A STRENGTHENING OF SOME RESULTS OF THE PAPER "STRUCTURED SUNFLOWERS AND CANONICAL RAMSEY PROPERTIES"

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Abstract. In this short note, we give an exposition of an argument (potentially LLM-generated by Budiyanto) which strengthens two theorems of the author and Winkel from the recent paper "Structured sunflowers and canonical Ramsey properties", also proving the main conjecture of that paper.

We assume familiarity with all definitions, notation and references from [SW26]. In this short note, we give results which improve upon the main Theorems A and B of [SW26]. We broaden the scope of Theorem A to all infinite first-order structures, and improve Theorem B by giving a proof of Conjecture 4.1(∗) from [SW26], which is the main conjecture of that paper.

Acknowledgements and attribution

The new argument in this note (the proof of the left-to-right direction of Theorem B$^+$(i) and the proof of (iii) ⇒ (i) in Theorem A$^+$, which are essentially the same argument) has a somewhat unusual provenance. I believe it was first LLM-generated by Agus Budiyanto (who, as far as I can tell from an online search, has not previously worked in this research area). Budiyanto submitted a paper with the argument to ZML, and the journal editor Stefan Geschke invited me to referee it. It seemed to me that the paper was direct, unedited output from an LLM, and as no LLM use had been disclosed to the journal, I recommended to Geschke to reject the paper, after discussion. (Geschke's LLM-detection tool also suggested the paper was LLM-generated, though these tools are presently not totally reliable.) I asked Geschke to suggest to Budiyanto to put a preprint on the arxiv first, but we were unsuccessful in contacting him despite some attempts. I do not know how to correctly attribute the argument in this note, and would like to ensure appropriate attribution if this were possible: the criteria for attribution and authorship in this new LLM-assisted era are not presently well-established. I only claim originality in exposition, and the mathematical ideas in this new proof are not my own. I do not intend to publish this note in any journal, and have uploaded it after first discussing and confirming the best course of action with Stefan Geschke.

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1. Previous results

Recall the following from [SW26]:

Fact 1. [SW26, Theorem A] *Let M be a relational Fraïssé structure with strong amalgamation. Then the following are equivalent:*

(i) M has the infinite sunflower property;
(ii) M has the infinite 2-sunflower property;
(iii) M has the canonical infinite point-Ramsey property.

Fact 2 ([SW26, Theorem B])**.** *Let $\mathcal{K}$ be a Fraïssé class. Then the following hold:*

(i) if $\mathcal{K}$ has the finite 2-sunflower property, then it has the canonical point-Ramsey property;
(ii) if $\mathcal{K}$ has the very *canonical point-Ramsey property, then it has the finite sunflower property.*

Conjecture 3 ([SW26, Conjecture 4.1])**.** *Let $\mathcal{K}$ be a Fraïssé class. We conjecture:*

$$(*) \quad \mathcal{K} \text{ has the canonical point-Ramsey property} \iff \mathcal{K} \text{ has the finite sunflower property.}$$

We specifically conjecture that $\mathcal{K}$ has the canonical point-Ramsey property if and only if $\mathcal{K}$ has the very canonical point-Ramsey property, which would imply (∗) by Fact 2. We also conjecture that $\mathcal{K}$ has the finite 2-sunflower property if and only if $\mathcal{K}$ has the finite sunflower property: note that Lemma 5 together with (∗) would imply this.

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2. New results

In this note, we show the following:

Theorem A$^+$. *Let M be an infinite first-order structure. Then the following are equivalent:*

(i) M has the infinite sunflower property;
(ii) M has the infinite 2-sunflower property;
(iii) M has the canonical infinite point-Ramsey property.

Theorem B$^+$. *Let $\mathcal{K}$ be a hereditary class of finite structures. Then:*

(i) $\mathcal{K}$ has the canonical point-Ramsey property $\Leftrightarrow$ $\mathcal{K}$ has the finite sunflower property;
(ii) $\mathcal{K}$ has the finite 2-sunflower property $\Leftrightarrow$ $\mathcal{K}$ has the finite sunflower property.

In [SW26, Definition 1.5], the canonical infinite point-Ramsey property was only defined for Fraïssé structures M (that is, countably infinite ultrahomogeneous structures). When we refer in Theorem A$^+$ to the canonical infinite point-Ramsey property for infinite first-order structures M in general, we mean the following:

Definition 4. Let M be an infinite first-order structure. We say that M has the *canonical infinite point-Ramsey property* if, for each colouring $\chi : M \to |M|$, there is a monochromatic or heterochromatic copy of M.

This generalises [SW26, Definition 1.5], where the set of colours was always ω.

Note that Theorem B$^+$ proves $(*)$ from Conjecture 3. We do not prove all statements in Conjecture 3: specifically, we do not show that the canonical point-Ramsey property is equivalent to the very canonical point-Ramsey property. However, the very canonical point-Ramsey property was only introduced by Winkel and the author in the first place because we were unable to prove Theorem B$^+$; we defined the very canonical point-Ramsey property as an ad-hoc strengthening of the canonical Ramsey property which held for a variety of examples and enabled us to prove Fact 2.

Also note that given Theorem B$^+$, Proposition 3.3 from [SW26] is not necessary – one can simply use the original result of Nešetřil-Rödl, [NR78, Theorem 2.1]. One can also directly apply various prior canonical Ramsey results from the literature to give more examples of classes with the finite sunflower property: see the references in the introduction of [SW26].

We only give the details of the proof of Theorem B$^+$ in this note. The proof of Theorem A$^+$ follows by an entirely analogous argument: one just works with the infinite structure M itself rather than taking finite witnesses of (canonical) point-Ramsey properties for finite structures.

3. Theorem B$^+$

The right-to-left direction of Theorem B$^+$(i) was proved in [SW26] as Fact 2(i) (see also [SW26, Lemma 2.16]); note that the proof given in the paper only uses that $\mathcal{K}$ is a hereditary class. We repeat the proof here in the below Lemma 5 for the convenience of the reader:

Lemma 5. *Let $\mathcal{K}$ be a hereditary class of finite structures with the finite 2-sunflower property. Then $\mathcal{K}$ has the canonical point-Ramsey property.*

Proof. Let $B \in \mathcal{K}$. Let $C \in \mathcal{K}$ be a witness for B of the finite 2-sunflower property. Let $\chi : C \to \omega$ be a colouring. Note that we may assume for convenience that the domain of C does not intersect the image of χ. Define a structure C' with domain $\{\{v, \chi(v)\} \mid v \in C\}$, with the first-order structure given by specifying that the map $f : C \to C'$, $f(v) = \{v, \chi(v)\}$ is an isomorphism. As C witnesses the finite 2-sunflower property for B, there is a structured sunflower $S \subseteq C'$ with $S \cong B$. If the core of the sunflower S is non-empty, then it consists of an integer r: thus $f^{-1}(S) \subseteq C$ is a copy of B of colour r. If the core of S is empty, then $f^{-1}(S)$ is a heterochromatic copy of B. □

We now prove Theorem B$^+$.

Proof of Theorem B$^+$. (i): $\Leftarrow$: See Lemma 5. $\Rightarrow$: Let $k \geq 2$. We show that $\mathcal{K}$ has the finite k-sunflower property. Let $B \in \mathcal{K}$. If $|B| \leq 2$, then B itself witnesses the finite k-sunflower property for B. Suppose $|B| \geq 3$. As $\mathcal{K}$ has the canonical point-Ramsey property, there is $W_2 \in \mathcal{K}$ which witnesses the canonical point-Ramsey property for B for colourings with 2 colours. Given a 2-colouring of W_2, as $|B| \geq 3$, the structure W_2 does not contain a heterochromatic copy of B, so must contain a monochromatic copy of B, and therefore W_2 is a witness of the point-Ramsey property for B for colourings with 2 colours. By induction, for all $r \geq 2$, there are $W_3, \cdots, W_r \in \mathcal{K}$ such that, for each $3 \leq i \leq r$, the structure W_i is a point-Ramsey witness for W_{i-1} for colourings with 2 colours; thus W_r witnesses the point-Ramsey property for B for colourings with r colours

(this is a standard classical argument: informally, given $\chi : W_r \to r$, one defines a colouring in the two colours "0 or 1 or $\cdots$ or $r-2$" and $r-1$, and one is then done by induction).

Take $C_0 \in \mathcal{K}$ witnessing the point-Ramsey property for B for colourings with $k(k-1)+1$ colours. Inductively define $C_1, \cdots, C_k$ as follows: for $i < k$, with $C_0, \cdots, C_i$ already defined, take $C_{i+1} \in \mathcal{K}$ to be a witness of the canonical point-Ramsey property for C_i. Let $C = C_k$. We now show that C is a witness of the finite k-sunflower property for B. Let C' be a structure on k-sets with $C' \cong C$. For each $v \in C'$, let $\varepsilon_0(v), \cdots, \varepsilon_{k-1}(v)$ be an enumeration of the elements of v; this defines, for each $i < k$, a function $\varepsilon_i : C' \to \bigcup C'$. By the construction of C, we have that C' contains a copy D of C_0 on which each of $\varepsilon_0|_D, \cdots, \varepsilon_{k-1}|_D$ is constant or injective. Let $I = \{i < k \mid \varepsilon_i|_D \text{ is constant}\}$, and let $J = \{0, \cdots, k-1\} \setminus I$. Let $n = |J|$. Let $Q = \{\varepsilon_i(v) \mid i \in I, v \in D\}$, and for each $v \in D$ let $\rho(v) = v \setminus Q$. As each $v \in D$ is a k-set, we have $|\rho(v)| = n$ for all $v \in D$.

For each $v \in D$ and $t \in \rho(v)$, as $\varepsilon_i|_D$ is injective for all $i \in J$, we have $|\{w \in D \setminus \{v\} \mid t \in w\}| \leq n-1$. Thus for each $v \in D$ we have:

$$|\{w \in D \setminus \{v\} \mid \rho(v) \cap \rho(w) \neq \varnothing\}| \leq \sum_{t \in \rho(v)} |\{w \in D \setminus \{v\} \mid t \in w\}| \leq n(n-1),$$

and hence D admits a colouring χ in $n(n-1)+1$ colours such that, for all distinct $v, w \in D$ with $\rho(v) \cap \rho(w) \neq \varnothing$, we have $\chi(v) \neq \chi(w)$. Each colour set is a sunflower with core Q, and as $C_0 \cong D$ witnesses the point-Ramsey property for B with $k(k-1)+1$ colours, we have a χ-monochromatic copy of B in D, giving a sunflower isomorphic to B. So C witnesses the finite k-sunflower property for B as required.

(ii): This follows immediately from Lemma 5 and Theorem B$^+$(i). □

References

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