A Chebyshev bound for the constant in the Prouhet–Tarry–Escott problem

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Abstract

Let (A, B) be an ideal solution of size n of the Prouhet–Tarry–Escott problem, with constant C defined by $\prod_i (x - a_i) - \prod_i (x - b_i) = C$, and let L be the diameter of $A \cup B$. We prove

$$|C| \;\le\; 4\left(\frac{L}{4}\right)^n,$$

with equality exactly for the configuration given by the roots of $T_n \mp 1$, where T_n is the Chebyshev polynomial of the first kind. Equality is attained by integer solutions for $n \le 3$ and, by Niven's theorem, is impossible for $n \ge 4$; for $n = 4$, $n = 5$ and $n = 6$, however, we exhibit infinite families of integer solutions, governed by a Pell equation, along which the ratio $4^{n-1}|C|/L^n$ tends to 1, so that the constant 4^{n-1} cannot be improved. The bounds on C in the standard references are p-adic throughout: they determine which prime powers divide C, and the present inequality appears to be the first of archimedean type. Two further consequences are recorded: an unconditional lower bound for the height of any ideal solution, in terms of the known divisors of the universal constant C_n, which yields the first such statements for sizes $17 \le n \le 20$, where no search has been published; and the observation that the extremal configuration is exactly the trigonometric configuration of Chen Shuwen, for which the analogous extremality statement for the individual variables is an open conjecture. The present result is its proven counterpart for the constant.

1 Introduction

For $n \ge 2$, two multisets of integers $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ with $A \ne B$ form an *ideal solution of size* n of the Prouhet–Tarry–Escott (PTE) problem if

$$\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} b_i^k, \qquad k = 1, 2, \dots, n-1. \tag{1}$$

Throughout, n denotes the *size*; the degree is $n - 1$. Ideal solutions are known for $n \le 10$ and $n = 12$; none is known for $n = 11$ or $n \ge 13$. We refer to [1, 5, 6, 9] for background.

Writing $F(x) = \prod_i (x - a_i)$ and $G(x) = \prod_i (x - b_i)$, condition (1) is equivalent to $F - G$ being a nonzero constant,

$$F(x) - G(x) = C. \tag{2}$$

The arithmetic of C has been studied since Kleiman [7], who proved $(n-1)! \mid C$, and Rees and Smyth [8]; the survey of Borwein and Ingalls [3] and the search of Borwein, Lisoněk and Percival [4] record the state of the question at the turn of the century; the current state of the divisibility conditions, for $n \le 20$, is tabulated by Coppersmith, Mossinghoff, Scheinerman and VanderKam [6, Table 1], and the analogous questions over imaginary quadratic rings are

treated by Caley [5]. These references are p-adic in nature: they assert that certain prime powers divide C. The only archimedean information in them is the trivial bound $|C| \le L^n$, immediate from the product formula $|C| = \prod_i |b_j - a_i|$, valid for every j [1, E5]; no bound limiting the *size* of C beyond it appears.[1]

Our result is such a bound. Set

$$L \;=\; \max(A \cup B) - \min(A \cup B),$$

the diameter of the solution; note that L is invariant under the translations $x \mapsto x + t$ under which (1) is invariant, and scales linearly under $x \mapsto \lambda x$, exactly as $|C|^{1/n}$ does.

Theorem 1. *Let (A, B) be an ideal solution of size n of the PTE problem, with constant C and diameter L. Then*

$$|C| \;\le\; 4\left(\frac{L}{4}\right)^n, \tag{3}$$

or equivalently, in integral form, $4^{n-1}|C| \le L^n$.

The bound is sharp. Equality holds precisely when

$$F(x) = \widetilde{T}(x) + \tfrac{|C|}{2}, \qquad G(x) = \widetilde{T}(x) - \tfrac{|C|}{2} \qquad \text{(or with } F \text{ and } G \text{ interchanged)},$$

where $\widetilde{T}$ is the monic Chebyshev polynomial of degree n for the interval $[\min(A\cup B), \max(A\cup B)]$. After normalising that interval to $[0, 1]$, the $2n$ elements of $A \cup B$ then take the $n + 1$ values

$$\cos^2\left(\frac{j\pi}{2n}\right), \qquad j = 0, 1, \ldots, n, \tag{4}$$

the extreme ones simply and the others doubly. For $n \le 3$ these values are rational and equality is attained by genuine integer solutions: for $n = 2$ by $\{0, 2\}$ versus $\{1, 1\}$ with $|C| = 1 = 4(2/4)^2$, and for $n = 3$ by $\{0, 3, 3\}$ versus $\{1, 1, 4\}$ with $|C| = 4 = 4(4/4)^3$. For $n \ge 4$ equality is impossible over $\mathbb{Q}$: it would force $\cos(\pi/n)$ to be rational, which by Niven's theorem happens only for $n \in \{1, 2, 3\}$. Thus (3) is strict for every ideal solution of size $n \ge 4$, although, as the numerical table of Section 6 shows, it can be approached closely.

Section 2 contains the proof, Section 3 discusses the extremal configuration and its relation to a conjecture of Chen Shuwen, Section 4 shows that the bound is approached arbitrarily closely by integer solutions, Section 5 records the resulting height bounds, and Section 6 the numerical verification.

2 Proof of Theorem 1

Write $m_0 = \min(A \cup B)$ and $M_0 = \max(A \cup B)$, so $L = M_0 - m_0$, and set

$$H \;=\; \tfrac{1}{2}(F + G), \qquad \text{so that} \qquad F = H + \tfrac{C}{2}, \qquad G = H - \tfrac{C}{2} \tag{5}$$

by (2). Thus H is monic of degree n, and both $H + \frac{C}{2}$ and $H - \frac{C}{2}$ split over $\mathbb{R}$ with all their roots in $[m_0, M_0]$. Also $F' = G' = H'$.

Lemma 2. *Let P be a monic polynomial of degree n with all roots real, and let $c_1 \le \cdots \le c_{n-1}$ be the roots of P'. Then $(-1)^{n-j}P(c_j) \ge 0$ for $1 \le j \le n-1$. Moreover, if r denotes the largest root of P then $P(x) \ge 0$ for $x \ge r$, and if s denotes the smallest root then $(-1)^n P(x) \ge 0$ for $x \le s$.*

[1] We have checked this directly in [1], [4], [5], [6], [8] and [9]; Chen Shuwen has also confirmed that no bound of this kind is known to him (personal communication). Theorem 1 improves the trivial bound by the factor 4^{n-1}, which is 262144 at $n = 10$.

Proof. Let $r_1 \le \cdots \le r_n$ be the roots of P. By Rolle's theorem and a count of multiplicities, $r_j \le c_j \le r_{j+1}$ for $1 \le j \le n-1$. On the open interval (r_j, r_{j+1}), exactly $n-j$ roots of P lie to the right, so P has constant sign $(-1)^{n-j}$ there; at the endpoints P vanishes. Hence $(-1)^{n-j}P(c_j) \ge 0$. The two final assertions are immediate. □

Lemma 3. *Let $c_1 \le \cdots \le c_{n-1}$ be the roots of H' and put $x_0 = m_0$, $x_j = c_j$ for $1 \le j \le n-1$, and $x_n = M_0$. Then $x_0 \le x_1 \le \cdots \le x_n$, all x_j lie in $[m_0, M_0]$, and*

$$(-1)^{n-j} H(x_j) \;\ge\; \frac{|C|}{2}, \qquad j = 0, 1, \ldots, n. \tag{6}$$

Proof. The roots of $H' = F'$ lie in the convex hull of the roots of F by the Gauss–Lucas theorem, hence in $[m_0, M_0]$; this gives the ordering.

For $1 \le j \le n-1$, Lemma 2 applied to F and to G — which have the same derivative, hence the same critical points — gives, with $\sigma = (-1)^{n-j}$,

$$\sigma F(c_j) \;\ge\; 0, \qquad \sigma G(c_j) \;\ge\; 0.$$

Since $F - G = C$ we have $\sigma F - \sigma G = \sigma C$. If $\sigma C \ge 0$ then $\sigma C = |C|$ and $\sigma F(c_j) = \sigma G(c_j) + |C| \ge |C|$; if $\sigma C < 0$ then $\sigma G(c_j) = \sigma F(c_j) + |C| \ge |C|$. In either case $\sigma F(c_j) + \sigma G(c_j) \ge |C|$, that is, $\sigma H(c_j) \ge |C|/2$ by (5).

At $x_n = M_0$ one of F, G vanishes and the other, being monic and evaluated at a point at least as large as all of its roots, is nonnegative by Lemma 2; its value there is $\pm C$, hence equals $|C|$. Therefore $H(M_0) = |C|/2$, which is (6) for $j = n$. Symmetrically, at $x_0 = m_0$ one of F, G vanishes and the other has sign $(-1)^n$ and absolute value $|C|$, so $H(m_0) = (-1)^n|C|/2$ and $(-1)^n H(m_0) = |C|/2$. □

Proof of Theorem 1. Let

$$\widetilde{T}(x) \;=\; 2\left(\frac{L}{4}\right)^n T_n\!\left(\frac{2x - m_0 - M_0}{L}\right)$$

be the monic Chebyshev polynomial of degree n for $[m_0, M_0]$, where T_n is the Chebyshev polynomial of the first kind; it satisfies

$$\max_{x \in [m_0, M_0]} |\widetilde{T}(x)| = \mu := 2\left(\frac{L}{4}\right)^n. \tag{7}$$

Assume, for contradiction, that $|C|/2 > \mu$. Put $Q = H - \widetilde{T}$; since both H and $\widetilde{T}$ are monic of degree n, $\deg Q \le n-1$. Let $\sigma_j = (-1)^{n-j}$. By Lemma 3 and (7), for each $j = 0, \ldots, n$,

$$\sigma_j Q(x_j) = \sigma_j H(x_j) - \sigma_j \widetilde{T}(x_j) \;\ge\; \frac{|C|}{2} - \mu \;>\; 0.$$

In particular $Q(x_j) \ne 0$ for every j, and consecutive values $Q(x_j), Q(x_{j+1})$ have opposite signs; this forces $x_0 < x_1 < \cdots < x_n$ and produces n sign changes of Q on $[m_0, M_0]$. Hence Q has at least n distinct real roots, while $Q \not\equiv 0$ and $\deg Q \le n-1$: a contradiction. Therefore $|C|/2 \le \mu$, which is (3). □

Remark 4. Equality in (3) forces $H = \widetilde{T}$. Indeed, if $|C|/2 = \mu$ then $\sigma_j Q(x_j) \ge 0$ for $j = 0, \ldots, n$, and the standard uniqueness argument in Chebyshev's equioscillation theorem (see e.g. [2, Ch. 2]) shows that a polynomial of degree at most $n-1$ with a weakly alternating sign pattern at $n+1$ points of an interval must vanish identically. Then $F = \widetilde{T} + \frac{C}{2}$ and $G = \widetilde{T} - \frac{C}{2}$ with $|C|/2 = \mu$, so $A \cup B$ is the set of solutions of $T_n(u) = \pm 1$ transported to $[m_0, M_0]$, which after normalisation to $[0, 1]$ is (4).

Remark 5. Only the real-rootedness of F and G and the constancy of $F - G$ were used; the integrality of A and B played no role. Theorem 1 therefore holds verbatim for ideal PTE solutions over any subring of $\mathbb{R}$.

3 The extremal configuration

The configuration (4) is not new to the subject. In 2021 Chen Shuwen discovered a trigonometric identity attached to the PTE problem which, in the notation of [9, Identity 10], reads as follows: if (A, B) is an ideal solution of size n normalised so that $\min(A \cup B) = 0$, and if $\max A > \max B$, then

$$\frac{\max B}{\max A} \;\geq\; \sin^2\!\left(\frac{(n-1)\pi}{2n}\right) \;=\; \cos^2\!\left(\frac{\pi}{2n}\right), \tag{8}$$

the right-hand side being exactly the second node in (4). The statement is restated in [9, Conjecture 2] together with an explicit description of the minimising configuration, which is precisely the pattern of double roots of $T_n \mp 1$. Inequality (8) has been verified on millions of numerical solutions; no proof is available.

Thus the configuration (4) is conjecturally extremal for the individual variables and, by Theorem 1, provably extremal for the constant. The two statements concern different functionals of the same solution, and they are tight at the same place: for the ideal solution of size 5

$$\{0, 664, 701, 1787, 1788\} \quad \text{versus} \quad \{188, 189, 1275, 1312, 1976\}$$

quoted in [9, (4.4)] — with A the second set, so that $\max A > \max B$ as (8) requires — the left-hand side of (8) exceeds the right-hand side by a factor 1.000387, while $|C|$ reaches 0.9981 times the bound (3).

4 Sharpness over the integers

Equality in (3) requires the configuration (4), which is irrational for $n \geq 4$. It is therefore natural to ask how closely integer solutions can approach the bound. The answer is: arbitrarily closely, at least for $n = 4$, $n = 5$ and $n = 6$.

The four examples of Table 1 were supplied by Chen Shuwen, who verified Theorem 1 on them independently.

n	solution	$4^{n-1}\lvert C\rvert/L^n$	dev.
4	$\{0, 288, 289, 577\}$ vs. $\{84, 85, 492, 493\}$	0.99996996	0.5
5	$\{0, 421, 422, 1103, 1104\}$ vs. $\{116, 117, 798, 799, 1220\}$	0.99998656	0.5
6	$\{0, 466, 467, 1399, 1400, 1866\}$ vs. $\{124, 126, 931, 935, 1740, 1742\}$	0.99995635	2.0
7	$\{0, 344, 431, 1227, 1270, 1923, 1966\}$ vs. $\{80, 123, 776, 819, 1615, 1702, 2046\}$	0.95984029	45.8

Table 1: Integer ideal solutions supplied by Chen Shuwen. The last column is the largest deviation, in units of the original scale, of an element from the nearest node of (4) after normalisation of $A \cup B$ to $[0, 1]$.

Their structure is visible once $A \cup B$ is normalised to $[0, 1]$: the two simple nodes 0 and 1 of (4) are hit exactly, and each of the $n - 1$ double nodes is straddled by a pair of elements. For $n = 4$ and $n = 5$ that pair is consecutive and the node is its exact midpoint, so the deviation from (4) is half a unit of the original scale; for $n = 6$ the largest deviation is two units, and for $n = 7$ it is close to 46, which is why that example is markedly less tight.

This is not an accident, and the first three cases extend to infinite families.

Proposition 6. (i) *Let u, w be positive integers with $u^2 - 2w^2 = 1$ and $u > 3$, and put $m = (u-1)/2$ and $a = (u-w-1)/2$. Then*

$$A = \{0,\, m,\, m+1,\, u\}, \qquad B = \{a,\, a+1,\, 2m-a,\, 2m+1-a\}$$

is an ideal solution of size 4 *with diameter* $L = u$.

(ii) *Let* v, w *be integers with* $w > v > 0$ *and* $2w^2 - 6vw + 2v^2 - v - w - 1 = 0$, *and put* $u = 2(w - v)$. *Then*

$$A = \{0,\ u - v - 1,\ u - v,\ u + w,\ u + w + 1\}, \qquad B = \{2u - x :\ x \in A\}$$

is an ideal solution of size 5 *with diameter* $L = 2u$.

(iii) *Let* d, h, e *be positive integers with* h *odd,* $e \equiv d \pmod 2$ *and* $e^2 - 3h^2 = 1 - 3d^2$, *and put* $a = (h - 1)/2$ *and* $b = (e - d)/2$. *Then*

$$A = \{\pm a,\ \pm(a + 1),\ \pm h\}, \qquad B = \{\pm d,\ \pm b,\ \pm(b + d)\}$$

is a symmetric ideal solution of size 6 *with diameter* $L = 2h$, *and*

$$\frac{4^5|C|}{L^6} \;=\; \Big(1 - \frac{1}{h^2}\Big)^2 \;-\; \frac{d^2\big(3h^2 + 1 - 4d^2\big)^2}{h^6} \;=\; 1 - \frac{2 + 9d^2}{h^2} + O(h^{-4}). \tag{9}$$

In all three cases the defining equation has infinitely many solutions, and along any of the families

$$\frac{4^{n-1}|C|}{L^n} \;\longrightarrow\; 1.$$

Proof. That each displayed pair satisfies (1) is a direct computation.

For (i), Pell's equation $u^2 - 2w^2 = 1$ has infinitely many solutions. For (ii), solving the defining conic for w gives discriminant $20v^2 + 20v + 9 = 5(2v + 1)^2 + 4$, so its integer points correspond to the solutions of $y^2 - 5X^2 = 4$ with $X = 2v + 1$; these are indexed by the Fibonacci numbers, $X = F_m$ with m even, and the integrality of w selects $m \equiv 2 \pmod 6$. There are infinitely many.

For (iii), both sides are symmetric about 0, so the odd power sums vanish identically and only $k = 2$ and $k = 4$ remain. Writing $S = (3h^2 + 1)/2$, which is an integer because h is odd, those two conditions read

$$b^2 + (b + d)^2 = S - d^2, \qquad b^4 + (b + d)^4 = \tfrac{1}{2}S^2 - d^4,$$

so that b^2 and $(b + d)^2$ are the roots of $z^2 - (S - d^2)z + (\frac{1}{2}S - d^2)^2$. Its discriminant is $d^2(3h^2 + 1 - 3d^2) = d^2e^2$, a square by the defining equation, and substituting $3h^2 = e^2 + 3d^2 - 1$ turns the two roots into $\big((e - d)/2\big)^2$ and $\big((e + d)/2\big)^2$, which are perfect squares; the only remaining condition is the Pell equation. For its infinitude, the map $(e, h) \mapsto (7e + 12h,\ 4e + 7h)$, multiplication by $(2 + \sqrt{3})^2$ in $\mathbb{Z}[\sqrt{3}]$, preserves $e^2 - 3h^2$ and the parities of e and of h; when d is odd the map $(e, h) \mapsto (2e + 3h,\ e + 2h)$ already serves, since it interchanges those two parities and both are then odd. Admissible seeds are $(e, h) = (1, 1)$ for $d = 1$, and $(e, h) = (4, 3)$ and $(8, 5)$ for $d = 2$. The seeds themselves give $A = B$ and only start the orbits; for $d = 2$ the two orbits under the first map give $h = 37, 515, 7173, \ldots$ and $h = 67, 933, 12995, \ldots$. Identity (9) is a direct computation, and it gives the limit for (iii) at once, d being fixed.

For the limit, note that $0 \in A$ in both cases, so $\prod_i a_i = 0$ and

$$\frac{4^{n-1}|C|}{L^n} \;=\; 4^{n-1} \prod_{i=1}^{n} \frac{b_i}{L},$$

a continuous function of the normalised configuration $(A \cup B - \min)/L$. Along family (i) one has $w/u \to 2^{-1/2}$, and along family (ii) the conic gives $w/v \to (3 + \sqrt{5})/2$; in either case the normalised configuration converges to (4), with 0 still in the part corresponding to A. The displayed quantity is therefore continuous at the limit and takes there the value 1, by the equality case of Theorem 1. □

Corollary 7. *For $n = 4$, $n = 5$ and $n = 6$, the supremum of $4^{n-1}|C|/L^n$ over all ideal solutions of that size over $\mathbb{Z}$ equals 1, and is not attained. In particular the constant 4^{n-1} in Theorem 1 cannot be replaced by any larger one.*

L	A	B	$4^{n-1}\|C\|/L^n$
size $n = 4$, from $u^2 - 2w^2 = 1$			
17	$\{0, 8, 9, 17\}$	$\{2, 3, 14, 15\}$	0.965505681206
99	$\{0, 49, 50, 99\}$	$\{14, 15, 84, 85\}$	0.998979789641
577	$\{0, 288, 289, 577\}$	$\{84, 85, 492, 493\}$	0.999969963647
3363	$\{0, 1681, 1682, 3363\}$	$\{492, 493, 2870, 2871\}$	0.999999115809
size $n = 5$, from $2w^2 - 6vw + 2v^2 - v - w - 1 = 0$			
68	$\{0, 23, 24, 61, 62\}$	$\{6, 7, 44, 45, 68\}$	0.995677733744
1220	$\{0, 421, 422, 1103, 1104\}$	$\{116, 117, 798, 799, 1220\}$	0.999986562781
21892	$\{0, 7563, 7564, 19801, 19802\}$	$\{2090, 2091, 14328, 14329, 21892\}$	0.999999958269
size $n = 6$, from $e^2 - 3h^2 = 1 - 3d^2$			
22	$\{0, 5, 6, 16, 17, 22\}$	$\{1, 2, 10, 12, 20, 21\}$	0.910383554391
82	$\{0, 20, 21, 61, 62, 82\}$	$\{5, 6, 40, 42, 76, 77\}$	0.993462997984
306	$\{0, 76, 77, 229, 230, 306\}$	$\{20, 21, 152, 154, 285, 286\}$	0.999530129935
74	$\{0, 18, 19, 55, 56, 74\}$	$\{4, 6, 35, 39, 68, 70\}$	0.972434781529
134	$\{0, 33, 34, 100, 101, 134\}$	$\{8, 10, 65, 69, 124, 126\}$	0.991552767685
1030	$\{0, 257, 258, 772, 773, 1030\}$	$\{68, 70, 513, 517, 960, 962\}$	0.999856730554

Table 2: The three families of Proposition 6. The examples of Table 1 are the third entry of the first block and the second of the second, and the third block is the family of part (iii), with $d = 1$ then $d = 2$.

Lemma 8. *Let (A, B) be an ideal solution of size 4. If A is symmetric about its mean, then so is B, about the same point.*

Proof. Translate the mean to 0, so that $\sum a_i = \sum b_i = 0$. For any four quantities of zero sum,

$$\sum_i a_i^3 \;=\; 3\,(a_1 + a_2)(a_1 + a_3)(a_1 + a_4).$$

If A is symmetric then $\sum a_i^3 = 0$, hence $\sum b_i^3 = 0$ by (1), so one of the three pairings of B has a vanishing pair sum; the complementary pair then sums to zero as well. □

Not every ideal solution of size 4 is symmetric: $\{0, 18, 18, 35\}$ against $\{3, 8, 30, 30\}$ is ideal, and neither side is symmetric about its mean.

Theorem 9. *Let (A, B) be an ideal solution of size 4 which is symmetric about its mean. Translate that mean to 0 and, if it is a half-integer, scale by 2, and write $A = \{-p, -q, q, p\}$ and $B = \{-r, -s, s, r\}$ with integers $p > q \geq 0$ and $r \geq s \geq 1$, the two sides being labelled so that $p = \max(A \cup B)$. Then $r^2 + s^2 = p^2 + q^2$, $r^2 s^2 > p^2 q^2$, and*

$$\frac{4^3|C|}{L^4} \;=\; \frac{4\,(r^2 s^2 - p^2 q^2)}{p^4} \;\leq\; \Big(1 - \frac{q^2}{p^2}\Big)^2,$$

with equality if and only if $r = s$; when $q = 0$ this equality is impossible, and the ratio equals $1 - \delta^2(2p^2 - \delta^2)/p^4$ with $\delta = r - s \geq 1$. Consequently every ideal solution of size 4 that is symmetric about its mean, of diameter L and in lowest terms, satisfies

$$\frac{4^3|C|}{L^4} \;\leq\; \Big(1 - \frac{1}{L^2}\Big)^2, \tag{10}$$

and equality holds precisely for

$$A = \Big\{0, \tfrac{L-1}{2}, \tfrac{L+1}{2}, L\Big\}, \qquad B = \{c, c, d, d\}, \qquad c = \tfrac{L-w}{2}, \quad d = \tfrac{L+w}{2},$$

where $L^2+1=2w^2$. For these, $cd=(L^2-1)/8$, so that $|C|=\left((L^2-1)/8\right)^2$ and the ratio equals $(1-1/L^2)^2$ exactly.

Proof. Both sides being symmetric, the odd power sums vanish and (1) reduces to its $k=2$ instance, $r^2+s^2=p^2+q^2=:T$. Interchanging A and B changes only the sign of C, so the labelling $p=\max(A\cup B)$ costs no generality, and $L=2p$. From $r\ge s$ we get $r^2\ge T/2$, and $p^2\ge T/2$ because $p>q$. On $[\sqrt{T/2},\sqrt{T}\,]$ the function $r\mapsto r\sqrt{T-r^2}$ is strictly decreasing and takes the value pq at $r=p$; since $\{r,s\}\ne\{p,q\}$ forces $r<p$, we obtain $rs>pq$. Next $C=p^2q^2-r^2s^2$, so $4^3|C|/L^4=4(r^2s^2-p^2q^2)/p^4$. Finally $rs\le T/2$, with equality if and only if $r=s$, so

$$r^2s^2-p^2q^2\;\le\;\frac{T^2}{4}-p^2q^2\;=\;\frac{(p^2-q^2)^2}{4},$$

which gives the bound.

If $q\ge 1$, the right-hand side is at most $(1-1/p^2)^2$, with equality exactly when $q=1$; together with $r=s$ that is $p^2+1=2r^2$.

If $q=0$, which can only happen when the mean is an integer, then $r=s$ would give $p^2=2r^2$, which is impossible, so $\delta=r-s\ge 1$; also $\delta<r<p$. Since $(r+s)^2=2p^2-\delta^2$ and $pq=0$,

$$\frac{4^3|C|}{L^4}\;=\;\frac{4r^2s^2}{p^4}\;=\;\frac{(r^2+s^2)^2-(r^2-s^2)^2}{p^4}\;=\;1-\frac{\delta^2(2p^2-\delta^2)}{p^4}.$$

The function $t\mapsto t(2p^2-t)$ is increasing on $[0,p^2]$ and $1\le\delta^2<p^2$, so the ratio is at most $1-(2p^2-1)/p^4=(1-1/p^2)^2$, with equality exactly when $\delta=1$; for instance $\{0,5,5,10\}$ against $\{1,2,8,9\}$ gives $576/625$.

In both cases the ratio is at most $(1-1/p^2)^2$. A solution symmetric about a half-integer has $L=p$ in lowest terms, which is (10); a solution symmetric about an integer has $L=2p$ and satisfies the sharper $(1-4/L^2)^2$. So (10) holds in both cases and only the half-integer case attains it. □

L	A	B	$4^3\lvert C\rvert/L^4$
7	$\{0,3,4,7\}$	$\{1,1,6,6\}$	0.959600166597
41	$\{0,20,21,41\}$	$\{6,6,35,35\}$	0.998810585892
239	$\{0,119,120,239\}$	$\{35,35,204,204\}$	0.999964986914
1393	$\{0,696,697,1393\}$	$\{204,204,1189,1189\}$	0.999998969311
8119	$\{0,4059,4060,8119\}$	$\{1189,1189,6930,6930\}$	0.999999969659

Table 3: The extremal family of Theorem 9; every entry attains (10) with equality. The first member has $|C|=36$, which is the universal constant C_4 itself.

Remark 10. The size 4 family of Proposition 6(i) and the extremal family of Theorem 9 are the two branches of one Pell equation: the first is $u^2-2w^2=1$, with $u=17,99,577,3363,\ldots$, the second is $u^2-2w^2=-1$, with $u=7,41,239,1393,\ldots$. Their diameters interleave, and at equal diameter the second is the better of the two: its ratio is $1-2/L^2+O(L^{-4})$ against $1-10/L^2+O(L^{-4})$.

Remark 11. Theorem 9 is proved for solutions symmetric about their mean. An exhaustive enumeration of the ideal solutions of size 4 with all entries in $[0,100]$, of which there are 30161, confirms that (10) holds for every one of them, symmetric or not, and that for each diameter the maximum of $4^3|C|/L^4$ is attained by a symmetric solution. For non-symmetric solutions the bound is therefore verified, not proved. The ancillary file reproduces this enumeration when its range is set to $[0,100]$, which takes about a gigabyte of memory; at its default range $[0,60]$ it finds 4878 solutions and the same two statements hold for all of them.

Remark 12 (priority)**.** Formulas of this kind for the sizes 4, 5 and 6 were obtained by Chen Shuwen in 2024, before the present work; the relevant results are in [9]. The three families of Proposition 6 were found independently, and the earlier work was learned of only afterwards, by personal communication.

Remark 13. The three families begin at $(u, w) = (17, 12)$, at $(v, w) = (10, 27)$ and at $(d, h, e) = (1, 11, 19)$; the examples of sizes 4, 5 and 6 in Table 1 are the third member of the first, the second of the second, and the member $(d, h, e) = (2, 933, 1616)$ of the third. That $n \leq 6$ should behave so well is explained by the nodes (4) lying in $\mathbb{Q}(\cos(\pi/n))$, a field of degree $\varphi(2n)/2$: that degree is 1 for $n \leq 3$, where the nodes are rational and equality is attained outright, and equals 2 exactly for $n = 4, 5, 6$, where the simultaneous approximation required reduces to a single Pell equation. For $n = 7$ the degree is 3 and the approximation is genuinely harder, which is what the last line of Table 1 records. Whether the supremum is 1 for every n remains open.

Remark 14. The constant 4 in (3) is the capacity normalisation: an interval of length L has logarithmic capacity $L/4$, and the monic Chebyshev polynomial of degree n on it has supremum norm $2\,(L/4)^n$, which is the quantity μ of (7). This is why the base 4 does not depend on n.

5 Lower bounds for the height

Let C_n denote the universal constant of the PTE problem of size n, that is, the greatest common divisor of C over all ideal solutions of size n, and let C'_n denote the corresponding quantity restricted to symmetric solutions. Divisors of C_n and of C'_n are tabulated in [6, Table 1] for $n \leq 20$. Since $C_n \mid C$ for every ideal solution, Theorem 1 inverts to:

Corollary 15. *Every ideal solution of size n satisfies*

$$L \;\geq\; 4\left(\frac{C_n}{4}\right)^{1/n}.$$

For a symmetric solution, whose elements may be taken symmetric about 0 so that $L = 2h$ with $h = \max|a_i|$, this reads $h \;\geq\; 2\,(C'_n/4)^{1/n}$.

The resulting values are listed in Table 4. They are weaker than the exhaustive searches of [6] wherever those have been carried out. For $17 \leq n \leq 20$ no search has been published, and Corollary 15 provides, as far as we are aware, the first unconditional lower bound on the magnitude of a hypothetical ideal solution for those n.

6 Numerical verification

Table 5 lists, for classical ideal solutions of size $n \leq 12$, the ratio of $|C|$ to the bound (3). The defining identities (1) are checked exactly for each entry. The computation is reproduced by the ancillary file `verify_pte_chebyshev.py`, which uses exact integer arithmetic and requires only the Python standard library. The same file replays the families of Proposition 6, the extremal family of Theorem 9, and the enumeration of Remark 11.

Status of the statements. Theorem 1, its equality case, Proposition 6, Corollary 7, Lemma 8, Theorem 9 and Corollary 15 are PROVED. The bound (10) is proved for solutions symmetric about their mean and, for the others, VERIFIED by the enumeration recorded above. The entries of Table 4 rest on the divisibility data of [6, Table 1], which we have VERIFIED independently for $n = 7, 9, 11, 13$; the columns L and h there are computed from the exact integer products. Inequality (8) is OPEN and is quoted here as a conjecture of [9].

n	$C_n \geq$	$L \geq$	$C'_n \geq$	$h \geq$	searched to
3	$10^{0.60}$	4	$10^{0.60}$	2	—
4	$10^{1.56}$	7	$10^{1.56}$	4	—
5	$10^{3.70}$	17	$10^{3.70}$	9	—
6	$10^{3.86}$	14	$10^{3.86}$	7	—
7	$10^{6.52}$	29	$10^{7.80}$	22	—
8	$10^{8.48}$	39	$10^{8.48}$	20	—
9	$10^{9.38}$	38	$10^{9.38}$	19	7000
10	$10^{10.81}$	42	$10^{10.81}$	21	2500
11	$10^{12.77}$	52	$10^{14.26}$	35	3500
12	$10^{12.70}$	41	$10^{14.16}$	27	1511
13	$10^{16.25}$	64	$10^{22.39}$	95	2000
14	$10^{17.36}$	64	$10^{20.42}$	53	900
15	$10^{18.51}$	63	$10^{25.00}$	85	1100
16	$10^{18.46}$	53	$10^{26.46}$	83	850
17	$10^{22.35}$	77	$10^{32.06}$	142	—
18	$10^{22.31}$	65	$10^{30.49}$	92	—
19	$10^{26.33}$	91	$10^{34.74}$	126	—
20	$10^{27.61}$	90	$10^{34.21}$	96	—

Table 4: Lower bounds from Corollary 15. The columns $C_n \geq$ and $C'_n \geq$ give the product of the divisors listed in [6, Table 1], displayed to two decimals; L and h are computed from the exact integer products and rounded up, L being the diameter and h the height of a symmetric solution. The last column is the height to which an exhaustive search has been published [6, Table 2].

n	solution	$\lvert C\rvert$	L	$4^{n-1}\lvert C\rvert/L^n$
2	$\{0,2\}$ vs $\{1,1\}$	1	2	1.0000
3	$\{0,3,3\}$ vs $\{1,1,4\}$	4	4	1.0000
3	$\{1,5,6\}$ vs $\{2,3,7\}$	12	6	0.8889
3	$\{1,8,8\}$ vs $\{2,5,10\}$	36	9	0.7901
5	Chen Shuwen [9, (4.4)]	117449348889600	1976	0.9981
6	Gloden 1944	216770400	86	0.5487
7	Escott 1910	13967553600	102	0.4981
7	symmetric, $h = 55$	37604952000	110	0.7904
7	symmetric, $h = 123$	11722885574400	246	0.8807
8	Chen Shuwen 1997	207048441600	96	0.4702
8	Tarry 1913	1210809600	50	0.5079
9	Letac 1942 (I)	3377425033382400	198	0.4732
9	Letac 1942 (II)	554721467677977600	348	0.4857
10	Borwein–Lisoněk–Percival (a)	5687272703608009113600	626	0.1613
10	Borwein–Lisoněk–Percival (b)	2628768883410597915187200	1030	0.5128
12	Kuosa–Meyrignac–Chen	67440294559676054016000	302	0.4915

Table 5: The bound (3) on classical ideal solutions of size $n \leq 12$, in the integral form $4^{n-1}|C| \leq L^n$. The first two rows are the equality cases.

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