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# Three Roles and One Bit:

## All Optimal Tables and Equality Graphs

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**Astra (GPT-6.2 Astra Max)** assisted with problem selection, the construction $F_0$, and the original $8/27$ minimax theorem.

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**Model.** Let $H$ be a simple graph on $n$ labeled vertices with $E(H) \neq \varnothing$. The vertices are assigned independent roles $X_v \in \mathbb{Z}_3$ with common distribution $p = (a, b, c)$. For a fixed symmetric table $F : \mathbb{Z}_3^2 \to \mathbb{F}_2$, define

$$P_{F,p}(H) = \mathbb{P}\left\{ \bigoplus_{uv \in E(H)} F(X_u, X_v) = 1 \right\}.$$

To compare two graphs, use the same labeling $X$ for both and let $H$ be the graph whose edge set is the symmetric difference of their edge sets. In the event log problem, an edge records the alternation of two letters in a word in which each letter occurs twice.

**Theorem 1.** For every $n \geq 4$, the value of $\max_{F,p} \min_{E(H)\neq\varnothing} P_{F,p}(H)$ is $8/27$. Every optimal pair has $p = (1/3, 1/3, 1/3)$. For $n \geq 5$, there are exactly 21 optimal tables with the role labels held fixed. For $n = 4$, the only additional optimal table is $I_3(x, y) = \mathbf{1}[x = y]$.

Set $s = 2k \bmod 3$ and $e_j(x) = \mathbf{1}[x = j]$, and define

$$T_{k,g,\varepsilon}(x, y) = \mathbf{1}[x + y = k \bmod 3] \oplus g(x) \oplus g(y) \oplus \varepsilon.$$

For each $k = 0, 1, 2$, take $g \in \{0, e_0, e_1, e_2\}$. For $\varepsilon = 0$, all four choices are allowed. For $\varepsilon = 1$, the three choices with $g \neq e_s$ are allowed. These are all 21 tables. Replacing $g$ by $g \oplus 1$ leaves the table unchanged.

**Theorem 2.** The following table lists the graphs up to isomorphism and the addition of isolated vertices.

| Rule | Number of tables | All graphs with $P = 8/27$ |
|---|---|---|
| Constant $g$ | 6 | $C_4$ |
| Nonconstant $g$, admissible $(g, \varepsilon)$ | 15 | $C_4$, $P_3 = K_{1,2}$ |
| Additional $I_3$ for $n = 4$ | 1 | $C_4$ |

**Theorem 3.** For the original table $F_0 = T_{0,0,0}$, under uniform roles, every graph outside the first equality case satisfies $P \geq 80/243$, with equality precisely for $K_{2,4}$ with isolated vertices. Thus the second smallest distinct probability is $80/243$ for $n \geq 6$ and $1/3$ for $n = 4, 5$. All three theorems also hold for symmetric differences of graphs associated with two event logs.

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## 1. Why the four-cycle determines the upper bound

Fix the roles of the two vertices in one part of $C_4 = K_{2,2}$. Adding their rows in $F$ modulo two gives a vector $D$. If the total $p$-mass of its ones is $z$, then the two independent roles in the other part select different bits with probability $2z(1-z)$. This is the conditional detection probability.

Regard two rows as equivalent if they agree or are complements. Within a single class the conditional detection probability is zero. If there are at most two classes, the probability of selecting distinct classes is at most $1/2$. Since $2z(1-z) \le 1/2$, we obtain $P(C_4) \le 1/4$.

Suppose there are three classes. For each unordered pair of rows, $D$ is, up to complementation, the indicator of one column. The three distinguished columns are distinct, since any coincidence would make two other rows equivalent. Relabel the roles so that $a \ge b \ge c$. Put $x = (ab, ac, bc)$ and $S = ab + ac + bc$. For the distinguished column of weight $p_i$, we have $p_i(1-p_i) = S - p_j p_k$. Counting both orders of selection in each part yields

$$P(C_4)/4 = S^2 - \sum_{i=1}^{3} x_i x_{\pi(i)}$$

for some permutation $\pi$. Since $x_1 \ge x_2 \ge x_3$, the rearrangement inequality gives $\sum_i x_i x_{\pi(i)} \ge 2(ab)(bc) + (ac)^2$. Consequently,

$$\frac{P(C_4)}{4} \le b^2(1-b)^2 + 2b(1-2b)ac \le \frac{b(1-b)^2}{2}.$$

Here $b \le 1/2$ and $ac \le (1-b)^2/4$. On this interval, $2b(1-b)^2$ attains its unique maximum $8/27$ at $b = 1/3$. Equality in the last inequality then requires $a = c$. Under uniform roles, three distinct rows modulo complementation give $P(C_4) = (2/3)(4/9) = 8/27$. This proves the upper bound and the necessity of uniformity.

**Table normalization.** All additions below are in $\mathbb{F}_2$. Define

$$B_{ij} = F(i,j) + F(i,0) + F(0,j) + F(0,0), \qquad i,j \in \{1,2\}.$$

The three row classes are distinct precisely when the two rows of $B$ are distinct and nonzero, that is, when $B$ is invertible. Two symmetric tables have the same $B$ precisely when their difference has the form $g(i) + g(j) + \varepsilon$. This follows by reconstructing the difference from the row indexed by zero and its diagonal entry 00.

There are exactly four invertible symmetric matrices of order two over $\mathbb{F}_2$. The entry triples $(B_{11}, B_{12}, B_{22})$ for $F_0, F_1, F_2, I_3$ are respectively $101, 011, 110, 010$, where $F_k = T_{k,0,0}$. Each class has size eight. Thus the four-cycle leaves $4 \cdot 8 = 32$ of the 64 tables. On the grid $(i,j,k)/60$ with $i+j+k = 60$, all 121 024 table–distribution pairs were checked. Equality was found only at $(20,20,20)/60$.

## 2. Finite obstructions and the exception at $n = 4$

In each of the three $F_k$ classes, the complemented table with $g = e_{2k}$, with the index taken modulo three, has two ones among its nine entries. On a single edge it gives $2/9$. Thus at most $3(8 - 1) = 21$ tables remain in these classes. In the $I_3$ class, role permutations allow us to use any one $j$ for nonconstant $g = e_j$. Write $I^g(x, y) = I_3(x, y) \oplus g(x) \oplus g(y)$.

| Table | Count | Witness | Probability |
|---|---|---|---|
| $I_3 \oplus 1$ | 1 | $K_3$ | $6/27$ |
| $I^{e_j} \oplus 1$ | 3 | $K_2$ | $2/9$ |
| $I^{e_j}$ | 3 | $K_4$ | $20/81$ |
| $I_3$ | 1 | $K_5$ | $60/243$ |

These numbers can be checked using triples of role counts. On $K_h$, with counts $b_0, b_1, b_2$, the number of equal-role pairs is $\sum_i \binom{b_i}{2}$. For $I^{e_j} \oplus \varepsilon$, add $(h - 1)b_j + \varepsilon\binom{h}{2}$. For each triple whose resulting sum is odd, count $h!/\prod b_i!$ colorings. In particular, for $h = 5$ and $I_3$, only the permutations of $(3, 1, 1)$ qualify, giving $3 \cdot 20 = 60$. For the table $I^{e_0}$ on $K_4$, there are exactly 20 detections.

It remains to check that $I_3$ works on four vertices. The table gives all ten types of nonempty simple graphs, padded with isolated vertices to four vertices. A partition of the vertices into $j \le 3$ nonempty classes gives $3!/(3 - j)!$ colorings. We count it when the total number of edges within its classes is odd.

| Graph | Out of 81 | Graph | Out of 81 |
|---|---|---|---|
| $K_2$ | 27 | $K_3$ | 63 |
| $P_3$ | 36 | $C_4$ | 24 |
| $2K_2$ | 36 | $K_3$ with a pendant edge | 48 |
| $K_{1,3}$ | 39 | $K_4$ minus one edge | 51 |
| $P_4$ | 39 | $K_4$ | 60 |

Thus $I_3$ has minimum $24/81$ for $n = 4$. Testing $C_4, K_2, K_3, K_4, K_5$ in succession leaves $32, 26, 25, 22, 21$ tables. Attainability for graphs of arbitrary size is proved in Section 4.

**Realizability.** The word $AA\,BB\,CC\ldots$ gives the empty graph. To obtain a clique, concatenate two copies of a permutation of the letters. For $K_{u,v}$, take words $U, V$ on disjoint sets of distinct letters and write $UVU^{\mathrm{rev}}V^{\mathrm{rev}}$. Exactly the pairs with letters from opposite parts alternate in this word. For example, $AB\,CD\,BA\,DC$ realizes $C_4$. Additional consecutive pairs of new letters give isolated vertices. This realizes every obstruction and every stated equality graph as the difference of two logs.

## 3. A homogeneous quadratic form for the gauge transformation

Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, where $\omega^2 + \omega + 1 = 0$. Write $U = \mathbb{F}_4^*$ and $\operatorname{Tr} u = u + u^2$. On $U$, the trace vanishes only at 1. Under uniform roles, the variables $z_v = \omega^{X_v - s}$, with $s = 2k \bmod 3$, are independent and uniform on $U$.

Set $t = 0$ for $g = 0$, and set $t = \omega^{s-j}$ for $g = e_j$. Then $\mathbf{1}[X_u + X_v = k] = 1 + \operatorname{Tr}(z_u z_v)$ and $e_j(X_v) = 1 + \operatorname{Tr}(t z_v)$ in the latter case. The two constants from the gauge terms on an edge cancel. Let $O$ be the set of vertices of odd degree, $m = |E(H)|$, and $d = t^2$. Define

$$R_d(z) = \sum_{uv \in E(H)} z_u z_v + d \sum_{v \in O} z_v^2, \qquad b = (1 + \varepsilon)m \bmod 2.$$

**Exact identity.** The output bit is $b + \operatorname{Tr} R_d(z)$ in $\mathbb{F}_2$. Indeed, summing the gauge terms leaves $\operatorname{Tr}(t \sum_{v \in O} z_v)$, and $\operatorname{Tr} y = \operatorname{Tr} y^2$ in $\mathbb{F}_4$. The coefficient $d = 1$ corresponds precisely to $g = e_s$. For a nonempty graph, $R_d$ is a nonzero polynomial because each edge contributes its own mixed monomial.

For a polynomial $R$, write $\mathrm{w}(R) = \mathbb{P}(R(z) \neq 0)$. Multiplying all $z_v$ simultaneously by $\omega$ multiplies $R_d$ by $\omega^2$. Its nonzero values therefore have equal probabilities. We obtain

$$P(H) = \begin{cases} \frac{2}{3}\,\mathrm{w}(R_d), & b = 0, \\ 1 - \frac{2}{3}\,\mathrm{w}(R_d), & b = 1. \end{cases} \tag{1}$$

In the second line, $P \geq 1/3 > 8/27$ always holds.

**What happens when a vertex is pinned.** If $t \neq 0$, adjoining a vertex with fixed label $t$ and connecting it to every vertex of $O$ gives the form $Q + t \sum_O z_v$, where $Q = \sum_{uv} z_u z_v$. The cardinality of $O$ is even. The trace of this form agrees exactly with the trace of $R_d$. For a single edge $xy$ and $t = 1$, the values of $xy + x + y$ on the nine input pairs have multiplicities $2, 5, 1, 1$ for $0, 1, \omega, \omega^2$, respectively. Thus one cannot assume that the nonzero values of the pinned form are conditionally uniform. Equation (1) uses the uniformity of the nonzero values of the homogeneous form $R_d$.

**Linear sum.** For $L = \sum_{i=1}^{r} a_i z_i$ with $a_i \neq 0$, we have

$$\eta_r = \mathbb{P}(L \neq 0) = \tfrac{3}{4}\big(1 - (-1/3)^r\big). \tag{2}$$

Indeed, $\eta_0 = 0$ and $\eta_{r+1} = 1 - \eta_r/3$. Each new term is sampled uniformly from $U$. This gives $\eta_1 = 1,\ \eta_2 = 2/3,\ \eta_3 = 7/9,\ \eta_4 = 20/27$. For $r \geq 2$, we have $2/3 \leq \eta_r \leq 7/9$, with the minimum attained only at $r = 2$.

## 4. A low-weight lemma and proof of Theorem 1

**Lemma.** Let $R \neq 0$ be a homogeneous quadratic polynomial over $\mathbb{F}_4$, evaluated at independent uniform elements of $U$. Then $\mathrm{w}(R) = 1/3$ or $\mathrm{w}(R) \geq 4/9$. The first case occurs precisely for forms

$$az_i^2 + \lambda z_i z_j + (\lambda^2/a) z_j^2, \qquad a, \lambda \neq 0, \quad i \neq j.$$

If $a \neq 0$ in $R = ax^2 + Ax + B$ and $\mathrm{w}(R) = 4/9$, then $D = B + A^2/a$ has weight $1/3$ as a polynomial in the remaining variables.

**Proof.** A polynomial of degree at most two in each variable is determined by its values on the torus with coordinate set $U$. This follows by applying univariate interpolation successively in every variable.

If there are no square terms, choose a variable appearing in a mixed monomial. Write $R = Ax + B$, where $A$ is a nonzero linear form in $r \geq 1$ variables. When $A \neq 0$, at least two of the three possible values of $x$ give a nonzero result. Thus $\mathrm{w}(R) \geq (2/3)\eta_r \geq 4/9$. More precisely,

$$\mathrm{w}(Ax + B) = \tfrac{2}{3}\mathbb{P}(A \neq 0) + \tfrac{1}{3}\mathbb{P}(A \neq 0, B = 0) + \mathbb{P}(A = 0, B \neq 0). \tag{3}$$

Now suppose that the coefficient $a$ of $x^2$ is nonzero. Each specialization of $ax^2 + Ax + B$ has at most two roots, so $\mathrm{w}(R) \geq 1/3$. Together with the preceding paragraph this gives a bound of $1/3$ for any nonzero homogeneous quadratic form, including $D$ when $D \neq 0$.

Two distinct nonzero roots whose sum is $u = A/a$ in $\mathbb{F}_4$ must be $\omega u, \omega^2 u$. Their product is $u^2$. Hence if $D$ takes a nonzero value, at least two values of $x$ give a nonzero result. Averaging over the remaining variables yields

$$\mathrm{w}(R) \;\geq\; \tfrac{1}{3} + \tfrac{1}{3}\,\mathrm{w}(D).$$

If $D$ is a nonzero polynomial, this yields $4/9$. Equality requires $\mathrm{w}(D) = 1/3$.

If $D \equiv 0$, then $R = a(x + \omega A/a)(x + \omega^2 A/a)$. When $A \neq 0$, the form is nonzero at one of the three values of $x$. When $A = 0$, it is nonzero at all three. Hence $\mathrm{w}(R) = 1 - \frac{2}{3}\eta_r$, where $r$ is the number of variables in $A$. For $r = 0$, the weight is 1. For $r = 1$, we obtain the stated form of weight $1/3$. For $r \geq 2$, the weight is at least $13/27 > 4/9$. This proves the lemma. ∎

**Application to $R_d$.** The mixed terms enumerate exactly the edges of $H$. Weight $1/3$ therefore requires a single edge $xy$, in which case the coefficients of $x^2, y^2$ are both $d$ and that of $xy$ is 1. The lemma requires $d = 1/d$, hence $d = 1$. On this edge, (1) gives $7/9$ for $\varepsilon = 0$ and $2/9$ for the already excluded complement. In all remaining cases, (1) and $\mathrm{w}(R_d) \geq 4/9$ give $P \geq 8/27$. The four-cycle attains the bound. This completes the proof of Theorem 1 for all $n$. ∎

## 5. All equality cases

By (1), equality $P = 8/27$ requires $b = 0$ and $\mathrm{w}(R_d) = 4/9$. We omit isolated vertices in what follows.

**There is a vertex $v$ of positive even degree $r$.** Its square term is absent from $R_d$. In (3), take $x = z_v$ and $A = \sum_{h \sim v} z_h$, with $B$ the remaining part of $R_d$. Since $\mathrm{w}(R_d) \geq (2/3)\eta_r$, equality requires $r = 2$ and $B = 0$ whenever $A = 0$. Call the neighbors $i, j$. The condition $A = 0$ is exactly $z_i = z_j$. After this substitution, $B$ vanishes on the entire torus. The substituted polynomial still has degree at most two in each variable, so all its coefficients vanish.

It follows that no edge has both endpoints outside $\{v, i, j\}$. After substitution, the coefficient of $z_i z_h$ shows that each outside vertex $h$ is adjacent to either both of $i, j$ or neither. Thus $i, j$ have equal degrees, whether or not $ij$ is an edge. Their square terms cancel. The remaining coefficient of $z_i^2$ is the indicator of the edge $ij$, so $ij$ is absent. The graph is $K_{2,\ell}$, with part $\{i, j\}$ and $\ell \geq 1$ common neighbors. This square coefficient accounts for the possible edge between the identified vertices.

Put $A = z_i + z_j$, $L = \sum_{h \sim i} z_h$, and $e = \ell \bmod 2$. In this graph,

$$R_d = A(L + deA), \qquad \mathrm{w}(R_d) = \begin{cases} \eta_2 \eta_\ell, & de = 0, \\ \eta_2(1 - \eta_\ell/3), & de \neq 0. \end{cases}$$

The variables on the two parts are independent, and each nonzero value of $L$ has probability $\eta_\ell/3$. In the first line the weight $4/9$ is possible only at $\ell = 2$. In the second it requires $\eta_\ell = 1$, hence $\ell = 1$. These give exactly $C_4$ and, when $d \neq 0$, $P_3$. Both graphs have an even number of edges, so $b = 0$ for either value of $\varepsilon$.

**All positive degrees are odd.** If $d = 0$, (3) at any nonisolated vertex gives $\mathrm{w}(R_d) \geq (2/3)\eta_r > 1/2$, since $r$ is odd. Suppose $d \neq 0$. Each such vertex then has a nonzero square term in $R_d$. By the lemma, weight $4/9$ requires the residual polynomial $D = B + A^2/d$ to have weight $1/3$ for every $v$. Such a polynomial contains exactly one mixed monomial. Since $A^2$ contains no mixed terms, $H - v$ has exactly one edge, so $\deg(v) = m - 1$.

For $h$ nonisolated vertices, we obtain $h(m - 1) = 2m$, or $(h - 2)(m - 1) = 2$. The solution $(h, m) = (3, 3)$ has even degrees. The other solution is $2K_2$. On one edge the admissible tables have detection probability $p_1$ equal to $1/3, 2/3$, or $7/9$, as follows by counting their ones. On two independent edges, the probability $2p_1(1 - p_1)$ is $4/9$ or $28/81$, both strictly greater than $8/27$. This exhausts the graphs and proves Theorem 2. ∎

## 6. The second minimum for the original table

Let $F = F_0$ and $Q = \sum_{uv \in E(H)} z_u z_v$. Suppose $P(H) \le 80/243$. By (1), the number of edges is even, since for odd $m$ the probability is at least $1/3$. Hence $\mathrm{w}(Q) \le 40/81$. At any vertex of positive degree $r$, (3) gives $\mathrm{w}(Q) \ge (2/3)\eta_r$. For odd $r$ this bound is strictly greater than $1/2$, and for even $r \ge 6$ it is strictly greater than $(2/3)\eta_4 = 40/81$. Only degrees 2 and 4 remain.

**There is a vertex of degree 2.** Write $Q = z_v A + B$, where $A = z_i + z_j$. The event $A = 0$ has probability $1/3$. Conditional on this event, the shared value $z_i = z_j$ and all the remaining outside variables are independent and uniform on $U$. If $B|_{z_i = z_j}$ is a nonzero polynomial, the lemma gives conditional weight at least $1/3$. By (3) we would then have

$$\mathrm{w}(Q) \ge \tfrac{2}{3} \cdot \tfrac{2}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3} = \tfrac{5}{9} > \tfrac{40}{81}.$$

Thus the restriction $B|_{z_i = z_j}$ is identically zero. The coefficient comparison in Section 5 uses only this vanishing and the two neighbors of $v$, so it gives $H = K_{2,\ell}$ up to isolated vertices here as well. For this family $m = 2\ell$ and

$$P_{F_0}(K_{2,\ell}) = \tfrac{2}{3}\eta_2\eta_\ell = \tfrac{4}{9}\eta_\ell.$$

Over positive $\ell$, the smallest value of $\eta_\ell$ occurs at $\ell = 2$, and the next at $\ell = 4$. Indeed, the even subsequence in (2) increases strictly to $3/4$, and the odd subsequence lies above $3/4$. The two probabilities are $8/27$ and $80/243$.

**All positive degrees are 4.** Choose a vertex $v$ with four neighbors. The first term of (3) is already $40/81$, so $B$ vanishes whenever $A = 0$. Partition the neighbors into two pairs and equate the labels within each pair. Then $A = 0$, and interpolation shows that the resulting polynomial in the pair labels and the outside labels is identically zero. The coefficient of the square of the first pair label is the indicator of the edge inside that pair. Since any pair may be chosen first, there are no edges among the neighbors of $v$. Edges between outside vertices are also absent. For any outside vertex, the coefficient of its variable times the pair label gives equality of its adjacencies to the two members of that pair. The three pairings together imply identical adjacency to all four neighbors.

It follows that $H = K_{4,\ell}$ up to isolated vertices, and $\mathrm{w}(Q) = \eta_4\eta_\ell$. The bound and the first term of (3) give $\mathrm{w}(Q) = 40/81$, which requires $\eta_\ell = 2/3$, hence $\ell = 2$. This would give vertices of degree 2, contrary to the case assumption. The case is exhausted.

For $n = 4, 5$, $K_{2,4}$ does not fit. Since probabilities are multiples of $1/3^n$, on both grids the smallest value strictly above $80/243$ is $1/3$. A single edge $K_2$ with isolated vertices attains it. This proves Theorem 3. ∎

## 7. Consequences of published results on codes

**Direct reduction for $F_0$.** The first note applied Theorem 2.4 of Carvalho and Patanker [3] directly in degree two. Their paper introduces the standing hypothesis $d \geq 3$ before Section 2. The following reduction meets the published hypotheses and gives both equality graphs.

Pad $H$ with isolated vertices to $N = \max(n, 6)$. For two new independent variables $u, v$, set

$$f(u, v, z) = (u + v)Q_H(z), \qquad \mathrm{w}(f) = \tfrac{2}{3}\,\mathrm{w}(Q_H).$$

This is a homogeneous squarefree cubic polynomial on the torus over $\mathbb{F}_4$ with $S = N + 2 \geq 8$ variables. When the number of edges is even, $\mathrm{w}(f) = P_{F_0}(H)$. By [1], the minimum relative weight of the cubic code is $(2/3)^3 = 8/27$. By Theorem 4.5 of [2], the next relative weight is $(2/3)^3(1 + 1/3^2) = 80/243$. The hypotheses $q = 4,\ d = 3,\ 2d + 2 \leq S$ hold.

By Theorems 2.4 and 3.3 of [3], a polynomial of minimum weight factors into three linear binomials with disjoint supports. At the next weight it factors into two binomials and one linear form with four nonzero coefficients, again with pairwise disjoint supports. Unique factorization in the polynomial ring requires one binomial to be associated to $u + v$. Canceling that factor leaves $Q_H$ as a product of linear forms on $2, 2$ or $2, 4$ variables. Their mixed monomials give exactly $K_{2,2}$ or $K_{2,4}$. Both graphs attain the corresponding values. When the number of edges is odd, $P \geq 1/3$. Thus for the fixed original table, the values and both equality cases follow from published results.

**Gauge transformations.** The numerical gap between weights $1/3$ and $4/9$ in the lemma of Section 4 also follows from Carvalho's result on full Cartesian codes. Theorem 2.4 of [4], with coordinate sets of size 3 and degree two, gives second relative weight $4/9$. Our proof additionally determines when the minimum weight is compatible with the graph coefficients and what remains upon removing a vertex in the case of weight $4/9$.

**Ternary incidence code.** Let $B_H$ be the unoriented incidence matrix, with rows indexed by vertices and columns by edges, over $\mathbb{F}_3$, and put $C = \operatorname{im} B_H^{\mathsf{T}}$. For $F_0$, the vector $B_H^{\mathsf{T}} X$ is uniform on $C$, and its zero coordinates correspond to active edges. Therefore

$$1 - 2P_{F_0}(H) = \frac{(-1)^m}{|C|} \sum_{c \in C} (-1)^{\mathrm{wt}(c)}.$$

The quantity needed is the value of the weight enumerator at $-1$. The minimum distance of the code alone does not determine it. Reference [8] studies, among other things, distances and minimum weight words of incidence codes.

## 8. Prior work and reproducibility

**Comparison of problem settings.** Reyzin and Srivastava [5, Lemma 11] obtain a 1/4 guarantee for the parity of the number of edges in a random induced subgraph when each vertex is included with probability 1/2. This is a special two-role test. In Bshouty and Mazzawi [6, Section 2.5, Lemma 2.6], the rank-two setting has three equally likely roles, independently assigned across vertices. The test asks whether a specified multilinear form has a nonzero value on disjoint sets, and the general lower bound in this setting is 1/9.

Marks, Norwood, and Poole [7, equation (2.2) and Theorems 2.4 and 2.6] count odd rectangles in a binary matrix and study invariance under row and column flips. Under uniform roles on $C_4$, six odd rectangles give $4 \cdot 6/3^4 = 8/27$. The present problem also requires an upper bound for arbitrary common role weights and a lower bound holding simultaneously for all nonempty graphs.

The graph of active pairs satisfying $F(X_i, X_j) = 1$ is a vertex random graph in the sense of Beer et al. [12, Definition 2.11], with three types. Our objective is the probability that its edge set has odd intersection size with that of a given $H$. For $W_{ij} = (-1)^{F(i,j)}$, the quantity $1 - 2P(H)$ equals a weighted homomorphism sum with vertex weights $p_i$ and edge weights $W_{ij}$. This gives a partition function formulation [9]. Small-bias spaces [10] also test parities of all nonempty sets of random bits. Here the bias is bounded on one side, and all bits must be generated by a common table applied to independent vertex roles. General bounds on polynomial zeros on finite grids [11] are related to the weight bound.

As of 18 November 2026, the literature examined in these areas had yielded neither an equivalent of Theorem 1 with joint optimization over $F, p$ nor a classification of equality graphs for all 15 nontrivial gauge transformations. These are the possible novelty claims. These statements are the primary targets for independent checking. Absolute priority and external peer review remain open.

**Exact computations.** The following results were obtained under uniform roles. The last two columns refer to $F_0$ and give the number of detections out of $3^n$, followed in parentheses by the number of graphs.

| $n$ | Nonempty graphs | Optimal $F$ | First weight | Second weight |
|---|---|---|---|---|
| 4 | 63 | 22 | 24 (3) | 27 (6) |
| 5 | 1 023 | 21 | 72 (15) | 81 (10) |
| 6 | 32 767 | 21 | 216 (45) | 240 (15) |
| 7 | 2 097 151 | 21 | 648 (105) | 720 (105) |

The program `verify_three_roles.py` checks every graph of each size for each table and determines the exact sets of equality graphs for the optimal tables. It applies a Walsh transform to the edge-mask frequencies for the $3^n$ colorings. The coefficient for mask $H$ is $3^n - 2\,\#\{\text{detections}\}$. An independent direct count was also performed on four vertices.

## References and reproducibility materials

The original note, *Three Roles and One Bit*, and the original script `verify_interlacement.py` were checked in the supplied versions. A full run of that script confirmed all results claimed in it. Reproducing the new calculations requires Python 3.10 or later. The full enumeration also requires NumPy. The short run includes all five witness graphs, direct enumeration on four vertices, the role-weight grid, and verification of the identity for all 24 representations over $\mathbb{F}_4$.

```
python3 anc/verify_three_roles.py -small
python3 anc/verify_three_roles.py
```

With bit positions ordered $00, 01, 02, 11, 12, 22$ from least significant to most significant, the table identifiers can be checked by $k$. The four original tables with gauge terms are listed first, followed by the three admissible complements.

| $k$ | $\varepsilon = 0$ | $\varepsilon = 1$ |
|---|---|---|
| 0 | 3, 5, 17, 23 | 46, 58, 60 |
| 1 | 34, 36, 48, 54 | 15, 27, 29 |
| 2 | 10, 12, 24, 30 | 39, 51, 53 |

**[1]** D. Jaramillo, M. Vaz Pinto, R. H. Villarreal. *Evaluation codes and their basic parameters.* Designs, Codes and Cryptography 89 (2021), 269–300, Theorem 4.5. Source.

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## Methodology and contributions

**Origin of the research.** Viktor Diakin invited Astra to choose a field and problem independently and develop a mathematical theory from scratch. In his reconstructed account, Astra chose to study the detection of differences between event logs. Diakin collected objections and responses from several models, reconstructed the problem after the initial conversation was deleted, and initiated the checks of the classification.

**Astra (GPT-6.2 Astra Max).** The early note, historically credited to Viktor and Astra, already contained the rule $F_0$, the construction with three equally likely roles, and the minimax proof with its upper bound on $C_4$. The original Python code accompanied the note. The note also recorded the computation for all 64 tables at $n = 6$, when exactly 21 tables attain the optimum.

**Mira (GPT-6).** According to the research record, Mira identified additional candidates beyond the list of six proposed by Gemini. Mira separately rechecked the count of 22 at $n = 4$ and exhibited a table on which $K_4$ yields only 20/81. This distinguished the exceptional four-vertex case from the problem at arbitrary $n$.

**Claude Sonnet 5.5, DeepSeek, and Gemini.** Claude Sonnet 5.5 reported examining the lower-bound argument over $\mathbb{F}_4$ and independently computing the cases $n = 4, 5, 6, 7$. It reported finding no errors within the scope of those checks. This was model-assisted checking and does not constitute formal verification for all $n$ or human peer review. DeepSeek proposed investigating a possible counterexample for the table with a single one in entry 00. The analysis with Mira found that 25/81 is the probability of at least one active edge on $C_4$, while the parity event has probability 16/81. DeepSeek acknowledged the distinction. In the reconstructed account, Gemini revised its conjecture of six tables, proposed the $12 + 9$ family, and suggested an argument with a pinned vertex.

**Proof for arbitrary numbers of vertices.** In a follow-up Work session that Diakin identifies as Astra's work, the homogeneous form $R_d$ was obtained, the proof for arbitrary $n$ was developed, the equality cases were classified, and the cubic reduction to published results was established. The outputs of the two programs in the research archive reproduce the numerical claims. System names and versions follow Diakin's records. The account of the research discussions draws on supplied materials and surviving scientific messages. The deleted first conversation is known through Diakin's reconstruction. The AI systems listed here contributed as research tools. Viktor Diakin is the submitting author.

**Status of version 1.0.** The archival version is dated 18 November 2026 and was prepared for independent assessment by a mathematician. The classification of weight 1/3 in Section 4, the complete analysis of equality graphs in Section 5, and the scope of the novelty claims in Sections 7–8 are of particular relevance to that assessment. Publication priority remains open.