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# Irrational values of Dirichlet series with periodic coefficients

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**Abstract**

Let $f$ be a nonzero rational-valued function of period $D$, and write $L_f(j) = \sum_{n\geq 1} f(n)n^{-j}$. For either prescribed parity we give an explicit lower bound of order $\sqrt{s \log\log s/\log s}$ for the number of exponents $2 \leq j \leq s$ of that parity for which $L_f(j)$ is irrational. The bound is uniform in $f$ for fixed $D$. It applies, for example, to every nonzero rational projection of a quartic Dirichlet character modulo five. If $f$ is even or odd, at most $D-1$ exponents $j \geq 2$ of the matching parity give zero. The nonzero values at distinct exponents of that parity are linearly independent over the algebraic numbers.

## 1 A uniform count in each parity

Fix a positive integer $D$. Assign a rational weight to each residue class modulo $D$, with at least one weight nonzero, and let $f(n)$ be the weight of the class of $n$. For real $x > 1$ define the Dirichlet series $L_f(x) = \sum_{n\geq 1} f(n)n^{-x}$. The constant choice $f \equiv 1$ gives the Riemann zeta function $\zeta$. A rational-valued Dirichlet character $f = \chi$ gives its $L$-function $L(x, \chi)$.

Weights of opposite signs allow cancellation between residue classes. Given an integer cutoff $s$, how many even exponents $2 \leq j \leq s$ must give an irrational value $L_f(j)$, and how many odd exponents must do so? We give a lower bound for each count that tends to infinity with $s$, uniformly over the weights for fixed $D$. For $\delta = 0$ (even) or $\delta = 1$ (odd), write

$$N_{f,\delta}(s) = \#\{2 \leq j \leq s : j \in \mathbb{Z},\ j \equiv \delta \pmod 2,\ L_f(j) \notin \mathbb{Q}\}, \qquad Y(s) = \sqrt{\frac{s \log\log s}{\log s}}.$$

The constants are defined by the exact prime powers $p^a \parallel D$. An empty product equals one.

$$\kappa = \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \qquad K_D = \kappa \prod_{p^a \parallel D} \frac{p^{2-a}}{p^2 - p + 1}, \qquad R_D = \prod_{p\mid D}(1 + p^{-1}).$$

**Theorem 1.1.** *For every $\epsilon > 0$ there is an $s_0(D, \epsilon)$ such that all integers $s \geq s_0(D, \epsilon)$, both parities $\delta$, and every such $f$ satisfy*

$$N_{f,\delta}(s) \ \geq\ (2\sqrt{K_D/R_D} - \epsilon)Y(s).$$

*If $f$ vanishes on all residues not coprime to $D$, the coefficient $2\sqrt{K_D/R_D}$ can be replaced by $2\sqrt{K_D}$. If $f(-n) = (-1)^\delta f(n)$ for every $n$, at most $D-1$ integers $j \geq 2$ of parity $\delta$ have $L_f(j) = 0$. The nonzero values at distinct such exponents are linearly independent over $\overline{\mathbb{Q}}$.*

One choice of weights applies throughout each count, though it may change with $s$. In the *matching parity*, defined by $f(-n) = (-1)^\delta f(n)$ for all $n$, a Fourier expansion gives $L_f(j) = \alpha_j \pi^j$ with $\alpha_j$ algebraic. The nonzero values are thus distinct and transcendental. When the reflection identity fails, a discrepancy between reflected weights makes a determinant nonzero.

## Examples and earlier results

Let $\chi$ be the quartic character modulo five with $\chi(2) = \sqrt{-1}$. For any $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, the coefficients of $a\Re L(j, \chi) + b\Im L(j, \chi)$ on residues $1, 2, 3, 4$ are $a, b, -b, -a$. Theorem 1.1 gives the coefficient $2\sqrt{5\kappa/21}$ at even exponents, including the imaginary part alone, whose first coefficient is zero. The two odd quadratic characters $\chi_4, \chi_{-8}$, viewed modulo eight, likewise give $2\sqrt{\kappa/6}$ for every nonzero rational combination of $\beta(j) = L(j, \chi_4)$ and $L(j, \chi_{-8})$. Their values at $1, 3, 5, 7$ are respectively $(1, -1, 1, -1)$ and $(1, 1, -1, -1)$.

Fischler proved a uniform result for complex periodic coefficients. His Theorem 2 gives at least $2^{(1-\epsilon)\log s/\log\log s}$ values in each parity outside any prescribed finite-dimensional rational vector space [3]. For real characters of conductor at least two, Kumar and Tosi obtained an irrationality count of order $\sqrt{s/\log s}$ in the opposite parity [6, Theorem 1]. For individual characters of modulus $D \geq 3$ with $4 \nmid D$, Mistiaen obtained a dimension bound of that order over the cyclotomic field $\mathbb{Q}(e^{2\pi\sqrt{-1}/D})$ [8, Théorème 1.0.5], again in the opposite parity. Our conclusion concerns a count uniform in rational coefficients. It attains the scale $Y(s)$ proved for odd zeta values by Fischler, Lai, Sprang and Zudilin (FLSZ) [4, Theorem 1.1], and for even beta values in [11, Theorem 1.1]. For $D = 1$ and for $\chi_4$ with $D = 4$ we recover their respective coefficients $2\sqrt{\kappa}$ and $2\sqrt{\kappa/3}$.

## The two counts behind the proof

When $f(n) = (-1)^\delta f(-n)$, a Fourier evaluation and a bound on real zeros give the last assertion of Theorem 1.1 (Section 6). For the other case, we use the elimination on rational grids of Fischler, Sprang and Zudilin [5, §5] and the totient selection of Lai and Yu [7, §2]. The Hankel determinant estimate and the criterion for a nonzero determinant come from FLSZ [4, §§3–5], in the form developed for the character modulo four in [11, §§3–6]. We determine the grid cost for general $D$ and show how a discrepancy between two reflected residue classes survives every nontrivial elimination.

Each grid $1/q, \dots, q/q$ supplies one weight. If fewer exponents give irrational values than there are grids, we can choose a nonzero vector of weights that cancels every irrational term. After clearing denominators, the resulting determinants would be integers of absolute value below one throughout a range of sizes, yet at least one would be nonzero. Proving this last assertion requires some care because the weights can have either sign.

Write $\varphi$ for Euler's totient. For $B > 1$ take the grid denominators $\mathcal{Q}_B = \{q \geq 1 : D \mid q,\ \varphi(q) \leq B\}$. For general support the allowed reduced denominators are $\mathcal{B}_B = \{b \geq 2 : \varphi(\operatorname{lcm}(b, D)) \leq B\}$. For support on residues coprime to $D$ take $\mathcal{B}_B = \{b \in \mathcal{Q}_B : b \geq 2\}$. Put

$$\Theta = \{a/b : b \in \mathcal{B}_B,\ 1 \leq a < b,\ (a, b) = 1\}, \quad h = |\Theta|.$$

Thus $|\mathcal{Q}_B|$ counts adjustable weights and $h$ counts fractions at which denominators may need clearing. Every contributing $j/q$ strictly between 0 and 1 reduces to some $a/b \in \Theta$ because $\operatorname{lcm}(b, D) \mid q$, and if $(j, D) = 1$ then $D \mid b$. The reflected fraction $1 - a/b$ has the same denominator. For instance $2/8 = 1/4$ requires a denominator smaller than the period on a general period-eight grid. Put

$$A = \prod_{b \in \mathcal{B}_B} b^{\varphi(b)} \prod_{b \in \mathcal{B}_B} \prod_{p \mid b} p^{\varphi(b)/(p-1)}. \tag{1.1}$$

The factor for numerator progressions of length $n$ will be $A^n$, at a logarithmic cost of $n \log A$. There are asymptotically $K_D B$ weights and $GB^2/2$ fractions, where $G = K_D R_D$ for general support and $G = K_D$ on units. Section 4 gives $\log A \sim (G/2)B^2 \log B$. For $B = \lambda Y(s)$ with $\lambda < 2/\sqrt{G}$, the determinant estimate still forces absolute value below one. Thus the count of irrational exponents must reach the number of weights, giving the coefficient $2K_D/\sqrt{G}$.

## 2 From exceptional values to an integer determinant

Take the integer $s \equiv \delta \pmod 2$ and set $e = (-1)^s$. It suffices to use these cutoffs with an arbitrarily small error, since $Y(s-1)/Y(s) \to 1$. Fix $s, B$ sufficiently large with $B^2 < s/100$. Replace $f$ by an integral multiple so that all $f(j)$ are integers. If $N_{f,\delta}(s) < |\mathcal{Q}_B|$, there are integers $w_q$, not all zero, such that

$$\sum_{q \in \mathcal{Q}_B} w_q q^i = 0 \quad (2 \le i \le s,\ i \equiv \delta \pmod 2,\ L_f(i) \notin \mathbb{Q}). \tag{2.1}$$

There are fewer homogeneous rational equations than weights. Choose a positive integer $A_0$ clearing the rational $L_f(i)$ on the same range and parity. Both choices are fixed as the even integer $n$ below grows. For $i > 1$, grouping the series by the residues of $qv + j$ gives

$$\sum_{j=1}^{q} f(j)\zeta(i, j/q) = q^i L_f(i), \qquad \zeta(i,\theta) = \sum_{v \ge 0} (v+\theta)^{-i}. \tag{2.2}$$

This is where the restriction $D \mid q$ is used.

Write $(t)_r = t(t+1)\cdots(t+r-1)$ and $(t)_0 = 1$. For even $n \ge 2$ use the rational function

$$R_n(t) = A^n (n!)^{s-h} \frac{\prod_{\theta \in \Theta} (t+\theta)_n}{(t)_{n+1}^s}.$$

The numerator vanishes at the reflected shifts needed in the finite Hurwitz tails. Since $n$ is even, reflection gives $R_n(-n-t) = eR_n(t)$. For a polynomial $U$ of degree at most $(n+1)s - nh - 2$, put

$$\mathcal{L}_n(U) = A_0 \sum_{q \in \mathcal{Q}_B} w_q \sum_{j=1}^{q} f(j) \sum_{v \ge 0} R_n(t) \left[U(t+n/2) + U(-t-n/2)\right]\big|_{t=v+j/q}. \tag{2.3}$$

The degree condition makes the series absolutely convergent. The two polynomial evaluations cancel for every odd test monomial, whatever $\delta$ is. For the Dirichlet exponent the reflection $R_n(-n-t) = (-1)^s R_n(t)$ forces the sum of the coefficients at each pole order of parity opposite to $s$ to vanish. Equation (2.2) turns the surviving Hurwitz terms into $L_f$-values.

For $2m \le (n+1)s - nh$ define the determinant in the monomial basis and the factors for a second basis by

$$\Delta_{n,m} = \det_{0 \le i,j < m} \mathcal{L}_n(t^{i+j}), \qquad d_n = \operatorname{lcm}(1, \dots, n), \quad E_{n,i} = (n!)^{\lfloor i/n \rfloor} (i \bmod n)!, \quad U_i(t) = (t)_i / E_{n,i}.$$

Dividing $U_i$ into blocks of length at most $n$ ensures that $d_n^u$ clears its Taylor coefficient of order $u$ at every integer. The integer matrix we shall use is

$$\boxed{d_n^s \mathcal{L}_n(U_i U_j) \in \mathbb{Z} \quad (0 \le i, j < m).} \tag{2.4}$$

Consequently its determinant is the integer

$$J_{n,m} = \det_{0 \le i,j < m} \left(d_n^s \mathcal{L}_n(U_i U_j)\right) = H_{n,m} \Delta_{n,m}, \qquad H_{n,m} = d_n^{sm} \prod_{i=0}^{m-1} E_{n,i}^{-2}. \tag{2.5}$$

Each $U_i$ has leading coefficient $1/E_{n,i}$. Applying the triangular change of basis on both sides gives exactly the last factor in (2.5).

## Proof of the entrywise assertion

We give the arithmetic argument since a weight on a nonunit residue can require a shift with a smaller denominator. It uses the partial-fraction divisibility argument of Sprang [10, Lemma 1.4] and Fischler, Sprang and Zudilin [5, Lemma 2], as applied in [4, Lemma 3.1] and [11, Lemmas 4.1–4.2].

For $U = U_i U_j$ the degree bound gives a proper partial fraction expansion

$$\widetilde{R}(t) = R_n(t)[U(t+n/2) + U(-t-n/2)] = \sum_{k=0}^{n} \sum_{\ell=1}^{s} \frac{a_{\ell,k}}{(t+k)^\ell}, \qquad d_n^{s-\ell} a_{\ell,k} \in \mathbb{Z}.$$

For the divisibility, take the coefficient of order $s-\ell$ at $t=-k$ after multiplication by $(t+k)^s$. Each of the $s$ pole factors $n!(t+k)/(t)_{n+1}$ has constant term $(-1)^k\binom{n}{k}$, and its other Taylor factors are geometric series with denominators powers of $k'-k$, $0 \le k' \le n$, $k' \ne k$. Hence $d_n^u$ clears order $u$. For each $a/b \in \Theta$ use the numerator block

$$b^n \prod_{p \mid b} p^{\lfloor n/(p-1) \rfloor} \frac{(t+a/b)_n}{n!}.$$

When $p \mid b$ its prime factor clears $v_p(n!)$. When $p \nmid b$, at an integer center the $n$ factors of $b^n(t+a/b)_n$ form an arithmetic progression with step prime to $p$. At least $\lfloor n/p^v \rfloor$ are multiples of $p^v$. Taking a coefficient of order $u$ deletes $u$ factors, losing at most $u$ at each level $1 \le v \le v_p(d_n)$. Thus $d_n^u$ again clears the coefficient. The product of the chosen numerator multipliers divides $A^n$, by grouping the $\varphi(b)$ fractions with denominator $b$. Each $U_i$ is a product of shifted blocks $(t+c)_r/r!$ with $c \in \mathbb{Z}$, $r \le n$. The same deletion count applies to them, also after the integer shifts by $n/2$. Multiplication preserves this Taylor-coefficient divisibility.

Put $\rho_\ell = \sum_k a_{\ell,k}$. The decay $\widetilde{R}(t) = O(t^{-2})$ gives $\rho_1 = 0$, including when $s$ is odd. Reflection gives $a_{\ell,n-k} = (-1)^{s+\ell} a_{\ell,k}$, so $\rho_\ell = 0$ on the other parity. For an active grid fraction $0 < \theta \le 1$ this yields

$$\sum_{v \ge 0} \widetilde{R}(v+\theta) = \rho_{0,\theta} + \sum_{\substack{2 \le \ell \le s \\ \ell \equiv s \ (2)}} \rho_\ell \zeta(\ell, \theta),$$

$$\rho_{0,\theta} = -\sum_{k=1}^{n} \sum_{v=0}^{k-1} \sum_{\ell=1}^{s} \frac{a_{\ell,k}}{(v+\theta)^\ell}. \tag{2.6}$$

One may first truncate at a common upper limit. The harmonic terms cancel there, and the upper boundary terms tend to zero.

We claim $d_n^s \rho_{0,\theta} \in \mathbb{Z}$. For $\theta = 1$ this follows from $1 \le v+1 \le n$. For a fractional shift, consider $n = k = 2$, $v = 1$, $\theta = 2/5$. Here $v + \theta = 7/5$, so its reciprocal has a factor of 7 in the denominator, although $d_2 = 2$. At $t_0 = -3/5$ the reflected block $(t+3/5)_2$ vanishes. The sum at pole $k = 2$ can therefore be recovered from the two other poles, whose offsets $-3/5$ and $2/5$ are both units at 7.

In general, for $\theta < 1$ fix $k, v$ in (2.6) and a prime $p$. If $v_p(v+\theta) \le v_p(d_n)$, the scaled terms are individually $p$-integral. Otherwise evaluate $\widetilde{R}$ at $t_0 = -k + v + \theta$. This is a regular point and the factor at index $k - v - 1$ in $(t+1-\theta)_n$ vanishes. Thus the sum over $\ell$ at pole $k$ equals the negative of the sum over all other poles. For $k' \ne k$,

$$v_p(t_0 + k') = v_p(k' - k) \le v_p(d_n).$$

Every term in the replacement is $p$-integral after scaling. This proves the claim at every prime. Finally (2.2) gives the coefficient $A_0 \rho_\ell \sum_q w_q q^\ell$ of $L_f(\ell)$. Equation (2.1) eliminates the irrational terms, and $A_0$ clears the rational values. This proves (2.4).[1]

[1] Of 960 finite-tail sums checked, 462 had at least one nonintegral scaled summand. In `checks/` (GitHub) or `anc/checks/` (Hexagon), run `python3 kernel_checks.py`.

## 3 How the largest grid gives a nonzero determinant

Zudilin used positive moment determinants in an irrationality criterion [12, Proposition 2]. FLSZ [4, §1] credit the construction of Fauzan [2, §2.4] as a precursor of their Hankel determinant. Here the weights can have both signs. To prove nonvanishing we will follow one residue class through the two tails of the moment functional. Define

$$X = \mathbb{Z} + (\Theta \cup \{1\}), \qquad W(x) = \sum_{\substack{q \in \mathcal{Q}_B \\ qx \in \mathbb{Z}}} w_q f(qx), \qquad \widehat{R}(z) = R_n(z - n/2).$$

We have $W(x+1) = W(x)$ since $D \mid q$. The signed measure representing $\mathcal{L}_n$ has masses on $X$ given by

$$a_x = \begin{cases} A_0 W(x)\widehat{R}(x), & x > n/2, \\ eA_0 W(-x)\widehat{R}(x), & x < -n/2, \\ 0, & |x| \le n/2. \end{cases} \tag{3.1}$$

Extend the masses by zero off $X$. The first line follows by collecting equal positive evaluation points in (2.3), and the second from $\widehat{R}(-x) = e\widehat{R}(x)$. In particular the mirror points $x$ and $-x$ have equal masses.

For $\theta \in \Theta \cup \{1\}$, periodicity compares $k + \theta$ with $-k + \theta$. These points are on opposite tails in the *same* class modulo one, at distance $2k$. For sufficiently large positive integers $k$, the kernel is regular and nonzero there. Dividing the masses in (3.1) by its value at each point gives $A_0 W(\theta)$ and $eA_0 W(-\theta)$. The determinant criterion below asks whether a difference equation has a rational solution. Such a solution would make these two numbers residues of a one-periodic meromorphic function, forcing them to agree.

Return to $f = \Im\chi$ modulo five, whose values at $1, 2, 3, 4$ are $0, 1, -1, 0$. Take even exponents, so $e = 1$, and give the 5- and 10-grids weights 4 and $-1$, with all other weights zero. Since $4 \cdot 5^2 - 10^2 = 0$, the coefficient of $L_f(2)$ cancels. Both grids hit $1/5 = 2/10$. The 5-grid samples the zero at 1, whereas the 10-grid samples the first discrepancy between $f(j)$ and $f(-j)$, at $j = 2$. Thus

$$W(1/5) = 4f(1) - f(2) = -1, \qquad W(-1/5) = 4f(-1) - f(-2) = 1.$$

A one-periodic function cannot have residues $-A_0$ and $A_0$ at $k + 1/5$ and $-k + 1/5$, respectively.

In general put $g(j) = f(j) - ef(-j)$. When $g$ is nonzero, let $j_0$ be the first positive index with $g(j_0) \neq 0$, and let $q_*$ be the largest grid denominator with $w_{q_*} \neq 0$. Set $\theta_* = j_0/q_*$, so $0 < \theta_* \le 1$ since $j_0 \le D \le q_*$. A smaller grid hits $\theta_*$ only if $q\theta_*$ is a positive integer below $j_0$, where the discrepancy is zero. Consequently

$$W(\theta_*) - eW(-\theta_*) = w_{q_*} g(j_0) \neq 0. \tag{3.2}$$

At least one of $f(j_0)$ and $f(-j_0)$ is nonzero, and the set of fractions is closed under reflection. Thus $X$ contains this class even if only one tail carries mass there. The endpoint $\theta_* = 1$ is included in $X$.

**Proposition 3.1.** *Suppose $g$ is not identically zero. Put $d = s + h + 2$ and $T = (n+1)s - nh - 1$. If $d \le T$ and $M \ge 1$ is an integer with $2(M + d - 1) - 2 < T$, then $\Delta_{n,m} \neq 0$ for some $M \le m \le M + d - 1$.*

*Proof.* The absolute moment of order $T - 1$ exists, since its summands are $O(|x|^{-2})$ on finitely many translates of $\mathbb{Z}$. The successive ratio of $\widehat{R}$ is $Q(z)/P(z)$, where

$$P(z) = (z + 1 + n/2)^{s+1}(z - n/2) \prod_{\theta \in \Theta} (z + \theta - n/2),$$
$$Q(z) = (z - n/2)^{s+1}(z + 1 + n/2) \prod_{\theta \in \Theta} (z + \theta + n/2).$$

These polynomials have degree $d$. Periodicity gives $P(x)a_{x+1} = Q(x)a_x$ along either tail. At a noninteger transition $x = n/2 - 1 + \theta$ or $x = -n/2 - \theta$, use the factor indexed by $1 - \theta$ in $P$ or by $\theta$ in $Q$, respectively. At the integer transitions $x = n/2$ and $x = -n/2 - 1$, use $z - n/2$ in $P$ and $z + 1 + n/2$ in $Q$. This proves the recurrence for every real $x$.

The Cauchy transform $F(z) = \sum_{x \in X} a_x/(z - x)$ consequently solves

$$P(z)F(z+1) - Q(z)F(z) = C(z), \quad C(z) = \sum_{x \in X} a_x \left( \frac{P(z) - P(x-1)}{z - (x-1)} - \frac{Q(z) - Q(x)}{z - x} \right).$$

The moment bound makes $C$ a polynomial, and the simple fractions cancel by the recurrence. The signed-measure criterion of FLSZ [4, Lemma 5.1 and Proposition 5.4] gives the stated nonzero principal determinant if this difference equation has no rational solution.

Suppose $r \in \mathbb{C}(z)$ solved the equation. Then $F - r$ solves its homogeneous part, and the successive-ratio identity makes $(F - r)/\widehat{R}$ one-periodic. For all sufficiently large $k$, $r$ is regular at both $k + \theta_*$ and $-k + \theta_*$. The residue of $F$ is the mass at each point, with residue zero when that mass vanishes. By (3.1) the residues of the periodic function there would therefore be $A_0 W(\theta_*)$ and $eA_0 W(-\theta_*)$. Their difference is nonzero by (3.2), which rules out the rational solution. □

A zero residue on one tail occurs already for the indicator of the class 2 mod 3 with $e = 1$, $w_3 = 1$, $w_6 = -1$ and all other weights zero. Here $j_0 = 1$, and only the 6-grid hits $\theta_* = 1/6$, giving $W(1/6) = 0$ and $W(-1/6) = -1$.[2]

## 4 The cost of the fractions

**Lemma 4.1.** *For fixed $D$ and the two denominator choices on page 2, as $B \to \infty$,*

$$|\mathcal{Q}_B| \sim K_D B, \qquad h \sim \tfrac{1}{2} G B^2, \qquad \log A \sim \tfrac{1}{2} G B^2 \log B, \qquad G = \begin{cases} K_D R_D & \text{for general support,} \\ K_D & \text{for support on units.} \end{cases}$$

*Proof.* Let $T_M(x)$ count positive multiples $q$ of $M$ with $\varphi(q) \le x$. Dressler's inverse-totient theorem gives $T_1(x) \sim \kappa x$ [1, Theorem 1.1]. To impose $p^a \mid q$ for a prime $p \nmid M$, suppose $T_M(x) \sim K_M x$, and let $g(x)$ count the same multiples with $p \nmid q$. Sorting by the valuation of $q$ gives

$$T_M(x) = g(x) + \sum_{k \ge 1} g(x/t_k), \quad t_k = (p-1)p^{k-1}, \quad \sigma = \sum_{k \ge 1} t_k^{-1} = \frac{p}{(p-1)^2}.$$

Then $g(x)/x \to c = K_M/(1 + \sigma)$. Indeed $g(x)/x$ is bounded for $x > 0$ (and is zero for $x < 1$). Subtract the proposed constant after dividing by $x$. For $p \ge 3$, the limsup of the absolute error is at most $\sigma$ times itself, with $\sigma < 1$; the summable weights bound the tail uniformly. For $p = 2$, $t_1 = 1$ gives $2g(x)$ on the right, and the remaining weights sum to one, giving a factor $1/2$. Restoring the terms $k \ge a$ multiplies $K_M$ by $\sum_{k \ge a} t_k^{-1}/(1 + \sigma) = p^{2-a}/(p^2 - p + 1)$. Induction gives $|\mathcal{Q}_B| \sim K_D B$.

For support on units, $h = \sum_{q \in \mathcal{Q}_B,\, q \ge 2} \varphi(q)$; partial summation gives $h \sim K_D B^2/2$. For general support, include temporarily $b = 1$ and group by $q = \operatorname{lcm}(b, D)$. One has

$$\sum_{\operatorname{lcm}(b,D) = q} \varphi(b) = \varphi(q) \prod_{\substack{p^a \| D \\ v_p(q) = a}} \frac{p}{p-1}.$$

[2] In `checks/` (GitHub) or `anc/checks/` (Hexagon), run `python3 profile_checks.py`. It checks the two-tail identity on 31,340 choices of profile, parity and weights, including 20,204 with exactly one zero tail residue.

At a prime with $v_p(q) = a = v_p(D)$, the local sum is $\sum_{j=0}^{a} \varphi(p^j) = p^a$. At the other primes the exponent of $b$ is fixed. For any subset of the primes dividing $D$, requiring $v_p(q) = v_p(D)$ at each selected prime has relative density $\prod_p(1 - 1/p)$ among $\mathcal{Q}_B$, by inclusion–exclusion and $K_{Dp} = K_D/p$ for $p \mid D$. Partial summation gives the same proportion with weight $\varphi(q)$. Expanding the displayed product as factors $1 + \mathbf{1}_{v_p(q)=a}/(p-1)$ therefore multiplies the leading coefficient by $\prod_{p|D}(1 + 1/p) = R_D$.

Finally every $b \in \mathcal{B}_B$ has $\varphi(b) \le B$. The full-grid estimates [7, (2.2) and Lemma 2.4], [4, Lemmas 2.1–2.2] give $b \ll B \log\log B$, $h + 1 \le B^2$ eventually, and a contribution $O(B^2(\log\log B)^2)$ to $\log A$ from the second product of (1.1). In the first product the terms with $\varphi(b) \le B/\log B$ have total $\varphi$-weight $O((B/\log B)^2)$ by the unrestricted count. On the remainder, $\log B - \log\log B \le \log b \le \log B + O(\log\log B)$. These bounds give the stated asymptotic. □

## 5 The size of the same integer

The analytic basis puts zeros at the earliest grid points, where $R_n(t)$ is largest. Since this basis is monic, changing to it preserves the unscaled determinant. This is the estimate of FLSZ [4, Proposition 4.1] and of the even-beta paper [11, Proposition 5.1]. We give the bounds here to keep track of the dependence on the elimination weights and on $f$.

Let $B_0 = 1 + \max \mathcal{Q}_B$ and $C_0 = 4A_0B_0^s \max_{1\le j\le D} |f(j)| \sum_{q\in\mathcal{Q}_B} q|w_q|$. If $2m \le (s - B^2 - 10)n$, then

$$|J_{n,m}| \le (C_0 m)^m A^{mn} \exp\{m^2 + (2s + 10B^2)mn\} \left(\frac{10B^2 n}{m}\right)^{(s-B^2)mn-2m^2}. \tag{5.1}$$

To see which matrix sizes are useful, take $B = \lambda Y(s)$ for fixed $\lambda > 0$ and consider $m/n \to C_s$ at each cutoff $s$. After taking logarithms in (5.1) and dividing by $mn$, the two terms that govern the leading rate are the arithmetic cost $\log A$ and $(s - B^2 - 2C_s)\log(10B^2/C_s)$. As $s$ grows we want $C_s/B^2$ large so that the logarithm is negative, and $C_s = o(s)$ so that its multiplier stays asymptotic to $s$. Taking

$$C_s = \frac{s}{\log\log s} \qquad \text{gives} \qquad \frac{C_s}{B^2} = \frac{\log s}{\lambda^2(\log\log s)^2}.$$

The logarithm of this ratio is asymptotic to $\log\log s$, giving decay on the $s\log\log s$ scale of the arithmetic cost.

To prove (5.1), observe that for each $v$, imposing zeros at both signs of all $h + 1$ candidate shifts costs $2(h + 1)$ degrees. A polynomial of degree $i$ can therefore remove the first $\nu_i = \lfloor i/(2h + 2)\rfloor$ levels. Use the monic polynomial

$$V_i(z) = z^{\,i-(2h+2)\nu_i} \prod_{\theta\in\Theta\cup\{1\}} \prod_{v=0}^{\nu_i-1} (z^2 - (v + \theta + n/2)^2).$$

It vanishes at both signs of $t + n/2$ for every active $t = v + j/q$ with $v < \nu_i$, including $j/q = 1$. For $v \ge \nu_i$ each root has absolute value at most $t + n/2$, giving $|V_i(\pm(t + n/2))| \le (t + n)^i$. Let $S_i$ be the sum for $\mathcal{L}_n(V_i^2)$ with absolute weights. Positivity of $R_n(t)$ for $t > 0$, Cauchy–Schwarz and the determinant expansion give

$$|\Delta_{n,m}| \le m! \prod_{i<m} S_i.$$

For $t \ge 1/(B_0 - 1)$ one has $(t)_{n+1} \ge (n!/B_0)(1 + t/n)^{n+1}$. Combine this with $(t + \theta)_n \le (t + n)^n$ and $n! \ge (n/\mathrm{e})^n$ to get

$$R_n(t) \le B_0^s A^n \exp(hn) \left(\frac{n}{t + n}\right)^{(n+1)s-nh}.$$

Put $e_i = (n+1)s - nh - 2i$ and $E = (s - B^2)n - 2m$. The range of $m$ gives $e_i - 1 > 10n$ and $e_i - 1 \geq E > 0$. Integral comparison of the tail $\sum_{v \geq \nu_i}(1 + (v + j/q)/n)^{-e_i}$ gives at most $2(1 + \nu_i/n)^{-(e_i - 1)}$. Since $h + 1 \leq B^2$, $1 + \nu_i/n > (i+1)/(2B^2 n)$. Consequently

$$S_i \leq C_0 A^n \exp(B^2 n) n^{2i} \left( \frac{2B^2 n}{i+1} \right)^E .$$

When the last base exceeds one we have simply bounded the preceding tail factor $(1 + \nu_i/n)^{-(e_i - 1)}$ by one. In the product of these estimates,

$$\prod_{i=0}^{m-1} n^{2i} \left( \frac{2B^2 n}{i+1} \right)^E = n^{m(m-1)} \left( \frac{(2B^2 n)^m}{m!} \right)^E .$$

Use $m! \geq (m/\mathrm{e})^m$ in this denominator. The factor $H_{n,m}$ supplies the reciprocal power of $n$. Indeed the bounds $d_n \leq \exp(2n)$ and $\log E_{n,i} \geq i \log n - i - n$ give $H_{n,m} \leq n^{-m(m-1)} \exp(m^2 + 2(s+1)mn)$. Using $m! \leq m^m$ in the determinant expansion now yields (5.1).

### Completing the count when the reflected profiles differ

Now fix $0 < \lambda < 2/\sqrt{G}$. With $B = \lambda Y(s)$ and $C_s = s/\log\log s$ as above, for all sufficiently large $s$,

$$B^2 < s/100, \qquad 2C_s < s - B^2 - 10.$$

Fix such an $s$ and a profile with $g \neq 0$, and assume as before that $N_{f,\delta}(s) < |\mathcal{Q}_B|$. For every sufficiently large even $n$, take $M = \lceil C_s n \rceil$. Since $h + 1 \leq B^2$ and $d$ is fixed, Proposition 3.1 gives a nonzero determinant of order $m \in [M, M + d - 1]$ in the range of (5.1). Thus $m/n \to C_s$. The integer $J_{n,m}$ has absolute value at least one, so

$$\begin{aligned} 0 &\leq \limsup_{\substack{n \to \infty \\ 2 \mid n}} \frac{\log |J_{n,m}|}{mn} \\ &\leq \log A + C_s + 2s + 10B^2 + (s - B^2 - 2C_s) \log(10B^2/C_s). \end{aligned}$$

In this first limit $\log(C_0 m)/n \to 0$, however large the rational denominators or the coefficients at this particular $s$ were. The right side therefore depends only on $s$ and the chosen denominator set. By Lemma 4.1,

$$\log A = (G\lambda^2/4 + o(1)) s \log\log s, \qquad \log(10B^2/C_s) = -\log\log s + 2\log\log\log s + O(1).$$

The upper bound is $(G\lambda^2/4 - 1 + o(1)) s \log\log s$, strictly negative for every sufficiently large $s$. This contradiction yields $N_{f,\delta}(s) \geq |\mathcal{Q}_B|$. Taking $\lambda$ arbitrarily close to $2/\sqrt{G}$ from below gives the coefficient $2K_D/\sqrt{G}$, uniformly in $f$. The two values of $G$ in Lemma 4.1 give the first two constants of Theorem 1.1.

## 6 Matching parity and the number of distinct values

Suppose $f(-n) = (-1)^\delta f(n)$. Here the two reflected coefficients agree with the prescribed sign and finite Fourier analysis gives more information. Put $\omega = \exp(2\pi\sqrt{-1}/D)$ and write $f(n) = \sum_{\ell=0}^{D-1} c_\ell \omega^{\ell n}$, with $c_\ell = D^{-1}\sum_{r=1}^{D} f(r)\omega^{-\ell r}$ algebraic. For $j \geq 2$ of parity $\delta$, the contributions of $n$ and $-n$ to the bilateral Dirichlet sum coincide. The absolutely convergent Fourier series of the Bernoulli polynomial $B_j$ [9, §24.8(i)] gives, also at $\ell = 0$,

$$2L_f(j) = -\frac{(2\pi\sqrt{-1})^j}{j!}\sum_{\ell=0}^{D-1} c_\ell B_j(\ell/D). \tag{6.1}$$

Thus $L_f(j) = \alpha_j \pi^j$ with $\alpha_j \in \overline{\mathbb{Q}}$. Related matching-parity independence for individual characters is noted in [8, Remarque 2]. For the present profiles, a finite relation among the nonzero values at distinct exponents with coefficients $\gamma_j \in \overline{\mathbb{Q}}$ would give $\sum_j \gamma_j \alpha_j \pi^j = 0$. Transcendence of $\pi$ over $\overline{\mathbb{Q}}$ forces every coefficient of this polynomial to vanish. Each participating $\alpha_j$ is nonzero, which proves linear independence. Each such value is itself transcendental by (6.1).

To count these values uniformly we must control the zero coefficients. For every nonzero $f$, of any parity, termwise integration gives

$$\Gamma(x)L_f(x) = \int_0^\infty t^{x-1}H(t)\,\mathrm{d}t, \qquad H(t) = \frac{\sum_{r=1}^{D} f(r)e^{-rt}}{1-e^{-Dt}} \qquad (x > 1).$$

The numerator is $e^{-t}$ times a nonzero polynomial of degree at most $D-1$ in $e^{-t}$. Thus $H$ changes sign at most $D-1$ times. Its Mellin transform has at most as many distinct real zeros on $x > 1$ as $H$ has sign changes. Indeed, a function of constant sign that is nonzero on an interval has a Mellin transform of strict sign. If $t_0$ is a sign-change point, $H(t)\log(t/t_0)$ has exactly one fewer sign change. Its Mellin transform, multiplied by $t_0^{-x}$, is the derivative of $t_0^{-x}\int_0^\infty t^{x-1}H(t)\,\mathrm{d}t$. Each differentiation is justified locally for $x > 1$ by $H(t) = O(1/t)$ at zero and $H(t) = O(e^{-t})$ at infinity, even after finitely many logarithmic factors. Induction and Rolle's theorem give the claimed bound.

Since $\Gamma(x) > 0$, there are at most $D-1$ real zeros of $L_f$ on $(1,\infty)$. A zero at a matching integer can occur. Take $f(n) = -8$ on multiples of three and $f(n) = 1$ otherwise. This even period-three function satisfies $L_f(x) = (1 - 9\cdot 3^{-x})\zeta(x)$, so $L_f(2) = 0$. In general, at least the number of eligible exponents minus $D-1$ give distinct transcendental values on the matching parity. This linear count completes Theorem 1.1, with a cutoff depending only on $D, \epsilon$.

**Counting distinct values for an arbitrary profile.** The same sign-change argument bounds repeated values. Every real value $c$ of $L_f(x)$ has at most $D$ preimages on $x > 1$. Indeed $\Gamma(x)(L_f(x)-c)$ has kernel $H(t) - ce^{-t}$. With $z = e^{-t}$, the numerator divided by $z$ is $\sum_{r=1}^{D} f(r)z^{r-1} - c(1-z^D)$, a nonzero polynomial of degree at most $D$. Hence there are at least $N_{f,\delta}(s)/D$ distinct irrational numbers uniformly for varying $f$.

For each fixed $f$ the full leading constants also count distinct irrational numbers. If $v \geq 2$ is the first nonzero coefficient index beyond 1, dominated convergence gives

$$v^j\bigl(L_f(j) - L_f(j+2)\bigr) \longrightarrow f(v)(1-v^{-2}) \neq 0.$$

Such a $v$ exists by periodicity. Each parity subsequence is eventually strictly monotone, so discarding finitely many initial terms suffices.

# 7 Prescribed support and simultaneous projections

Restricting the support of $f$ can reduce the number of fractions while leaving all the grid weights available. Consider an odd period-eight profile, which vanishes at 0 and 4. Its allowed numerator classes have $(j, 8) \in \{1, 2\}$. A reduced denominator $b$ with $(b, 8) = 4$ requires an even numerator on the original grid. Its smallest possible grid denominator is $2b$, so $\varphi(2b) = 2\varphi(b) \le B$. For $8 \mid b$, the unit class already permits the grid $q = b$ and the full cutoff $\varphi(b) \le B$. The fraction count grows quadratically with the cutoff. Halving it for the first class therefore quarters that class's contribution.

More generally, let $\mathcal{G}$ be any nonempty set of divisors of $D$ containing $(j, D)$ whenever $f(j) \neq 0$. For each $d \mid D$, $c_d$ below is the density coefficient for denominators with $(b, D) = d$, and $B/m_d$ is their totient cutoff. Put

$$c_d = \sum_{e \mid D/d} \mu(e) K_{de}, \qquad m_d = \min_{\substack{g \in \mathcal{G} \\ D/g \mid d}} g \prod_{\substack{p \mid g \\ p \nmid d}} (1 - 1/p), \qquad G_{\mathcal{G}} = \sum_{d \mid D} \frac{c_d}{m_d^2}.$$

Here $\mu$ is the Möbius function. An empty minimum is $+\infty$ and contributes zero.

**Corollary 7.1.** *For these profiles the coefficient in Theorem 1.1 can be taken to be* $2K_D/\sqrt{G_{\mathcal{G}}}$.

*Proof.* The denominators required for whole classes $\mathcal{G}$ are exactly the $b \ge 2$ for which $D/g \mid b$ and $\varphi(gb) \le B$ for some $g \in \mathcal{G}$. In fact $j/q = (j/g)/(q/g)$ with $g = (j, D)$. The integer $j/g$ is coprime to $D/g$. Thus the reduced denominator $b$ is divisible by $D/g$, and $gb \mid q$. Conversely each reduced numerator $a$ of such a $b$ occurs as $ga/(gb)$ with $(ga, D) = g$. In particular the reduced fractions remain closed under reflection.[3]

For $d = (b, D)$ the ratio $\varphi(gb)/\varphi(b)$ is $g \prod_{p \mid g,\, p \nmid d}(1 - 1/p)$. The cutoff is therefore $\varphi(b) \le B/m_d$. By Möbius inversion from Lemma 4.1, the count of $b$ with $(b, D) = d$ and $\varphi(b) \le x$ is $c_d x + o(x)$. Partial summation gives $h = (G_{\mathcal{G}}/2 + o(1))B^2$. This coefficient is positive since all multiples $b$ of $D$ with $\varphi(b) \le B/g$ occur for any fixed $g \in \mathcal{G}$. The proof of the bound for $\log A$ and the two determinant estimates now apply with $G = G_{\mathcal{G}}$. Only finitely many supports occur at fixed $D$, so a common cutoff suffices. □

In the period-eight example $c_4 = K_4 - K_8 = \kappa/6$ and $c_8 = K_8 = \kappa/6$. The two cutoffs give $G_{\{1,2\}} = c_4/4 + c_8 = 5\kappa/24$ and hence the coefficient $2\sqrt{2\kappa/15}$. For unit support alone the coefficient is $2\sqrt{\kappa/6}$.

**Simultaneous detection for all rational directions.** Let $\mathcal{V}$ be an $r$-dimensional rational vector space of $D$-periodic functions, $r \ge 1$. Suppose the uniform bound for the chosen parity at a fixed cutoff guarantees at least $M \ge 1$ irrational indices for every nonzero $f \in \mathcal{V}$, where $M$ is an integer. There are $\lceil M/r \rceil$ disjoint sets of at most $r$ exponents each, such that *on each set* every nonzero $f \in \mathcal{V}$ has an irrational value. The sets are common to all $f$, although the exponent witnessing irrationality within a set may depend on $f$. Indeed, after deleting fewer than $M$ indices, the remaining rational-linear maps $f \mapsto L_f(j) + \mathbb{Q}$ into $\mathbb{R}/\mathbb{Q}$ have trivial common kernel. Choose maps successively that lower the dimension of the intersection of kernels, until it reaches zero. This uses at most $r$ indices. Remove those indices and repeat. For the quartic character modulo five this produces disjoint pairs or singletons that work simultaneously for all rational directions $(a, b)$.

[3] In `checks/` (GitHub) or `anc/checks/` (Hexagon), run `python3 grid_checks.py`. It compares the denominator description with direct enumeration on 3,110 combinations of support and cutoff.

**Verification.** An independent AI audit of the 18 November 2026 working proof verified the uniform constants for both parities, the signed nonvanishing criterion, the arithmetic normalization and the matching-parity conclusion. A separate exposition review examined the ten-page manuscript of the same date.

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