Old and new Drinfeld cusp forms of level t in characteristic 2: the weights $k = 2^m + 1$

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Abstract

Let $q = 2$, $A = \mathbb{F}_2[t]$, and let U_t be the Atkin operator on the space $S_k(\Gamma_0(t))$ of Drinfeld cusp forms of level t. For every weight $k = 2^m + 1$ we prove that $S_k(\Gamma_0(t))$ is the direct sum of its oldforms and its newforms, that $k/2$ is the largest slope of U_t, and that no oldform has slope $k/2$. This is Conjecture 1.2(3) of Bandini and Valentino, and their conjectured old/new decomposition, for this infinite family of weights in characteristic 2, where the known methods do not apply. We also obtain the characteristic polynomial of U_t in closed form in terms of that of the Hecke operator T_t in level one, and the bound: every slope of T_t on $S_k(\mathrm{GL}_2(A))$ is at most $J_m = (2^m - (-1)^m)/3$. The proof rests on a combinatorial theorem: the leading principal minors of the Bandini–Valentino matrix are nonzero exactly at the Jacobsthal numbers; equivalently, the number of perfect matchings of the graph of nonempty subsets of $\{1, \dots, m\}$ joined when disjoint or nested is odd exactly when the number of vertices plus one is a Jacobsthal number.

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Status. Statements labelled PROVED carry a complete proof in this text. Statements labelled VERIFIED have been checked by computer only, over the range indicated; statements labelled OPEN are neither. Every computational assertion is reproduced by the ancillary script `anc/verif_all.py` (pure Python with numpy, 52 checks, about one minute). The text has not been read by any person other than the author. Prepared with Claude (Anthropic), which carried out the computations and drafted the proofs under the author's direction. The author is responsible for the content.

1 Introduction

1.1 Setting

Let q be a power of a prime p and $A = \mathbb{F}_q[t]$. Bandini and Valentino [2, 3] gave an explicit formula for the Atkin operator U_t on Drinfeld cusp forms of level $\Gamma_0(t)$, and in [4] they extracted from it a matrix M over $\mathbb{F}_q$ such that, on the block of dimension n,

$$U_t = MD, \qquad D = \mathrm{diag}(t^{s_1}, \dots, t^{s_n}), \qquad s_i = j+1+(i-1)(q-1), \quad k = 2j+2+(n-1)(q-1)$$

[4, (11), (12)]. For $q = 2$ one has $j = 0$, hence $s_i = i$ and $k = n + 1$: the whole space $S_k(\Gamma_0(t))$ has dimension $n = k - 1$. Throughout,

$$q = 2, \qquad n = 2^m, \qquad k = 2^m + 1 \qquad (m \geq 1).$$

We write $J_r = (2^r - (-1)^r)/3$ for the Jacobsthal numbers $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \dots$; they satisfy

$$J_r + J_{r+1} = 2^r, \qquad J_{r+1} = J_r + 2J_{r-1}. \tag{1}$$

1.2 Known results

(K1) Every newform of level t has slope $k/2$ [4, Section 2.6.2]: the Fricke involution gives $U_t^2 = t^k$ on the space of newforms.

(K2) Bandini and Valentino conjectured that $S_k(\Gamma_0(t))$ is the direct sum of oldforms and newforms [4, Conj. 2.6(3)], and that $k/2$ is the maximal slope, with no oldform of slope $k/2$ [6, Conj. 1.2(3)]. These are equivalent [13, Thm. 3.15]; and [4, Thm. 5.1, Rem. 5.3] reduces them to the absence of an eigenvalue $t^{k/2}$ (up to sign) for the Hecke operator T_t in level one. Their own slope bound [4, Thm. 6.4] is of order n^2.

(K3) The decomposition is known when $\dim S_k(\mathrm{GL}_2(A)) = 1$ [5], when it is at most 2 [8], when every Hecke eigenvalue in level one has multiplicity less than p [9], and, for q *odd*, when $\dim S_k(\Gamma_0(t))$ is even [7]. For $q = 2$ and $k = 2^m + 1$ with $m \geq 3$ none of these applies: $\dim S_k(\mathrm{GL}_2(A)) = J_m \geq 3$, the characteristic is even, and the level-one eigenvalues are multiple roots (Remark 5.5). In characteristic 2 the trace method of [9] determines the eigenvalues of odd multiplicity and gives no information on those of even multiplicity. The recent strong Ramanujan bound of de Vries [10] is a bound on traces; as stated there, it does not imply the corresponding bound for repeated Hecke eigenvalues in characteristic 2.

(K4) For $q = 2$ the slopes have been computed and tabulated: [4, Ex. 6.10] for $k = 9$, 17, 33, 65, 129 and [11, Section 4] for $k \leq 23$. The elementary lower bound (8) below is [11, Lem. 2.3(3)].

1.3 Results

Theorem A (PROVED)**.** *Let S be the symmetric matrix over $\mathbb{F}_2$, indexed by the integers $p, q \geq 0$ written in binary, with*

$$S_{pq} = [p \,\&\, q = 0] + [p \subseteq q] + [q \subseteq p] \pmod 2.$$

Then $\det S[0{:}i]$ *is nonzero if and only if i is a Jacobsthal number, and* $\operatorname{corank} S[0{:}i]$ *is the distance from i to the set $\{J_r\}$. Equivalently: let L_N be the graph whose vertices are the integers $1, \ldots, N$, viewed as nonempty finite sets of positive integers through their binary digits, two of them being adjacent when the sets are disjoint or nested. Then L_N has an* odd *number of perfect matchings if and only if $N + 1$ is a Jacobsthal number.*

Theorem C (PROVED)**.** *For $k = 2^m + 1$, every slope of the Hecke operator T_t on $S_k(\mathrm{GL}_2(A))$ is at most J_m. More precisely* $\dim S_k(\mathrm{GL}_2(A)) = J_m$*, and* $v_t(\det T_t) = J_m(J_m + 1)/2$.

Main Theorem (PROVED)**.** *For $q = 2$ and $k = 2^m + 1$:*

(i) $S_k(\Gamma_0(t)) = S_k^{\mathrm{old}}(\Gamma_0(t)) \oplus S_k^{\mathrm{new}}(\Gamma_0(t))$*, with* $\dim S^{\mathrm{old}} = 2J_m$ *and* $\dim S^{\mathrm{new}} = 2J_{m-1}$*;*

(ii) the characteristic polynomial of U_t is $X^{J_m} \cdot P_m(X) \cdot (X^2 + t^k)^{J_{m-1}}$, where P_m is the characteristic polynomial of T_t on $S_k(\mathrm{GL}_2(A))$;

(iii) $k/2$ is the largest slope of U_t, it occurs with multiplicity exactly $2J_{m-1}$, carried by the newforms, and every nonzero slope of an oldform is at most $J_m < k/2$.

In particular Conjecture 1.2(3) of [6], and the decomposition conjecture [4, Conj. 2.6(3)], hold for these weights.

We recall that the decomposition fails in general at levels which are not powers of a prime [8], [6, Section 4]; the level t considered here is prime.

1.4 What is new

To our knowledge, after a literature search in November 2026, Theorem A, Theorem C and the Main Theorem are new. We claim nothing on the numerical pattern of the slopes, which is in [4, Ex. 6.10] and [11]; on the slope of newforms, which is (K1); or on the lower bound (8), which is Hattori's. Remark 5.4 records, as VERIFIED only, the finer statement that the slopes of T_t in weight 2^m+1 are exactly the slopes of U_t in weight $2^{m-1}+1$, in line with [4, Conj. 6.9].

2 The matrix and its group-algebra structure

2.1 The matrix

Specialising [4, (11)] at $q=2$ (so $j=0$) and reducing modulo 2, the matrix M has entries, for $1 \le a, b \le n$,

$$m_{a,b} = \binom{n-a}{n-b} + \binom{n-a}{b-1} \ (a \neq b), \qquad m_{a,a} = \binom{n-a}{a-1}.$$

Index rows and columns by $p = a-1$ and $q = b-1$ in $[0, 2^m)$, written in binary, and put

$$U_{pq} = [p \subseteq q], \qquad V = U^{\mathsf{T}}, \qquad T_{pq} = [p \,\&\, q = 0], \qquad \Pi_{pq} = [q = 2^m - 1 - p],$$

where $2^m - 1 - p$ is the complement of p in m bits, and I is the identity.

Proposition 2.1 (PROVED)**.** *$M = I + T + U$, and $D = \mathrm{diag}(t^{q+1})$.*

Proof. By Lucas' theorem $\binom{x}{y}$ is odd iff $y \subseteq x$. With $x = n-a = \bar{p}$ (the complement of p in m bits, since $n-1-p = \bar{p}$) the condition $n-b \subseteq \bar{p}$ reads $\bar{q} \subseteq \bar{p}$, i.e. $p \subseteq q$; and $b-1 \subseteq \bar{p}$ reads $p \,\&\, q = 0$. The diagonal is $\binom{\bar{p}}{p} = [p \,\&\, p = 0] = [p = 0]$, which is the diagonal of $I+T+U$. The off-diagonal entries are those of $T+U$. □

[Checked entry by entry against [4, (11)] for $m \le 6$.]

2.2 Tensor structure

Put $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $t_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_2)$. Since the conditions defining U, V, T, Π are bitwise,

$$U = u^{\otimes m}, \quad V = v^{\otimes m}, \quad \Pi = \pi^{\otimes m}, \quad T = t_0^{\otimes m},$$

and the relations $u^2 = v^2 = \pi^2 = 1$, $t_0 = u\pi = vu$, $\pi u \pi = v$, $t_0^3 = 1$, checked on 2×2 matrices, give

$$U^2 = V^2 = \Pi^2 = I, \quad T = U\Pi = VU, \quad \Pi U \Pi = V, \quad T^3 = I, \quad \Pi U = T^{-1} = T^2, \quad \Pi = UVU. \tag{2}$$

Thus $G = \langle U, \Pi \rangle$ is a quotient of $\mathrm{GL}_2(\mathbb{F}_2) = S_3$, acting on $\mathbb{F}_2^{2^m}$ as the m-th tensor power of the natural module $N = \mathbb{F}_2^2$. The matrix M is the image of $z = 1 + t_0 + u$ in $\mathbb{F}_2[S_3]$.

2.3 Identities (PROVED)

In $\mathbb{F}_2[S_3]$ one checks directly $z^3 = z$ and $z^2 = \pi z$. Hence

$$M^3 = M, \qquad M^2 = \Pi M. \tag{3}$$

Moreover $S := T + U + V$ is symmetric (T is, and $V = U^{\mathsf{T}}$), and

$$M = VS \tag{4}$$

since $V(T+U+V) = U+T+I$ by (2). V is lower unitriangular in the numerical order, because $q \subseteq p$ implies $q \le p$. Finally the row and the column of index 0 of S are e_0 ($S_{0q} = 1+[q=0]+1$), and $S_{pp} = 0$ for $p \ge 1$. Hence, for every $i \ge 1$,

$$S[0{:}i] = (1) \oplus S[1{:}i], \qquad \text{with } S[1{:}i] \text{ symmetric with zero diagonal,}$$

i.e. alternating; so $\operatorname{rank} S[0{:}i]$ is *odd*, and since adding one row and one column changes the rank by at most 2,

$$\operatorname{corank} S[0{:}i] \text{ changes by exactly } +1 \text{ or } -1 \text{ at each step } i \to i+1 \text{ with } i \ge 1. \tag{5}$$

3 Theorem A

Write $S_m, U_m, \dots$ for the matrices of size 2^m, $W_i = \operatorname{span}(e_0, \dots, e_{i-1})$, and note that S_m is the leading block of S_{m+1}, so that $\det S[0{:}i]$ does not depend on m once $2^m \ge i$. Splitting the indices of size 2^{m+1} by the leading bit, the bitwise definitions give

$$S_{m+1} = \begin{pmatrix} S & T+U \\ T+V & U+V \end{pmatrix}, \qquad V_{m+1} = \begin{pmatrix} V & 0 \\ V & V \end{pmatrix}, \qquad \Pi_{m+1} = \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix}, \tag{6}$$

with S, T, U, V, Π at level m.

Lemma 3.1 (PROVED)**.** *Let $m \ge 1$ and $X := \Pi(I+U)(I+\Pi) = I+V+T^2+\Pi$. Then (y,z) lies in $\ker S_{m+1}$ if and only if*

$$y = (\Pi + T^2)z \qquad \text{and} \qquad Xz = 0.$$

Consequently $\operatorname{rank} S_{m+1} = 2^m + \operatorname{rank} X$.

Proof. The equality of the two expressions of X follows from (2). The equations are $Sy + (T+U)z = 0$ and $(T+V)y + (U+V)z = 0$. Multiplying the second by V and using $VT = U$, $VU = T$ gives $(U+I)y + (T+I)z = 0$. Adding $(U+T)$ times this to the first equation, and using $(U+T)(U+I) = I+U+V+T$ and $(U+T)(T+I) = \Pi+T^2+T+U$, one gets $y = (\Pi+T^2)z$. Substituting, $(U+I)(\Pi+T^2) + (T+I) = T+V+\Pi+T^2+T+I = X$. These operations are invertible, so $z \mapsto (y,z)$ is a bijection $\ker X \to \ker S_{m+1}$, and $\operatorname{corank} S_{m+1} = \operatorname{corank} X$. □

Proposition 3.2 (dyadic case, PROVED)**.** $\operatorname{rank} S_m = J_{m+1}$ *and* $\operatorname{corank} S_m = J_m$ *for all* $m \ge 0$.

Proof. For $m = 0, 1$ this is direct ($S_1 = \operatorname{diag}(1,0)$). Let $m \ge 1$. Since Π is a fixed-point-free involution, $I+\Pi$ has image $\operatorname{Fix}(\Pi)$, of dimension 2^{m-1}, hence

$$\operatorname{rank} X = \operatorname{rank}(I+U)(I+\Pi) = 2^{m-1} - \dim\big(\operatorname{Fix}(\Pi) \cap \operatorname{Fix}(U)\big) = 2^{m-1} - \dim \operatorname{Fix}(G).$$

Now $\mathbb{F}_2^{2^m} = N^{\otimes m}$ as a G-module. The restriction of N to $\langle u \rangle$ is the regular module, so $N^{\otimes m}$ is free over the Sylow 2-subgroup $\langle U \rangle$, hence projective over $\mathbb{F}_2[S_3]$ (Higman's criterion [12]). The projective indecomposables are N (simple, on which t_0 acts without fixed vector) and the projective cover P of the trivial module (dimension 2, on which t_0 acts trivially); thus $N^{\otimes m} = P^{a_m} \oplus N^{b_m}$, $\dim \operatorname{Fix}(t_0) = 2a_m$ and $\dim \operatorname{Fix}(G) = a_m$. Over $\mathbb{F}_4$ the element t_0 acts on N with eigenvalues ω, ω^2 ($\omega^2 + \omega + 1 = 0$), so it has a 2-dimensional fixed space on $N \otimes N$ and none on $P \otimes N$; as these modules are projective, $N \otimes N = P \oplus N$ and $P \otimes N = N \oplus N$. Hence $a_{m+1} = b_m$, $b_{m+1} = 2a_m + b_m$, $a_1 = 0$, $b_1 = 1$, which by (1) gives $a_m = J_{m-1}$, $b_m = J_m$. By Lemma 3.1, $\operatorname{rank} S_{m+1} = 2^m + 2^{m-1} - J_{m-1} = 2^m + J_m = J_{m+2}$. □

(Independently, $\dim \operatorname{Fix}(T) = (2^m + 2(-1)^m)/3 = 2J_{m-1}$ is the multiplicity computed by Bacher and Chapman for the symmetric Pascal matrix [1].)

Theorem 3.3 (PROVED)**.** $\ker S_m \cap W_{J_{m+1}} = 0$ *for every* $m \geq 0$.

Proof. Induction in steps of two; $m = 0, 1$ are immediate. Let $m \geq 2$ and suppose $v \neq 0$ lies in $\ker S_m \cap W_{J_{m+1}}$. Split $v = (y, z)$ along the leading bit (level $m-1$). As $J_{m+1} - 2^{m-1} = J_{m-1}$ by (1), z is supported in $[0, J_{m-1})$. By Lemma 3.1 at level $m-1$, $y = (\Pi + T^2)z$, so $z \neq 0$, and $Xz = 0$, i.e. $c := (I + \Pi)z$ lies in $\mathrm{Fix}(U) \cap \mathrm{Fix}(\Pi)$. The support of Πz is contained in $[2^{m-1} - J_{m-1}, 2^{m-1}) = [J_m, 2^{m-1})$, disjoint from that of z, so $c \neq 0$, and c vanishes on $[J_{m-1}, J_m)$.

Now split $c = (a, b)$ along the leading bit of level $m-1$, with level $m-2$ blocks marked by a prime. By (6), $\Pi c = c$ gives $b = \Pi' a$, and $Vc = c$ (which holds since $Uc = c$ and $V = \Pi U \Pi$) gives $V'a = a$ and $(V' + I)b = a$. As $\Pi' U' \Pi' = V'$, one has $(V' + I)\Pi' = \Pi'(U' + I)$, so $a = \Pi'(U' + I)a$, i.e. $\Pi' a = (U' + I)a$, and then

$$a = (V' + I)\Pi' a = (V' + I)(U' + I)a = (S' + I)a,$$

so $S'a = 0$. Since $J_{m-1} \leq 2^{m-2} \leq J_m$, the vector a is supported in $[0, J_{m-1})$, and $a \neq 0$ because $b = \Pi' a$ and $c \neq 0$. Thus a is a nonzero element of $\ker S_{m-2} \cap W_{J_{m-1}}$, contradicting the induction. □

Corollary 3.4 (PROVED)**.** $\det S[0{:}J_r] \neq 0$ *for every* r*, and* $\operatorname{corank} S[0{:}i] = \operatorname{dist}(i, \{J_r\})$ *for every* $i \geq 1$.

Proof. S_m is symmetric with radical $\ker S_m$ of dimension J_m (Proposition 3.2). By Theorem 3.3 and $J_m + J_{m+1} = 2^m$, $\mathbb{F}_2^{2^m} = W_{J_{m+1}} \oplus \ker S_m$, so the form restricted to $W_{J_{m+1}}$ is nondegenerate: $\det S[0{:}J_{m+1}] \neq 0$. For $r \geq 2$ one has $J_r \leq 2^{r-1} \leq J_{r+1}$ with $2^{r-1} - J_r = J_{r+1} - 2^{r-1} = J_{r-1}$, and $\operatorname{corank} S[0{:}2^{r-1}] = J_{r-1}$. By (5) the corank moves by ± 1 at each step, so on $[J_r, 2^{r-1}]$ it must rise by 1 at every step and on $[2^{r-1}, J_{r+1}]$ fall by 1 at every step. □

Remark 3.5 (the combinatorial form, PROVED). For p, q nonzero and distinct, "$p \,\&\, q = 0$" and "$p \subseteq q$" exclude each other, so $S_{pq} = 1$ exactly when the sets p, q are disjoint or nested: $S[1{:}i]$ is the adjacency matrix of L_{i-1}. By (5), $\det S[0{:}i] = \det S[1{:}i]$, the determinant of an alternating matrix, which over $\mathbb{F}_2$ equals its Pfaffian, i.e. the number of perfect matchings of L_{i-1} modulo 2. Corollary 3.4 is therefore the matching statement of Theorem A. [Matchings counted directly for $N \leq 16$.]

4 Level one

Let $W_0 = \ker(M^{\mathsf{T}})$, the left kernel of M. By (4) and Proposition 3.2, $\operatorname{rank} M = \operatorname{rank} S = J_{m+1}$ and $\dim W_0 = J_m$. Put

$$f(i) = \dim(W_0 \cap W_i), \qquad g(i) = \dim\big(W_0 \cap \operatorname{span}(e_i, \ldots, e_{n-1})\big).$$

Lemma 4.1 (PROVED)**.** $\Pi W_0 = W_0$.

Proof. x is in ΠW_0 iff $M^{\mathsf{T}} \Pi x = 0$. By (3), $M^{\mathsf{T}}\Pi = (\Pi M)^{\mathsf{T}} = (M^{\mathsf{T}})^2$, and $\ker(M^{\mathsf{T}})^2 = \ker M^{\mathsf{T}}$ because $(M^{\mathsf{T}})^3 = M^{\mathsf{T}}$. □

Lemma 4.2 (PROVED)**.** $f(i) = 0$ *for* $i \leq J_{m+1}$*, and* $g(i) = f(n - i)$ *for all* i*. In particular* $g(J_m) = f(J_{m+1}) = 0$.

Proof. $f(i) = i - \operatorname{rank}(\text{first } i \text{ rows of } M)$, and the first i rows of M are $V[0{:}i, 0{:}i]$ times the first i rows of S, V being lower unitriangular; these are independent for $i \leq J_{m+1}$ by Corollary 3.4. The identity $g(i) = f(n - i)$ follows from Lemma 4.1, since Π maps $\operatorname{span}(e_i, \ldots, e_{n-1})$ onto W_{n-i}. Finally $n - J_m = J_{m+1}$. □

Theorem 4.3 (PROVED)**.** $\operatorname{rank}(M+\Pi) = J_m$ *and* $\det(M+\Pi)[0{:}J_m] \neq 0$.

Proof. (a) $\ker(M+\Pi) = \operatorname{im}(M)$. If $Mx = \Pi x$ then, by (3), $x = \Pi\Pi x = \Pi M x = M^2 x = M(\Pi x)$, so x is in $\operatorname{im}(M)$; conversely $(M+\Pi)M = M^2 + \Pi M = 0$. Hence $\operatorname{rank}(M+\Pi) = 2^m - J_{m+1} = J_m$.

(b) By (2) and (4), $M+\Pi = V(S+V\Pi) = V(S+T^2)$, and $S+T^2$ is symmetric ($T^2 = UV$ is, as T is symmetric). Since V is invertible and lower unitriangular, $\det(M+\Pi)[0{:}i] = \det(S+T^2)[0{:}i]$ for all i, and the radical of $S+T^2$ is $\ker(S+T^2) = \ker(M+\Pi) = \operatorname{im}(M) = W_0^\perp$.

(c) $W_{J_m} \cap W_0^\perp = 0$. Indeed its orthogonal is $\operatorname{span}(e_{J_m}, \ldots, e_{n-1}) + W_0$, a sum of spaces of dimensions J_{m+1} and J_m whose intersection has dimension $g(J_m) = 0$ by Lemma 4.2; it is therefore the whole space.

(d) Since $\dim W_{J_m} + \dim W_0^\perp = J_m + J_{m+1} = 2^m$, (c) gives $\mathbb{F}_2^{2^m} = W_{J_m} \oplus \operatorname{rad}(S+T^2)$, so the form $S+T^2$ is nondegenerate on W_{J_m}, i.e. $\det(S+T^2)[0{:}J_m] \neq 0$. □

4.1 The Hecke operator in level one

By [4, (14), (15)], the Fricke matrix is (up to a power of t) the antidiagonal matrix F with $Fe_b = (-t)^{s_b} e_{n+1-b}$, the trace is $T_r = I + MA$ with A antidiagonal of entries ± 1, and $AF = D$. Modulo 2, $A = \Pi$ and $F = \Pi D$, so

$$T_r F = (I + M\Pi)\Pi D = (M+\Pi)D.$$

(For odd q, Dalal [7, Prop. 3.2] writes the trace matrix as $M + A$.) By [4, Rem. 5.3], $T_r F$ maps $S_k(\Gamma_0(t))$ into the image of δ_1, which is the image of the idempotent T_r, and acts there as T_t. Hence $\dim S_k(\mathrm{GL}_2(A)) = \operatorname{rank} T_r = \operatorname{rank}(M+\Pi) = J_m$, and

$$\operatorname{char}\big((M+\Pi)D\big) = X^{n-J_m} \cdot P_m(X), \tag{7}$$

where P_m is the characteristic polynomial of T_t on $S_k(\mathrm{GL}_2(A))$.

5 Slopes, and the Main Theorem

5.1 The elementary bound

For any matrix B over $\mathbb{F}_2$, the coefficient of X^{n-j} in $\operatorname{char}(BD)$ is $E_j = \sum_{|\sigma|=j} \det B[\sigma,\sigma] \cdot t^{\sum_{q\in\sigma}(q+1)}$, and the least exponent occurring is reached only for $\sigma = \{0, \ldots, j-1\}$; hence

$$v_t(E_j) \geq j(j+1)/2, \qquad \text{with equality iff } \det B[0{:}j] \neq 0 \tag{8}$$

([11, Lem. 2.3(3)] in the framework of glissando matrices).

Proof of Theorem C. Apply (8) to $B = M + \Pi$. By (7) the coefficients of P_m are $E_0, \ldots, E_{J_m}$, and by Theorem 4.3 $v_t(E_{J_m}) = J_m(J_m+1)/2$. The largest slope of the Newton polygon of P_m is the maximum over $j < J_m$ of $(v(E_{J_m}) - v(E_j))/(J_m - j)$, which by (8) is at most

$$\max_{j<J_m} \frac{J_m(J_m+1) - j(j+1)}{2(J_m - j)} = \max_{j<J_m} \frac{J_m + j + 1}{2} = J_m. \qquad \square$$

Proof of the Main Theorem. (i) By [4, Thm. 5.1] the decomposition holds iff $I - t^{-k}(T_r F)^2$ is invertible, i.e. iff no eigenvalue λ of $T_r F$ satisfies $\lambda^2 = t^k$. The eigenvalues of $T_r F$ are 0 and those of T_t, whose valuations are at most J_m by Theorem C, while $\lambda^2 = t^k$ forces $v(\lambda) = k/2$; and $2J_m < 2^m + 1 = k$. So the decomposition holds. The map δ is injective [4, Section 2.6.1], so $\dim S^{\mathrm{old}} = 2J_m$, and $\dim S^{\mathrm{new}} = n - 2J_m = J_{m+1} - J_m = 2J_{m-1}$.

(ii) On $S^{\mathrm{old}} = \delta_1 S_k(\mathrm{GL}_2(A)) \oplus \delta_t S_k(\mathrm{GL}_2(A))$ one has $U_t \delta_t = 0$ (in characteristic p the q translates add up to zero), and $U_t \delta_1(\varphi) = \delta_1(T_t\varphi) + c \cdot \delta_t(\varphi)$ for a power c of t [4, Section 2.6.1].

In the basis $\delta_1(\varphi_i), \delta_t(\varphi_i)$ the matrix of U_t is block lower triangular with diagonal blocks T_t and 0, so its characteristic polynomial on S^{old} is $X^{J_m} P_m(X)$. On S^{new}, $U_t^2 = t^k$ [4, Section 2.6.2]; as k is odd, X^2+t^k is irreducible over $\mathbb{F}_2(t)$, so the characteristic polynomial there is $(X^2+t^k)^{J_{m-1}}$.

(iii) The roots of $X^2 + t^k$ have valuation $k/2$; the nonzero roots of P_m have valuation at most $J_m < k/2$ by Theorem C. □

Remark 5.1 (Theorem C is far from sharp). The bound J_m is about $k/3$, while the largest slope of T_t found by computation in weight $2^m + 1$ is $(2^{m-1} + 1)/2$, about $k/4$, for $m = 2, \ldots, 5$ (VERIFIED). The bound suffices here because the new slope $k/2$ is much larger than J_m.

Remark 5.2 (consequences for U_t, PROVED). By (ii), the Newton polygon of U_t has its last edge from J_m to J_{m+1}, of slope $(n+1)/2$ and length $2J_{m-1}$; and by (8) and Corollary 3.4 ($B = M$) its vertices include every Jacobsthal index $j \le J_{m+1}$, the vertex at j having ordinate $j(j+1)/2$. Between two consecutive vertices J_r and J_{r+1} there are $2J_{r-1}$ slopes, of mean $(2^r + 1)/2$.

Remark 5.3 (VERIFIED, $m = 2, \ldots, 5$). The identity (ii) has been checked directly on the characteristic polynomials over $\mathbb{F}_2[t]$. The slopes of U_t are 1 (×1), 5/2 (×2), 9/2 (×2), 17/2 (×6), 33/2 (×10) for $m = 5$, in agreement with [4, Ex. 6.10].

Remark 5.4 (VERIFIED, $m = 2, \ldots, 5$; OPEN in general). The slopes of T_t in weight $2^m + 1$ coincide with the slopes of U_t in weight $2^{m-1} + 1$. If this holds for all m, the slopes of U_t in weight $2^m + 1$ are exactly $(2^r + 1)/2$ with multiplicity $2J_{r-1}$ ($r = 2, \ldots, m$), together with 1 (×1); this is the tower case of [4, Conj. 6.9]. It is *not* proved here.

Remark 5.5 (why [9] does not apply). With $q = 2$ the hypothesis of [9] asks that all Hecke eigenvalues in level one be simple. In fact $P_m(X) = (X + t) \cdot Q_m(X^2)$ for some $Q_m \in \mathbb{F}_2[t][X]$ (VERIFIED for $m = 2, \ldots, 5$), and in characteristic 2 every root of a polynomial in X^2 is multiple.

Remark 5.6 (neighbouring weights, VERIFIED). For $q = 2$ and any n, put $r = \mathrm{rank}(M + \Pi) = \dim S_{n+1}(\mathrm{GL}_2(A))$. If $\det(M + \Pi)[0{:}r] \neq 0$ and $2r < n + 1$, the proof of the Main Theorem applies verbatim in weight $n + 1$. For $n \le 160$ both conditions hold exactly when n is of the form $2^e - 1$, 2^e or $2^e + 1$. Only the case $n = 2^e$ is proved here.

6 Verification

The ancillary script `anc/verif_all.py` (pure Python with numpy; no parameter to enter) checks, and aborts its summary on the first failure:

[1] $M = I+T+U$ against formula (11) of [4] ($m \le 6$); identities (2), (3), (4), the triangularity of V and the diagonal of S ($m \le 9$);

[2] Lemma 3.1 and every step of the proof of Proposition 3.2 ($m \le 8$);

[3] Theorem 3.3 ($m \le 9$) and the restriction map of its proof (levels ≤ 8);

[4] Corollary 3.4 for $i \le 512$;

[5] the combinatorial form of Theorem A, by counting perfect matchings directly ($N \le 16$);

[6] Lemmas 4.1, 4.2 and Theorem 4.3, and $I + M\Pi = M + \Pi$ ($m \le 9$);

[7] the Main Theorem (ii)–(iii) and Theorem C on exact characteristic polynomials over $\mathbb{F}_2[t]$ ($m \le 5$), with the observations of Remarks 5.4 and 5.5;

[8] the observation on neighbouring weights ($n \le 160$).

All 52 checks pass.

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