POWER-SAVING BOUNDS FOR BALANCED RADICAL TRIPLES VIA SMALL PELL DISCRIMINANTS

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Abstract. For $0 < \lambda \le 1$, let $\mathcal{B}_\lambda(N)$ be the set of integers $n \le N$ for which $\mathrm{rad}(n(n+1)(2n+1)) < (2n+1)^\lambda$. The elementary radical-box estimate gives $|\mathcal{B}_\lambda(N)| \ll N^{\lambda/3+\varepsilon}$. We prove a strict power-saving refinement of this exponent. There is an absolute constant $C > 0$ such that $|\mathcal{B}_\lambda(N)| \ll_{\lambda,\varepsilon} N^{\lambda/3-C\lambda^3+\varepsilon}$. The argument uses Diao's uniform bound for the number of small fundamental solutions to Pell equations, together with a squarefree-kernel transfer for the three unit-difference equations, a fixed-support radical-box lemma, and a three-branch dyadic optimization. We also formulate the resulting estimate for high-quality values and for Chinese-remainder orbits, and quantify the variable-right-hand-side obstruction to a direct extension to arbitrary abc triples. No assertion of the abc or Szpiro conjecture is used, and no pointwise abc inequality is claimed. The only non-elementary external input is Diao's Theorem 1.3 in arXiv:2610.28001v1, whose statement and preprint status are recorded explicitly.

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1. Introduction

For a positive integer m, write $\mathrm{rad}(m) = \prod_{p\mid m} p$ for its radical. The three integers

$$m_1(n) = n, \qquad m_2(n) = n+1, \qquad m_3(n) = 2n+1 \tag{1.1}$$

are pairwise coprime and satisfy the three unit-difference relations

$$m_2 - m_1 = 1, \qquad m_3 - 2m_1 = 1, \qquad m_3 - 2m_2 = -1. \tag{1.2}$$

These relations make their joint radical a rigid one-dimensional Diophantine object, distinct from the full three-variable abc exceptional set. Background on the abc problem and its radical formulation can be found in Granville–Tucker [2].

For $0 < \lambda \le 1$ and $N \ge 2$, define

$$\mathcal{B}_\lambda(N) = \left\{1 \le n \le N : \ R_n < (2n+1)^\lambda\right\}, \qquad R_n = \mathrm{rad}(n(n+1)(2n+1)). \tag{1.3}$$

Put $r_i(n) = \mathrm{rad}(m_i(n))$. Pairwise coprimality gives $R_n = r_1 r_2 r_3$. On $\mathcal{B}_\lambda(N)$, at least one r_i is at most $N^{\lambda/3+o(1)}$. Since a coordinate determines n, the standard radical-box bound (Corollary 2.2) yields the elementary estimate

$$|\mathcal{B}_\lambda(N)| \ll_{\lambda,\varepsilon} N^{\lambda/3+\varepsilon}. \tag{1.4}$$

The purpose of this paper is to give all steps of a strict improvement of the exponent $\lambda/3$.

The main new mechanism is a transfer from squarefree kernels to Pell discriminants. Write each coordinate uniquely as

$$m_1 = ax^2, \qquad m_2 = by^2, \qquad m_3 = cz^2, \tag{1.5}$$

where a, b, c are the squarefree products of the primes occurring to odd valuation in their respective coordinates. Each pair in (1.2) then produces a solution of an ordinary Pell equation. For example,

$$(2n+1)^2 - ab(2xy)^2 = 1. \tag{1.6}$$

Thus, if ab is small compared with N, the corresponding Pell discriminant has an unusually small fundamental solution. A uniform power-saving bound for such discriminants, proved by Diao [1, Theorem 1.3], supplies the saving exactly in the dyadic range where the two elementary covers balance.

There are three technical points that must not be suppressed. First, for $m = ax^2$ with a squarefree, the correct radical formula is

$$\mathrm{rad}(m) = a\,\mathrm{rad}_{a^c}(x),$$

where $\mathrm{rad}_{a^c}(x)$ retains only prime divisors of x coprime to a; in general it is not $a\,\mathrm{rad}(x)$. We prove a fixed-support box estimate that controls the primes shared by a and x. Second, a fixed Pell discriminant can contribute more than one n, but its positive solutions grow exponentially, so the fibre size is only $O(\log N)$. Third, the square-part cover uses two-sided relations $x_i^2 \asymp N/U_i$. We therefore prove the correlation estimates first on $N/2 < n \le N$ and recover the full interval $n \le N$ by a convergent dyadic sum. These facts make the passage from discriminant counts to counts of n uniform.

Our main result is the following.

Theorem 1.1 (Main counting theorem)**.** *There is an absolute constant $C > 0$ such that, for every $0 < \lambda \le 1$ and every $\varepsilon > 0$,*

$$\boxed{|\mathcal{B}_\lambda(N)| \ll_{\lambda,\varepsilon} N^{\lambda/3 - C\lambda^3 + \varepsilon}.} \tag{1.7}$$

More precisely, if c_D is the constant in Diao's Theorem 3.3, then one may take $C = c_D/2160$ after harmlessly reducing c_D so that $c_D \le 1$.

The improvement is small but strict. Its cubic shape is not an arbitrary choice: in the critical range the Pell discriminant has size N^s with $s \asymp \lambda$, while the fundamental solution is at most N, so the relevant Diao parameter is $\theta \asymp s^{-1}$. The saving $c_D\theta^{-2}$ therefore becomes $O(s^3)$ on the N-exponent scale.

For $\eta > 0$, define the quality

$$q(n) = \frac{\log(2n+1)}{\log R_n}. \tag{1.8}$$

Then $q(n) > 1 + \eta$ is equivalent to $R_n < (2n+1)^{1/(1+\eta)}$. Theorem 1.1 consequently gives

$$\boxed{\#\{n \le N : q(n) > 1 + \eta\} \ll_{\eta,\varepsilon} N^{\frac{1}{3(1+\eta)} - \frac{C}{(1+\eta)^3} + \varepsilon}.} \tag{1.9}$$

A Chinese-remainder orbit version is stated and proved in Section 7.

Status of the input and scope. Theorem 1.1 uses no conjectural hypothesis. Its only non-elementary external ingredient is Diao's preprint theorem on small Pell solutions [1]; we do not claim to reprove that theorem here. The new arguments are the support-adjusted radical covering, the uniform three-pair Pell transfer, the dyadic synthesis, and the passage to (1.7). The result is a counting theorem for a special one-dimensional family. It neither proves a pointwise abc inequality for this family nor implies the full abc or Szpiro conjecture.

Organization. Section 2 proves the fixed-support and radical-box estimates. Section 3 derives all three Pell equations and controls their fibres. Section 4 establishes the three-branch dyadic bound. Section 5 proves the diagonal theorem and deduces Theorem 1.1. Section 6 analyzes the variable-square obstruction for arbitrary abc triples and records the resulting open problem. Section 7 treats CRT orbits and discusses limitations. Appendix A collects the exponent bookkeeping and the linear-progression details used in the main proof.

2. Fixed-support and radical-box estimates

Throughout, $N \ge 2$ is the global height parameter. The notation $A \lesssim B$ is never used; $A \ll B$ has its usual meaning, and a subscript indicates permitted dependence of the implied constant. We write $f(q, X)$ for the number of integers $1 \le m \le X$ all of whose prime factors divide q, and $\Psi(X, Y)$ for the number of Y-friable integers $m \le X$.

2.1. A uniform fixed-support estimate.

Lemma 2.1 (Fixed-support box)**.** *Uniformly for integers $1 \le q, X \le N$,*

$$f(q, X) = N^{o(1)}. \tag{2.1}$$

More precisely, for every $\varepsilon > 0$ there is C_ε such that $f(q, X) \le C_\varepsilon N^\varepsilon$ in this range.

Proof. We may replace q by its radical without changing $f(q, X)$. Let $q_1 < \cdots < q_k$ be the prime divisors of q, and let $p_1 < \cdots < p_k$ be the first k primes. The exponent-preserving map

$$\prod_{i=1}^{k} q_i^{e_i} \longmapsto \prod_{i=1}^{k} p_i^{e_i} \tag{2.2}$$

is injective by unique factorization. Since $p_i \le q_i$, an integer at most X is mapped to an integer at most X. Hence

$$f(q, X) \le \Psi(X, p_k). \tag{2.3}$$

Because $q \le N$, the elementary upper bound for $\omega(q)$ gives

$$k \le (1+o(1))\frac{\log N}{\log\log N},$$

and the prime number theorem yields $p_k \le 2\log N$ for large N. Thus

$$f(q,X) \le \Psi(X, 2\log N).$$

Lemma A.1 in Appendix A.1 now gives $\Psi(X, 2\log N) = N^{o(1)}$ uniformly for $X \le N$. If N is bounded, the assertion is trivial; enlarging the implied constant proves the ε formulation. □

2.2. **Radical boxes and support-adjusted boxes.**

Corollary 2.2 (Radical-box bound)**.** *Uniformly for* $1 \le R, X \le N$,

$$\#\{1 \le m \le X : \mathrm{rad}(m) \le R\} \ll_\varepsilon RN^\varepsilon. \tag{2.4}$$

In particular, a coordinate of size at most $2N+1$ *whose radical is at most* N^γ *determines at most* $N^{\gamma+\varepsilon}$ *values of* n.

Proof. Group the integers according to $q = \mathrm{rad}(m)$. For each fixed squarefree q, their number is $f(q,X)$. Lemma 2.1 therefore gives

$$\#\{m \le X : \mathrm{rad}(m) \le R\} \le \sum_{\substack{q \le R \\ \mu^2(q)=1}} f(q,X) \ll_\varepsilon RN^\varepsilon.$$

For the last sentence, apply (2.4) with $2N+1$ in place of the global parameter N when the coordinate is $2n+1$; this changes the implied constant and the N^ε factor only by a bounded factor. Finally, n is recovered from n, $n+1$, or $2n+1$ uniquely. □

For integers a, x, define the part of the radical coprime to a by

$$\mathrm{rad}_{a^c}(x) = \prod_{\substack{p \mid x \\ p \nmid a}} p. \tag{2.5}$$

The following lemma is the support correction needed for the squarefree-kernel decomposition.

Lemma 2.3 (Support-adjusted radical box)**.** *Uniformly for* $1 \le a, R, X \le N$,

$$\#\{1 \le x \le X : \mathrm{rad}_{a^c}(x) \le R\} \ll_\varepsilon RN^\varepsilon. \tag{2.6}$$

Proof. Every x has a unique factorization $x = uv$, where

$$u = \prod_{p \mid a} p^{v_p(x)}, \qquad v = x/u.$$

Thus every prime factor of u divides a, while $\gcd(v,a) = 1$ and $\mathrm{rad}(v) = \mathrm{rad}_{a^c}(x)$. The number of possible $u \le X$ is $f(a,X) = N^{o(1)}$ by Lemma 2.1. For each fixed u, the number of $v \le X/u$ with $\mathrm{rad}(v) \le R$ is at most $RN^{o(1)}$ by Corollary 2.2. Multiplying the two boxes proves (2.6). □

2.3. **Squarefree kernels and exact radical formulae.** Every positive integer m has a unique decomposition

$$m = ax^2, \qquad a = \prod_{\substack{p \mid m \\ 2 \nmid v_p(m)}} p, \tag{2.7}$$

with a squarefree. We call a the *odd-valuation squarefree kernel*. The integer x need not be coprime to a. Nevertheless, unique factorization gives the exact identity

$$\mathrm{rad}(m) = a\,\mathrm{rad}_{a^c}(x). \tag{2.8}$$

Indeed, the primes dividing $\mathrm{rad}(m)$ are precisely the primes dividing a, together with the primes dividing x but not a.

For the three coordinates in (1.1), write

$$n = ax^2, \qquad n+1 = by^2, \qquad 2n+1 = cz^2 \tag{2.9}$$

as in (2.7). Pairwise coprimality of the coordinates implies

$$\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1, \qquad \gcd(x,y) = \gcd(x,z) = \gcd(y,z) = 1. \tag{2.10}$$

Moreover, c and z are odd. The corresponding radicals are

$$r_1 = a\,\mathrm{rad}_{a^c}(x), \qquad r_2 = b\,\mathrm{rad}_{b^c}(y), \qquad r_3 = c\,\mathrm{rad}_{c^c}(z). \tag{2.11}$$

These identities, rather than the generally false formula $\mathrm{rad}(ax^2) = a\,\mathrm{rad}(x)$, are what enter the dyadic estimates below.

3. The three Pell transfers

3.1. **Algebraic identities.** For a nonsquare positive integer D, let

$$\varepsilon_D = t_1 + u_1\sqrt{D} \tag{3.1}$$

be the fundamental solution of the Pell equation $t^2 - Du^2 = 1$ in positive integers t_1, u_1. The three coordinate pairs lead to the following uniform data.

Table 1. Pell discriminants attached to the three unit-difference pairs.

Pair (i,j)	Kernel product w_{ij}	D_{ij}	T_{ij}	U_{ij}
$(1,2)$	ab	ab	$2n+1$	$2xy$
$(1,3)$	ac	$2ac$	$4n+1$	$2xz$
$(2,3)$	bc	$2bc$	$4n+3$	$2yz$

Lemma 3.1 (Three Pell identities)**.** *With the notation of Table 1, every $n \geq 1$ gives*

$$T_{ij}^2 - D_{ij}U_{ij}^2 = 1, \qquad 1 \leq i < j \leq 3. \tag{3.2}$$

Moreover, D_{ij} is nonsquare and

$$\varepsilon_{D_{ij}} < 2T_{ij} \leq 8N+6 \tag{3.3}$$

whenever $1 \leq n \leq N$.

Proof. For $(1,2)$, use $(2n+1)^2 - 1 = 4n(n+1)$:

$$(2n+1)^2 - 1 = 4abx^2y^2 = ab(2xy)^2.$$

For $(1,3)$, use $(4n+1)^2 - 1 = 8n(2n+1)$:

$$(4n+1)^2 - 1 = 8acx^2z^2 = (2ac)(2xz)^2.$$

For $(2,3)$, use $(4n+3)^2 - 1 = 8(n+1)(2n+1)$:

$$(4n+3)^2 - 1 = 8bcy^2z^2 = (2bc)(2yz)^2.$$

This proves (3.2). All U_{ij} are positive. If D_{ij} were a square, then $T_{ij}^2 - D_{ij}U_{ij}^2$ would be a difference of two positive integral squares equal to 1. The only consecutive integral squares are 0 and 1, which is incompatible with $T_{ij} > 1$ and $U_{ij} > 0$. Thus D_{ij} is nonsquare.

Finally, $\eta = T_{ij} + U_{ij}\sqrt{D_{ij}}$ is a Pell unit greater than 1, so $\varepsilon_{D_{ij}} \leq \eta$. Since $U_{ij}\sqrt{D_{ij}} = \sqrt{T_{ij}^2 - 1} < T_{ij}$ and $T_{ij} \leq 4N+3$, we obtain (3.3). □

The underlying generalized unit equations are also needed for the alternative dyadic cover. They are

$$by^2 - ax^2 = 1, \tag{3.4}$$

$$cz^2 - 2ax^2 = 1, \tag{3.5}$$

$$cz^2 - 2by^2 = -1. \tag{3.6}$$

3.2. **Fibre cardinality.** Exponential growth of Pell orbits is also the mechanism behind the fixed-support consecutive-integer count of Lebowitz-Lockard [3, Lemma 8]; here we require the analogous fibre bound for our three varying discriminants.

Lemma 3.2 (Logarithmic Pell fibres)**.** *Fix $1 \le i < j \le 3$ and a nonsquare integer $D \ge 2$. Then*

$$\#\{1 \le n \le N : D_{ij}(n) = D\} \ll \log N. \tag{3.7}$$

The implied constant is absolute.

Proof. Every positive solution of $t^2 - Du^2 = 1$ satisfies

$$t + u\sqrt{D} = \varepsilon_D^k$$

for some integer $k \ge 1$. This is the standard structure theorem for the positive Pell unit group. Lemma 3.1 shows that each n in the fibre gives such a solution with $T_{ij} \le 4N + 3$. Since $\varepsilon_D > \sqrt{D} \ge \sqrt{2}$,

$$k \log \varepsilon_D \le \log(2T_{ij}) \ll \log N,$$

and hence there are $O(\log N)$ possible exponents k. Finally, $T_{12} = 2n + 1$, $T_{13} = 4n + 1$, and $T_{23} = 4n + 3$ each determine n uniquely. □

3.3. **Diao's uniform small-solution bound.** For $\theta > 0$ and $X \ge 2$, define

$$E_\theta(X) = \#\{0 < D < X : \ D \text{ is nonsquare and } \varepsilon_D \le D^\theta\}. \tag{3.8}$$

The following is the sole non-elementary external input.

Theorem 3.3 (Diao, [1, Theorem 1.3])**.** *There is an absolute constant $c_D > 0$ such that, for all sufficiently large X,*

$$\boxed{E_\theta(X) \ll X^{1-c_D/\theta^2}} \tag{3.9}$$

uniformly in the range

$$3 \le \theta \le c_D \sqrt{\frac{\log X}{\log\log X}}. \tag{3.10}$$

The implied constant is absolute.

Remark 3.4. Theorem 3.3 is quoted from arXiv:2610.28001v1, dated 20 October 2026. The present paper uses it as a cited preprint result and does not assert an independent verification or a peer-reviewed publication status. Diao's Section 8.1 has already passed from the squarefree Proposition 8.1 to general nonsquare discriminants by using the squarefree estimate with parameter 2θ and putting $c = c_0/4$. Our c_D denotes the resulting constant in the final Theorem 1.3, so no further halving or other reduction is required. Diao explicitly leaves c unoptimized, and the auxiliary absolute constants in the later propositions are not assigned numerical values; accordingly, we claim an absolute positive constant but no numerical value for it. Reducing c_D weakens (3.9) and shrinks the upper endpoint in (3.10); this is harmless in our application because only bounded values of θ are used. We therefore assume $c_D \le 1$ when convenient.

The next lemma converts Theorem 3.3 directly into a count of n in a dyadic kernel-product box.

Lemma 3.5 (Small-discriminant branch)**.** *Fix $1 \le i < j \le 3$ and a compact set $K \subset (0, 2/3]$. Uniformly for $s \in K$ and $W = N^s$, the integers $n \le N$ with $w_{ij}(n) \in [W, 2W)$ satisfy*

$$\#\{n \le N : w_{ij}(n) \in [W, 2W)\} \ll_{K,\varepsilon} N^{s-(c_D/4)s^3+\varepsilon}. \tag{3.11}$$

The implied constant is independent of the particular $s \in K$.

Proof. We have $D_{ij} = \kappa_{ij} w_{ij}$ with $\kappa_{ij} \in \{1, 2\}$, so $D_{ij} \asymp W = N^s$. By Lemma 3.1, $\varepsilon_{D_{ij}} \leq 8N + 6$. Choose

$$\theta = \frac{2}{s} \geq 3. \tag{3.12}$$

For sufficiently large N,

$$D_{ij}^{\theta} \geq W^{2/s} = N^2 > 8N + 6,$$

so every discriminant in the box belongs to $E_\theta(CW)$ for an absolute $C > 0$. The upper restriction in (3.10) is satisfied eventually because $\theta = 2/s$ is bounded on the fixed compact set, whereas the permitted upper endpoint tends to infinity. Theorem 3.3 gives

$$E_\theta(CW) \ll W^{1-c_D/\theta^2} = W^{1-(c_D/4)s^2}.$$

Lemma 3.2 multiplies this by only $O(\log N)$. Rewriting the result in terms of N proves (3.11); the dyadic constant C contributes only $N^{o(1)}$. □

4. The three-branch dyadic estimate

For $1 \leq i < j \leq 3$ and $A, B \geq 1$, define the dyadic-height counting function

$$\mathcal{C}_{ij}(A, B; N) = \#\{N/2 < n \leq N : r_i(n) \leq A,\ r_j(n) \leq B\}. \tag{4.1}$$

We also write

$$\mathcal{C}_{ij}(A, B; N; W) = \#\{n \in \mathcal{C}_{ij}(A, B; N) : w_{ij}(n) \in [W, 2W)\}, \tag{4.2}$$

where, by a slight abuse of notation, $n \in \mathcal{C}_{ij}$ means that n belongs to the set counted by (4.1).

4.1. The two elementary covers.

Lemma 4.1 (Coefficient and square-part covers)**.** *Let $1 \leq A, B \leq N$. For every $\varepsilon > 0$,*

$$\mathcal{C}_{ij}(A, B; N; W) \ll_\varepsilon W N^\varepsilon. \tag{4.3}$$

If in addition $W \leq N$, then

$$\mathcal{C}_{ij}(A, B; N; W) \ll_\varepsilon \frac{AB}{W} N^\varepsilon. \tag{4.4}$$

The implied constants are independent of i, j.

Proof. The first bound is immediate from the Pell transfer. There are $O(W)$ possible discriminants $D_{ij} = \kappa_{ij} w_{ij}$ in the box, and each has $O(\log N)$ preimages by Lemma 3.2.

For the second bound, dyadically localize the odd-valuation kernels:

$$u_i \in [U_i, 2U_i), \qquad u_j \in [U_j, 2U_j),$$

where $(u_1, u_2, u_3) = (a, b, c)$ and $(x_1, x_2, x_3) = (x, y, z)$. Since $w_{ij} = u_i u_j \in [W, 2W)$, we have $U_i U_j \asymp W$. There are $O((\log N)^2)$ such pairs (U_i, U_j), which are absorbed by N^ε after replacing ε.

The exact radical formula (2.11) gives

$$\mathrm{rad}_{u_i^c}(x_i) \leq \frac{A}{u_i} \ll \frac{A}{U_i}, \qquad \mathrm{rad}_{u_j^c}(x_j) \ll \frac{B}{U_j}.$$

Because $N/2 < n \leq N$, each coordinate $m_i(n)$ is comparable to N, and hence $x_i^2 = m_i(n)/u_i \asymp N/U_i$, with the analogous two-sided relation for x_j. In particular, the support-adjusted lemma is applied with $x_i, x_j \leq N$. Lemma 2.3 therefore yields

$$\#x_i \ll_\varepsilon \frac{A}{U_i} N^\varepsilon, \qquad \#x_j \ll_\varepsilon \frac{B}{U_j} N^\varepsilon. \tag{4.5}$$

It remains to show that fixed (x_i, x_j) leaves $O(1)$ coefficient pairs when $W \leq N$.

Consider first $(i, j) = (1, 2)$. Equation (3.4) and $\gcd(x, y) = 1$ imply that all coefficient solutions with fixed (x, y) have the form

$$(a, b) = (a_0 + y^2 k,\ b_0 + x^2 k), \qquad k \in \mathbb{Z}, \tag{4.6}$$

relative to any one solution. In the dyadic boxes, $a \asymp U_1$ and $b \asymp U_2$, while

$$y^2 = \frac{n+1}{b} \asymp \frac{N}{U_2}, \qquad x^2 = \frac{n}{a} \asymp \frac{N}{U_1}.$$

Thus the number of possible k is

$$\ll 1 + \min\left\{\frac{U_1}{y^2}, \frac{U_2}{x^2}\right\} \ll 1 + \frac{U_1U_2}{N} \ll 1 + \frac{W}{N} \ll 1.$$

For $(1,3)$ and $(2,3)$, equations (3.5)–(3.6) give respectively the homogeneous steps

$$(\Delta a, \Delta c) = (z^2, 2x^2), \qquad (\Delta b, \Delta c) = (z^2, 2y^2).$$

Coprimality and the oddness of z show that these describe all integer solutions. The same height calculation gives $O(1 + U_iU_j/N) = O(1)$ possibilities; see Appendix A.2. Consequently,

$$\mathcal{C}_{ij}(A, B; N; W) \ll_\varepsilon \frac{A}{U_i}\frac{B}{U_j}N^\varepsilon \ll_\varepsilon \frac{AB}{W}N^\varepsilon,$$

after again enlarging ε to absorb the dyadic multiplicities. □

4.2. **Synthesis with the small-discriminant branch.** The three estimates play complementary roles. If $W = N^s$ is small, (4.3) is strong. If W is large relative to A, B, (4.4) is strong. Near the balance $W \approx \sqrt{AB}$, both elementary exponents meet, and Lemma 3.5 supplies a strict saving.

Proposition 4.2 (Three-branch bound)**.** *Let $1 \le A, B \le N$, and let $W = N^s$ with s in a fixed compact subset $K \subset (0, 2/3]$. Then*

$$\mathcal{C}_{ij}(A, B; N; W) \ll_{K,\varepsilon} N^\varepsilon \min\left\{W,\ \frac{AB}{W},\ N^{s-(c_D/4)s^3}\right\}, \tag{4.7}$$

provided $W \le N$ for the middle term. In particular, near $W \asymp \sqrt{AB}$ the third term improves the elementary balance by $N^{-(c_D/4)s^3}$.

Proof. Combine Lemmas 4.1 and 3.5. □

For orientation, when $A = B = N^\rho$ and $W = N^\rho$, the first two bounds are both $N^{\rho+\varepsilon}$, whereas the third is

$$N^{\rho-(c_D/4)\rho^3+\varepsilon}.$$

The rest of the paper is the rigorous covering of all off-diagonal dyadic ranges and the conversion of this local saving into a saving for $\mathcal{B}_\lambda(N)$.

5. The diagonal theorem and the global exponent

Throughout this section,

$$\rho = \frac{\lambda}{3}, \qquad 0 < \lambda \le 1, \tag{5.1}$$

so that $0 < \rho \le 1/3$. We also set

$$c_0 = \frac{c_D}{4}. \tag{5.2}$$

After reducing c_D if necessary, we assume $c_0 \le 1$. For the dyadic-height argument, put

$$\mathcal{B}_\lambda^{\mathrm{dy}}(N) = \{N/2 < n \le N : R_n < (2n+1)^\lambda\}. \tag{5.3}$$

We first bound these dyadic pieces. The full set $\mathcal{B}_\lambda(N)$ is recovered by a geometric sum at the end of the proof.

5.1. **Uniform power saving near the diagonal.**

Theorem 5.1 (Diagonal correlation bound)**.** *For every $0 < \lambda \le 1$, put*

$$\tau = \tau_\lambda = \frac{c_0 \rho^3}{10}. \tag{5.4}$$

Then, uniformly for $1 \le i < j \le 3$ and

$$\alpha, \beta \in [\rho - \tau, \rho + \tau], \tag{5.5}$$

one has

$$\boxed{\mathcal{C}_{ij}(N^\alpha, N^\beta; N) \ll_{\lambda,\varepsilon} N^{\rho-\tau+\varepsilon}.} \tag{5.6}$$

Equivalently, in the notation $\mathbf{Diag}_\lambda(\vartheta, \tau)$ *from the program notes, this proves* $\mathbf{Diag}_\lambda(\tau, \tau)$ *with τ as above.*

Proof. Write $A = N^\alpha$, $B = N^\beta$. Since a squarefree kernel divides the corresponding radical,

$$u_i \le r_i \le A, \qquad u_j \le r_j \le B.$$

Consequently,

$$w_{ij} = u_i u_j \le AB = N^{\alpha+\beta} \le N^{2\rho+2\tau} < N. \tag{5.7}$$

The last inequality uses $\rho \le 1/3$ and $\tau \le 1/270$. Thus (4.4) applies to every dyadic w-box.

Decompose the range $1 \le w_{ij} \le AB$ into $O(\log N)$ dyadic boxes $w_{ij} \in [W, 2W)$, and write $W = N^s$. It suffices to prove the stated bound in each box, with a uniform implied constant; the logarithmic number of boxes is absorbed by N^ε.

Case 1: $s \le \rho - \tau$. The coefficient cover (4.3) gives

$$\mathcal{C}_{ij}(A, B; N; W) \ll_\varepsilon N^{s+\varepsilon} \le N^{\rho-\tau+\varepsilon}. \tag{5.8}$$

Case 2: $s \ge \rho + 4\tau$. The square-part cover (4.4) and $\alpha + \beta \le 2\rho + 2\tau$ give

$$\mathcal{C}_{ij}(A, B; N; W) \ll_\varepsilon N^{\alpha+\beta-s+\varepsilon} \le N^{2\rho+2\tau-\rho-4\tau+\varepsilon} = N^{\rho-2\tau+\varepsilon}. \tag{5.9}$$

Case 3: $\rho - \tau < s < \rho + 4\tau$. Here

$$0 < \frac{\rho}{2} < s < \rho + 4\tau < \frac{2}{5}.$$

Indeed, $\tau < \rho/2$ and $\rho + 4\tau < 1/3 + 4/270 < 2/5$. Hence s lies in a fixed compact subset of $(0, 2/5)$ determined by λ. Lemma 3.5 yields

$$\mathcal{C}_{ij}(A, B; N; W) \ll_{\lambda,\varepsilon} N^{s-c_0 s^3+\varepsilon}. \tag{5.10}$$

The function $f(s) = s - c_0 s^3$ is increasing on the relevant range because

$$f'(s) = 1 - 3c_0 s^2 > 0$$

for $s < 2/5$ and $c_0 \le 1$, since then $3c_0 s^2 < 12/25$. Therefore

$$\begin{aligned} s - c_0 s^3 &\le \rho + 4\tau - c_0(\rho + 4\tau)^3 \\ &\le \rho + 4\tau - c_0 \rho^3 = \rho + 4\tau - 10\tau = \rho - 6\tau. \end{aligned} \tag{5.11}$$

This is stronger than the required $\rho - \tau$.

Combining (5.8), (5.9), and (5.11), and summing over the dyadic boxes, proves (5.6). □

5.2. **From two small radicals to the balanced family.** The remaining step is a combinatorial amplification. If the smallest coordinate radical is already below the improved threshold, Corollary 2.2 suffices. Otherwise, the radical budget forces the two smallest radicals into the diagonal window of Theorem 5.1.

Lemma 5.2 (Diagonal amplification)**.** *Suppose that for some $\vartheta, \tau > 0$ and all $i < j$,*

$$\mathcal{C}_{ij}(N^{\alpha}, N^{\beta}; N) \ll_{\lambda,\varepsilon} N^{\rho-\vartheta+\varepsilon} \qquad (\alpha, \beta \in [\rho-\tau, \rho+\tau]). \tag{5.12}$$

Then for every $0 < \delta < \min(\vartheta, \tau)$,

$$|\mathcal{B}_{\lambda}^{\mathrm{dy}}(N)| \ll_{\lambda,\delta,\varepsilon} N^{\rho-\delta+\varepsilon}. \tag{5.13}$$

Proof. Let $n \in \mathcal{B}_{\lambda}^{\mathrm{dy}}(N)$ and order its coordinate radicals:

$$r_{(1)} \le r_{(2)} \le r_{(3)}.$$

Since $r_1 r_2 r_3 = R_n < (2n+1)^{\lambda} \le (2N+1)^{\lambda}$,

$$r_{(1)} r_{(2)} r_{(3)} \le N^{\lambda+o(1)}. \tag{5.14}$$

First consider the set where $r_{(1)} \le N^{\rho-\delta}$. A coordinate determines n, so Corollary 2.2 bounds this set by $N^{\rho-\delta+\varepsilon}$.

Now suppose $r_{(1)} > N^{\rho-\delta}$. Dyadically localize $r_{(1)}$ and $r_{(2)}$, and write

$$r_{(1)} = N^{a+o(1)}, \qquad r_{(2)} = N^{b+o(1)}. \tag{5.15}$$

The $o(1)$ terms include both the dyadic localization and the N-dependence of the boxes. Because $r_{(1)}^3 \le r_1 r_2 r_3$,

$$a \le \rho + o(1), \tag{5.16}$$

while the assumption on $r_{(1)}$ and the ordering give

$$a \ge \rho - \delta + o(1), \qquad b \ge a + o(1). \tag{5.17}$$

Moreover, $r_{(3)} \ge r_{(2)}$ implies $r_{(1)} r_{(2)}^2 \le N^{\lambda+o(1)}$, hence

$$a + 2b \le \lambda + o(1) = 3\rho + o(1). \tag{5.18}$$

Combining (5.17) and (5.18) gives

$$b \le \frac{3\rho - a}{2} + o(1) \le \rho + \frac{\delta}{2} + o(1). \tag{5.19}$$

After shrinking the dyadic grid by a further $o(1)$ allowance, the two exponents lie in

$$[\rho-\delta, \rho+\delta/2] \subset [\rho-\tau, \rho+\tau],$$

because $\delta < \tau$. Thus the pair of coordinates attaining $r_{(1)}$ and $r_{(2)}$ is covered by (5.12). There are only three pairs and $O((\log N)^2)$ dyadic boxes, so this contribution is

$$\ll_{\lambda,\varepsilon} N^{\rho-\vartheta+\varepsilon} \le N^{\rho-\delta+\varepsilon}$$

after replacing ε appropriately. Adding the small-$r_{(1)}$ contribution proves (5.13). □

5.3. **Proof of the main theorem and the quality corollary.**

Proof of Theorem 1.1. Theorem 5.1 verifies the hypothesis of Lemma 5.2 with

$$\vartheta = \tau = \frac{c_0 \rho^3}{10}.$$

Choose $\delta = \tau/2 = c_0\rho^3/20$. The dyadic conclusion is therefore

$$|\mathcal{B}_{\lambda}^{\mathrm{dy}}(M)| \ll_{\lambda,\varepsilon} M^{\rho - c_0\rho^3/20 + \varepsilon}, \tag{5.20}$$

uniformly in the scale M.

Apart from a bounded residual interval, the intervals $(M/2, M]$ with $M = N/2^j$ and $j \geq 0$ partition $[1, N]$. Applying (5.20) on each interval gives

$$|\mathcal{B}_\lambda(N)| \ll_{\lambda,\varepsilon} 1 + \sum_{j\geq 0} \left(\frac{N}{2^j}\right)^{\rho - c_0\rho^3/20+\varepsilon}.$$

The exponent $\gamma = \rho - c_0\rho^3/20$ is positive: indeed $c_0 \leq 1$ and $\rho \leq 1/3$, so $c_0\rho^2/20 < 1$. Applying the dyadic estimate first with $\varepsilon < \gamma/2$ makes the geometric series convergent with sum $\ll N^{\gamma+\varepsilon}$; enlarging ε afterwards yields the usual formulation for every $\varepsilon > 0$. Substituting $\rho = \lambda/3$ gives the exponent

$$\frac{\lambda}{3} - \frac{c_0}{540}\lambda^3 = \frac{\lambda}{3} - \frac{c_D}{2160}\lambda^3,$$

as claimed. □

Corollary 5.3 (High-quality values)**.** *For every $\eta > 0$ and $\varepsilon > 0$,*

$$\#\{n \leq N : q(n) > 1+\eta\} \ll_{\eta,\varepsilon} N^{\frac{1}{3(1+\eta)} - \frac{C}{(1+\eta)^3}+\varepsilon}, \tag{5.21}$$

where C is the constant of Theorem 1.1.

Proof. Since $R_n \geq 6$ for $n \geq 1$, the quality $q(n)$ in (1.8) is defined. The inequality $q(n) > 1+\eta$ is equivalent to $R_n < (2n+1)^\lambda$ with $\lambda = (1+\eta)^{-1}$. Theorem 1.1 applies, and $\lambda^3 = (1+\eta)^{-3}$. □

Remark 5.4. Ignoring the negative term $C\lambda^3$ recovers (1.4). The proof shows that the entire improvement comes from the middle dyadic range in Theorem 5.1; the two outer ranges are already covered by the elementary estimates.

6. Variable right-hand sides and the general abc obstruction

This section explains why Theorem 1.1 does not extend to arbitrary abc triples by a change of parameters. The distinction is structural rather than numerical. Nothing in this section is used in the proof of the balanced-family theorem.

6.1. **The variable-square identity.** Let $a + b = c$ be positive integers with $a \neq b$, and write

$$a = \alpha x^2, \qquad b = \beta y^2$$

using the odd-valuation squarefree kernels from (2.7). Then

$$c^2 - \alpha\beta(2xy)^2 = (a-b)^2. \tag{6.1}$$

Thus, with $D = \alpha\beta$, $t = c$, $u = 2xy$, and $k = |a-b|$,

$$t^2 - Du^2 = k^2. \tag{6.2}$$

The balanced coordinates $(n, n+1, 2n+1)$ force $k = 1$ in the corresponding identity, and (6.2) becomes an ordinary Pell equation. In general, however, k is an additional height variable. Factorization now gives

$$(t-k)(t+k) = Du^2, \qquad \gcd(t-k, t+k) \mid 2k. \tag{6.3}$$

When k is unbounded, this does not lead to Diao's bounded list $h \in \{\pm 1, \pm 2\}$ in the factorization (2.1) of [1]. Theorem 3.3 consequently cannot be quoted for (6.2) merely by replacing the right-hand side.

6.2. **Generalized Pell fibres can be linear.** The next proposition makes the loss of logarithmic fibres explicit. It also shows that an assertion of exact order $X \log^2 X$ is unnecessary for the obstruction.

Proposition 6.1 (Linear generalized fibres)**.** *Fix a positive nonsquare integer D, and put*

$$\mathcal{S}_D(X) = \#\{(t,u,k) \in \mathbb{Z}^3_{>0} : t, u, k \leq X,\ t^2 - Du^2 = k^2\}.$$

Then, for fixed D,

$$X \ll_D \mathcal{S}_D(X) \leq X^{1+o(1)}. \tag{6.4}$$

Proof. For the upper bound, note that $u \le X/\sqrt{D}$. For each fixed u, the factorization (6.3) determines (t,k) from a same-parity ordered factorization of Du^2. The divisor bound gives $X^{o(1)}$ such factorizations, uniformly for $Du^2 \le X^2$. Summing over u proves the upper bound.

For the lower bound, consider positive integers r, s in the ellipse

$$Dr^2 + s^2 \le X. \tag{6.5}$$

Define

$$t = Dr^2 + s^2, \qquad u = 2rs, \qquad k = |Dr^2 - s^2|.$$

Then $t^2 - Du^2 = k^2$ and $t \le X$. Moreover, $2rs \le X/\sqrt{D}$ by $Dr^2 + s^2 \ge 2\sqrt{D}\, rs$. The diagonal $Dr^2 = s^2$ is empty because D is nonsquare, and hence $k < t$. The lattice region (6.5) contains $\gg_D X$ points. It remains to check injectivity. The values t and k determine the unordered pair $\{Dr^2, s^2\}$. If two parameter pairs gave the same triple with these components interchanged, we would have $Dr^2 = (s')^2$, forcing $D = (s'/r)^2$ to be the square of a rational number, contrary to D being a nonsquare integer. Thus the components are not interchanged, and (r,s) is recovered uniquely. This proves the lower bound. □

The parametrization in the proof is also the abc identity in disguise:

$$(Dr^2) + s^2 = Dr^2 + s^2.$$

It exhibits the two free parameters present in the general problem. By contrast, the ordinary Pell equation has one orbit parameter, and its bounded-height fibre is logarithmic.

6.3. **Fixed right-hand squares do not directly invoke Diao's theorem.** One might next consider the fixed-κ coordinates

$$(n, n+\kappa, 2n+\kappa).$$

Writing $n = \alpha x^2$ and $n + \kappa = \beta y^2$ gives

$$(2n+\kappa)^2 - \alpha\beta(2xy)^2 = \kappa^2. \tag{6.6}$$

Nevertheless, a small solution of (6.6) need not imply a small fundamental solution ε_D. In particular, Theorem 3.3 cannot by itself produce a power-saving estimate for this fixed-κ family.

Remark 6.2 (An explicit fixed-κ obstruction)**.** Take $\kappa = 3$ and $n = 61$. Then

$$(n, n+\kappa, 2n+\kappa) = (61, 64, 125),$$

and (6.6) becomes

$$125^2 - 61 \cdot 16^2 = 3^2. \tag{6.7}$$

However, the fundamental solution of the ordinary Pell equation is

$$\varepsilon_{61} = 1766319049 + 226153980\sqrt{61}, \tag{6.8}$$

so $\varepsilon_{61} > 3.5 \times 10^9$. For a minimality certificate, the continued fraction is

$$\sqrt{61} = [7; \overline{1,4,3,1,2,2,1,3,4,1,14}],$$

with period length 11, and the convergent of index 21 (the initial term having index 0) is $1766319049/226153980$. The standard continued-fraction parametrization of Pell solutions therefore gives the fundamental solution above. Direct multiplication also verifies both (6.7) and $1766319049^2 - 61 \cdot 226153980^2 = 1$. Thus the small generalized solution in the fixed-κ family does not place $D = 61$ on the small-ε_D branch to which Diao's theorem applies. Also, $(125 + 16\sqrt{61})/3$ is not an algebraic integer, so it is not itself a Pell unit.

For the same reason, the fixed-κ assertion

$$\#\{a \le N : \mathrm{rad}(a(a+\kappa)(2a+\kappa)) < (2a+\kappa)^\lambda\} \ll_\kappa N^{\lambda/3 - C_\kappa \lambda^3 + \varepsilon}$$

is *not* a corollary of the argument in this paper. It remains an open construction with the methods currently available here. Nor do we assume that all solutions of (6.6) form one Pell coset: in general, orbit multiplicities must be proved separately.

6.4. **Revised status of G-B86d6-GenPell.** The preceding counterexample also forces a correction to the proposed reduction. Adjoining the condition $\varepsilon_D \le D^\theta$ to (6.2) does not capture the general family: Diao already counts the discriminants satisfying that condition, while Remark 6.2 shows that the generalized solutions of interest need not satisfy it.

A useful G-B86d6-GenPell statement must instead estimate generalized solutions themselves, with the fibre cardinality included. For example, the fibre-weighted object is

$$\mathcal{G}(W,X;\kappa)=\#\left\{(D,t,u):\ \begin{array}{l} W\le D<2W,\ D \text{ nonsquare},\ t,u\le X,\\ t^2-Du^2=\kappa^2\end{array}\right\}. \tag{6.9}$$

A power saving must be proved for (6.9), uniformly in the required κ-ranges and after comparison with the elementary dyadic covers; a discriminant-only saving multiplied by an $X^{o(1)}$ fibre is incompatible with Proposition 6.1. Moreover, the exponent ledger for the full two-dimensional abc problem must be rebuilt rather than copied from the one-dimensional balanced family.

Status. G-B86d6-GenPell is an open construction; no ready-made paper supplying the required uniform bound has been located. The balanced-family result G-B86d5-Diag remains proved modulo the cited Diao preprint theorem, but no fixed-κ or full-abc extension is asserted.

6.5. **The external general-abc benchmark.** For orientation, let $N_\lambda(X)$ denote the number of coprime positive triples $a+b=c$ in $[1,X]^3$ with $\mathrm{rad}(abc)<c^\lambda$. Bernert–Browning–Lichtman–Teräväinen prove

$$N_\lambda(X)\ll_\varepsilon X^{2\lambda/3+\varepsilon},\qquad N_\lambda(X)\ll_{\lambda,\varepsilon} X^{(23\lambda+3)/40+\varepsilon}, \tag{6.10}$$

and, for $0<\lambda<1$,

$$N_\lambda(X)\ll_{\lambda,\varepsilon} X^{3/5+\varepsilon}; \tag{6.11}$$

see [4, Proposition 1.1 and Theorems 1.2–1.3]. Consequently, a hypothetical general bound

$$N_\lambda(X)\ll X^{2\lambda/3-\delta+\varepsilon}$$

improves the minimum of these three benchmarks precisely under the following thresholds:

$$\delta>\begin{cases}0, & 0<\lambda\le 9/11,\\ \dfrac{11\lambda-9}{120}, & 9/11<\lambda\le 21/23,\\ \dfrac{2\lambda}{3}-\dfrac{3}{5}, & 21/23<\lambda<1.\end{cases} \tag{6.12}$$

The second and third expressions agree at $\lambda=21/23$, and the last threshold at $\lambda=1$ is $1/15$. These are comparison thresholds for the full two-dimensional problem only; they play no role in Theorem 1.1.

7. Chinese-remainder orbits and concluding remarks

7.1. **CRT normalization.** Let (a_0,b_0,c_0) be a pairwise coprime source triple with a_0 even and b_0,c_0 odd. Put

$$M=a_0b_0c_0,\qquad D_0=\mathrm{rad}(M),\qquad S=\mathrm{Supp}(M). \tag{7.1}$$

The subscripts distinguish the source data from the squarefree kernels a,b,c used earlier. Consider the simultaneous congruences

$$n\equiv 0 \pmod{a_0},\qquad n\equiv -1 \pmod{b_0},\qquad n\equiv -2^{-1} \pmod{c_0}. \tag{7.2}$$

They are pairwise compatible because a_0,b_0,c_0 are pairwise coprime and c_0 is odd. Let $0<n_0<M$ be the least positive solution and define

$$n_t=n_0+Mt,\qquad t\ge 0. \tag{7.3}$$

Then there are integers u_t,v_t,w_t such that

$$n_t=a_0u_t,\qquad n_t+1=b_0v_t,\qquad 2n_t+1=c_0w_t. \tag{7.4}$$

The radical separates into a fixed source part and a varying part. Indeed, the three coordinates are pairwise coprime, and each is coprime to the two source moduli attached to the other coordinates. Consequently,

$$R_{n_t} = D_0 \operatorname{rad}_{S^c}(u_t v_t w_t), \tag{7.5}$$

where rad_{S^c} retains exactly the prime divisors outside S. This identity is exact; primes of u_t, v_t, w_t lying in S are already accounted for by D_0.

Define the orbit quality

$$q_s(t) = \frac{\log(2n_t + 1)}{\log R_{n_t}}. \tag{7.6}$$

The following corollary is uniform in the source triple.

Corollary 7.1 (CRT orbit count)**.** *Let $\eta > 0$, and put*

$$X = 2M(T + 1). \tag{7.7}$$

Then

$$\boxed{\#\{0 \le t \le T : q_s(t) > 1 + \eta\} \ll_{\eta,\varepsilon} X^{\frac{1}{3(1+\eta)} - \frac{C}{(1+\eta)^3} + \varepsilon}.} \tag{7.8}$$

The implied constant is independent of (a_0, b_0, c_0) and T.

Proof. Since $0 < n_0 < M$, for $0 \le t \le T$ we have

$$n_t < M + M(T) = M(T + 1) = X/2.$$

Moreover, $q_s(t) = q(n_t)$, and distinct t give distinct n_t. Applying Corollary 5.3 with $N = X$ proves (7.8). Theorem 1.1 has no dependence on a source triple, which yields the asserted uniformity. □

Remark 7.2. The proof of Corollary 7.1 uses only the membership $n_t \le X$ and the exact definition of q_s. It does not require a pointwise radical lower bound for the varying cubic polynomial $u_t v_t w_t$, nor any uniform Szpiro estimate.

7.2. **Conclusion and limitations.** We have obtained a strict power-saving improvement over the elementary exponent for the balanced radical family $\mathcal{B}_\lambda(N)$. The proof isolates a precise mechanism: squarefree kernels convert unit differences into Pell solutions; fixed-support radical boxes provide the two outer dyadic covers; and Diao's small-fundamental-solution bound breaks the balance in the middle range.

Three limitations are essential. First, the result is a cardinality estimate, not a pointwise lower bound for R_n. Second, it concerns the special coordinates $(n, n + 1, 2n + 1)$ and does not assert a comparable theorem for arbitrary coprime $a + b = c$ triples. Section 6 makes this distinction quantitative: the right-hand side becomes a variable square, and generalized Pell fibres can have linear size. Third, the argument does not control the square part of an individual coordinate beyond the averaged boxes used here. Accordingly, Theorem 1.1 does not prove the abc conjecture, Szpiro's conjecture, or a uniform radical conjecture for polynomial values.

Within the research program that motivated this note, G-B86d5-Diag is closed for the balanced family relative to Diao's cited preprint theorem. G-B86d6-GenPell remains an open construction in the revised form explained in Section 6; no fixed-κ extension is claimed. G-B86d4-Select and the corresponding uniform degeneration step also remain unresolved. A separate pointwise selection or uniform degeneration mechanism would still be required before any abc–Szpiro consequence could be contemplated.

Appendix A. Friable estimates and exponent bookkeeping

A.1. **The uniform friable input.** The following elementary Rankin argument makes the fixed-support estimate self-contained. For Rankin's method in analytic number theory, see Tenenbaum [5].

Lemma A.1. *Uniformly for* $1 \le X \le N$,

$$\Psi(X, 2\log N) = N^{o(1)}. \tag{A.1}$$

Proof. For N bounded there is nothing to prove. Put $Y = 2\log N$ and choose

$$\delta = (\log\log N)^{-2}. \tag{A.2}$$

For every $0 < \delta < 1$, Rankin's device gives

$$\begin{aligned}\Psi(X, Y) &\le X^{\delta} \sum_{m : p \mid m \Rightarrow p \le Y} m^{-\delta} \\ &= X^{\delta} \prod_{p \le Y} (1 - p^{-\delta})^{-1}. \end{aligned} \tag{A.3}$$

Since $X \le N$, the first factor satisfies

$$\delta \log X \le \frac{\log N}{(\log\log N)^2} = o(\log N). \tag{A.4}$$

For every $p \le Y$,

$$0 < \delta \log p \le \delta \log(2\log N) = O\big((\log\log N)^{-1}\big).$$

Consequently $1 - p^{-\delta} \gg \delta \log p$ for large N, and hence

$$-\log(1 - p^{-\delta}) \ll \log \frac{1}{\delta \log p} + 1 \ll \log \frac{1}{\delta} + 1 \ll \log\log\log N$$

because $\log p \ge \log 2$. Chebyshev's elementary upper bound $\pi(Y) \ll Y/\log Y$ therefore gives

$$\begin{aligned}\log \prod_{p \le Y} (1 - p^{-\delta})^{-1} &\ll \pi(Y) \log\log\log N \\ &\ll \frac{\log N}{\log\log N} \log\log\log N = o(\log N). \end{aligned} \tag{A.5}$$

Combining (A.4) and (A.5) proves that the logarithm of the Rankin upper bound is $o(\log N)$. □

A.2. **Linear progressions for fixed square parts.** We give the full detail behind the $O(1 + W/N)$ coefficient multiplicity in Lemma 4.1. The three generalized unit equations have the following homogeneous solution steps:

Pair	Fixed square parts	Homogeneous step
$(1,2)$	(x, y)	$(\Delta a, \Delta b) = (y^2, x^2)$
$(1,3)$	(x, z)	$(\Delta a, \Delta c) = (z^2, 2x^2)$
$(2,3)$	(y, z)	$(\Delta b, \Delta c) = (z^2, 2y^2)$

For $(1,2)$, if (a_0, b_0) and (a_1, b_1) solve $by^2 - ax^2 = 1$, subtraction gives

$$(b_1 - b_0)y^2 = (a_1 - a_0)x^2.$$

Since $\gcd(x^2, y^2) = 1$, there is $k \in \mathbb{Z}$ with $\Delta a = y^2 k$ and $\Delta b = x^2 k$. For $(1,3)$, subtraction gives $(\Delta c)z^2 = 2(\Delta a)x^2$; since $\gcd(z^2, 2x^2) = 1$, the stated step follows. The third case is identical. Thus the coefficients in each equation form a single arithmetic progression once the square parts are fixed.

Now let U_i, U_j be the dyadic sizes of the two kernels. The definition (4.1) restricts to $N/2 < n \le N$, so each of n, $n+1$, and $2n+1$ is comparable to N. Since $u_i \in [U_i, 2U_i)$ and $u_j \in [U_j, 2U_j)$, in every case we genuinely have the two-sided height relations

$$x_i^2 \asymp \frac{N}{U_i}, \qquad x_j^2 \asymp \frac{N}{U_j}.$$

This dyadic restriction is essential: the upper bound $n \le N$ alone would give only one-sided estimates for these square parts. The step in either coefficient is therefore comparable to N/U_j

or N/U_i, up to the absolute factor 2. Since the coefficient interval has length U_i or U_j, the number of progression points is

$$\ll 1 + \frac{U_i U_j}{N} \ll 1 + \frac{W}{N}.$$

This completes the justification of (4.4).

A.3. **Explicit exponent ledger.** For convenient checking, Table 2 records the three exponents in the proof of Theorem 5.1 when $A = N^\alpha$, $B = N^\beta$, and $W = N^s$.

TABLE 2. Exponent ledger for the diagonal proof.

Branch	Input	Exponent of N	Range used
Coefficient	Lemma 4.1	s	$s \le \rho - \tau$
Square parts	Lemma 4.1	$\alpha + \beta - s$	$s \ge \rho + 4\tau$
Small Pell	Lemma 3.5	$s - c_0 s^3$	$\rho - \tau < s < \rho + 4\tau$

In the middle range, $s < \rho + 4\tau < 2/3$, so $\theta = 2/s > 3$. The parameter θ is bounded above because $s > \rho/2$; hence Diao's growing upper restriction on θ is automatic for large N. With $\tau = c_0\rho^3/10$, the middle exponent is at most $\rho - 6\tau$. The outer exponents are at most $\rho - \tau$ and $\rho - 2\tau$, respectively. This verifies the uniform diagonal saving without leaving an uncovered dyadic interval.

Finally, the amplification lemma is applied with $\delta = \tau/2$. Since

$$\tau = \frac{c_0\rho^3}{10}, \qquad c_0 = \frac{c_D}{4}, \qquad \rho = \frac{\lambda}{3},$$

the final negative term is

$$\delta = \frac{c_0\rho^3}{20} = \frac{c_D}{2160}\lambda^3.$$

This is the source of the explicit constant in Theorem 1.1.

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