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Uniform intervals for irrational even beta values

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Abstract

For every positive integer q, at least one of the Dirichlet beta values $\beta(2q+2), \beta(2q+4), \ldots, \beta(2\lfloor 33q/2 \rfloor)$ is irrational. This gives an explicit interval from every even starting index at least four. For every positive odd m, some even s with $m+1 \le s \le (299/20)m + C$ has $\beta(s)$ irrational, where C is absolute. This reduces the previous upper-endpoint multiplier 18.58 to 14.95 up to a bounded additive term. For every odd $m \ge 4{,}000{,}000{,}001$, the explicit endpoint $s < 15m$ suffices.

1 Locating the next irrational value

Write $\beta(s) = \sum_{h \ge 0} (-1)^h/(2h+1)^s$ for $s > 0$. Given an even starting index, how far along the list of even beta values must one go to encounter an irrational number? Knowing that one of $\beta(4), \beta(6), \ldots, \beta(32)$ is irrational leaves open the possibility that $\beta(4)$ is the only irrational member of that list. To answer the question starting at $\beta(10)$, or at a starting index chosen arbitrarily far out, one needs control of both ends of the interval.

Theorem 1.1. *For every integer $q \ge 1$, some even integer s satisfies*

$$2q+2 \le s \le 2\lfloor 33q/2 \rfloor, \qquad \beta(s) \text{ is irrational.}$$

The first three upper endpoints are $32, 66, 98$, at starting indices $4, 6, 8$, respectively. The first is known from [2], and the next two from [3]. Starting at 10 gives the list $\beta(10), \beta(12), \ldots, \beta(132)$. The theorem gives an endpoint for each subsequent starting index as well.

Theorem 1.2. *There is an absolute constant C such that for every positive odd integer m there is an even s with*

$$m+1 \le s \le \frac{299}{20}m + C, \qquad \beta(s) \text{ irrational.}$$

For every odd $m \ge 4{,}000{,}000{,}001$, one can take $m+1 \le s < 15m$.

The earlier uniform result gives the explicit upper endpoint $m-1+2\lceil (879m-50)/100 \rceil < 18.58m$ for every positive odd m [1, Proposition B.1]. Theorem 1.1 gives an asymptotic multiplier 16.5 on writing $m = 2q+1$. The first assertion of Theorem 1.2 improves this to 14.95, and the second gives a specified range for a multiplier of 15.

Ball and Rivoal used reflection in a rational function to isolate odd zeta values [4, Section 1]. For even beta values, Rivoal and Zudilin used half-integer poles and an alternating sum [5, Lemma 2]. Rivoal used a second derivative to remove $\zeta(3)$ from odd-zeta forms [7, Section 2, Lemma 1]. Krisman adapted that step to remove $\beta(2)$, giving the integer zeros multiplicity three so that they survive differentiation [6, Section 1.2]. We use the order-31 denominator refinement of [2]. Taking a power before differentiating makes the starting index adjustable [8, 3]. Section 2 treats two choices

of derivative order. In each case irrationality in the interval follows from the nonvanishing of a particular rational number, which we call its *seed*.

Two observations let finitely many computations control infinitely many intervals. First, a successful power can serve several starting points. For example, the ninth power with twenty derivatives gives the interval from 22 to 298. It fits inside the interval from 22 to 330 requested at $q = 10$. Its multiples have lower and upper indices $20j + 2$ and $299j$, which explains the ratio $299/20$. Second, the normalized seeds are coefficients of an algebraic function. Its behavior at infinity forces bounded gaps between nonzero coefficients. Reduction modulo primes lets a single rational seed certify almost every prime exponent. We use bounded gaps for the sharpest slope and primes for the explicit ranges.

2 From a rational seed to an irrational value

The denominators consist of blocks of consecutive poles of different lengths. Zudilin used such blocks for even beta values [9, Section 3]. Lai and Zhou changed their lengths and factorial normalization to reach the interval from $\beta(2)$ to $\beta(10)$ [10, Theorem 6.1]. Here we take the 31 blocks specified in [2, Section 2].

Let $96 \mid n$, $n > 0$. For $0 \le h \le 30$ put $I_h = [hn/96, (1 - h/96)n] \cap \mathbb{Z}$, and let $B_a(n)$ count the sets I_h containing a. With $\Lambda_n = [-7n/4 - 1, 3n/4] \cap \mathbb{Z}$, the kernel is

$$R_n(t) = \frac{(2t + n + 1) \prod_{v \in \Lambda_n} (t - v)^3}{\prod_{a=0}^{n} (t + a + \frac{1}{2})^{B_a(n)}}.$$

Its symmetry is $R_n(-n - 1 - t) = R_n(t)$. For fixed $A \ge 1$ and even $d \ge 2$ we use the form

$$L_n(A, d) = \sum_{t \ge 0} (-1)^t (R_n^A)^{(d)}(t).$$

The two pairs throughout the proof are

$$(A, d) = (j, 2j) \quad \text{or} \quad (9j, 20j), \qquad j \in \mathbb{Z}_{\ge 1}. \tag{2.1}$$

At a pole of order i, differentiation raises the order to $i + d$. Reflection cancels the odd i in the beta coefficients. Consequently the even beta indices range from $d + 2$ to at most $31A + d$. In both pairs $d < 3A$, so the derivative still vanishes at each integer zero. This inequality will also be needed to clear denominators.

The seed is computed at just 97 pole addresses. Set

$$u_a = 96 - 2a, \quad B_a = B_a(96) = 1 + \min(a, 96 - a, 30) \quad (0 \le a \le 96), \qquad \sigma_a(s) = \sum_{h=0}^{a-1} \frac{(-1)^h}{(2h + 1)^s}.$$

Define the even rational function f and its polar coefficients by

$$f(z) = \frac{z \prod_{|v| \le 239} (z - v)^3}{\prod_{a=0}^{96} (z - u_a)^{B_a}}, \qquad f(z)^A = \sum_{a=0}^{96} \sum_{i=1}^{AB_a} \frac{c_{A,a,i}}{(z - u_a)^i}. \tag{2.2}$$

The cutoff 239 comes from the zeros divisible by the dilation prime, as in [2, Section 5]. A nominal coefficient can vanish at the central pole because the numerator contains z. The normalized seed is

$$b(A, d) = -\sum_{a=0}^{96} (-1)^a \sum_{i=1}^{AB_a} \binom{i + d - 1}{d} c_{A,a,i}\, \sigma_a(i + d). \tag{2.3}$$

Multiplying by $d!$ restores the rising factorial $(i)_d = i(i + 1) \cdots (i + d - 1)$ from differentiation.

Proposition 2.1. *For either pair in (2.1), if $b(A,d)\neq 0$, some even $s\in[d+2,31A+d]$ has $\beta(s)$ irrational.*

Zudilin bounded the polar coefficients by differentiating the individual factors [9, proof of Lemma 4]. Krisman's valuation bound for the series at each pole [6, Lemma 4.3] is the starting point for the isolated-pole estimate for these 31 blocks. We use that estimate and the real phase bounds from [2, Sections 3–5 and Appendix A]. The case $d=2A$ after powering is proved in [3, Sections 3–5].

2.1 The cost of the extra derivatives

Write $D_n=\prod_{p\le n}p^{e_p}$ for the integer of [2, Section 3 and Appendix A], and put

$$\lambda_p=\lfloor \log_p(7n)\rfloor \quad (p \text{ odd}), \qquad H_n=\prod_{\substack{p\le n\\ p \text{ odd prime}}} p^{\lambda_p}.$$

We claim that the rational coefficients of $L_n(A,d)$, including its constant term, are integers after multiplication by

$$D_n^A H_n^{d-2A}. \tag{2.4}$$

To see where the extra cost enters, write $a_{n,A;a,i}$ for the coefficient of $(t+a+\frac12)^{-i}$ in $R_n(t)^A$. Partial fractions express the form as

$$L_n(A,d)=\sum_{a,i}(-1)^a 2^{i+d}(i)_d\, a_{n,A;a,i}\{\beta(i+d)-\sigma_a(i+d)\}.$$

The coefficients at a and $n-a$ differ by the factor $(-1)^i$. This proves the parity and endpoint assertions above. All the shifts and differentiations involve absolutely convergent series.

At an odd p with $p^2\le 7n$, write

$$\nu_p(a)=3\sum_{v\in\Lambda_n} v_p(2a+1+2v)-\sum_{b\neq a}B_b(n)v_p(b-a).$$

By [2, Appendix A], the degree-h coefficient of the series obtained by removing the nominal pole of R_n has valuation at least $\nu_p(a)-h\lambda_p$, and $e_p\ge -\nu_p(a)+33\lambda_p$. Multiplication of series gives

$$v_p(a_{n,A;a,i})\ge A\nu_p(a)-\lambda_p(AB_a(n)-i).$$

A harmonic denominator costs at most $(i+d)\lambda_p$. Since $B_a(n)\le 31$, the combined loss is at most $Ae_p+(d-2A)\lambda_p$. Such primes are at most n. At 2 all harmonic denominators are odd, so the inherited local bound and D_n^A already suffice.

For odd $p\le n$ with $p^2>7n$, addresses are grouped by their residue modulo p. In a repeated class the local estimate is $v_p(a_{n,A;a,i})\ge i-AF_p(a)$, where the local pole excess is

$$F_p(a)=\sum_{\substack{0\le b\le n\\ b\equiv a \bmod p}} B_b(n)-3\#\{v\in\Lambda_n : p\mid 2a+1+2v\}-\mathbf{1}_{p\mid n-2a}.$$

If M_p is its maximum over the repeated classes, then $e_p=\max(0,M_p+2)$ [2, Section 3]. An evaluation denominator divisible by p costs $i+d$ powers, so the total valuation is at least $-AM_p-d$. This is covered by Ae_p+d-2A, and here $\lambda_p=1$. In a class with only one address the local coefficients are p-integral.

For the last assertion to help with the harmonic denominators, group the constant term by its evaluation point $-\ell$, $1 \le \ell \le n$. The term at this point uses addresses $a \ge \ell$ with denominators $(a-\ell+\frac{1}{2})^{i+d}$. Summing over *all* addresses gives $(R_n^A)^{(d)}(-\ell) = 0$, since $d < 3A$. Thus the sum over $a \ge \ell$ equals the negative of the sum over $a < \ell$. Zudilin used zeros in the pole interval to shift the summation limit in the rational term [9, proofs of Lemmas 1 and 5]. Krisman obtained the two complementary sums after two derivatives [6, Lemma 4.6]. Here $d < 3A$ preserves the same choice for the powered kernel. Only the class $2(a-\ell)+1 \equiv 0 \pmod p$ can have a bad denominator. If it has just one address, choose the side which omits that address. All other terms then have unit denominators and the stated local coefficient bounds apply. For $p > n$ every class has at most one address. This finishes the proof of (2.4).

First powers contribute $\sum_{2<p\le n} \log p$ to $\log H_n$, and higher powers contribute $O(\sqrt{n}\log n)$. Thus $\log H_n = (1+o(1))n$ by the prime number theorem. We compare this cost with the decay of the same form. The first nonzero summand can occur only at $t > 3n/4$. On these integers logarithmic differentiation gives $|(R_n^A)^{(d)}(t)| \le C_{A,d} n^d R_n(t)^A$, since all the linear factors have distance at least one from their zeros or poles and there are $O(n)$ factors counted with multiplicity. The Stirling and prime-density estimates of [2, Section 4] give the following upper bound with $\delta = 41/5 + 15\log 2 - 18.83 < -0.23$:

$$\limsup_{\substack{n\to\infty \\ 96\mid n}} \frac{1}{n} \log|D_n^A H_n^{d-2A} L_n(A,d)| \;\le\; A\delta + (d-2A) < 0. \tag{2.5}$$

Indeed the $G\log(n!)$ term in $\log D_n$, $G = 221/16$, cancels the corresponding term from the kernel. The remaining denominator cost is at most $41/5 + 15\log 2 + o(1)$ per n, and the real phase is at most -18.83. Beyond $t = 3n$ its additional bound $(3n/t)^{3An}$ absorbs the polynomial factors from Stirling's formula and differentiation. All these estimates take A, d fixed. For the ninth-power family the two extra derivatives per j cost $2j$, leaving $9j\delta + 2j < -0.07j$.

2.2 How the seed prevents vanishing

Lai and Sprang's real auxiliary-prime criterion singles out the constant coefficient by its strictly smaller valuation at primes tending to infinity [11, Lemma 2.3]. For this kernel, the dilation developed in [8, Section 4] and used in [2, 3, Section 5] produces that separation. Set $n = 96P$ with $P > 337$ prime and use the centered variable $x = 2t + n + 1$. In this variable the kernel, up to the factor 2^{Gn+25}, is

$$F_n(x) = \frac{x \prod_{v \text{ odd}, |v|\le 5n/2+1} (x-v)^3}{\prod_{a=0}^{n} (x-n+2a)^{B_a(n)}}.$$

Write $\gamma_{n,A;a,i}$ for the polar coefficient of F_n^A at address a. After removal of the common nonzero factor $2^{A(Gn+25)+d}$, the form $L_n(A,d)$ is

$$\sum_{a,i} (-1)^a (i)_d \, \gamma_{n,A;a,i} \{\beta(i+d) - \sigma_a(i+d)\}.$$

The dilation lemma of [3, Section 5] applies directly to the undifferentiated power. It gives

$$P^{1356A-i}\gamma_{96P,A;Pa,i} \equiv (-1)^A c_{A,a,i} \pmod P, \qquad v_P(\gamma_{96P,A;b,i}) \ge i - 1326A \quad (P \nmid b).$$

The first congruence includes integrality at P. Multiply the centered form by $P^{1356A+d}$. At a selected address Pa, the order-i term has $i+d$ powers of P available. Its beta coefficient vanishes modulo P. In its harmonic correction only the terms $2h+1 = P(2v+1)$ survive, giving

$$P^{i+d}\sigma_{Pa}(i+d) \equiv (-1)^{(P-1)/2}\sigma_a(i+d) \pmod P.$$

For b outside the selected addresses, each harmonic denominator contains at most one factor P, since $2n < P^2$. Even after this loss at least $30A$ further powers remain. Thus the scaled rational constant reduces to $(-1)^{A+(P-1)/2}d!\,b(A,d)$, and every scaled beta coefficient is divisible by P.

Conclusion of Proposition 2.1. Suppose that the beta values in the indicated interval are rational with common denominator T. If $b(A,d) \neq 0$, every sufficiently large prime P avoiding T and the numerator and denominator of $d!\,b(A,d)$ gives a nonzero form at $n = 96P$. Indeed its scaled constant is a unit at P while every other scaled term vanishes modulo P. By (2.4) and (2.5), the numbers $TD_n^A H_n^{d-2A} L_n(A,d)$ along that subsequence are nonzero integers tending to zero. This contradiction proves the proposition. □

3 Why the successful powers have bounded gaps

We use the same argument for both derivative orders. Fix $(k,r) = (1,2)$ or $(9,20)$, and set $b_j = b(kj, rj)$ for $j \geq 1$ and $b_0 = 0$. The first term is nonzero in each case. For the first sequence $2!\,b(1,2) \equiv 3738 \pmod{10007}$ [2, Section 5]. For the second the following finite-field calculations use the normalization in (2.3).

Prime	$b(9,20)$	$20!\,b(9,20)$
10007	1491	3058
10009	7947	4140

At each prime we compute the $9\sum_a B_a = 18{,}693$ nominal polar coefficients both by multiplying truncated local factors and by the logarithmic derivative recurrence of [3, Section 6]. We then sum (2.3) in its original order and in the order used in (3.1) below. All inversions occur below the characteristic or at unit pole differences. The program `checks/derivative_seed.py` in the accompanying `uniform-even-beta-intervals` source directory reproduces the table with the option `-cross-check`. The same directory convention is used for the other programs below. The original kernel certificates are in the project repository [14].

Theorem 3.1. *For each of the two sequences there is an integer $L \geq 1$ such that every L consecutive positive indices contain a j with $b_j \neq 0$. Moreover, for every prime $p > 192$ and every $j \geq 1$,*

$$b_{pj} \equiv b_j \pmod{p} \qquad \text{in } \mathbb{Z}_{(p)}.$$

In particular, if $b_j \neq 0$, then for all but finitely many primes p all coefficients b_{jp^e}, $e \geq 0$, are nonzero.

3.1 A residue representation with a fixed rational function

The difficulty with b_j in (2.3) is that both the order of the pole and the order of differentiation grow with j. The normalization by $(rj)!$ allows us to put both changes in one ordinary power. For $1 \leq \ell \leq 96$ define

$$y_\ell = 97 - 2\ell, \qquad g_\ell(z) = \frac{f(z)^k}{(y_\ell - z)^r}.$$

In (2.3), change the harmonic summation index h to $\ell = a - h$. Then $y_\ell - u_a = 2h + 1$ and $(-1)^{a+h} = (-1)^\ell$. Expanding $(y_\ell - z)^{-rj-1}$ around u_a gives, as its coefficient of $(z-u_a)^{i-1}$, $\binom{i+rj-1}{rj}(y_\ell - u_a)^{-i-rj}$. Thus

$$b_j = -\sum_{\ell=1}^{96}(-1)^\ell \sum_{a=\ell}^{96} \operatorname{Res}_{z=u_a} \frac{g_\ell(z)^j}{y_\ell - z}\,dz. \tag{3.1}$$

This holds also at $j = 0$, when every residue is zero. Each y_ℓ is a triple zero of f. Thus g_ℓ has a zero there of order $3k - r > 0$. Its other finite poles are among the u_a. At infinity f has a zero of order 1356, by counting the factors of (2.2). Consequently g_ℓ has a zero at infinity of order $1356k + r$.

3.2 Algebraicity and propagation of zeros

Consider $B(T) = \sum_{j\geq 0} b_j T^j$. For each ℓ, take disjoint small circles around the u_a occurring in (3.1), with y_ℓ outside. For $|T|$ sufficiently small the geometric series converges uniformly on all the circles. Summing under their integrals replaces the differential in (3.1) by

$$\frac{dz}{(y_\ell - z)(1 - Tg_\ell(z))}.$$

For a generic nonzero T near zero the original poles u_a are regular points of this differential. The poles now enclosed are the simple roots ξ of $g_\ell(\xi) = 1/T$ tending to the chosen centers. Indeed, writing $g_\ell = P/Q$ in lowest terms, Rouché's theorem on each circle counts the roots of $Q - TP$ there with their multiplicities. The critical parameter values form a finite set. The residue at such a simple root is

$$h_\ell(\xi) = \frac{g_\ell(\xi)}{(\xi - y_\ell)g'_\ell(\xi)}. \tag{3.2}$$

Thus $B(T)$ is a finite signed sum of algebraic functions over $\mathbb{Q}(T)$, and is itself algebraic.

To show that the series has infinitely many nonzero coefficients, follow the roots to large $|T|$. They approach zeros of g_ℓ on the Riemann sphere. At a finite zero v of order e, we have $g_\ell/g'_\ell = (\xi - v)/e + O((\xi - v)^2)$. The residue (3.2) tends to $1/e$ if $v = y_\ell$, and to zero for every other finite zero. At infinity it tends to $-1/(1356k + r)$. Since every root lies near one of these zeros for sufficiently large $|T|$, all branches of the sum are bounded there. A polynomial $B(T)$ would therefore be constant. Its linear coefficient b_1 is nonzero, so its coefficients cannot eventually vanish.

The coefficients of an algebraic power series satisfy a recurrence with polynomial coefficients [12, Theorems 1.5 and 2.1]. We recall the argument to see why a run of zero coefficients propagates. The finite extension $\mathbb{Q}(T, B)$ is closed under differentiation. For a minimal polynomial $A(T, X)$, implicit differentiation gives $B' = -A_T(T, B)/A_X(T, B)$, since characteristic zero gives separability. The derivatives of B are therefore linearly dependent over $\mathbb{Q}(T)$. Clearing denominators and comparing coefficients gives a recurrence, after translating the index by a fixed amount,

$$\sum_{i=0}^{L} P_i(j) b_{j+i} = 0 \quad (j \text{ sufficiently large}), \qquad P_i \in \mathbb{Q}[j], \quad P_L \neq 0. \tag{3.3}$$

Indeed, the terms $T^v B^{(u)}$ of the differential equation contribute shift $u - v$. Among terms of any fixed shift, the term of highest derivative order has strictly highest polynomial degree in j. Thus some coefficient polynomial survives. The case $L = 0$ would force eventual vanishing. Beyond the last integer root of P_L, a run of L zeros propagates forwards by (3.3) and would also force eventual vanishing. Enlarging L covers the finite initial segment. This proves the bounded-gap assertion.

3.3 Prime exponents

Fix $p > 192$. Distinct pole positions differ by at most 192, and $y_\ell - u_a$ is a nonzero odd integer of absolute value at most 191. Consequently the Laurent expansion of g_ℓ^j at $w = z - u_a$ has p-integral coefficients. Write its negative part as $\sum_{i\geq 1} a_{-i} w^{-i}$, a finite sum. In characteristic p, taking the pth

power sends w to w^p and fixes the reduced coefficients. With $v = y_\ell - u_a$, the two residues after division by $y_\ell - z = v - w$ reduce respectively to

$$\sum_{i\geq 1} a_{-i} v^{-i}, \qquad \sum_{i\geq 1} a_{-i} v^{-pi}.$$

They agree by Fermat's theorem. Summing (3.1) proves the congruence and hence Theorem 3.1. For any fixed nonzero rational b_j, every sufficiently large prime is a unit prime for b_j, so the congruence can be iterated at that same prime for all e. The corresponding unnormalized seed $(rjp^e)!\, b_{jp^e}$ is nonzero over $\mathbb{Q}$.

Proof of the first assertion of Theorem 1.2. Take L for the family $(9, 20)$. Given a positive odd m, set $J = \max(1, \lceil (m-1)/20 \rceil)$. There is a successful $J \leq j \leq J + L - 1$. Proposition 2.1 gives an irrational beta value at an even index between $20j + 2$ and $299j$. The first endpoint is at least $m + 1$. We can take $C = 299L$, since

$$299j \leq \frac{299}{20}(m-1) + 299L. \qquad \square$$

For the original family $k_j = (2j)!\, b(j, 2j)$, Theorem 3.1 also supplies bounded gaps and all powers of every sufficiently large prime. Whether $k_j \neq 0$ at every sufficiently large j remains open. Covering a prescribed interval leaves us free to use any successful pair whose two endpoints fit inside it.

4 An effective supply of prime exponents

The bounded-gap argument gives no numerical value for L. We can obtain explicit endpoints by choosing the exponent j to be a prime. Write $B_* = b(9, 20)$. For every prime $p > 192$ at which B_* is a unit, Theorem 3.1 supplies the interval $[20p + 2, 299p]$. To find such a prime in a prescribed range, we compare the logarithmic weight of *all* possible exceptional primes with the weight of the primes in the range.

Lemma 4.1. *There is a nonzero integer M with $\log |M| < 180{,}000$ divisible by every prime $p > 192$ at which B_* is not a unit.*

Proof. Put $D = \operatorname{lcm}(1, \ldots, 192)$ and

$$D_a = \prod_{b \neq a} |u_a - u_b|^{9B_b}, \qquad Q = D^{299} \operatorname{lcm}_{0\leq a\leq 96} D_a.$$

Then $M = QB_*$ is a nonzero integer. Indeed, expand $w^{9B_a} f(u_a + w)^9$. In degree h, the other pole factors introduce denominators dividing $D_a D^h$, by the integral binomial expansion of each inverse linear factor. For the pole coefficient of order i we take $h = 9B_a - i$. Each accompanying harmonic denominator divides D^{i+20}, and $h + i + 20 \leq 299$. This proves integrality. All prime divisors of Q are at most 192.

On $|w| = 1$, every other pole factor has modulus at least one. The bounds $|u_a + w| \leq 97$ and $|u_a + w - v| \leq 336$ show by Cauchy's coefficient estimate that $|c_{9,a,i}| \leq 97^9 336^{6480}$. Each of the 18,693 terms has harmonic sum of magnitude at most 96 and binomial factor at most 2^{299}. Hence

$$1 \leq |M| \leq Q \cdot 18{,}693 \cdot 96 \cdot 2^{299} 97^9 336^{6480}.$$

This upper bound is an explicitly specified integer of bit length 256,201, as reproduced by `checks/explicit_threshold.py`. Since $\log 2 < 7/10$, its logarithm is less than 180,000. $\square$

Proposition 4.2. *For $x \geq 4{,}000{,}000$, the interval $(x, 11x/10]$ contains a prime p with $b(9p, 20p) \neq 0$. For $x \geq 200{,}000{,}000$ the same holds in $(x, 1003x/1000]$.*

Proof. Let $\vartheta(x) = \sum_{p \leq x} \log p$. Dusart's estimate [13, Theorem 5.2, table on p. 4] gives

$$|\vartheta(x) - x| \leq \frac{x}{5 \log^2 x} \qquad (x > 3{,}594{,}641).$$

If $x \geq x_0$ and $\log x_0 > t$, subtracting the two estimates at x and $(1+u)x$ gives a strict lower bound $x[u - (2+u)/(5t^2)]$ for the logarithmic weight in the interval. In the two cases its value at x_0 is as follows.

x_0	t	u	$x_0[u - (2+u)/(5t^2)]$
4,000,000	14	1/10	2,740,000/7
200,000,000	18	3/1000	28,570,000/81

The comparisons $(11/4)^t < x_0$ and $e < 11/4$ verify the log hypotheses. Both lower bounds exceed 180,000. By Lemma 4.1, some prime p in each such interval does not divide M. At that prime $b(9p, 20p) \equiv B_* \neq 0 \pmod{p}$. □

Explicit endpoints for large starting indices. For $q \geq 40{,}000{,}000$ take $x = q/10$ in the first prime interval. Its successful exponent p satisfies

$$20p + 2 > 2q + 2, \qquad 299p \leq (299 \cdot 11/100)q < 33q.$$

The even index supplied by Proposition 2.1 is at most $2\lfloor 33q/2 \rfloor$. For odd $m \geq 4{,}000{,}000{,}001$, take $x = (m-1)/20$ in the second interval. Now $20p + 2 > m + 1$ and $299p \leq (299 \cdot 1003/20000)(m-1) < 15m$. This proves the second assertion of Theorem 1.2. In both applications the exponent prime p has been fixed before the dilation prime P in Proposition 2.1 tends to infinity. □

5 Covering the finite initial range

It remains to prove Theorem 1.1 below $q = 40{,}000{,}000$. One successful pair $(9j, 20j)$ works for every q satisfying

$$\left\lceil \frac{299j}{33} \right\rceil \leq q \leq 10j. \tag{5.1}$$

Indeed, these inequalities place $[20j + 2, 299j]$ inside $[2q + 2, 33q]$, and rounding both upper endpoints down to even integers preserves containment. This is why forty million starting points can be covered by a short list of certificates. We describe the selection rules and the exact arithmetic verifying them. All step-by-step residues and endpoints accompany the programs in `checks/`.

The middle range. Start with the prime 193. Given the last prime p, append the largest prime at most $\lfloor 27p/25 \rfloor$, stopping when $10p \geq 40{,}000{,}000$. For each entry test the same rational number $B_* = b(9, 20)$ modulo that prime. The computation `prime_bridge.py` verifies primality by trial division and strict increase. It evaluates B_* modulo the product of the distinct primes and the two reference primes $10007, 10009$. The denominator factors inverted in the truncated products are units in every component, since all primes exceed 192. The binomial coefficients in (2.3) are formed as integers before reduction. Thus the product calculation gives the individual residues exactly, even for primes smaller than 299.

	Exponent p	B_* mod p	Range in q from (5.1)
First	193	27	1749 to 1930
Last	4,047,157	660,746	36,669,696 to 40,471,570

All 135 residues are nonzero. The recorded neighboring intervals satisfy $\text{left}_{i+1} \le \text{right}_i + 1$. Theorem 3.1 therefore gives complete coverage from 1749 through 40,471,570.

The first starting points. Begin with $q = 1$, and always take the first uncovered q. Set $j = \lfloor 33q/299 \rfloor$. If $j \ge 1$ and $q \le 10j$, test $b(9j, 20j)$ and cover through $10j$. Otherwise test $b(q, 2q)$ and cover q itself. Stop once the next uncovered q is at least 1749. The resulting list consists of 89 seeds, 45 of the original type and 44 of the ninth-power type. Every residue is nonzero modulo 65537, and the intervals cover 1 through 1890. The largest $31A + d$ used is 56,511.

To certify these Taylor coefficients, first extract the central zero. At u_a the remaining products P, Q of linear factors from (2.2) have unit constant terms modulo 65537. Thus

$$w^{B_a} f(u_a + w) = w^{\epsilon} P(w)/Q(w), \qquad \epsilon = \mathbf{1}_{u_a=0}.$$

For A in the list put $l = A(B_a - \epsilon)$. The coefficients of $Y = (P/Q)^A$ through degree $l - 1$ are uniquely determined in the field by

$$Y(0) = (P(0)/Q(0))^A, \qquad PQ\,Y' = A(P'Q - PQ')Y \pmod{w^{l-1}}.$$

In degree $h - 1$ the new coefficient of Y is multiplied by $hP(0)Q(0)$, a unit for $1 \le h < l < 65537$. The polar coefficient of order i is $[w^{l-i}]Y$ for $1 \le i \le l$, and is zero for $l < i \le AB_a$. The program `initial_intervals.py` verifies these identities on 21,187,413 coefficients before using the symmetry $c_{A,96-a,i} = (-1)^i c_{A,a,i}$ for the remaining addresses. It evaluates (2.3) with the finite harmonic sums.

As a check on the computation, `interval_replay.py` computes all 89 seeds modulo a second prime, 100003, expanding every contributing pole directly and summing the harmonic correction in reverse order. All 89 are again nonzero. It also checks the 135 prime residues one field at a time. The accompanying tables can be recomputed together with `python3 -B checks/verify.py` from the `uniform-even-beta-intervals` directory, using Python and `python-flint`.

Conclusion of Theorem 1.1. The two finite ranges $[1, 1890]$ and $[1749, 40{,}471{,}570]$ overlap. The latter overlaps the range $q \ge 40{,}000{,}000$ proved in Section 4. Hence every positive integer q has a successful pair in (2.1) whose interval fits inside the prescribed one. Proposition 2.1 supplies an irrational beta value there. □

Verification. Independent AI reviewers checked the mathematical arguments in the 17 November 2026 proof records and reproduced the seed, coverage and threshold checks. The effective-threshold argument also received a separate review of its exposition.

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