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The alternating sum of reciprocal Fibonacci cubes is irrational

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Abstract

We prove that the alternating sum of the reciprocal cubes of the Fibonacci numbers is irrational. The result extends to the Lucas sequences of the first kind associated with real quadratic units greater than one, including the Pell numbers. For every such sequence and every fixed positive integer k, the reciprocal cubes with indices $k, 2k, 3k, \ldots$, taken with signs $+, -, +, \ldots$, also have an irrational sum.

1 The alternating sum

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Write

$$A_F^+ = \sum_{n\geq 1} \frac{1}{F_n^3}, \qquad A_F^- = \sum_{n\geq 1} \frac{(-1)^{n-1}}{F_n^3}.$$

Absolute convergence gives $A_F^+ + A_F^- = 2\sum_{n\geq 1} F_{2n-1}^{-3}$. Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa proved that the odd-index sum is transcendental [1][2, Theorem 4, p. 95]. Thus at least one of A_F^+ and A_F^- is transcendental. Krisman proved A_F^+ irrational [4] and extended that result to ordinary reciprocal cube sums of Lucas sequences with a positive dominant unit [5, Theorem 1.1]. These results leave open whether A_F^- is rational [5, p. 3].

For integers P, Q, let $U_n = U_n(P, Q)$ be the Lucas sequence of the first kind,

$$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+2} = PU_{n+1} - QU_n.$$

Theorem 1.1. *Suppose that $Q = -1$ and $P \geq 1$, or that $Q = 1$ and $P \geq 3$. Then*

$$A_{P,Q}^- = \sum_{n\geq 1} \frac{(-1)^{n-1}}{U_n(P,Q)^3} \quad \textit{is irrational.}$$

Changing the sign of the trace gives, directly from the recurrence,

$$U_n(-P, Q) = (-1)^{n-1} U_n(P, Q) \qquad (n \geq 1).$$

Cubing preserves this sign. Thus $A_{P,Q}^-$ is the ordinary reciprocal cube sum for trace $-P$. In the Fibonacci case $(P, Q) = (1, -1)$, the negative-trace sequence begins $1, -1, 2, -3, 5, \ldots$. Its dominant root is negative, explaining the signed reciprocal root in Section 2. The pair $(2, -1)$ gives the Pell case. For $Q = -1$, Elsner, Shimomura and Shiokawa treat alternating reciprocal *even* powers of first- and second-kind Lucas sequences [3, Theorems 2 and 4]. Here the power is three.

Corollary 1.2. *Under the hypotheses of Theorem 1.1, for each integer $k \geq 1$,*

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{U_{kn}(P,Q)^3} \quad \textit{is irrational.}$$

The determinant method produces polynomials in a single unknown value. Under a rationality assumption, clearing their denominators gives algebraic integers in a real quadratic field. A nonzero algebraic integer has absolute norm at least one, so estimates at *both* real embeddings can yield a contradiction. Zudilin used polynomial Hankel determinants in an irrationality criterion [6, Proposition 2]. Uvarov related the orthogonal polynomials before and after a rational change of weight [7, §1]. Krattenthaler gave a determinant identity for the modified moments [8, Theorem 1]. The particular moving denominator and interpolation argument used here come from the q-zeta construction [10, Sections 2 and 5] and Krisman's adaptation of it to reciprocal cubes [4, 5].

The alternating sign amounts to changing the sign of the small unit in Krisman's construction. His cyclotomic denominator is an identity in a formal variable, so it still clears every coefficient. The main analytic point is at the conjugate embedding. The conjugate pole values turn out to be evaluations of a *single* holomorphic function formed from the original signed moments. A contour integral combines them before their sizes are estimated. For a determinant of size K, measure logarithms in units of $K^3 \log \eta$, where $\eta > 1$ is the dominant unit. The first embedding saves $2/9$, the conjugate embedding saves 1, and clearing the denominators costs at most $4393/3675$. The margin is

$$\frac{2}{9} + 1 - \frac{4393}{3675} = \frac{296}{11025}. \tag{1.1}$$

We prove the estimates for the actual signed value and show that infinitely many of the corresponding determinants are nonzero.

2 The signed unit and its moments

Put $D = P^2 - 4Q$, $\eta = (P + \sqrt{D})/2$, and $u = \eta^{-1}$. The two roots of $x^2 - Px + Q$ are $\eta > 1$ and Q/η. They are algebraic units. A rational root would be an integer dividing $Q = \pm 1$, so the field $E = \mathbb{Q}(\sqrt{D})$ is quadratic. Conversely, the trace of a real quadratic unit $\eta > 1$ of norm Q is the integer $\eta + Q/\eta$, which is positive for $Q = -1$ and exceeds two for $Q = 1$. This gives exactly the parameter ranges in Theorem 1.1. Let σ denote the nontrivial automorphism of E and put

$$\tau = -u, \qquad w = Q\tau^2, \qquad \rho = Q\tau.$$

Here $|w| < 1$, $|\rho| < 1$, $w = \rho\tau$, and $\rho w = \tau^3$. Changing τ from u to $-u$ fixes the nodes w^n and multiplies the weights ρ^n by $(-1)^n$. Binet's formula gives $U_n = \eta^n(1 - w^n)/\sqrt{D}$. Thus all the U_n with $n \geq 1$ are positive and

$$\psi_N = \frac{\tau^{3N}}{(1 - w^N)^3}, \qquad S_a = \sum_{N=1}^{a} \psi_N, \qquad \Xi = \sum_{N \geq 1} \psi_N, \qquad A^-_{P,Q} = -D\sqrt{D}\,\Xi. \tag{2.1}$$

Here $\tau^{3N} = (-1)^N \eta^{-3N}$, which accounts for the minus sign in the formula for $A^-_{P,Q}$. The series is absolutely convergent. The finite quantities satisfy

$$\sigma\tau = Q/\tau, \qquad \sigma w = w^{-1}, \qquad \sigma\rho = \tau^{-1}, \qquad \sigma\psi_N = -\psi_N. \tag{2.2}$$

For the last identity, factor $-w^{-N}$ out of $1 - w^{-N}$. We shall use $\sigma\Xi = -\Xi$ only under the hypothesis $A^-_{P,Q} = v/m$, $v \in \mathbb{Z}$, $m \geq 1$. In that case

$$mD^2\Xi = -v\sqrt{D} \in \mathcal{O} = \mathbb{Z}[\eta], \qquad \sigma\Xi = -\Xi, \tag{2.3}$$

since $\sqrt{D} = 2\eta - P$. This argument uses the value of the convergent series as a whole.

We want polynomial moments in E and simple-pole moments that recover tails of Ξ. The Euler operator $\theta = z\,d/dz$ supplies the cubed denominator in (2.1) through the identity $\theta(\theta+1)(1-cz)^{-1} = 2cz/(1-cz)^3$. At the node w^n, weighting the numerator by ρ^n supplies $(\rho w)^n = \tau^{3n}$. Thus for g holomorphic on a neighborhood of $|z| \le |w|$, set

$$L(g) = \sum_{n\ge 1} \rho^n [\theta(\theta+1)g](w^n).$$

The derivatives of g are bounded on the indicated disk, so the sum converges absolutely. With $\alpha_j = \rho w^j$, summing gives

$$L(z^j) = \nu_j = \frac{j(j+1)\alpha_j}{1-\alpha_j} \qquad (j \ge 0), \tag{2.4}$$

$$L\big((1-w^a z)^{-1}\big) = 2\rho^{-a}(\Xi - S_a) \qquad (a \ge 1). \tag{2.5}$$

Indeed, in the second identity put $N = n + a$ and use $w^a\tau^{-3a} = \rho^{-a}$. This is the signed specialization of Krisman's moment identities [5, Section 2].

For $K \ge 2$ let $D_K(z) = \prod_{a=1}^{K}(1-w^a z)$. On rational functions of the form p/D_K, define L_X by ordinary partial fractions. Use (2.4) on the polynomial quotient and (2.5) with Ξ replaced by X on the simple fractions. Set

$$\Delta_{K,\tau}(X) = \det_{0\le i,j<K} L_X\big(z^{i+j}/D_K(z)\big). \tag{2.6}$$

Each entry is affine in X, hence this polynomial has degree at most K. At $X = \Xi$, all its entries are values of the actual functional L. Replacing τ in the finite formulas by an indeterminate T defines $\Delta_{K,T}(X) \in \mathbb{Q}(T)[X]$. This formal version will let us use the same denominator at both embeddings.

3 The denominator and the first embedding

Let Φ_d denote the dth cyclotomic polynomial and φ Euler's totient. The arithmetic input to our proof is [5, Proposition 4.4 and the proof of Proposition 4.6]. For each fixed $Q = \pm 1$ it supplies finitely supported nonnegative integers $e_{K,d}$ such that

$$C_K(T) = \prod_d \Phi_d(T)^{e_{K,d}}, \qquad C_K(T)\Delta_{K,T}(X) \in \mathbb{Z}[T, T^{-1}][X], \tag{3.1}$$

$$\sum_d e_{K,d} = O(K^2 \log K), \qquad \sum_d \varphi(d) e_{K,d} \le \frac{4393}{3675}K^3 + O(K^2). \tag{3.2}$$

The weight $\varphi(d)$ in (3.2) is the degree of Φ_d. The coefficient $4393/3675$ comes from adding the contribution of colliding poles, at most $\frac{11}{8}\prod_{p\le 7}(1-p^{-2})$ with the product over primes, to the contribution $1/3$ from monomial moments. Since (3.1) holds before evaluating T, we may substitute the negative unit τ and its conjugate. Each evaluation has coefficients in $\mathcal{O}$. The accompanying programs check (3.1) coefficientwise for $2 \le K \le 5$ and both signs of Q [11].

Lemma 3.1. *For each of the units in Theorem 1.1,*

$$\log|C_K(\tau)C_K(\sigma\tau)| \le \frac{4393}{3675}K^3 \log\eta + O(K^2 \log K).$$

Proof. For $|x| = u$, the product identity $\Phi_d(x) = \prod_{a|d}(x^a - 1)^{\mu(d/a)}$ gives

$$\big|\log|\Phi_d(x)|\big| \le E_u, \qquad E_u = \sum_{a\ge 1} -\log(1-u^a) < \infty.$$

Here μ is the Möbius function and $\bigl|\log|1-x^a|\bigr| \le -\log(1-u^a)$ for both real signs of x. For $|x|=\eta$, factor x^a out of x^a-1. The same bound applies after subtracting $\varphi(d)\log\eta$. The error in the logarithm of each factor's norm is therefore at most $2E_u$. Sum with weights $e_{K,d}$ and use (3.2). □

All subsequent error constants may depend on the fixed unit. In logarithmic upper bounds a zero determinant has logarithm $-\infty$.

Proposition 3.2. *At the first real embedding,*

$$\log|\Delta_{K,\tau}(\Xi)| \le -\frac{2}{9}K^3\log\eta + O(K^2).$$

Proof. The functional uses derivatives through order two. A polynomial with a triple zero at a node contributes zero there, even after multiplication by a factor holomorphic at that node. This explains the monic basis from [5, Proposition 3.1],

$$V_i(z) = \prod_{\ell=1}^{i}(z - w^{\lceil \ell/3\rceil}), \qquad R_i = \lceil i/3\rceil, \qquad b_i = \sum_{\ell=1}^{i}\lceil \ell/3\rceil.$$

Using V_i, V_j in the rows and columns of (2.6) preserves the determinant. When $n < \max(R_i, R_j)$, at least one factor V_i or V_j has a triple zero at w^n. At every other node, each factor $z - w^b$ at $z = w^n$ has modulus at most $2|w|^b$. Applying θ once or twice to that factor gives z, of modulus at most $|w|^b$. The reciprocal $1/D_K$ and its first two Euler derivatives are bounded uniformly on $|z| \le |w|$. This follows from Cauchy's estimates on the unit circle, where $|D_K(z)| \ge \prod_{a\ge1}(1-|w|^a) > 0$.

The product rule and $\sum_n |\rho|^n < \infty$ now give

$$|L(V_iV_j/D_K)| \le C(i+j+1)^2\,2^{i+j}|w|^{b_i+b_j}.$$

Extract $|w|^{b_i}$ and $|w|^{b_j}$ from the row and column bounds. The remaining factors contribute $O(K^2)$ to the logarithm by Hadamard's inequality. As $b_i = i^2/6 + O(i)$, we have $2\sum_{i<K} b_i = K^3/9 + O(K^2)$. Finally, $\log|w| = -2\log\eta$, which gives the asserted coefficient. □

4 A contour for the conjugate determinant

The conjugate of w has modulus greater than one. We therefore apply σ to the finite moment formulas, keeping X formal, and use their partial fractions to estimate $\Delta_{K,\sigma\tau}$. Denote the resulting functional by L_X^{σ}. The conjugated denominator vanishes at the points $r_a = w^a$ inside the unit circle. Indeed,

$$\sigma D_K(z) = (-1)^K w^{-K(K+1)/2} F_K(z), \qquad F_K(z) = \prod_{a=1}^{K}(z - r_a). \tag{4.1}$$

Passing to the reciprocal denominator gives a factor of absolute value $|w|^{K(K+1)/2}$ in each entry. Extracting it from every row will give a cubic saving in the logarithm, provided the remaining moments with denominator F_K are small enough.

Lemma 4.1. *Define $\Gamma(r) = L_s((1-rs)^{-1})$, with L acting on the variable s. This function is holomorphic on $|r| < |w|^{-1}$, and for $1 \le a \le K$,*

$$L^{\sigma}_{-\Xi}\bigl((z-r_a)^{-1}\bigr) = \Gamma(r_a).$$

Proof. Use $(z-r_a)^{-1} = -r_a^{-1}(1-w^{-a}z)^{-1}$. By (2.2) the conjugate of the right side of (2.5), with Ξ replaced by the indeterminate X, is $2\tau^a(X+S_a)$. At $X=-\Xi$ this gives

$$L^{\sigma}_{-\Xi}\big((z-r_a)^{-1}\big) = \frac{2\tau^a}{r_a}(\Xi - S_a) = 2\rho^{-a}(\Xi - S_a) = L_s\big((1-r_a s)^{-1}\big).$$

The middle equality uses $r_a = (\rho\tau)^a$. Direct differentiation in the original functional gives

$$\Gamma(r) = 2r\sum_{n\geq 1}\frac{\tau^{3n}}{(1-rw^n)^3},$$

locally uniformly on the indicated disk. □

Take $K=2$ and $j=0$, so the rational function is $1/F_2$, with poles $r_1 = w$ and $r_2 = w^2$. By Lemma 4.1 the two simple fractions contribute

$$\frac{\Gamma(r_1)}{r_1 - r_2} + \frac{\Gamma(r_2)}{r_2 - r_1} = \frac{\Gamma(r_1) - \Gamma(r_2)}{r_1 - r_2}.$$

For a fixed holomorphic Γ, this divided difference tends to $\Gamma'(r_1)$ as $r_2 \to r_1$. Estimating the two fractions separately loses that cancellation. For K nodes, a single contour will combine all the pole contributions as the nodes accumulate at zero.

Proposition 4.2. *At the prescribed value of the conjugate variable,*

$$\log|\Delta_{K,\sigma\tau}(-\Xi)| \leq -K^2(K+1)\log\eta + O(K\log K).$$

Proof. Write $Y_j = L^{\sigma}_{-\Xi}(z^j/F_K(z))$ for $0 \leq j \leq 2K-2$. By Lemma 4.1 and the residue theorem, the contribution from the simple fractions is the counterclockwise contour integral

$$\sum_{a=1}^{K}\frac{r_a^j\Gamma(r_a)}{F_K'(r_a)} = \frac{1}{2\pi i}\int_{|z|=1}\frac{z^j\Gamma(z)}{F_K(z)}\,dz. \tag{4.2}$$

On this circle $|F_K(z)| \geq \prod_{a\geq 1}(1-|w|^a) > 0$, and Γ is bounded there. Thus (4.2) is bounded independently of j, K. This uses the signed analogue of the common pole function in [5, Proposition 5.1].

It remains to bound the polynomial quotient when $j \geq K$. On expanding $1/F_K(z)$ at infinity its coefficients after the leading z^{-K} are $h_\ell = h_\ell(r_1,\dots,r_K)$, defined by $\sum_{\ell\geq 0}h_\ell t^\ell = \prod_{a=1}^{K}(1-r_a t)^{-1}$. The polynomial quotient is $\sum_{\ell=0}^{j-K}h_\ell z^{j-K-\ell}$, and $\sum_{\ell\geq 0}|h_\ell| \leq \prod_{a\geq 1}(1-|w|^a)^{-1}$. Also $\sigma\alpha_i = Q/\alpha_i$, so $\sigma\nu_i = Qi(i+1)/(\alpha_i - Q)$ is $O((i+1)^2)$, using $|\alpha_i| \leq u < 1$. Consequently the entire moment Y_j is $O((K+1)^2)$.

Returning to (4.1) multiplies the absolute determinant of the Y_{i+j} by $|w|^{K^2(K+1)/2}$. Hadamard's inequality and $|w| = \eta^{-2}$ give the result. □

5 Nonvanishing at arbitrarily large sizes

The Cauchy transform of L is the function from Lemma 4.1 in a reciprocal variable. For $|z| > |w|$,

$$\mathcal{C}(z) = L_s\big((z-s)^{-1}\big) = \frac{1}{z}L_s\big((1-s/z)^{-1}\big) = \frac{\Gamma(1/z)}{z}.$$

The points $r_a = w^a$ of Section 4 correspond to $x_a = w^{-a}$. These exterior points are the zeros of D_K for $1 \leq a \leq K$, and $|x_a| \to \infty$.

Kronecker proved that the generating function of a fixed moment sequence is rational exactly when its Hankel determinants eventually vanish [9, Arts. VII and IX]. Zudilin recalls this criterion in the irrationality setting [6, Section 5]. Here the moments change with K, but every zero of D_K is also a zero of D_{K+1}. We use these shared points in a rational interpolation argument, following [10, Proposition 5.1] and [5, Proposition 6.1], for the actual signed functional.

Proposition 5.1. *For arbitrarily large K, one has $\Delta_{K,\tau}(\Xi) \neq 0$.*

Proof. The series for Γ in Lemma 4.1 gives the continuation

$$\mathcal{C}(z) = \sum_{n\geq 1} \frac{2\rho^n w^n z}{(z-w^n)^3}.$$

The series converges normally on compact subsets away from the nodes and their accumulation point 0. Its cubic pole coefficient at w^n is $2\rho^n w^{2n} \neq 0$. It follows that $\mathcal{C}$ has infinitely many poles and is not rational.

Suppose now that all determinants from some size onward vanish. At each such size k, take a nonzero polynomial R_k of degree $< k$ in the kernel. Thus $L_s(s^j R_k(s)/D_k(s)) = 0$ for $0 \leq j < k$. We seek a polynomial T_k such that the error $\mathcal{C}R_k - T_k$ vanishes at $x_1, \ldots, x_k$ and is $O(1/z)$ at infinity. Consecutive sizes share the first k points. With this decay those shared zeros will force $T_k/R_k = T_{k+1}/R_{k+1}$. To obtain both properties, set

$$T_k(z) = L_s\left(\frac{R_k(z)D_k(s) - D_k(z)R_k(s)}{(z-s)D_k(s)}\right), \qquad E_k(z) = \mathcal{C}(z)R_k(z) - T_k(z). \tag{5.1}$$

The numerator in (5.1) vanishes at $z = s$, so T_k is a polynomial of degree $< k$. Subtracting in (5.1) gives

$$E_k(z) = D_k(z)L_s\left(\frac{R_k(s)}{(z-s)D_k(s)}\right).$$

Thus E_k vanishes at x_a, $1 \leq a \leq k$, where the functional on the right is analytic. At infinity expand $(z-s)^{-1}$ in powers of $1/z$. The expansion converges with two s-derivatives on a disk containing the nodes, and the first k terms give zero by the kernel equations. The factor $D_k(z)$ has degree k, so $E_k(z) = O(1/z)$.

The polynomial

$$T_{k+1}R_k - T_kR_{k+1} = R_{k+1}E_k - R_kE_{k+1}$$

has degree at most $k-1$ by its growth at infinity, and has the k common zeros $x_1, \ldots, x_k$. Hence it vanishes identically. All sufficiently large k therefore give the same rational function $T_k/R_k = T/R$, written in lowest terms. Write $R_k = V_kR$, $T_k = V_kT$ with V_k a nonzero polynomial. At a zero of V_k, cancelling this common factor can lose an interpolation condition. The decay at infinity bounds how many can be lost. Indeed, the fixed error $E = \mathcal{C}R - T$ is nonzero since $\mathcal{C}$ is not rational. Write $E(z) \sim cz^{-q}$ at infinity, with $c \neq 0$. As $E_k = V_kE = O(1/z)$, we have $\deg V_k \leq q-1$. But $E(x_a) \neq 0$ for all sufficiently large a, since $|x_a| \to \infty$. Every such x_a with $a \leq k$ must be a zero of V_k. Their number eventually exceeds the fixed bound $q-1$, a contradiction. □

6 The norm and the fixed-stride sums

Proof of Theorem 1.1. Suppose $A^{-}_{P,Q} = v/m$ with $v \in \mathbb{Z},\ m \geq 1$. By (2.3), $mD^2\Xi \in \mathcal{O}$. The degree bound after (2.6) and the coefficient clearing in (3.1) show that

$$y_K = (mD^2)^K C_K(\tau)\Delta_{K,\tau}(\Xi) \in \mathcal{O}.$$

No cyclotomic factor vanishes at τ, since $|\tau| < 1$. By Proposition 5.1 we can choose arbitrarily large K with $y_K \neq 0$. For these indices $\mathrm{N}_{E/\mathbb{Q}}(y_K)$ is a nonzero integer. Under our rationality assumption $\sigma\Xi = -\Xi$ by (2.3), so its absolute value is

$$|\,\mathrm{N}_{E/\mathbb{Q}}(y_K)| = (mD^2)^{2K}|C_K(\tau)C_K(\sigma\tau)|\,|\Delta_{K,\tau}(\Xi)|\,|\Delta_{K,\sigma\tau}(-\Xi)|.$$

The first factor contributes only $2K\log(mD^2)$ to the logarithm. For the other three factors use Lemma 3.1 and Propositions 3.2 and 4.2, respectively. This gives

$$0 \leq \log|\,\mathrm{N}_{E/\mathbb{Q}}(y_K)| \leq -\frac{296}{11025}K^3\log\eta + O(K^2\log K) + 2K\log(mD^2),$$

by (1.1). The right-hand side tends to $-\infty$ because $\eta > 1$ and m, D are fixed. This proves irrationality. □

Proof of Corollary 1.2. Write $\bar{\eta} = Q/\eta$ and $P_k = \eta^k + \bar{\eta}^k$. This is an integer, by the recurrence for the traces of the powers of η. Binet's formula gives

$$U_{kn}(P,Q) = U_k(P,Q)\,U_n(P_k,Q^k).$$

If $Q^k = 1$, then $P_k = \eta^k + \eta^{-k} > 2$. If $Q^k = -1$, then $P_k = \eta^k - \eta^{-k} > 0$. Thus the new pair is admissible. Apply Theorem 1.1 and divide by the nonzero integer U_k^3. For Fibonacci, $U_2(1,-1) = 1$ and $P_2 = 3$, so $F_{2n} = U_n(3,1)$. This gives the irrational sum $F_2^{-3} - F_4^{-3} + F_6^{-3} - \cdots$. □

Verification

Another AI model independently audited the mathematics and exposition of the 18 November 2026 working proof, including the fixed-stride result. Two independent reviews subsequently examined the exposition of the seven-page manuscript of the same date.

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