690, with 20 binding instances. One binding instance is

$$A = a, \quad (j, k) = (2, 3), \quad B = c, \quad I_B = \{1, 2, 3\}, \quad O_C = \{3, 5\}.$$

Thus every required right-hand side is at least D. The independent script and output listed in Appendix A reproduce these assertions without floating-point arithmetic. □

For comparison, the old-family control tree has 49,033 leaves, 5,584 geometric nodes and 5,431 collision nodes. Its independently checked maximum leaf bound is

$$D^*_{\mathrm{old}} \leq \frac{11697}{20012} = 0.5844993004197482\ldots. \tag{8.3}$$

Thus the old optimum is tightly bracketed, although its exact value is not needed here.

8.2. **The new family cuts this point.** Evaluate the all-pair FullOC family at the same point (8.1). Exact enumeration of all 181,440 instances gives minimum collision right-hand side

$$\frac{53}{92} = 0.5760869565217391\ldots, \qquad \frac{403}{690} - \frac{53}{92} = \frac{11}{1380}. \tag{8.4}$$

There are eight binding instances. One is

$$A = c, \quad (j,k) = (3,4), \quad B = a, \quad C = b, \quad I_B = \{1,2,3,4\}, \quad O_C = \{3,4,5\}. \tag{8.5}$$

We display the substitution to make the separation independently checkable by hand. At (8.5),

$$f_A = 0, \qquad f_B = \frac{9}{230}, \qquad f_C = \frac{27}{115}, \qquad \alpha_C = \frac{37}{115},$$

$$p = \frac{3}{23}, \qquad q = \frac{79}{690}, \qquad r = \frac{77}{690}, \qquad w_B = \frac{9}{46}, \qquad h_B = \frac{37}{46}, \qquad w_C = \frac{39}{46}.$$

Consequently

$$(2p - w_B)_+ = \frac{3}{46}, \qquad (2q - w_B)_+ = \frac{1}{30},$$

$$\kappa(p,q;w_B) = \frac{203}{690}, \qquad K = 2r + \kappa = \frac{119}{230}.$$

Since $h_B - w_C = -1/23$, the whole-value saving is $\delta_{\text{value}} = 0$, and therefore the minimum saving is $\delta = 0$. The diagonal expression is

$$f_A + f_B + \alpha_C = \frac{83}{230},$$

whereas the off-diagonal expression is

$$f_A + f_B + f_C + \frac{\alpha_C - f_C + K + \delta}{2} = \frac{63}{230} + \frac{1}{2}\left(\frac{20}{230} + \frac{119}{230}\right) = \frac{53}{92}.$$

Thus (4.6) gives $D \le 53/92$, which the old feasible point violates.

Theorem 8.2 (Strict FullOC separation)**.** *The old and new program optima satisfy*

$$\boxed{D^*_{\text{FullOC}} \le U < \frac{73}{125} < \frac{403}{690} \le D^*_{\text{old}} \le \frac{11697}{20012}.} \tag{8.6}$$

In particular, FullOC gives an independent exponent improvement rather than merely improving the numerical solver.

Proof. Theorem 6.2 gives the first inequality. The exact target identity gives $U < 73/125$, and

$$\frac{403}{690} - \frac{73}{125} = \frac{1}{17250} > 0.$$

Proposition 8.1 gives $403/690 \le D^*_{\text{old}}$, and the old control certificate gives the last inequality. Strictness can also be read directly from (8.4): a point feasible for the old program is infeasible for FullOC. □

The improvement over the old upper certificate is exactly

$$\frac{11697}{20012} - U = \frac{24980035}{50030030018} = 0.0004993008197479\ldots.$$

8.3. **A lower obstruction for the FullOC program.** Now consider the symmetric point

$$\alpha_{s,\bullet} = \left(\frac{19}{396}, \frac{1}{22}, \frac{3}{22}, \frac{1}{11}, 0, 0\right), \qquad \alpha_s = \frac{1}{3} \quad (s \in \{a, b, c\}),$$
$$F_a = \frac{5}{22}, \qquad F_b = F_c = \frac{3}{22}, \qquad D = \frac{7}{12}. \tag{8.7}$$

Each letter has tail mass

$$\frac{1}{3} - \left(\frac{19}{396} + \frac{1}{22} + \frac{3}{22} + \frac{1}{11}\right) = \frac{5}{396},$$

and $L = 1$.

Proposition 8.3 (Exact FullOC lower obstruction)**.** *The point* (8.7) *satisfies every base row, all* 64^3 *geometric inequalities and all* 181,440 *FullOC collision instances with all* 15 *exponent pairs. Therefore*

$$\boxed{\frac{7}{12} \le D^*_{\mathrm{FullOC}} \le U = \frac{11680007}{20000012}.} \tag{8.8}$$

Exact rational computation. The base rows again have zero maximum violation. The minimum geometric right-hand side is 7/12, attained for example by

$$I_a = \{1\}, \qquad I_b = I_c = \{1, 3\}.$$

The minimum over all FullOC collision right-hand sides is also 7/12, with 576 binding instances. A binding choice is

$$A = a, \quad (j, k) = (3, 4), \quad B = b, \quad I_B = \{3\}, \quad O_C = \varnothing.$$

The independent rational evaluator checks all instances, rather than only these witnesses. □

The remaining gap inside the present program is

$$U - \frac{7}{12} = \frac{10000}{15000009} = 0.0006666662666669\ldots.$$

Proposition 8.3 rules out every bound below 7/12 from this program. It does not assert that $D^*_{\mathrm{FullOC}} = 7/12$, nor that any further positive part of the displayed gap is attainable by additional search. Most importantly, neither (8.1) nor (8.7) is a lower-bound construction for $N_\lambda(X)$.

9. Scope, limitations and relation to abc–Szpiro

9.1. **What the theorem does and does not say.** Theorem 1.1 is a general upper bound for the exceptional-set counting function. It does not prove the abc conjecture, which requires boundedness of $N_\lambda(X)$ for every fixed $\lambda < 1$, and it does not prove any pointwise inequality of the form $\mathrm{rad}(abc) \gg_\varepsilon c^{1-\varepsilon}$. A polynomial upper bound on the number of exceptions is weaker than finiteness.

Likewise, the paper gives no new bound for the discriminant–conductor ratio of an elliptic curve and does not prove Szpiro's conjecture. No Frey-curve specialization is used to convert the counting theorem into a pointwise Arakelov statement. In particular, the present argument constructs neither a cyclotomic-tower degeneration functor nor an identification between exponent-program variables and arithmetic degrees of such a functor.

The exact feasible points in Section 8 have a deliberately restricted meaning. They certify lower bounds for the values of finite exponent programs. They do not provide sequences of anatomic boxes, still less sequences of primitive abc triples. Thus $7/12 \le D^*_{\mathrm{FullOC}}$ is not an arithmetic lower bound for $N_\lambda(X)$.

9.2. **Limitations of the present computation.**

(1) The new tree is run at radical budget $\lambda = 1$, with uniformity for $0 < \lambda \leq 1$ obtained by inclusion. The inherited anatomic estimates are available for $0 < \lambda \leq 2$, but no new fixed-budget table beyond 1 is claimed here.
(2) The upper certificate uses 11 coprime exponent pairs. All 15 pairs are arithmetically legal by Lemma 3.3 and are included in the exact obstruction checks, but adding the four noncoprime pairs to the upper search is not claimed to improve U.
(3) Completeness of the 32 modulus sets is proved for the whole-value lock and the grouped savings specified in Proposition 5.1. It is not a completeness theorem for every possible coordinate-lock extension.
(4) The upper bound U is not asserted to equal D^*_{FullOC}. The exact interval is $7/12 \leq D^*_{\mathrm{FullOC}} \leq U$. Closing the remaining gap, or beating $7/12$, requires additional valid inequalities or new arithmetic input.
(5) The BBLT anatomic estimates are external inputs. The collision mechanism and arbitrary-degree weighted-difference theorem are inherited from Li's rStation manuscripts, whose public preprint status has been stated. The new contribution is the exhaustive certified deployment, elimination, separation and obstruction analysis, not a rebranding of those inputs.

9.3. **Conclusion.** The complete whole-value modulus family lowers the certified exceptional-set exponent to the exact value $U < 0.584$. The improvement is proved twice at different levels: arithmetically, by a complete exact-rational tree for $N_\lambda(X)$, and structurally, by a strict separation between the old and new exponent programs. The old family has a feasible obstruction above 0.584, while FullOC excludes it and closes below 0.584. The symmetric FullOC obstruction at $7/12$ delineates precisely what remains inside the current method. Reproducing the checker, rather than rerunning the floating-point search, is sufficient to audit every finite implication used in the proof.

Appendix A. Certificate data and reproduction

A.1. **Data schema.** The main files use schema `break213-v1`. The root records the budget, target, declared maximum, leaf count and tree. A leaf has keys `type`, `y` and `bound`; `y` maps row identifiers to positive rational multiplier strings. A geometric internal node has `I`, `J`, `K` and `children`. A collision node has `A`, `pA`, `B`, `IB`, `kB`, `OC` and `children`. No solver matrix is trusted as input: the checker reconstructs every row from the mathematical formulas.

The raw search label `all` is normalized to `all_deg` in the main certificate. This metadata change tells the checker to reconstruct δ_e for $e = 2, \ldots, 6$; it does not alter the tree, multipliers or leaf bounds. The normalization script is included.

A.2. **Verification commands.** Verification of the supplied certificates and obstructions requires only the Python standard library:

```
python3 -B check_extended.py cert_fulloc_5840.json.gz --faults
python3 -B check_extended.py cert_old_5845.json.gz --faults
python3 -B check_obstructions.py
sha256sum -c SHA256SUMS
```

The first two commands must end in `VERIFIED`, and the third in `ALL EXACT OBSTRUCTION CHECKS PASSED`. The fault option tests a changed multiplier, a deleted child and an altered declared maximum; all three corrupted certificates must be rejected.

The recorded obstruction check gives

```
old point / old family:      minimum RHS = 403/690, binding = 20
old point / FullOC:          minimum RHS = 53/92,   binding = 8
symmetric point / FullOC:    minimum RHS = 7/12,    binding = 576
```

The same script checks the root rows and all 64^3 geometric instances for each point.

Optional regeneration requires NumPy and SciPy and is not needed to audit the proof:

```
BREAK213_SECONDS=2400 python3 -B solve_fulloc_5805.py 1.00 73/125 full5840
python3 -B normalize_fulloc.py cert_full5840_1p00.json cert_fulloc_5840.json
python3 -B check_extended.py cert_fulloc_5840.json --faults
```

Branch choices and runtime may vary with LP versions. The supplied exact certificate, rather than a fresh optimizer run, is the proof artifact.

A.3. **Principal SHA-256 hashes.** The uncompressed normalized main certificate has hash

`d29def66fa03ddc512b7ba7938a26295a23a278ce90cfa49ab9f0456a33275c8`.

The principal files in the reproduction package have the following hashes.

- `cert_fulloc_5840.json.gz`
  `8b81b3f0e8030f0b98445970dbf95763dd6725feb12d981738bb0ad4c64cdbfd`
- `cert_old_5845.json.gz`
  `603cd0be5dd912a3389fcd292ffec36ddd07099d3257504e4d08f149e9106291`
- `check_extended.py`
  `d3e4256b6ddb9004292519b2a1e1429e39a5203c8998d00ee7bdd9f1a0bfefe7`
- `check_obstructions.py`
  `89f616f79f50b20dbb8371db3b8ff6bbfe34ebd8a47ff00c97981060a9af0eb6`
- `solve_fulloc_5805.py`
  `6450573601c2d4cab92327ae1652e2692301fff5c68323a76ef81b6ddcd0f36e`
- `verify_fulloc_5840.log`
  `f731ff73c9255cef2de206c243575304182fe3cdb9bf269ba80408f156dba9d1`

The package manifest also covers the old generator, engine, normalization script, raw point diagnostics and all recorded logs.

A.4. **Audit boundary.** A `TypeError` or candidate-exhaustion event in the search is not treated as a mathematical lower bound. The lower obstructions used in Section 8 are proved only after their rational coordinates have been checked against every constraint by `check_obstructions.py`. Conversely, an LP display such as `max_D=0.5840000` is not treated as an upper-bound proof; the upper theorem uses only the completed tree and its independently checked rational identities.

Data availability. All rational certificates, linear-programming witness files and verification scripts for the exponent computations are archived in a separate Zenodo record (DOI: 10.5281/zenodo.24354214). This manuscript contains only the mathematical exposition; the computational reproduction package is hosted externally.

References

[1] Christian Bernert, Tim Browning, Jared Duker Lichtman and Joni Teräväinen, *Bounds on the exceptional set in the abc conjecture*, arXiv:2410.12234v2, version 2 submitted 9 May 2026 (version 1: 16 October 2024). https://arxiv.org/abs/2410.12234v2.

[2] Hongmin Li, *Collision inequalities and the radical budget for the exceptional set in the abc conjecture*, rStation, Mathematics paper 17, version 2 (2026). https://rstation.ai/problems/math/papers/17/v2.

[3] Hongmin Li, *Squares and twisted energy in the exceptional set of the abc conjecture*, rStation, Mathematics paper 28, version 3, 13 November 2026. https://rstation.ai/problems/math/papers/28/v3.