Sums of three composite triangular numbers

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Abstract

A triangular number $T_k = k(k+1)/2$ is composite for every $k \geq 3$, the only non-composite ones being 0, 1 and 3. We ask which integers are sums of three composite triangular numbers, a restricted form of Gauss's Eureka theorem; equivalently, when $8n+3$ is a sum of three squares none of which is 1, 9 or 25. We prove that every integer $n > 194$ is a sum of three composite triangular numbers, with at most three possible exceptions, all larger than $8.4 \cdot 10^9$. Such exceptions can occur only if Tatuzawa's form of Siegel's theorem, with $\varepsilon = 1/20$, has an exceptional discriminant $-N_*$, and then only for $8n+3 \in \{N_*, 9N_*, 25N_*\}$. Up to $8.4 \cdot 10^9$ the integers that are not such sums are exactly 41 explicit values, the largest being 194. Under the Generalized Riemann Hypothesis these 41 values are the complete list. The proof combines an elementary lemma, a union bound for the representations that use a forbidden square, the class number formula for sums of three squares, Tatuzawa's theorem, and explicit bounds for a divisor-type function. The analytic range begins at $8n+3 = 3.4561 \cdot 10^{10}$ and overlaps the computed range by a factor 1.944. The computed range is certified by two independent algorithms, supplied as ancillary files together with a verifier for every constant.

1 Introduction

Gauss's Eureka theorem states that every non-negative integer is a sum of three triangular numbers $T_k = k(k+1)/2$, $k \geq 0$; it is equivalent to the representation of every $N \equiv 3 \pmod 8$ as a sum of three squares. Since $T_k = \frac{k}{2}(k+1)$ or $k\frac{k+1}{2}$ is a product of two integers larger than 1 as soon as $k \geq 3$, the only triangular numbers that are not composite are $T_0 = 0$, $T_1 = 1$ and $T_2 = 3$. It is natural to ask how much of Gauss's theorem survives when these three are removed.

Definition 1.1. Let $\mathcal{E}$ be the set of integers $n \geq 0$ that cannot be written $n = T_a + T_b + T_c$ with $a, b, c \geq 3$.

Theorem 1.2. *(i) The elements of $\mathcal{E}$ not exceeding $8.4 \cdot 10^9$ are exactly*

$$0, 1, \ldots, 17, \quad 19, 20, 21, 23, 24, 25, 28, 29, 32, 34, 38,$$
$$39, 43, 47, 50, 54, 60, 68, 69, 73, 95, 113, 194,$$

forty-one values, the largest being 194.

(ii) Every element of $\mathcal{E}$ larger than $8.4 \cdot 10^9$ is of the form $(f^2 N_* - 3)/8$ with $f \in \{1, 3, 5\}$, where $-N_*$ is the exceptional discriminant allowed by Lemma 5.1 with $\varepsilon = 1/20$, if there is one. In particular $\mathcal{E}$ has at most 44 elements, and every $n > 194$ is a sum of three composite triangular numbers with at most three possible exceptions.

The exceptional discriminant may well not exist; if it exists, it need not be $\equiv 5 \pmod 8$, in which case part (ii) gives no exception at all. What the theorem cannot do is decide this, which is the familiar limitation of every effective form of Siegel's theorem. Under the Generalized Riemann Hypothesis the question disappears.

Corollary 1.3. *Assume the Generalized Riemann Hypothesis, in the form used in [8] and [9]. Then $\mathcal{E}$ consists exactly of the 41 values listed in Theorem 1.2; every integer $n > 194$ is a sum of three composite triangular numbers.*

The closest result in the literature concerns sums of three *nonzero* triangular numbers, that is, the exclusion of T_0 alone. Kim, Lee and Oh [8] proved, under the Generalized Riemann Hypothesis, that every positive integer except $1, 2, 4, 6, 11, 20, 29$ is such a sum. Excluding T_0, T_1 and T_2 amounts to excluding the three odd squares 1, 9, 25 instead of the single square 1. Their result enters our proof of Corollary 1.3, through Proposition 2.3. We are not aware of an earlier treatment of the present question. The recent historical survey of Aygin and Williams [1] is the natural place where one would be recorded; its abstract concerns formulas for the number of representations, and we have not been able to consult the full text.

Plan. Section 2 states the problem in terms of sums of three squares and treats every $8n+3$ divisible by the square of an odd $q \geq 7$. Section 3 shows that the remaining cases follow from the inequality $H(N) > 3D\big((N-1)/2\big)$ between a class number and a divisor-type function. Section 4 bounds that function, and Section 5 recalls Tatuzawa's lower bound for the class number. Section 6 closes the chain of ranges, Section 7 proves Corollary 1.3, and Section 8 describes the computation and the ancillary files. Section 9 states precisely what rests on what.

2 Reformulation and the elementary range

Lemma 2.1. *Put $N = 8n + 3$. Then $n = T_a + T_b + T_c$ with $a, b, c \geq 0$ if and only if $N = x^2 + y^2 + z^2$ with x, y, z odd, the correspondence being $x = \pm(2a + 1)$, and similarly for y, z. Every representation of N as a sum of three squares has x, y, z odd. Consequently $n \notin \mathcal{E}$ if and only if $N = x^2 + y^2 + z^2$ with $|x|, |y|, |z| \geq 7$, that is, with none of x^2, y^2, z^2 equal to 1, 9 or 25.*

Proof. The identity $8T_k+1 = (2k+1)^2$ gives $8(T_a+T_b+T_c)+3 = (2a+1)^2+(2b+1)^2+(2c+1)^2$. Squares are 0, 1 or 4 modulo 8, and three of them sum to 3 modulo 8 only as $1 + 1 + 1$, so x, y, z are odd. An odd x is $\pm(2a + 1)$ with $a \geq 0$, and $a \geq 3$ exactly when $|x| \geq 7$; the odd squares below 49 are 1, 9 and 25. □

Proposition 2.2. *If $q^2 \mid 8n + 3$ for some odd $q \geq 7$, then $n \notin \mathcal{E}$.*

Proof. Every odd square is $\equiv 1 \pmod 8$, so $M = (8n + 3)/q^2 \equiv 3 \pmod 8$. By Gauss's theorem $M = u^2+v^2+w^2$, and by Lemma 2.1 u, v, w are odd, hence non-zero. Then $8n+3 = (qu)^2 + (qv)^2 + (qw)^2$ with $|qu|, |qv|, |qw| \geq q \geq 7$. □

Proposition 2.3. *Let $q \in \{3, 5\}$ and suppose $8n + 3 = q^2(8m + 3)$. If m is a sum of three nonzero triangular numbers, then $n \notin \mathcal{E}$.*

Proof. If $m = T_a + T_b + T_c$ with $a, b, c \geq 1$, then $8m + 3 = u^2 + v^2 + w^2$ with $u = 2a + 1 \geq 3$, and similarly for v, w. Hence $8n + 3 = (qu)^2 + (qv)^2 + (qw)^2$ with $qu \geq 3q \geq 9$. □

Write $N = 8n + 3 = f^2 N_0$ with N_0 squarefree and $f \geq 1$; f is odd. If $f \notin \{1, 3, 5\}$, then $f \geq 7$ and Proposition 2.2 applies with $q = f$. Since $N_0 \equiv N \equiv 3 \pmod 8$, the number $-N_0 \equiv 5 \pmod 8$ is a fundamental discriminant. *From now on $f \in \{1, 3, 5\}$.*

3 The counting inequality

Let $t_3(n)$ be the number of ordered triples (a,b,c) with $a,b,c \geq 0$ and $T_a + T_b + T_c = n$, and $\rho_2(m)$ the number of ordered pairs (b,c) with $b,c \geq 0$ and $T_b + T_c = m$ (with $\rho_2(m) = 0$ for $m < 0$).

Lemma 3.1. *Let $N = 8n+3 = f^2 N_0$ as above, with $N_0 > 3$. Then*

$$t_3(n) = \tfrac{1}{8}\, r_3(N) = 3H(N), \qquad H(N) = \sum_{d \mid f} h(-d^2 N_0),$$

where r_3 counts representations as a sum of three squares and $h(-d^2N_0)$ is the class number of primitive positive definite binary quadratic forms of discriminant $-d^2N_0$. In particular

$$H(N) = h(-N_0) \cdot \begin{cases} 1 & \text{if } f = 1, \\ 4 - \chi(3) & \text{if } f = 3, \\ 6 - \chi(5) & \text{if } f = 5, \end{cases}$$

where χ is the Kronecker character of $-N_0$; hence $H(N) \geq f\, h(-N_0)$.

Proof. By Lemma 2.1 each triple (a,b,c) corresponds to exactly 8 representations of N, one for each choice of signs, so $t_3(n) = r_3(N)/8$. For $N \equiv 3 \pmod 8$, Gauss's theorem on primitive representations [4, art. 291–292], see also [5], gives $24\,h(-N)$ primitive representations of $N > 3$; summing over the squares $e^2 \mid N$, that is over $N/e^2 = d^2N_0$ with $d \mid f$, gives $r_3(N) = 24H(N)$. The class number of the order of conductor d is $h(-d^2N_0) = h(-N_0)\, d \prod_{p \mid d}\bigl(1 - \chi(p)/p\bigr)$, the unit index being 1 since $N_0 > 3$ [2, Section 7]; for $d = 3$ and $d = 5$ this is $h(-N_0)(3 - \chi(3))$ and $h(-N_0)(5 - \chi(5))$. □

Lemma 3.2. *For $m \geq 0$, $\rho_2(m) = d_1(4m+1) - d_3(4m+1)$, where $d_i(M)$ is the number of divisors of M congruent to i modulo 4. Consequently $\rho_2(m) \leq \prod_{p \mid 4m+1,\ p \equiv 1 (4)} (e_p + 1)$, where $p^{e_p} \parallel 4m+1$.*

Proof. $T_b + T_c = m$ is $(2b+1)^2 + (2c+1)^2 = 8m+2$. Every representation of $8m+2$ as a sum of two squares has both terms odd, so the pairs (b,c) are the representations with both terms positive, that is $r_2(8m+2)/4$. By Jacobi's formula [6, Thm. 278], $r_2(8m+2) = 4\bigl(d_1(8m+2) - d_3(8m+2)\bigr)$, and the odd divisors of $8m+2$ are those of $4m+1$. The last assertion is the usual evaluation of $d_1 - d_3$ as a multiplicative function: it vanishes unless every prime $\equiv 3 \pmod 4$ occurs to an even power, and then equals the product. □

Proposition 3.3. *If $H(N) > \rho_2(n) + \rho_2(n-1) + \rho_2(n-3)$, then $n \notin \mathcal{E}$.*

Proof. For $i \in \{1,2,3\}$ and $j \in \{0,1,2\}$, let $A(i,j)$ be the set of triples counted by $t_3(n)$ whose i-th entry is j. Then $|A(i,j)| = \rho_2(n - T_j)$, and $T_0, T_1, T_2 = 0, 1, 3$. A triple outside $\bigcup A(i,j)$ has all entries ≥ 3. Such a triple exists as soon as

$$t_3(n) = 3H(N) \;>\; \sum_{i,j} |A(i,j)| = 3\bigl[\rho_2(n) + \rho_2(n-1) + \rho_2(n-3)\bigr]. \qquad \square$$

Definition 3.4. For $Y \geq 1$ let

$$D(Y) = \max_{1 \leq M \leq Y} \prod_{p \mid M,\ p \equiv 1\,(4)} (e_p + 1), \qquad p^{e_p} \parallel M.$$

The three integers $4n+1$, $4n-3$, $4n-11$ attached by Lemma 3.2 to n, $n-1$, $n-3$ are at most $4n+1 = (N-1)/2$. Since D is non-decreasing, Proposition 3.3 gives:

Corollary 3.5. *If $H(N) > 3\,D\big((N-1)/2\big)$, then $n \notin \mathcal{E}$.*

Remark 3.6. The union bound in Proposition 3.3 is crude: a triple with two forbidden entries is counted twice. Inclusion–exclusion would improve the constants of Section 6, but the chain closes without it, and the cruder form is easier to check.

4 The function D

Let $5 = q_1 < q_2 < q_3 < \cdots$ be the primes $\equiv 1 \pmod 4$, and write $v(M) = \prod_{p \mid M,\, p \equiv 1 (4)} (e_p + 1)$.

Lemma 4.1. *For every Y, $D(Y)$ is the maximum of $v(M)$ over the integers*

$$M = q_1^{e_1} q_2^{e_2} \cdots q_r^{e_r} \le Y, \qquad e_1 \ge e_2 \ge \cdots \ge e_r \ge 1.$$

Proof. Given $M \le Y$, delete its prime factors $\not\equiv 1 \pmod 4$, then assign the exponents of the remaining primes, in non-increasing order, to $q_1, q_2, \ldots$. The value v does not change and, by the rearrangement inequality, the integer does not increase. □

The integers of Lemma 4.1 up to 10^{13} involve at most eight primes, since $q_1 \cdots q_9 > 10^{13}$, and are enumerated exhaustively by the ancillary verifier. The records that matter are in Table 1; the enumeration is checked against a brute-force scan of all $M \le 10^6$.

D	smallest M	factorisation
96	243 061 325	$5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41$
128	1 215 306 625	$5^3 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41$
144	3 159 797 225	$5^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 37 \cdot 41$
160	6 076 533 125	$5^4 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41$
192	12 882 250 225	$5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53$
216	53 716 552 825	$5^2 \cdot 13^2 \cdot 17^2 \cdot 29 \cdot 37 \cdot 41$

Table 1: Records of D: $D(Y)$ first reaches the value in the first column at $Y = M$.

In particular

$$D(Y) \le 192 \qquad \text{for } Y < 53\,716\,552\,825. \tag{1}$$

Lemma 4.2. *For $t > 0$ put $g(t) = \prod_{p \equiv 1 (4),\ p^t < 2} \max_{e \ge 0} (e+1) p^{-et}$. Then $D(Y) \le g(t)\, Y^t$ for all $Y \ge 1$. Moreover $g(0.2) < 2.18407$, the maxima being attained at $e = 2, 1, 1, 1$ for $p = 5, 13, 17, 29$.*

Proof. For $M \le Y$, $v(M) = \prod_p (e_p+1) p^{-e_p t} \cdot \prod_p p^{e_p t} \le \prod_p (e_p+1) p^{-e_p t} \cdot M^t$, the products running over $p \equiv 1 \pmod 4$. If $p^t \ge 2$ then $(e+1)p^{-et} \le (e+1)2^{-e} \le 1$. The numerical value is computed by the verifier. □

5 Tatuzawa's theorem

Lemma 5.1 (Tatuzawa)**.** *Let $0 < \varepsilon < 1/2$. There is at most one fundamental discriminant $D < 0$ with $|D| \ge \max\big(e^{1/\varepsilon}, e^{11.2}\big)$ such that*

$$L(1, \chi_D) \;\le\; 0.655\, \varepsilon\, |D|^{-\varepsilon}.$$

This is [10, Theorem 2], proved there from Lemma 9; it is restated by Elsenhans, Klüners and Nicolae [3, Lemma 2] and used by Hopkins and Stopple [7]. We take $\varepsilon = 1/20$, so that $\max(e^{20}, e^{11.2}) = e^{20} = 4.8517 \cdot 10^8$. If $-N_0 \neq -N_*$ is a fundamental discriminant with $N_0 \geq e^{20}$, then, by the class number formula $h(-N_0) = \sqrt{N_0}\, L(1,\chi)/\pi$,

$$h(-N_0) \;>\; A\, N_0^{0.45}, \qquad A = \frac{0.655}{20\pi} = 0.01042464877\ldots \tag{2}$$

Here and below, $-N_*$ denotes the exceptional discriminant of Lemma 5.1 for $\varepsilon = 1/20$, if there is one.

Remark 5.2. The proof requires the form of Lemma 5.1, with the constant $0.655\,\varepsilon$, as given in [10, Theorem 2].

6 Proof of Theorem 1.2

Let $N = 8n+3 = f^2 N_0$ with $f \in \{1,3,5\}$, N_0 squarefree, and suppose $N_0 \neq N_*$. By Lemma 3.1 and (2), as long as $N_0 \geq e^{20}$,

$$H(N) \;\geq\; f\, h(-N_0) \;>\; f \cdot f^{-0.9}\, A\, N^{0.45} \;\geq\; A\, N^{0.45},$$

since $3^{0.1} = 1.116\ldots$ and $5^{0.1} = 1.174\ldots$. By Corollary 3.5 it is therefore enough to have

$$A\, N^{0.45} \;\geq\; 3\, D\big((N-1)/2\big). \tag{3}$$

The condition $N_0 \geq e^{20}$ holds throughout the analytic ranges below, which start at $N > 3.4 \cdot 10^{10}$, so that $N_0 \geq N/25 > 1.38 \cdot 10^9 > e^{20}$.

Range 1: $N > 8.9635 \cdot 10^{10}$. By Lemma 4.2 with $t = 0.2$, $3D((N-1)/2) \leq 3g(0.2)(N/2)^{0.2}$, so (3) holds as soon as

$$N^{0.25} \;\geq\; \frac{3\, g(0.2)}{A \cdot 2^{0.2}}, \qquad \text{that is} \qquad N \;\geq\; 8.9635 \cdot 10^{10}.$$

Range 2: $3.4561 \cdot 10^{10} < N \leq 1.0743 \cdot 10^{11}$. Here $(N-1)/2 < 53\,716\,552\,825$, so $D((N-1)/2) \leq 192$ by (1), and (3) holds as soon as $A\, N^{0.45} \geq 576$, that is

$$N \;\geq\; \Big(\frac{576}{A}\Big)^{1/0.45} = 3.456051 \cdot 10^{10}.$$

Range 2 extends beyond the start of Range 1, so Ranges 1 and 2 together cover every $N > 3.4561 \cdot 10^{10}$, that is every $n > 4.3201 \cdot 10^9$.

Range 3: $N \leq 6.72 \cdot 10^{10}$. By direct computation (Section 8), the elements of $\mathcal{E}$ up to $8.4 \cdot 10^9$, that is with $N \leq 8 \cdot 8.4 \cdot 10^9 + 3 = 67\,200\,000\,003$, are exactly the 41 values of part (i).

Closure. Ranges 2 and 3 overlap, by the factor

$$\frac{67\,200\,000\,003}{3.456051 \cdot 10^{10}} \;=\; 1.9444.$$

Hence every $n > 8.4 \cdot 10^9$ with $N_0 \neq N_*$ lies outside $\mathcal{E}$. The remaining integers have $N \in \{N_*, 9N_*, 25N_*\}$, which is part (ii). Part (i) is the computation. □

Remark 6.1. The factor 1.9444 is the whole of the slack between the analytic and the computed ranges: a factor 2 lost in (3) would separate them. Every constant entering the chain is recomputed in the ancillary verifier in 50-digit decimal arithmetic, with π replaced by an upper bound so that (2) is never overstated, and the closure of the chain is checked by assertions.

7 Proof of Corollary 1.3

By Theorem 1.2 it suffices to show that, under GRH, no $n > 8.4 \cdot 10^9$ with $8n+3 \in \{N_*, 9N_*, 25N_*\}$ lies in $\mathcal{E}$.

If $8n+3 = q^2 N_*$ with $q \in \{3,5\}$, then $N_* \equiv 3 \pmod 8$, say $N_* = 8m+3$, and $m > (e^{20}-3)/8 > 29$. By the theorem of Kim, Lee and Oh [8, Cor. 3.11], valid under GRH, m is a sum of three nonzero triangular numbers, and Proposition 2.3 gives $n \notin \mathcal{E}$.

If $8n+3 = N_*$, then $N_* > 6.72 \cdot 10^{10}$. Under GRH, Lamzouri, Li and Soundararajan [9, Thm. 1.5] prove, for every primitive character χ modulo $q \geq 10^{10}$,

$$\frac{1}{|L(1,\chi)|} \;\leq\; \frac{12e^{\gamma}}{\pi^2}\Big(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}\Big).$$

At $q = 6.72 \cdot 10^{10}$ this gives $L(1,\chi) \geq 0.0898$, while the exceptional inequality of Lemma 5.1 would require $L(1,\chi) \leq 0.655 \cdot \frac{1}{20} q^{-1/20} = 0.00942$; the ratio between the two bounds is at least 9.54 for every $q \geq 6.72 \cdot 10^{10}$, since the first decreases only like $1/\log\log q$ while the second decreases like $q^{-1/20}$. Hence N_* does not exist in this range. □

8 Computation

The ancillary file `VERIFY_range.py` recomputes $\mathcal{E} \cap [0, 8.4 \cdot 10^9]$ by two algorithms that share no code except an exact test used on rare survivors. Both work on segments of 2^{26} consecutive integers.

Method A. On each segment, the set of sums $T_a + T_b$ with $3 \leq a \leq b$ is built on a window reaching just below the segment, and n is declared covered as soon as $n - T_k$ lies in it for one of the 64 composite triangular numbers $T_3, \ldots, T_{66}$.

Method B. The set of sums $T_a + T_b$ with $3 \leq a \leq b$ is built once on $[0, 4 \cdot 10^6]$, and n is declared covered as soon as $n - T_c$ lies in it for some c. For $n \leq 4 \cdot 10^6$ this test is exhaustive.

Every integer that is not declared covered is passed to an exact test, which tries every $c \geq 3$ and decides whether $n - T_c$ is a sum of two composite triangular numbers; the exceptions are therefore certified by exhaustion. As a guard against a marking error, 24 integers declared covered are drawn at random in each segment and re-tested exactly. Up to $8.4 \cdot 10^9$, method A passes 368 integers to the exact test and method B passes 16 341; both return exactly the 41 values of Theorem 1.2 (i). The two runs together take about six and a half minutes on two cores.

The file `VERIFY_composite_triangular.py` recomputes the exceptional set up to $2 \cdot 10^7$ by a third implementation, the records of D (Table 1) by the enumeration of Lemma 4.1 checked against a brute-force scan up to 10^6, the constants A, $g(t)$, the thresholds and the overlap of Section 6, and the identities of Lemma 3.1 for $f = 1$ ($11 \leq N < 2000$), $f = 3$ ($11 \leq N_0 < 400$) and $f = 5$ ($11 \leq N_0 < 200$), with class numbers computed by counting reduced forms. The files `CHECK_classnumbers.magma` and `CHECK_exceptions.magma` repeat the class number identities and the exceptional set up to $2 \cdot 10^5$ under Magma, with Magma's own class number routine. Both Python files require only Python 3 and `numpy`, and take no input.

As consistency checks, none of the 41 exceptions has $8n+3$ divisible by q^2 with $q \geq 7$ odd, as Proposition 2.2 requires; and the six exceptions with $9 \mid 8n+3$ or $25 \mid 8n+3$, namely $n = 3, 9, 12, 21, 34, 39$, give $m = 0, 0, 1, 2, 1, 4$ in Proposition 2.3, all of them on the list of Kim, Lee and Oh.

9 Status of the argument

Proved here: Lemmas 2.1, 3.2, 4.1, 4.2, Propositions 2.2, 2.3, 3.3, and the deduction of Theorem 1.2 (ii) and Corollary 1.3 from the inputs below.

Verified by computation: the 41 exceptions up to $8.4 \cdot 10^9$; the records of D; the constants and the closure of the chain; the identities of Lemma 3.1 on the ranges stated.

Quoted: Gauss's formula for r_3 and the class number of orders (Lemma 3.1); Jacobi's formula for r_2; Tatuzawa's theorem (Lemma 5.1), checked against the original [10, Theorem 2]; and, for Corollary 1.3 only, [8, Cor. 3.11] and [9, Thm. 1.5], both conditional on GRH.

Open: whether the exceptional discriminant can be excluded unconditionally, which would make the list of 41 values complete without any reserve.

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