hexagon:2611.00614v2

THE INEQUALITY $I_2 < 2I_1$ FOR THE SKYRME PROBLEM AND THE EXISTENCE OF A MINIMISER OF DEGREE TWO

AIDEN SMITH

ABSTRACT. Let I_k be the infimum of the Skyrme energy $\int_{\mathbb{R}^3} |\nabla\phi|^2 + |A(\phi)|^2 \,\mathrm{d}x$ among finite-energy maps $\phi : \mathbb{R}^3 \to S^3$ of degree k. By a theorem of Esteban, I_k is achieved as soon as $I_k < \sum_i I_{d_i}$ for every decomposition of k into at least two nonzero integers d_i; this is known for $k = \pm 1$. We prove, with computer assistance, that

$$I_2 \;\le\; 2.358835 \cdot 12\pi^2 \;<\; 2.378324 \cdot 12\pi^2 \;\le\; 2\, I_1,$$

so that I_2 is achieved. The upper bound is the energy of an explicit axially symmetric map. The lower bound, $I_1 \ge 1.189162 \cdot 12\pi^2$ (Faddeev's bound is $12\pi^2$), comes from an inequality on the level sets of the distance between ϕ and the antipode of its value at infinity, which combines the coarea formula, the degree and the isoperimetric inequality of $\mathbb{R}^3$ and reduces the question to a one-dimensional variational problem. The analytic part of the proof is written out in full; the two numbers are evaluated with interval arithmetic. The proof has not yet been checked by anyone but the author.

1. INTRODUCTION

1.1. **The problem.** We use the setting and the notation of Esteban [3]. For $\phi : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$ let

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^3} |\nabla\phi|^2 + |A(\phi)|^2 \,\mathrm{d}x, \qquad A(\phi) = (\partial_i\phi \wedge \partial_j\phi)_{1\le i<j\le 3}, \qquad X = \{\phi :\ \mathcal{E}(\phi) < \infty\},$$

$$d(\phi) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \det(\phi, \nabla\phi) \,\mathrm{d}x, \qquad I_k = \inf\{\mathcal{E}(\phi) :\ \phi \in X,\ d(\phi) = k\}.$$

The degree $d(\phi)$ is an integer for every $\phi \in X$ [4]. A reflection of $\mathbb{R}^3$ changes the sign of d and not $\mathcal{E}$, so $I_{-k} = I_k$. By [3, Proposition 2],

$$12\pi^2|k| \;\le\; I_k \;\le\; 12\sqrt{2}\,\pi^2|k|; \tag{1}$$

the lower bound is Faddeev's. The existence theory rests on the following criterion.

Theorem 1 (Esteban [3, Theorem 1])**.** *Let $k \neq 0$. If $I_k < \sum_{i=1}^{J} I_{d_i}$ for every decomposition $k = d_1 + \cdots + d_J$ with $J \ge 2$ and $d_i \in \mathbb{Z} \setminus \{0\}$, then I_k is achieved.*

For $k = \pm 1$ the hypothesis follows from (1), so $I_{\pm 1}$ is achieved [3, Corollary 3]; see also Lin and Yang [8]. For $|k| \ge 2$ the hypothesis was left open in [3].

1.2. **Result.**

Theorem 2 (computer-assisted)**.**

$$I_1 \;\ge\; 1.189162 \cdot 12\pi^2 \qquad \textit{and} \qquad I_2 \;\le\; 2.358835 \cdot 12\pi^2.$$

In particular $I_2 < 2I_1$, with a margin of $0.0194 \cdot 12\pi^2$.

Corollary 3. *I_2 and I_{-2} are achieved: there is $\phi \in X$ with $d(\phi) = 2$ and $\mathcal{E}(\phi) = I_2$.*

Date: 16 November 2026. Programs, data and logs: see Appendix A.

Proof. Let $2 = d_1 + \cdots + d_J$ with $J \geq 2$, $d_i \neq 0$. If all d_i are positive, the decomposition is $1 + 1$ and $I_2 < 2I_1$ by Theorem 2. Otherwise some $d_i \leq -1$, hence $\sum_i |d_i| \geq 4$ and, by (1), $\sum_i I_{d_i} \geq 4 \cdot 12\pi^2 > I_2$. Theorem 1 applies. The case $k = -2$ follows by reflection. □

The two inequalities of Theorem 2 are proved independently of each other: the first is Theorem 15 (Sections 2 and 3), the second is Proposition 19 (Section 4). Nothing is assumed, or learnt, about the minimisers themselves.

1.3. **Idea of the proof.** *Upper bound.* We write down a Lipschitz map of degree two, given by two piecewise bilinear functions of the polar coordinates (r, ϑ) and equal to a constant outside a ball, and bound its energy from above.

Lower bound. Faddeev's bound follows from the pointwise inequality $|\nabla\phi|^2 + |A(\phi)|^2 \geq 6|\det(\phi, \nabla\phi)|$, in which equality requires $\mathrm{d}\phi$ to be an isometry. It is not attained, because $\mathbb{R}^3$ and S^3 are not locally isometric [10], but the pointwise argument does not say by how much it fails. We quantify the failure through the isoperimetric inequality. Let $\phi \in X$ have degree one and value e at infinity, and let $\theta(x)$ be the distance in S^3 between $\phi(x)$ and $-e$. For almost every $\tau \in (0, \pi)$ three things are true of the level set $\{\theta = \tau\}$: it is mapped onto the geodesic sphere of radius τ about $-e$ with degree one, so that the Jacobian of ϕ along it integrates to at least $c(\tau) = 4\pi \sin^2 \tau$; it bounds the set $\{\theta < \tau\}$, so that its area $A(\tau)$ is at least that of the ball with the same volume $v(\tau)$; and the volume grows at least at the rate $M(\tau) = \int_{\{\theta=\tau\}} |\nabla\theta|^{-1}$. The energy carried by the level set, per unit of τ, is bounded below in terms of A, c and M only (Theorem 5). If one could take $A = M = c$, one would recover exactly Faddeev's bound. This is what the isoperimetric inequality forbids: then $v(\tau) = \int_0^\tau c$ would be the volume of a geodesic ball of S^3, and a region of $\mathbb{R}^3$ with that volume has boundary area strictly larger than $c(\tau)$. What remains is a variational problem for the single function $v(\tau)$. We bound its value from below by a dynamic-programming recursion (Section 3), each step of which is an elementary inequality proved by hand (Lemma 13, Proposition 14); a program evaluates the recursion.

1.4. **What the computer does, and what has to be trusted.** Proved by hand, in this paper: Corollary 3; the level-set inequality (Theorem 5); all inequalities used for the one-dimensional problem (Lemmas 10–13, Proposition 14); the degree and the energy density of the trial map (Lemmas 17, 18). Done by machine:

(i) the recursion of Proposition 14 on a grid of 261 levels and 7877 cells (Theorem 15), about $3 \cdot 10^7$ applications of Lemma 13;
(ii) an upper bound for the integral of an explicit function of two variables over 472 704 boxes (Proposition 19).

Both use interval arithmetic: after every $+, -, \times, /$ on double precision numbers the result is moved outward by one unit in the last place, which encloses the exact result if these operations are correctly rounded (IEEE 754). Sines, cosines, π and cube roots are not taken from a library on trust (Section 4.2). What has to be trusted is the correctness of about 1100 lines of Python (four files), of the floating-point unit, and of the function `nextafter`. The numerical optimisations that *found* the trial map and the multipliers of the recursion are not part of the proof: any other choice would also give valid, possibly weaker, bounds. The runs listed in Appendix A were repeated from clean copies of the programs on 16 November 2026 and gave the same output.

1.5. **Related work and status.** Esteban [2, 3] and Lin and Yang [8] reduce existence to strict inequalities between the I_k and obtain it for $k = \pm 1$. Manton, Schroers and Singer [12] show that two well-separated Skyrme solitons attract each other after a suitable relative rotation when the orders of their leading multipoles differ by at most two, and Schroers [14] discusses

to what extent this, together with Esteban's criterion, gives the existence of minima. These arguments start from solutions of the Euler–Lagrange equations that are regular enough to have a multipole expansion, which, as far as the author could find, is not known for minimisers in X. A recent paper on a related model describes existence for the Skyrme energy as established for degrees ± 1 [13]. To the author's knowledge the inequality $I_2 < 2I_1$ and the existence of a minimiser of degree two had not been proved; the search of the literature was limited to web searches in November 2026 and may have missed relevant work. For the lower bound, the author found only Faddeev's bound in this role for the energy without potential (Harland [6] improves it when a pion mass term is present). The idea of slicing and using a sharp inequality on each slice was suggested by a recent paper of S. Li on p-harmonic maps [7], which concerns a different problem.

How this paper was produced. The mathematics, the programs and the text were produced by the author working with an AI assistant (Claude, Anthropic). No other person has checked them. "Proved" below means that a complete argument is written down here, not that it has survived independent scrutiny.

2. The level-set inequality

2.1. **Notation and statement.** Let $e = (1,0,0,0)$ and $\Theta(u) = \operatorname{dist}_{S^3}(u,-e) \in [0,\pi]$. For $u \neq \pm e$ let $\nu(u) \in T_u S^3$ be the unit vector $\nabla_{S^3}\Theta(u)$, and $H_u = \nu(u)^\perp \subset T_u S^3$ the tangent plane of the geodesic sphere $\{\Theta = \Theta(u)\}$, which is a round sphere of area

$$c(\tau) = 4\pi \sin^2 \tau \qquad (\tau = \Theta(u)).$$

Further let

$$g(v) = (36\pi)^{1/3} v^{2/3}, \qquad h_\kappa(D) = 2\sqrt{(1+2D)(D^2+2D+\kappa)} \qquad (v, D, \kappa \geq 0);$$

$g(v)$ is the area of the ball of volume v in $\mathbb{R}^3$.

Definition 4. A *minorant* is a triple $(\alpha,\beta,\kappa) \in [0,\infty)^3$ with $\alpha + \beta D \leq h_\kappa(D)$ for all $D \geq 0$.

Examples: $(2\sqrt{\kappa}, 0, \kappa)$ for every $\kappa \geq 0$, because h_κ is increasing; and $(0,\beta,\kappa)$ for $0 \leq \beta \leq 6$, $\kappa \geq 0$, because $h_\kappa(D)^2 \geq h_0(D)^2 = 36D^2 + 8D(D-1)^2$.

Theorem 5 (level-set inequality)**.** *Let $\phi \in X$ with $d(\phi) = 1$ and $\phi - e \in L^6(\mathbb{R}^3)$, let $\theta = \Theta \circ \phi$ and $v(\tau) = |\{\theta < \tau\}|$. Then $v(\tau) < \infty$ for $\tau < \pi$, and there are measurable functions $\varepsilon, A, M : (0,\pi) \to [0,\infty)$ such that*

(a) $\mathcal{E}(\phi) \geq \int_0^\pi \varepsilon(\tau)\,\mathrm{d}\tau$;
(b) $v(\tau_2) - v(\tau_1) \geq \int_{\tau_1}^{\tau_2} M(\tau)\,\mathrm{d}\tau$ *for* $0 < \tau_1 < \tau_2 < \pi$;
(c) $A(\tau) \geq g(v(\tau))$ *for a.e.* τ;
(d) *for a.e.* τ: $\varepsilon(\tau) \geq \alpha A(\tau) + \beta c(\tau) - \kappa M(\tau)$ *for every minorant* (α,β,κ).

For a.e. τ, $A(\tau)$ is the area of the level set $\{\theta = \tau\}$, $M(\tau)$ the integral of $|\nabla\theta|^{-1}$ over it, and $\varepsilon(\tau)$ the integral of $(|\nabla\phi|^2 + |A(\phi)|^2)/|\nabla\theta|$ over it.

Remark 6. (1) Every $\phi \in X$ of degree ± 1 can be brought into the form required in Theorem 5 without changing $\mathcal{E}$: since $\nabla\phi \in L^2$ there is a constant ϕ_∞ with $\phi - \phi_\infty \in L^6$, and $|\phi_\infty| = 1$ because $|\phi| = 1$; a rotation of S^3 moves ϕ_∞ to e and changes neither $\mathcal{E}$ nor d, and a reflection of $\mathbb{R}^3$ changes the sign of d. (2) With the minorant $(0,6,0)$, (a) and (d) give $\mathcal{E}(\phi) \geq 6\int_0^\pi c = 12\pi^2$, Faddeev's bound. Any improvement comes from (b) and (c). (3) For the hedgehog, (a), (b) and (c) are equalities.

2.2. **A pointwise inequality.**

Lemma 7. *Let $\Pi \subset \mathbb{R}^4$ be a three-dimensional subspace, $\nu \in \Pi$ a unit vector, $H = \Pi \cap \nu^\perp$, and $F : \mathbb{R}^3 \to \Pi$ linear with $a := F^T\nu \neq 0$. Write $|\Lambda^2 F|^2 = \sum_{i<j} |Fe_i \wedge Fe_j|^2$, so that $|\Lambda^2 \nabla\phi|^2 = |A(\phi)|^2$. Then F maps $T := a^\perp$ into H. Let $D \geq 0$ be the area factor of $F|_T : T \to H$. Then $|\det F| = |a|\, D$ (determinant with respect to orthonormal bases of $\mathbb{R}^3$ and Π) and*

$$|F|^2 + |\Lambda^2 F|^2 \;\geq\; |a|^2(1+2D) + 2D + D^2. \tag{2}$$

Consequently, for every $\kappa \geq 0$,

$$\frac{|F|^2 + |\Lambda^2 F|^2}{|a|} \;\geq\; h_\kappa(D) - \frac{\kappa}{|a|}. \tag{3}$$

Proof. Let $n = a/|a|$ and τ_1, τ_2 an orthonormal basis of T. Since $\langle \nu, F\xi\rangle = a\cdot\xi$, we have $F\tau_i \in H$ and $Fn = |a|\nu + h$ with $h \in H$. Write $G = F|_T$. In the bases (n, τ_1, τ_2) and $(\nu, \cdot, \cdot)$ the matrix of F is block triangular, so $|\det F| = |a|\,|\det G| = |a|D$. Next $|F|^2 = |G|^2 + |a|^2 + |h|^2$ and

$$|\Lambda^2 F|^2 = |G\tau_1 \wedge G\tau_2|^2 + \sum_{i=1,2} \big||a|\, G\tau_i \wedge \nu + G\tau_i \wedge h\big|^2 = D^2 + |a|^2|G|^2 + \sum_{i=1,2} |G\tau_i \wedge h|^2,$$

because $G\tau_i \wedge \nu$ is orthogonal to $\Lambda^2 H$ and $|G\tau_i \wedge \nu| = |G\tau_i|$. Dropping the terms that contain h and using $|G|^2 \geq 2|\det G| = 2D$ gives (2). Dividing (2) by $|a|$ and adding $\kappa/|a|$, the arithmetic–geometric mean inequality gives $|a|(1+2D) + (D^2 + 2D + \kappa)/|a| \geq h_\kappa(D)$. □

2.3. **The degree formula.** Let ω be the volume form of S^3, so that $\det(\phi, \nabla\phi)\,\mathrm{d}x = \phi^*\omega$. It is integrable for $\phi \in X$, because $|\det(\phi, \nabla\phi)| \leq |\nabla\phi|\,|A(\phi)|$.

Lemma 8. *Let $\phi \in X$ with $\phi - e \in L^6(\mathbb{R}^3)$. Then for every $\psi \in C(S^3)$*

$$\int_{\mathbb{R}^3} \psi(\phi)\, \det(\phi, \nabla\phi)\,\mathrm{d}x = d(\phi) \int_{S^3} \psi\,\omega.$$

Proof. By density we may take ψ smooth. Let $\bar\psi$ be its mean. Since $H^3(S^3) = \mathbb{R}$, $(\psi - \bar\psi)\,\omega = \mathrm{d}\beta$ for a smooth 2-form β on S^3; subtracting an exact form we may assume $\beta(e) = 0$. Extend β to a neighbourhood of S^3 in $\mathbb{R}^4$ by pulling it back under the radial projection $u \mapsto u/|u|$; the extension $\sum_{i<j} b_{ij}(u)\,\mathrm{d}u^i \wedge \mathrm{d}u^j$ is smooth, vanishes at e, so $|b_{ij}(u)| \leq C|u-e|$, and ϕ^* of it and of its differential are $\phi^*\beta$ and $\phi^*(\mathrm{d}\beta)$.

Claim: $\mathrm{d}(\phi^*\beta) = \phi^*(\mathrm{d}\beta)$ in the sense of distributions. Indeed $\phi \in W^{1,2}_{\mathrm{loc}} \cap L^\infty$, so $\mathrm{d}\phi^i \wedge \mathrm{d}\phi^j = \mathrm{d}(\phi^i\,\mathrm{d}\phi^j)$ (mollify ϕ); these forms are closed, and they belong to L^2 because $A(\phi) \in L^2$. If $\eta \in L^2_{\mathrm{loc}}$ is a closed 2-form and $f \in W^{1,2}_{\mathrm{loc}} \cap L^\infty$, then $\mathrm{d}(f\eta) = \mathrm{d}f \wedge \eta$: this is true for smooth f, and if $f_\delta \to f$ in $W^{1,2}_{\mathrm{loc}}$ boundedly then $f_\delta\eta \to f\eta$ in L^2_{loc} and $\mathrm{d}f_\delta \wedge \eta \to \mathrm{d}f \wedge \eta$ in L^1_{loc}. Apply this with $f = b_{ij}(\phi)$.

Let χ_R be a cut-off function with $\chi_R = 1$ on B_R, support in B_{2R} and $|\nabla\chi_R| \leq 2/R$. By the claim and Hölder's inequality with exponents $6, 2, 3$,

$$\Big|\int \chi_R\,\phi^*(\mathrm{d}\beta)\Big| = \Big|\int \mathrm{d}\chi_R \wedge \phi^*\beta\Big| \leq \frac{C}{R}\int_{B_{2R}\setminus B_R} |\phi - e|\,|A(\phi)|\,\mathrm{d}x \leq C'\,\|\phi - e\|_{L^6(\mathbb{R}^3\setminus B_R)}\,\|A(\phi)\|_{L^2},$$

which tends to 0 as $R \to \infty$. Hence $\int \phi^*(\psi\omega) = \bar\psi \int \phi^*\omega = d(\phi)\int_{S^3}\psi\,\omega$. □

This is a variant of the argument of Esteban and Müller [4]; integrality of d is not needed here.

2.4. **Proof of Theorem 5.** *Step 1: the function θ.* Θ is Lipschitz on S^3, so $\theta = \Theta \circ \phi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$. We use its precise representative. For $\delta > 0$ let Θ_δ be a smooth function on S^3 with $\Theta_\delta = \Theta$ on $\{\delta \le \Theta \le \pi - \delta\}$. Two Sobolev functions have the same gradient a.e. on the set where they coincide; applying this to θ and $\Theta_\delta \circ \phi$ we get

$$\nabla\theta = (\nabla\phi)^T \nu(\phi) \quad \text{a.e. on } \{0 < \theta < \pi\}, \qquad \nabla\theta = 0,\ \nabla\phi = 0 \quad \text{a.e. on } \{\phi = \pm e\}. \tag{4}$$

Put $N = \{\nabla\theta = 0\}$. At a.e. $x \notin N$, Lemma 7 applies to $F = \nabla\phi(x)$, $\Pi = T_{\phi(x)}S^3$, $\nu = \nu(\phi(x))$, $a = \nabla\theta(x)$; let $D(x)$ be the corresponding area factor. Moreover $\det(\phi, \nabla\phi) = |\nabla\theta|\, J$ with a measurable J, $|J| = D$, a.e. outside N, and $\det(\phi, \nabla\phi) = 0$ a.e. on N (on $N \cap \{0 < \theta < \pi\}$ the image of $\nabla\phi$ lies in the plane H_ϕ).

Step 2: coarea. By the co-area formula for precisely represented $W^{1,p}_{\mathrm{loc}}$ maps into $\mathbb{R}^m$, $p > m$ [11] (here $m = 1$, $p = 2$), for every measurable $f \ge 0$

$$\int_{\mathbb{R}^3} f\, |\nabla\theta| \, \mathrm{d}x = \int_0^\pi \int_{\theta^{-1}(\tau)} f \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}\tau. \tag{5}$$

Hence $\mathcal{H}^2(\theta^{-1}(\tau) \cap N) = 0$ for a.e. τ, and a property that holds a.e. in $\mathbb{R}^3 \setminus N$ holds $\mathcal{H}^2$-a.e. on $\theta^{-1}(\tau)$ for a.e. τ. Write $\Sigma_\tau = \theta^{-1}(\tau) \setminus N$ and define, for a.e. τ,

$$A(\tau) = \mathcal{H}^2(\Sigma_\tau), \qquad M(\tau) = \int_{\Sigma_\tau} \frac{\mathrm{d}\mathcal{H}^2}{|\nabla\theta|}, \qquad \varepsilon(\tau) = \int_{\Sigma_\tau} \frac{|\nabla\phi|^2 + |A(\phi)|^2}{|\nabla\theta|} \, \mathrm{d}\mathcal{H}^2.$$

Formula (5) with $f = (|\nabla\phi|^2 + |A(\phi)|^2)\, |\nabla\theta|^{-1} \mathbf{1}_{\mathbb{R}^3 \setminus N}$ gives (a).

Step 3: volume. On $\{\theta < \tau\}$ we have $|\phi - e| = 2\cos(\theta/2) \ge 2\cos(\tau/2)$, so $v(\tau) \le (2\cos(\tau/2))^{-6} \|\phi - e\|^6_{L^6} < \infty$. Formula (5) with $f = |\nabla\theta|^{-1} \mathbf{1}_{\{\tau_1 \le \theta < \tau_2\} \setminus N}$ gives $|\{\tau_1 \le \theta < \tau_2\} \setminus N| = \int_{\tau_1}^{\tau_2} M$, which is (b); in particular $M < \infty$ a.e.

Step 4: isoperimetry. Since $\theta \in W^{1,1}_{\mathrm{loc}}$, the set $E_\tau = \{\theta < \tau\}$ has locally finite perimeter for a.e. τ [5, §5.5]. Let Z be the set of points that are not Lebesgue points of θ; it has zero 2-capacity, hence $\mathcal{H}^2(Z) = 0$ [5, §4.7, §4.8]. At a Lebesgue point where the precise value of θ differs from τ, the set E_τ has density 0 or 1, whereas it has density $\frac{1}{2}$ at every point of its reduced boundary [5, §5.7]. Therefore $\partial^* E_\tau \subset \theta^{-1}(\tau) \cup Z$ and, for a.e. τ,

$$A(\tau) = \mathcal{H}^2(\theta^{-1}(\tau)) \ \ge\ \mathcal{H}^2(\partial^* E_\tau) = P(E_\tau) \ \ge\ g(|E_\tau|) = g(v(\tau)),$$

by the isoperimetric inequality for sets of finite perimeter and finite measure [9, Chapter 14]. This is (c).

Step 5: flux. Let $\chi \in C_c(0, \pi)$. By Lemma 8 with $\psi = \chi \circ \Theta$, Step 1 and (5),

$$\int_0^\pi \chi(\tau)\, c(\tau)\, \mathrm{d}\tau = \int_{S^3} \chi(\Theta)\, \omega = \int_{\mathbb{R}^3} \chi(\theta) \det(\phi, \nabla\phi)\, \mathrm{d}x = \int_0^\pi \chi(\tau) \int_{\Sigma_\tau} J \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}\tau.$$

Hence $\int_{\Sigma_\tau} J \, \mathrm{d}\mathcal{H}^2 = c(\tau)$ and $\int_{\Sigma_\tau} D \, \mathrm{d}\mathcal{H}^2 \ge c(\tau)$ for a.e. τ.

Step 6: proof of (d). Let τ be such that the conclusions of Steps 2–5 hold and $\varepsilon(\tau), M(\tau) < \infty$, and let (α, β, κ) be a minorant. By (3), $\mathcal{H}^2$-a.e. on Σ_τ,

$$\frac{|\nabla\phi|^2 + |A(\phi)|^2}{|\nabla\theta|} \ \ge\ h_\kappa(D) - \frac{\kappa}{|\nabla\theta|} \ \ge\ \alpha + \beta D - \frac{\kappa}{|\nabla\theta|}.$$

Integrating over Σ_τ and using Step 5 and $\beta \ge 0$ we get $\varepsilon(\tau) \ge \alpha A(\tau) + \beta c(\tau) - \kappa M(\tau)$. With the minorant $(2, 0, 1)$ this shows $A(\tau) \le \frac{1}{2}(\varepsilon(\tau) + M(\tau)) < \infty$. □

3. The one-dimensional problem

3.1. Definition and first properties.

Definition 9. Let $0 < \tau < \pi$ and $v \geq 0$. A *profile from* (τ, v) is a triple (ε, A, M) of measurable functions $(\tau, \pi) \to [0, \infty)$ such that $\tilde{v}(s) := v + \int_\tau^s M$ is finite for every $s < \pi$ and, for a.e. s,

$$A(s) \geq g(\tilde{v}(s)) \qquad \text{and} \qquad \varepsilon(s) \geq \alpha A(s) + \beta c(s) - \kappa M(s) \ \text{ for every minorant } (\alpha, \beta, \kappa).$$

We put $\mathcal{W}(\tau, v) = \inf \int_\tau^\pi \varepsilon$, the infimum over all profiles from (τ, v).

In the language of optimal control, s is the time, $\tilde{v}$ the state, $M = \tilde{v}'$ the control, and the number κ in a minorant plays the role of the costate (the price of volume).

Lemma 10. (i) $\mathcal{W}(\tau, \cdot)$ *is nondecreasing.* (ii) *If* (ε, A, M) *is a profile from* (τ, v) *and* $\tau < \tau' < \pi$, *then* $\int_\tau^\pi \varepsilon \geq \int_\tau^{\tau'} \varepsilon + \mathcal{W}(\tau', \tilde{v}(\tau'))$. (iii) $I_1 \geq \mathcal{W}(\tau, 0)$ *for every* $\tau \in (0, \pi)$.

Proof. (i) g is increasing, so a profile from (τ, v_2) is a profile from (τ, v_1) if $v_1 \leq v_2$. (ii) The restriction to (τ', π) is a profile from $(\tau', \tilde{v}(\tau'))$. (iii) Let $\phi \in X$, $d(\phi) = \pm 1$, normalised as in Remark 6(1). By Theorem 5(b), $v(s) \geq v(\tau) + \int_\tau^s M$ for $s > \tau$, so by (c) and (d) the functions ε, A, M of that theorem, restricted to (τ, π), form a profile from $(\tau, v(\tau))$. By (a) and (i), $\mathcal{E}(\phi) \geq \mathcal{W}(\tau, v(\tau)) \geq \mathcal{W}(\tau, 0)$. □

Lemma 11 (exterior bound)**.** *For* $0 < \tau < \pi$ *and* $v \geq 0$,

$$\mathcal{W}(\tau, v) \ \geq \ \Xi(\tau, v) := 4\pi(\pi - \tau)^2\, r(v), \qquad r(v) = \Big(\frac{3v}{4\pi}\Big)^{1/3}.$$

$\Xi(\tau, \cdot)$ *is concave and nondecreasing.*

Proof. For $v = 0$ there is nothing to prove. Let $v > 0$ and let (ε, A, M) be a profile from (τ, v). Then $\rho(s) := r(\tilde{v}(s)) \geq r(v) > 0$ is nondecreasing and locally absolutely continuous, with $\rho' = M/(4\pi\rho^2)$ a.e. For a.e. s and every $\kappa > 0$ the minorant $(2\sqrt{\kappa}, 0, \kappa)$ gives $\varepsilon \geq 2\sqrt{\kappa}\, g(\tilde{v}) - \kappa M = 8\pi\sqrt{\kappa}\,\rho^2 - \kappa M$. If $M(s) = 0$ this is impossible for large κ; so $M > 0$ a.e., and the choice $\sqrt{\kappa} = 4\pi\rho^2/M$ gives $\varepsilon \geq (4\pi\rho^2)^2/M = 4\pi\rho^2/\rho'$. For $\tau < s_1 < \pi$, by the Cauchy–Schwarz inequality,

$$(s_1 - \tau)^2 \leq \int_\tau^{s_1} \frac{\rho^2}{\rho'}\,\mathrm{d}s \int_\tau^{s_1} \frac{\rho'}{\rho^2}\,\mathrm{d}s \leq \frac{1}{4\pi}\int_\tau^\pi \varepsilon \ \cdot \Big(\frac{1}{\rho(\tau)} - \frac{1}{\rho(s_1)}\Big) \leq \frac{1}{4\pi\, r(v)}\int_\tau^\pi \varepsilon.$$

Let $s_1 \to \pi$. □

(Ξ is the Dirichlet energy needed to bring θ from τ to π outside a ball of volume v.)

3.2. Minorants, and one step of the recursion.

Lemma 12. *For all real* $\alpha, \beta, \kappa, m, q$ *and* D,

$$h_\kappa(D)^2 - (\alpha + \beta D)^2 = 8(D - m)^2(D + q) + q_2 D^2 + q_1 D + q_0,$$

$$q_2 = 20 - \beta^2 - 8(q - 2m), \qquad q_1 = 8 + 8\kappa - 2\alpha\beta - 8(m^2 - 2mq), \qquad q_0 = 4\kappa - \alpha^2 - 8m^2 q.$$

If $\alpha, \beta, \kappa, m, q \geq 0$, $q_2 \geq 0$, $q_0 \geq 0$, *and* $q_1 \geq 0$ *or* $q_1^2 \leq 4q_2q_0$, *then* (α, β, κ) *is a minorant.*

Proof. Expand $h_\kappa^2 = 8D^3 + 20D^2 + (8 + 8\kappa)D + 4\kappa$. Under the hypotheses the right-hand side is nonnegative for $D \geq 0$, and h_κ and $\alpha + \beta D$ are nonnegative there. □

Lemma 13 (one step). *Let $0 < \tau_a = s_0 < s_1 < \cdots < s_P = \tau_b < \pi$, $\Delta_l = s_l - s_{l-1}$ and $C_l = \int_{s_{l-1}}^{s_l} c$. Let $(\alpha_l, \beta_l, \kappa_l)$, $l = 1, \dots, P$, be minorants, $a_l = \sum_{m>l} \alpha_m \Delta_m$, and let $\kappa \ge 0$, $\bar{w} > 0$, $0 \le \sigma \le g'(\bar{w})$ be such that*

$$\kappa_l \le \kappa + \sigma\, a_l \qquad (l = 1, \dots, P).$$

Then for every $v \ge 0$ and every profile (ε, A, M) from (τ_a, v) with $w := \tilde{v}(\tau_b) \le \bar{w}$,

$$\int_{\tau_a}^{\tau_b} \varepsilon \;\ge\; g(v) \sum_{l=1}^{P} \alpha_l \Delta_l + \sum_{l=1}^{P} \beta_l C_l - \kappa\,(w - v). \tag{6}$$

Proof. Let $w_l = \tilde{v}(s_l)$, so $v = w_0 \le w_1 \le \cdots \le w_P = w$. On (s_{l-1}, s_l), $\varepsilon \ge \alpha_l g(\tilde{v}) + \beta_l c - \kappa_l M \ge \alpha_l g(w_{l-1}) + \beta_l c - \kappa_l M$, and $g(w_{l-1}) \ge g(v) + \sigma(w_{l-1} - v)$ because g is concave and $v \le w_{l-1} \le \bar{w}$. Integrating and summing,

$$\int_{\tau_a}^{\tau_b} \varepsilon \;\ge\; g(v) \sum_{l} \alpha_l \Delta_l + \sum_{l} \beta_l C_l + \sigma \sum_{l} \alpha_l \Delta_l\,(w_{l-1} - v) - \sum_{l} \kappa_l\,(w_l - w_{l-1}).$$

Since $w_{l-1} - v = \sum_{j<l}(w_j - w_{j-1})$, the third sum equals $\sum_j (w_j - w_{j-1})\, a_j$; and $\sigma a_j - \kappa_j \ge -\kappa$, $w_j - w_{j-1} \ge 0$. □

For $P = 1$ this is the obvious bound with g frozen at the beginning of the step. With P pieces the earlier pieces are allowed larger multipliers κ_l, which recovers most of the growth of $g(\tilde{v})$ during the step. The point of (6) is that its right-hand side is affine in w and concave in v.

3.3. **The recursion.** Fix levels $0 < \tau_0 < \tau_1 < \cdots < \tau_N < \pi$ and nodes $0 = v_0 < v_1 < \cdots < v_n$. Let $\mathcal{P}_i$ be the set of functions $W : [0, \infty) \to \mathbb{R}$ that are continuous and affine on each cell $[v_j, v_{j+1}]$, nondecreasing on $[0, v_n]$, equal to $\Xi(\tau_i, \cdot)$ on (v_n, ∞), and satisfy $W(v_n) \le \Xi(\tau_i, v_n)$. Such a function is nondecreasing on $[0, \infty)$.

Proposition 14 (induction step). *Let $W_{i+1} \in \mathcal{P}_{i+1}$ with $W_{i+1} \le \mathcal{W}(\tau_{i+1}, \cdot)$, and $C = \int_{\tau_i}^{\tau_{i+1}} c$. Fix a cell $[v_j, v_{j+1}]$ and integers $j = J_0 < J_1 \le J_2 \le \cdots \le J_K \le n$. For $k = 1, \dots, K$ let $\bar{\alpha}_k, \bar{\beta}_k, \kappa_k \ge 0$ be such that*

$$\int_{\tau_i}^{\tau_{i+1}} \varepsilon \;\ge\; \bar{\alpha}_k\, g(v) + \bar{\beta}_k - \kappa_k\,(w - v) \tag{7}$$

for every $v \ge 0$ and every profile from (τ_i, v) with $w = \tilde{v}(\tau_{i+1}) \le v_{J_k}$ (Lemma 13 provides such numbers). Put $\psi_k(j') = W_{i+1}(v_{j'}) - \kappa_k v_{j'}$ and, for $v \in [v_j, v_{j+1}]$,

$$\begin{aligned} b_1(v) &= \bar{\alpha}_1 g(v) + \bar{\beta}_1 + \min\Big\{ W_{i+1}(v),\; \kappa_1 v + \min_{j < j' \le J_1} \psi_1(j') \Big\}, \\ b_k(v) &= \bar{\alpha}_k g(v) + \bar{\beta}_k + \kappa_k v + \min_{J_{k-1} \le j' \le J_k} \psi_k(j') \qquad (2 \le k \le K), \\ b_\infty(v) &= 6C + W_{i+1}(v_{J_K}). \end{aligned}$$

Then $\mathcal{W}(\tau_i, v) \ge \min\{b_1(v), \dots, b_K(v), b_\infty(v)\}$ on the cell. This remains true if b_1 is replaced by $6C + W_{i+1}(v)$, or b_k $(k \ge 2)$ by $6C + W_{i+1}(v_{J_{k-1}})$. All these functions are concave on the cell. Hence an affine function that lies below each of them at v_j and at v_{j+1} lies below $\mathcal{W}(\tau_i, \cdot)$ on the cell.

Proof. Let (ε, A, M) be a profile from (τ_i, v) and $w = \tilde{v}(\tau_{i+1}) \ge v$. By Lemma 10(ii), $\int_{\tau_i}^{\pi} \varepsilon \ge \int_{\tau_i}^{\tau_{i+1}} \varepsilon + W_{i+1}(w)$. If $w \le v_{J_1}$, (7) gives $\int_{\tau_i}^{\pi} \varepsilon \ge \bar{\alpha}_1 g(v) + \bar{\beta}_1 + \kappa_1 v + [W_{i+1}(w) - \kappa_1 w]$. The bracket is continuous and piecewise affine in w with breaks at the nodes, so on $[v, v_{J_1}]$ it is at least its value at $w = v$ or at a node $v_{j'}$, $j < j' \le J_1$; this is $b_1(v)$. If $v_{J_{k-1}} \le w \le v_{J_k}$ for some $k \ge 2$, the same argument with the nodes $J_{k-1}, \dots, J_k$ gives $b_k(v)$. If $w \ge v_{J_K}$: $(0, 6, 0)$ is a

minorant, so $\int_{\tau_i}^{\tau_{i+1}} \varepsilon \geq 6C$, and $W_{i+1}(w) \geq W_{i+1}(v_{J_K})$ because W_{i+1} is nondecreasing; this is b_∞. The alternatives follow in the same way from $w \geq v$, respectively $w \geq v_{J_{k-1}}$. Concavity: g is concave, W_{i+1} is affine on the cell, and a minimum of affine functions is concave. □

Theorem 15 (computer-assisted). *$\mathcal{W}(\tau_0, 0) \geq 140.83872$ for $\tau_0 = 0.1$. Consequently, by Lemma 10*(iii)*,*

$$I_1 \;\geq\; 140.83872 \;>\; 1.189162 \cdot 12\pi^2.$$

Proof (program `step_cert.py`). *Grid.* Levels $\tau_0 = 0.1 < \tau_1 < \cdots < \tau_N = \pi - 0.04$, $N = 261$, with $\tau_{i+1} - \tau_i = \min(0.0289,\ 0.025\,\tau_i,\ 0.025\,(\pi - \tau_i))$; nodes $v_j = 10^{-5}(1.004^j - 1)/0.004$, $j \leq n = 7877$, so that $r(v_n) \geq 3000$.

Start. W_N is the piecewise affine interpolant of lower bounds of $\Xi(\tau_N, v_j)$ (the program checks that they are nondecreasing in j). Since $\Xi(\tau_N, \cdot)$ is concave, $W_N \in \mathcal{P}_N$ and $W_N \leq \mathcal{W}(\tau_N, \cdot)$ by Lemma 11.

Step. Given W_{i+1}, for each cell and each range k the program chooses J_k (with $J_k - j = 1, 2, 3, 4, 6, 8, 11, 16, \dots$) and a multiplier κ_k, cuts the step into $P = 6$ equal pieces, takes a lower bound σ of $g'(v_{J_k}) = \frac{2}{3} g(v_{J_k})/v_{J_k}$, defines κ_l as $\kappa_k + \sigma a_l$ rounded down ($l = P, \dots, 1$), and obtains minorants $(\alpha_l, \beta_l, \kappa_l)$ by verifying the hypotheses of Lemma 12 in interval arithmetic for a proposed $(\alpha_l, \beta_l, m, q)$; if the verification fails it uses $(0, 6(1 - 10^{-7}), \kappa_l)$. This gives (7) with $\bar\alpha_k, \bar\beta_k$ lower bounds of $\sum_l \alpha_l \Delta_l$, $\sum_l \beta_l C_l$. It then evaluates the functions of Proposition 14 (or their alternatives) at both ends of the cell, rounding down; a cell stops at the first k for which $6C + W_{i+1}(v_{J_k})$ is not smaller than the bounds found so far. The value of W_i at the node v_j is *defined* as the minimum of the lower-end values for the cell $[v_j, v_{j+1}]$ and the upper-end values for the cell $[v_{j-1}, v_j]$; the value at v_n is then replaced by its minimum with a lower bound of $\Xi(\tau_i, v_n)$, and the nodal values are made nondecreasing by lowering. By Proposition 14 and Lemma 11, $W_i \in \mathcal{P}_i$ and $W_i \leq \mathcal{W}(\tau_i, \cdot)$.

Result. The program returns $W_0(0) = 140.838721\dots$; and $1.189162 \cdot 12\pi^2 < 140.83871$ because $12\pi^2 < 118.435253$. □

The choices of J_k, of κ_k (a floating-point maximisation of the lower-end value of b_k) and of the proposed minorants (tangents to the convex envelope of h_{κ_l}) are heuristic. They influence the size of $W_0(0)$, not its validity: whatever they are, the program only stores numbers for which the hypotheses of Proposition 14 have been verified. No sampling is involved: Proposition 14 covers every volume v in a cell, every arrival volume $w \geq v$, and Lemma 13 covers the whole interval $[\tau_i, \tau_{i+1}]$.

Remark 16 (checks that are not part of the proof). (1) A second program, written earlier and sharing no code with the first (other interval routines, minorants certified by Sturm sequences in exact rational arithmetic, a different step lemma, discontinuous piecewise affine functions on a moving grid), gives $I_1 \geq 1.19204 \cdot 12\pi^2$. (2) Floating-point solutions of the one-dimensional problem by shooting and by dynamic programming give $\inf_v \mathcal{W} \approx 1.1921 \cdot 12\pi^2$. The hedgehog has energy $\approx 1.23145 \cdot 12\pi^2$; the difference arises in (d) of Theorem 5 only, where the best minorant yields the convex envelope of the level energy and not the level energy itself. (3) On coarser grids the program gives smaller values (1.1869, 1.1823, rounded down), increasing with refinement.

4. The upper bound

4.1. **The map.** Let $0 = r_0 < \cdots < r_N = R$ and $0 = t_0 < \cdots < t_K = 1$, and let φ, γ be continuous functions on $[0, R] \times [0, 1]$, bilinear in (r, t) on every cell $[r_i, r_{i+1}] \times [t_k, t_{k+1}]$, with

$$\varphi(0, t) = 1, \qquad \varphi(R, t) = 0, \qquad \gamma(r, 0) = 0, \qquad \gamma(r, 1) = 1. \tag{8}$$

In polar coordinates $x = (r\sin\vartheta\cos\chi,\ r\sin\vartheta\sin\chi,\ r\cos\vartheta)$ put, for $r \le R$ and $\vartheta = \frac{\pi}{2}t \le \frac{\pi}{2}$,

$$f = \pi\,\varphi(r,t), \qquad g = \tfrac{\pi}{2}\,\gamma(r,t), \qquad f(r,\pi-\vartheta) = f(r,\vartheta), \quad g(r,\pi-\vartheta) = \pi - g(r,\vartheta),$$

$$U = \begin{cases} \big(\cos f,\ \sin f\sin g\cos 2\chi,\ \sin f\sin g\sin 2\chi,\ \sin f\cos g\big) & (|x| \le R), \\ (1,0,0,0) & (|x| > R). \end{cases} \tag{9}$$

(In this section g denotes this angle, not the function of Section 2.) The nodes and the nodal values are the double precision numbers stored in the file `trial_c_exact.npz` ($N = 58$, $K = 33$, $R = 12$); they were found by minimising a quadrature approximation of the energy, and how they were found plays no role. The energy density of U is concentrated on a torus.

Lemma 17. *U is Lipschitz, $U \in X$, and $d(U) = -2$. Hence $I_2 = I_{-2} \le \mathcal{E}(U)$.*

Proof. Away from the axis and the origin, U is the composition of a smooth map with piecewise bilinear functions. On the first row of cells $g = \frac{\pi}{2}\gamma(r,t_1)\,t/t_1$, so $|\sin g|/\sin\vartheta$ is bounded; on the first column $\pi - f = \pi(1-\varphi(r_1,t))\,r/r_1$ and $\partial_t f = O(r)$, so $|\sin f|/r$ and $|\partial_\vartheta f|/r$ are bounded. Hence $|\mathrm{d}U|$ is bounded and U, being continuous, is Lipschitz; it is constant outside B_R, so $U \in X$. In the coordinates $(f,g,\psi) \mapsto (\cos f, \sin f\sin g\cos\psi, \sin f\sin g\sin\psi, \sin f\cos g)$ of S^3 the volume form is $\sin^2 f\,\sin g\,\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}\psi$, and $\psi = 2\chi$. With $F(f) = \frac{1}{2}(f - \sin f\cos f)$ and $G(g) = -\cos g$,

$$2\pi^2 d(U) = 2\cdot 2\pi \int_{[0,R]\times[0,\pi]} \mathrm{d}(F\circ f)\wedge\mathrm{d}(G\circ g) = 4\pi \oint F(f)\,\mathrm{d}G(g) = 4\pi\cdot\tfrac{\pi}{2}\big[G(0) - G(\pi)\big] = -4\pi^2$$

(Stokes' formula for Lipschitz functions on the rectangle in the (r,ϑ) plane): $F(f) = 0$ on the edge $r = R$, $G(g)$ is constant on the edges $\vartheta = 0$ and $\vartheta = \pi$, and on the edge $r = 0$, run from $\vartheta = \pi$ to $\vartheta = 0$, $F(f) = \pi/2$. □

Lemma 18. $\mathcal{E}(U) = \dfrac{12\pi^2}{6}\displaystyle\int_0^R\int_0^1 \Gamma\,\mathrm{d}t\,\mathrm{d}r$, *where, with $s = \sin f$, $\varsigma = \sin g$, $w = \sin\vartheta$,*

$$\Gamma = (P + s^2 Q)\Big(w + \frac{4s^2\varsigma^2}{r^2 w}\Big) + \frac{4s^2\varsigma^2}{w} + w\,s^2 L^2,$$
$$P = r^2 f_r^2 + f_\vartheta^2, \qquad Q = r^2 g_r^2 + g_\vartheta^2, \qquad L = f_r g_\vartheta - f_\vartheta g_r.$$

Proof. The pull-back of the metric of S^3 is $\mathrm{d}f^2 + s^2(\mathrm{d}g^2 + 4\varsigma^2\mathrm{d}\chi^2)$. In the orthonormal coframe $(\mathrm{d}r,\ r\,\mathrm{d}\vartheta,\ r\sin\vartheta\,\mathrm{d}\chi)$ its matrix is block diagonal: a 2×2 block B with $\operatorname{tr} B = (P + s^2Q)/r^2$ and $\det B = s^2L^2/r^2$, and the entry $\lambda = 4s^2\varsigma^2/(r^2w^2)$. Therefore $|\nabla U|^2 + |A(U)|^2 = \operatorname{tr} B + \lambda + \det B + \lambda\operatorname{tr} B$. Multiply by the volume element $2\pi r^2 w\,\mathrm{d}r\,\mathrm{d}\vartheta$, and use $\mathrm{d}\vartheta = \frac{\pi}{2}\mathrm{d}t$ and the symmetry $\vartheta \mapsto \pi - \vartheta$. □

Proposition 19 (computer-assisted)**.** $\mathcal{E}(U) \le 2.358835\cdot 12\pi^2$.

Proof (program `verify_b2.py`). Every cell is cut into boxes $\mathcal{B}$. On the cells with $i \ge 1$ and $k \ge 1$ the function Γ is real analytic, and with z the centre of $\mathcal{B}$,

$$\int_{\mathcal{B}}\Gamma \ \le\ |\mathcal{B}|\,\Gamma(z) + \nabla\Gamma(z)\cdot\int_{\mathcal{B}}(x-z) + \tfrac{1}{2}\sup_{\mathcal{B}}\Gamma_{rr}\int_{\mathcal{B}}(r-z_r)^2 + \sup_{\mathcal{B}}|\Gamma_{rt}|\int_{\mathcal{B}}|r-z_r||t-z_t| + \tfrac{1}{2}\sup_{\mathcal{B}}\Gamma_{tt}\int_{\mathcal{B}}(t-z_t)^2.$$

$\Gamma(z)$ is enclosed by interval arithmetic and the suprema by interval arithmetic on second-order jets (automatic differentiation with interval coefficients); φ, γ and their derivatives are enclosed

exactly on $\mathcal{B}$ because they are bilinear. On the first column ($r \le r_1$) and the first row ($t \le t_1$) we use $\int_{\mathcal{B}} \Gamma \le |\mathcal{B}| \sup_{\mathcal{B}} \Gamma$, with the two removable singularities bounded by

$$\frac{\sin^2 f}{r^2} \le \frac{\pi^2 (1-\varphi(r_1,t))^2}{r_1^2}, \qquad \frac{\sin^2 g}{\sin\vartheta} \le \frac{\frac{\pi}{2}\,\gamma(r,t_1)^2\, t}{t_1^2\,(1-\vartheta^2/6)}$$

($|\sin x| \le |x|$, $\sin\vartheta \ge \vartheta - \vartheta^3/6$). With 16×16 boxes per regular cell and 8×8 per singular cell, the sum of the centre terms is 2.358823, of the remainders $2.6 \cdot 10^{-6}$, of the singular cells $9.2 \cdot 10^{-6}$ (all divided by 6), and the rigorous total is 2.3588347. □

Remark 20. (1) *Checks that are not part of the proof.* An evaluation that does not use Lemma 18 (U from (9) in Cartesian coordinates, derivatives by central differences) gives $\mathcal{E}(U) \approx 2.35876 \cdot 12\pi^2$ and the degree integral -1.99994. Two coarser trial maps give the rigorous bounds 2.362512 and 2.360025. These values are consistent with the energy of about $1.18 \cdot 12\pi^2$ per unit of degree reported in the numerical literature for the axially symmetric solution of degree two [1]. (2) *Degree one.* The same program with χ in place of 2χ in (9) and a trial map close to the hedgehog ($N = 68$, $K = 33$, $R = 30$) gives $I_1 \le 1.231865 \cdot 12\pi^2$. With Theorem 15 this confines $I_1/12\pi^2$ to $[1.189162,\ 1.231865]$, compared with $[1, \sqrt{2}]$ in (1).

4.2. **The arithmetic.** Intervals have double precision end points; after every $+, -, \times, /$ the end points are moved outward by one unit in the last place (file `ival.py`, 309 lines). sin and cos are enclosed by Taylor polynomials whose coefficients are enclosed with exact rational arithmetic, plus the Lagrange remainder; π is enclosed from Machin's formula with exact rationals; cube roots are verified by cubing with directed rounding; the final sum in Proposition 19 uses a correctly rounded summation followed by one outward step.

5. What is not proved

(1) Nothing is said about the shape, the symmetry or the uniqueness of the minimiser of degree two, nor about whether the hedgehog achieves I_1. By Remark 20(2), I_1 lies within 3.5 per cent of the hedgehog energy.
(2) Nothing follows for $|k| \ge 3$. For $k = 3$ one would need a lower bound for $I_1 + I_2$ above the energy of a trial map of degree three, and the present lower bounds are too weak for that.
(3) The measure-theoretic facts used in Steps 2 and 4 of the proof of Theorem 5 are quoted from [11, 5, 9] and not re-proved. The existence statement is exactly as reliable as Theorem 1, which is quoted, and as the two programs.
(4) Independent verification is missing. The most useful checks would be: reading Sections 2 and 3 (five pages); re-implementing the recursion of Proposition 14, which needs only Lemmas 11–13; and re-evaluating the integral of Lemma 18 for the stored map by any rigorous quadrature.

Appendix A. Programs, data and logs

The supplementary archive contains the following files (Python 3.13, NumPy 2.5; no other dependency for the proofs).

file	lines	role
`ival.py`	309	interval arithmetic, enclosures of sin, cos, π (Section 4.2)
`lsb.py`	157	lower bounds of g, r, Ξ, C_l; minorants by Lemma 12
`step_cert.py`	312	the recursion; Theorem 15
`verify_b2.py`	283	upper bound for the stored map; Proposition 19
`trial_c_exact.npz`		the map U of Section 4 (nodes and nodal values)
`trial_h0_exact.npz`		the degree-one map of Remark 20(2)
`*.log`, `SHA256SUMS.txt`		output of the runs of 16 November 2026; checksums

command	last line of output	time
`python3 step_cert.py 104 0.004 0.025 0.025 0.1 0.04 1e-5 3000 1e9 6 0`	$I_1/(12\pi^2) \geq 1.189162$	10 min
`python3 verify_b2.py trial_c_exact.npz 16 8 0 2`	$\mathcal{E}(U)/(12\pi^2) \leq 2.358834681$	10 s
`python3 verify_b2.py trial_h0_exact.npz 16 8 0 1`	$\mathcal{E}/(12\pi^2) \leq 1.231864728$	13 s

The second program of Remark 16(1), the floating-point solvers of Remark 16(2), the optimiser that produced the trial maps and the Cartesian cross-check are in the same archive, in a separate folder; they are not needed for the proofs.

References

[1] R. A. Battye, P. M. Sutcliffe, Skyrmions, fullerenes and rational maps, Rev. Math. Phys. 14 (2002), 29–85.
[2] M. J. Esteban, A direct variational approach to Skyrme's model for meson fields, Comm. Math. Phys. 105 (1986), 571–591.
[3] M. J. Esteban, Erratum: Existence of 3D Skyrmions. Complete version, arXiv:math-ph/0401042 (2004); see also Comm. Math. Phys., doi:10.1007/s00220-004-1139-y.
[4] M. J. Esteban, S. Müller, Sobolev maps with integer degree and applications to Skyrme's problem, Proc. Roy. Soc. London A 436 (1992), 197–201.
[5] L. C. Evans, R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
[6] D. Harland, Topological energy bounds for the Skyrme and Faddeev models with massive pions, Phys. Lett. B 728 (2014), 518–523.
[7] S. Li, The hedgehog uniquely minimises every p-energy, arXiv:2609.13979 (2026).
[8] F. Lin, Y. Yang, Existence of energy minimizers as stable knotted solitons in the Faddeev model, Comm. Math. Phys. 249 (2004), 273–303.
[9] F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems, Cambridge University Press, 2012.
[10] N. S. Manton, Geometry of Skyrmions, Comm. Math. Phys. 111 (1987), 469–478.
[11] J. Malý, D. Swanson, W. P. Ziemer, The co-area formula for Sobolev mappings, Trans. Amer. Math. Soc. 355 (2003), 477–492.
[12] N. S. Manton, B. J. Schroers, M. A. Singer, The interaction energy of well-separated Skyrme solitons, Comm. Math. Phys. 245 (2004), 123–147.
[13] C. B. Muratov, T. M. Simon, V. V. Slastikov, Existence of higher degree minimizers in the magnetic skyrmion problem, arXiv:2409.07205 (2024).
[14] B. J. Schroers, On the existence of minima in the Skyrme model, arXiv:hep-th/0212076 (2002).