hexagon:2611.00602v1

COUNTING POWERFUL PRODUCTS OF CONSECUTIVE INTEGERS

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Abstract. We show that the number of $n \le x$ for which n and $n+1$ are both powerful is $O_\varepsilon(x^{1/4+\varepsilon})$, improving the exponent 29/100 of Reuss. For each fixed $k \ge 3$ we show that the number of $n \le x$ for which $(n+1)(n+2)\cdots(n+k)$ is powerful is $O_{k,\varepsilon}(x^{77/320+\varepsilon})$, and we prove the same exponent uniformly over all widths $k \ge 3$. The new ingredient is a per-interval count in Reuss's form of Heath-Brown's approximate determinant method: in lattice-reduced coordinates, the solutions on an auxiliary curve of bidegree (α, β) become integral points of height at most $L_1^3 T_1^2$ on an affine curve of degree $2\alpha + 3\beta$, to which the uniform bound of Castryck, Cluckers, Dittmann and Nguyen applies. The proofs are formalized in Lean 4 with Mathlib and checked by Lean's kernel and an independent kernel; the formal statements take six theorems from the literature as explicit hypotheses. The results and exposition were produced and reviewed using AI systems; no human mathematical review has been performed. Section 10 describes the verification and its scope.

1. Introduction

A positive integer m is *powerful* (or square-full) if $p^2 \mid m$ for every prime $p \mid m$. Every powerful number can be written uniquely as d^3u^2 with d squarefree, so there are $\asymp x^{1/2}$ powerful numbers up to x.

1.1. **Consecutive powerful numbers.** Pairs of consecutive powerful numbers exist in abundance: every solution of $X^2 - 8Y^2 = 1$ gives the pair $8Y^2$, X^2, and this Pell family alone contributes $\asymp \log x$ pairs up to x; see [10, 18] for other families. Erdős conjectured that there are no three consecutive powerful numbers [7, 6][2, Problem #364], and it has been conjectured that the number of consecutive pairs up to x is $(\log x)^{O(1)}$; see [7], [8, p. 68], and [17, (1.15)], [2, Problem #365]. Erdős also asked whether a product of $k \ge 3$ consecutive integers can be powerful [2, Problem #137]; by the Erdős–Selfridge theorem such a product is never a perfect power [9]. Under the *abc* conjecture the number of consecutive pairs is $x^{o(1)}$ and consecutive triples are finite in number; see [17, (1.16) and the following discussion].

Unconditionally, the trivial bound for

$$N(x) := \#\{n \le x : \ n \text{ and } n+1 \text{ are powerful}\}$$

is $O(x^{1/2})$. Chan [5] proved that the number of $n \le x$ with n and $n+l$ both powerful is $\ll d_3(l)x^{2/5}(\log x)^2$ uniformly in $l \ge 1$ [5, Theorem 3], and $\ll_l x^{7/19}\log x$ for fixed l [5, Theorem 4]. Aktaş and Murty [1] gave an $O(x^{2/5})$ bound for $N(x)$. Tao records $x^{2/5}$ as the best known bound [17, (1.16)] and proves an estimate of the same strength, $x^{2/5+o(1)}$, for the non-powerful integers lying in intervals of length at least two with powerful product [17, Theorem 1.8]. Reuss [14, Theorem 4] proved $N(x) \ll_\varepsilon x^{29/100+\varepsilon}$ with his extension of Heath-Brown's approximate determinant method [13, 11]; the exponent 29/100 is also recorded in [19]. We are not aware of a published improvement of 29/100.

Date: Version 1.0.1, 24 October 2026.

1.2. **Results.**

Theorem 1.1. *For every $\varepsilon > 0$,*

$$\#\{n \le x : \ n \text{ and } n+1 \text{ are powerful}\} \ll_\varepsilon x^{1/4+\varepsilon}.$$

The exponent $1/4$ is the exact limit of our method for $k = 2$ (Remark 9.1). For $k \ge 2$ and $x \ge 1$ put

$$N_k(x) := \#\{0 \le n \le x : \ (n+1)(n+2)\cdots(n+k) \text{ is powerful}\}.$$

For $k \ge 3$ the product can be powerful without any single factor being powerful, because primes $p < k$ may divide two factors. Every n counted by $N_k(x)$ has, for $1 \le i \le k$, a decomposition $n+i = f_i d_i^3 u_i^2$, where d_i is the product of the primes $p \ge k$ that divide $n+i$ to an odd power (Lemma 2.1). Define

$$N_k^{\mathrm{cen}}(x) := \#\{n \le x : \ (n+1)\cdots(n+k) \text{ is powerful, and } (n+3)^{19/100} < d_i < (n+3)^{21/100} \text{ for } i = 1,2,3\}.$$

For $k = 3$ this is the "central box" of the three-term problem, the region in which elementary pairwise estimates are weakest.

Theorem 1.2. *Let $k \ge 3$ be fixed. For every $\varepsilon > 0$,*

$$N_k(x) \ll_{k,\varepsilon} x^{77/320+\varepsilon} \qquad \text{and} \qquad N_k^{\mathrm{cen}}(x) \ll_{k,\varepsilon} x^{38059/160000+\varepsilon}.$$

Here $77/320 = 0.240625$ and $38059/160000 = 0.23786875$. We know of no bound for $N_k(x)$, $k \ge 3$, stated in the literature, but two bounds follow at once from the literature: Chan's fixed-shift bound [5, Theorem 4] applied to the pair equation of Lemma 2.2 gives $N_k(x) \ll_k x^{7/19} \log x$, and [17, Theorem 1.8 and (1.16)] give $N_k(x) \ll x^{2/5+o(1)}$. Both deductions are ours. Theorem 1.2 improves both exponents.

Theorem 1.3 (Uniformly over all widths)**.** *For every $\varepsilon > 0$,*

$$\#\{0 \le n \le x : \exists k \ge 3, \ P(n,k) \text{ powerful}\} \ll_\varepsilon x^{77/320+\varepsilon}.$$

With $k \ge 2$ in place of $k \ge 3$, the bound is $O_\varepsilon(x^{1/4+\varepsilon})$. The number of non-powerful positive integers at most x which belong to an interval of consecutive positive integers with powerful product is $O_\varepsilon(x^{77/320+\varepsilon})$. The implied constants are not explicit: they depend on the threshold in Theorem 3.6.

Here $P(n,k) = (n+1)\cdots(n+k)$. The first bound counts the union of starts over all widths, with one constant independent of k. The last assertion bounds the quantity in Tao's Theorem 1.8.

1.3. **Method.** Write $n+i = f_i d_i^3 u_i^2$ and $n+j = f_j d_j^3 u_j^2$. After absorbing the bounded coefficients f_i, f_j, each pair of factors gives a solution of

$$e^3 v^2 - d^3 u^2 = h, \tag{1.1}$$

with $h \ne 0$ bounded in terms of k (Lemma 2.2). Reuss [14, §2] splits the range of $\rho = d/e$ into $\asymp M$ intervals I, with $\log M = \frac{9}{8}(1+\delta)\log(DE)\log(UV)/\log x$, and shows by the determinant method that on each interval all solutions lie on one auxiliary curve of bounded degree (Lemma 5.1). He then passes to lattice-reduced coordinates $(\lambda, \tau) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with $|\lambda_i| \le L_i$, $|\tau_i| \le T_i$, and counts the points of the auxiliary curve using one factor at a time, which gives $\ll x^\varepsilon \min(L_1, T_1)$ solutions per interval.

Our new step (Lemma 5.5) also uses the equation (1.1) itself. If $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a component of bidegree (α, β) of the auxiliary curve, the twelve monomials of bidegree $(3,2)$ in (λ, τ) map the solutions on Γ, at most two-to-one, to integral points of height at most $L_1^3 T_1^2$ on the affine curve obtained by slicing the cone over the Segre–Veronese image of Γ with the hyperplane $\{\ell = h\}$, where ℓ represents the left-hand side of (1.1). This curve has degree $2\alpha + 3\beta$, and the bound of Castryck,

Cluckers, Dittmann and Nguyen [4] gives $\ll (L_1^3T_1^2)^{1/(2\alpha+3\beta)}\log(2+L_1T_1)$ solutions, independently of the coefficients. Components with $\min(\alpha,\beta)=1$ are rational curves; they carry $x^{o(1)}$ solutions by Stewart's bound for Thue equations [16] (Proposition 5.3), once a Riemann–Hurwitz argument shows that the relevant binary form has at least three distinct roots (Lemma 5.2). Summing over intervals with tail bounds for the reduced lattices and the Cauchy–Schwarz inequality gives a per-box exponent $B_{\mathrm{red}} = m + \frac{3}{10}\ell^* + \frac{1}{5}t^*$ (Theorem 6.3); Reuss's corresponding exponent is $m + \min(\ell^*, t^*)$ (Remark 9.2). The final exponents are the suprema of explicit piecewise-quadratic functionals, bounded by exact certificates (Section 7).

For $k \geq 3$ the proof uses only three pairwise inputs: the Pell count, the Thue count and B_{red}. For $k = 2$ it also uses Reuss's Theorem 1, on windows whose cube-part scales are both at least a fixed power of x, and a Heath-Brown-type bound from [14, §5].

1.4. **What is not claimed.** The exponent $1/4$ of Theorem 1.1 is exactly the supremum of the functional behind it, attained at a degenerate corner where three bounds that were known before this paper meet (Remark 9.1). We obtain nothing below $1/4$ for $k = 2$. A uniform saving over the exponent $1/10$ for the components of bidegree $(2,2)$ would lower $77/320$, for example to $89/384$ (Remark 9.3); we do not prove such a saving. The constants of Theorem 1.3 are not explicit, because the threshold in Tao's Lemma 3.1 is not; we do not claim explicit constants anywhere. The formal proofs assume the six results from the literature stated in Section 3; they do not prove them.

1.5. **Formal status.** All three theorems, and the propositions and lemmas of Sections 2 to 8, are proved in Lean 4 (version 4.33.1) with Mathlib. The Lean statements are the theorems above, with the implied constants written as explicit existential quantifiers, and with the results of Section 3 as named hypotheses (`let`-bound propositions assumed in the statement; no axiom is added). Their axiom sets are exactly `propext`, `Classical.choice` and `Quot.sound`. Appendix A lists the Lean declaration for each statement, and Section 10 describes the checks.

Organisation. Section 2 sets up the pair equations, and Section 3 states the results from the literature with their Lean forms. Sections 4 to 8 give the proof. The steps that carry the new argument (Lemma 5.2, Proposition 5.3, Lemma 5.5, Proposition 5.6, Theorem 6.3 and Lemma 8.1) are proved in full, following the route of the corresponding Lean proof; the remaining steps are sketched. The Lean declarations listed in Appendix A are the reference for every statement. Section 9 collects remarks, and Section 10 describes verification and the use of AI systems.

2. Set-up

2.1. **Notation.** We write $y \sim Y$ for $Y < y \leq 2Y$; dyadic scales Y run over $\{2^j : j \geq -1\}$, so that $y = 1$ has the scale $Y = 1/2$. Implied constants may depend on k, on ε and on the auxiliary parameters δ, c introduced below; any further dependence is indicated. We put $P_k := \prod_{p<k} p$. For a polynomial G with integer coefficients, $H(G)$ is the maximum absolute value of its coefficients.

2.2. **The decomposition.**

Lemma 2.1. *Let $k \geq 2$ and $n \geq 0$ be such that $(n+1)\cdots(n+k)$ is powerful. For $1 \leq i \leq k$ put*

$$f_i := \prod_{\substack{p<k \\ v_p(n+i)\ \text{odd}}} p, \qquad d_i := \prod_{\substack{p\geq k \\ v_p(n+i)\ \text{odd}}} p.$$

Then $n+i = f_i d_i^3 u_i^2$ for a positive integer u_i; $f_i \mid P_k$; d_i is squarefree and coprime to P_k. For $i \neq j$ we have $\gcd(d_i, d_j) = 1$, $\gcd(d_j, f_i u_i) = 1$ and $\gcd(u_i, u_j)^2 \mid (j-i)$.

A prime $p \geq k$ divides at most one factor, so its odd valuation there is at least three; removing $f_i d_i^3$ leaves a square.

Lemma 2.2 (Pair equation)**.** *In the situation of Lemma 2.1, let $1 \le i < j \le k$ and put $d := f_i d_i$, $e := f_j d_j$, $u := f_j u_i$, $v := f_i u_j$ and $h := f_i^2 f_j^2 (j-i)$. Then $e^3v^2 - d^3u^2 = h$ and $d^3u^2 = f_i^2 f_j^2 (n+i)$. Moreover $1 \le h \le kP_k^4$, $\gcd(d,e) \mid P_k$ and $\gcd(u,v) \le k^{1/2} P_k^2$. For fixed (i, j, f_i, f_j) the quadruple (d, e, u, v) determines n.*

For $k = 2$ this is the set-up of [14, §5], with $h = 1$.

2.3. **Windows and exponents.** Fix a nonzero integer h. Following [14, Theorem 1], for $D, E \ge 1/2$ and $x > 1$ put $U := x^{1/2} D^{-3/2}$, $V := x^{1/2} E^{-3/2}$, and call $W = (D, E, x)$ a *window*. Let $\mathcal{S}_W$ be the set of $(d, e, u, v) \in \mathbb{N}^4$ with $d \sim D$, $e \sim E$, $U < u < 2U$, $V < v < 2V$ and $e^3v^2 - d^3u^2 = h$. We use the exponents

$$a = \frac{\log D}{\log x}, \quad b = \frac{\log E}{\log x}, \quad s = a + b, \quad t = \frac{\log(UV)}{\log x} = 1 - \frac{3s}{2}, \quad \bar{u} = \frac{1-3a}{2}, \quad \bar{v} = \frac{1-3b}{2}, \tag{2.1}$$

so $U = x^{\bar{u}}$ and $V = x^{\bar{v}}$, and

$$m = m(s) := \frac{9}{8}\, st = \frac{9}{8}\, s\Big(1 - \frac{3s}{2}\Big), \tag{2.2}$$

which is concave with maximum $m(1/3) = 3/16$.

Lemma 2.3 (Covering by windows)**.** *Let $D, E \ge 1/2$ and let x be large. Every $(d, e, u, v) \in \mathbb{N}^4$ with $d \sim D$, $e \sim E$, $x/2 < d^3u^2 \le 2x$ and $e^3v^2 - d^3u^2 = h$ lies in $\mathcal{S}_{W'}$ for one of $O(1)$ windows $W' = (D', E', x')$ with $D' \asymp D$, $E' \asymp E$ and $x' \asymp x$. This holds uniformly in shifts with $|h| \le x/100$.*

A grid $\{2^{j/8}\}$ for D', E', x' places the ratios D'^3u^2/x' and E'^3v^2/x' strictly inside $(1, 4)$; for $|h| \le x/100$ the same grid works uniformly in h.

3. Results from the literature

Six results from the literature enter the Lean statements as named hypotheses: four from journals and books, and two, Reuss's Theorem 1 and Tao's Lemma 3.1, from arXiv preprints. For each we give the statement as used, its source, and the Lean form. For Reuss's paper we use arXiv:1212.3150v2 and specialise to $k = 3$, $l = 2$ in his notation. For Theorems 3.1, 3.2, 3.3, 3.5 and 3.6 the Lean hypothesis is an instance or a weakening of the source statement, compared with the source text, so it holds if the source theorem holds. For Theorem 3.4 it is an instance of the statement in Heath-Brown's manuscript of [12], applied to $2F$ as explained after the theorem; the published text has not been compared (Section 10). The set counted in Theorem 1.1 is infinite (the Pell family above); for $k \ge 3$ the counted sets are not known to be nonempty.

Theorem 3.1 (Reuss [14, Theorem 1], $k = 3$, $l = 2$, $h = 1$)**.** *Let $h = 1$ and let $W = (D, E, x)$ be a window with $D, E > 1$ and $\log(DE) \asymp \log(UV) \asymp \log x$. Define $M > 1$ by $\log M = \frac{9}{8} \log(DE) \log(UV) / \log x$. Then for x large in terms of ε, the number of $(d, e, u, v) \in \mathbb{N}^4$ with $D < d < 2D$, $E < e < 2E$, $U < u < 2U$, $V < v < 2V$ and $e^3v^2 - d^3u^2 = 1$ is*

$$\ll_\varepsilon x^\varepsilon \min\{(DEM)^{1/2} + D + E,\ (UVM)^{1/2} + U + V\}.$$

Reuss writes $d \sim D$ for $D < d < 2D$. The Lean hypothesis fixes the comparability constants as $\frac{1}{20} \log x \le \log(DE), \log(UV) \le 20 \log x$ and requires the bound only on windows with $D, E \ge x^c$, for each fixed $c > 0$, with constants depending on c and ε; so it is an instance of Theorem 3.1. The declaration `reuss_window_bridge` proves that this open-endpoint form implies the closed-endpoint form $d \sim D$, $e \sim E$ of $\mathcal{S}_W$: for fixed (d, e) the equation has at most one solution with $u \in (U, 2U)$, so the endpoints $d = 2D$ and $e = 2E$ add $\ll x^\varepsilon \min(E, U) + x^\varepsilon \min(D, V)$. Reuss states his Theorem 1 for every fixed $h \ne 0$; his proof begins by reducing to $(ev, du) = 1$, a reduction that is immediate for

$h = \pm 1$. We use the theorem only with $h = 1$, that is, only for $k = 2$. In exponents, Theorem 3.1 bounds $\#\mathcal{S}_W$ by $x^{R(a,b)+\varepsilon}$ with

$$R(a,b) := \min\Big\{ \max\Big(\frac{s+m}{2}, a, b\Big),\ \max\Big(\frac{t+m}{2}, \bar{u}, \bar{v}\Big)\Big\}. \tag{3.1}$$

Theorem 3.2 (Castryck–Cluckers–Dittmann–Nguyen [4, Theorem 3]). *For each $N > 1$ there is $c(N)$ such that for all $d \geq 1$, all integral affine curves $C \subset \mathbb{A}^N_{\overline{\mathbb{Q}}}$ of degree d and all $B \geq 1$,*

$$N_{\mathrm{aff}}(C, B) := \#\{P \in C(\mathbb{Q}) \cap \mathbb{Z}^N :\ |P_i| \leq B\} \leq c(N)\, d^3 B^{1/d}(\log B + d).$$

The Lean hypothesis is the case $N = 12$, for every prime ideal $I \subset \mathbb{Q}[x_1, \ldots, x_{12}]$ whose quotient has Krull dimension one; the degree d is the leading coefficient of the Hilbert polynomial of the quotient. It therefore covers curves that are irreducible over $\mathbb{Q}$ but not geometrically irreducible. This is the extension stated in [4, Corollary 4.1.4 and Remark 4.1.5]: the rational points of such a curve lie on the intersection with a Galois conjugate, at most d^2 points.

Theorem 3.3 (Stewart [16, p. 796, consequence of Theorem 1]). *Let $Q \in \mathbb{Z}[r, s]$ be a binary form of degree $q \geq 3$ with content 1 and nonzero discriminant, and let m be a nonzero integer. Then*

$$\#\{(r,s) \in \mathbb{Z}^2 :\ \gcd(r,s) = 1,\ Q(r,s) = m\} \leq 2800\, q^{1+\omega(|m|)}.$$

Stewart's Theorem 1 assumes only content 1 and nonzero discriminant, so reducible forms are included, and it counts pairs (r, s), so (r, s) and $(-r, -s)$ are counted separately; the displayed bound is his consequence of Theorem 1 with $g = |m|$, the factor $1 + 1/(8\varepsilon q)$ disappearing as $\varepsilon \to \infty$ [16, pp. 795–796]. The Lean hypothesis is this statement, with "nonzero discriminant" expressed as squarefreeness of the form over $\mathbb{Q}$.

Theorem 3.4 (Heath-Brown [12, Theorem 15], $n = 2$; see also [15, Theorem 2]). *Let $F \in \mathbb{Z}[x_1, x_2]$ be absolutely irreducible of degree $d \geq 2$, let $\eta > 0$ and $B_1, B_2 \geq 2$. Let $T = \max B_1^{e_1} B_2^{e_2}$, the maximum over the exponent vectors of the monomials occurring in F. Then the integral zeros of F with $|x_i| \leq B_i$ all lie on the zero sets of at most $\ll_{d,\eta} T^{\eta} \exp(\log B_1 \log B_2 / \log T) \log(2\|F\|)$ polynomials of degree $O_{d,\eta}(1)$, each coprime to F.*

In [12, Theorem 15] (Section 2.6 of the author's manuscript of the notes), and in its restatement [15, Theorem 2], the degree is arbitrary, $B_i \geq 1$, the auxiliary polynomials have integer coefficients, are coprime to F and have degree bounded in terms of d and η, and the last factor is $\log \|F\|$, with $\|F\|$ the largest modulus of a coefficient. Applied to $2F$, which has the same integral zeros, degree, monomials and irreducibility as F, the theorem gives the same conclusion with the factor $\log \|2F\| = \log(2\|F\|)$. The Lean hypothesis asks only for complex polynomials not divisible by F, implied by integer polynomials coprime to F, and assumes $d \geq 2$ and $B_i \geq 2$, so it is an instance of the theorem applied to $2F$.

Theorem 3.5 (Gelfond's inequality, consequence; see [3, Lemma 1.6.11]). *For each n there is $c_n \geq 0$ such that if $G \mid F$ in $\mathbb{Z}[x_1, \ldots, x_n]$ with $F \neq 0$, then $H(G) \leq e^{c_n \deg F} H(F)$.*

Lemma 1.6.11 of [3] states that if $f = f_1 \cdots f_m$ for complex polynomials in n variables, then $2^{-d} \prod_j \ell_\infty(f_j) \leq \ell_\infty(f) \leq 2^d \prod_j \ell_\infty(f_j)$, where ℓ_∞ is the largest absolute value of a coefficient and d is the sum of the partial degrees of f. If $F = GK$ in $\mathbb{Z}[x_1, \ldots, x_n]$ then $\ell_\infty(K) \geq 1$ and $d \leq n \deg F$, which gives the displayed statement with $c_n = n \log 2$; the Lean hypothesis is this statement.

Theorem 3.6 (Tao [17, Lemma 3.1]). *If $(N+1)\cdots(N+H)$ is powerful, then $H < N$, and $H \leq \exp((\log N)^{2/3+o(1)})$ as $N \to \infty$.*

The Lean hypothesis is the consequence that for some N_0, every $N \geq N_0$ and $3 \leq k < N$ with $P(N, k)$ powerful has $k \leq \exp((\log N)^{7/10})$. It is used only in Theorem 1.3. No explicit value of N_0 is given in [17].

4. Elementary pair bounds

Lemma 4.1. *Let $h \neq 0$ and $G_0 \geq 1$ be fixed.*

(i) *For positive integers d, e and $Z \geq 2$, the number of $(u, v) \in \mathbb{N}^2$ with $u, v \leq Z$ and $e^3v^2 - d^3u^2 = h$ is $\ll_\varepsilon (deZ)^\varepsilon$.*

(ii) *For positive integers u, v, the number of $(d, e) \in \mathbb{N}^2$ with $\gcd(d, e) \leq G_0$ and $e^3v^2 - d^3u^2 = h$ is at most $G_0 \cdot 2800 \cdot 3^{1+\omega(|h|)}$.*

Consequently $\#\mathcal{S}_W \ll x^{s+\varepsilon}$ for every window W, and $\ll x^t$ if also $\gcd(d, e) \leq G_0$ on the counted set.

Part (i) is a Pell-type count: the solutions correspond to elements of norm he^3 in a real quadratic order, $\tau(|he^3|)$ ideal classes times $O(\log x)$ units. Part (ii) applies Theorem 3.3 to the binary cubic $v^2e'^3 - u^2d'^3$ after removing $\gcd(d, e)$.

Lemma 4.2. *Let $h = 1$ and let W be a window. The number of $(d, e, u, v) \in \mathcal{S}_W$ with d and e squarefree is*

$$\ll_\varepsilon x^\varepsilon EU\Big(1 + \frac{D}{E^3}\Big)^{1/2} \quad \text{and} \quad \ll_\varepsilon x^\varepsilon DV\Big(1 + \frac{E}{D^3}\Big)^{1/2};$$

in exponents, $H_1(a, b) := b + \bar{u} + \frac{1}{2}\max(0, a - 3b)$ and $H_2(a, b) := a + \bar{v} + \frac{1}{2}\max(0, b - 3a)$.

This is the estimate displayed after (30) in [14, §5] and its mirror image. Fixing e and u, the cube congruence $d^3 \equiv -u^{-2} \pmod{e^3}$ leaves $3^{\omega(e)}$ residue classes c; writing $d = c + e^3t$, the pairs (t, v) are integral zeros of the absolutely irreducible cubic $e^3w^2 - u^2(c + e^3t)^3 - 1$, and Theorem 3.4 with Bézout's theorem gives $\ll x^\varepsilon(1 + D/E^3)^{1/2}$ of them.

5. The per-interval count

Throughout this section $h \neq 0$ is fixed, $W = (D, E, x)$ is a window with $s, t \geq c$ for a fixed $c \in (0, 1)$, $\delta \in (0, 1)$ is fixed, x is large in terms of h, c and δ, and $M := \lceil x^{(1+\delta)m} \rceil$. Then $x^c \leq DE$, $x^c \leq UV$, $x^{c^2} \leq M \leq x^{1/2}$ and $\log M \geq \frac{9}{8}(1 + \delta)\log(DE)\log(UV)/\log x$. The range $D/(2E) < \rho \leq 2D/E$ of $\rho = d/e$ is covered by at most $3M + 1$ intervals $I = (\rho_0, \rho_0(1 + M^{-1})]$ with $\rho_0 \in \{\frac{D}{2E}(1 + M^{-1})^j : j \geq 0\} \cap [\frac{D}{2E}, \frac{2D}{E}]$. We fix a finite set $\mathcal{G}$ of positive integers and count solutions with $\gcd(d, e), \gcd(u, v) \in \mathcal{G}$; by Lemma 2.2 this covers the solutions coming from Lemma 2.1. Implied constants may depend on h, c, δ, $\mathcal{G}$ and ε.

5.1. The determinant step.

Lemma 5.1 (Reuss's Lemma 7 for fixed $h \neq 0$)**.** *Let $h \neq 0$, $c \in (0, 1)$ and $\delta \in (0, 1)$ be fixed. There are an integer $K_0 \geq 1$ and a real $\kappa > 0$, depending only on c and δ, with the following property. Let x be large in terms of h, c and δ, let W be a window with $DE \geq x^c$ and $UV \geq x^c$, and let M be an integer with $x^{c^2} \leq M \leq x$ and $\log M \geq \frac{9}{8}(1 + \delta)\log(DE)\log(UV)/\log x$. Let $I = (\rho_0, \rho_0(1 + M^{-1})]$ with $D/(2E) \leq \rho_0 \leq 2D/E$. Then there is a nonzero $C_I \in \mathbb{Z}[\rho, \omega]$ with $\deg_\rho C_I, \deg_\omega C_I \leq K_0$ and coefficients at most x^κ, such that $C_I(d/e, v/u) = 0$ for all $(d, e, u, v) \in \mathcal{S}_W$ with $d/e \in I$.*

This is [14, Lemma 7], whose proof begins with the reduction to $(ev, du) = 1$ recalled in Section 3; the reduction is not needed. The approximation $|v/u - (d/e)^{3/2}| \ll |h|(D/E)^{3/2}/x$ and a Taylor expansion make every maximal minor of the matrix $(\rho_j^a\omega_j^b)_{a \leq A, b \leq B}$ smaller than the reciprocal of its integral denominators, by Cauchy–Binet and a rearrangement bound for $\sum(i\ell_1 + k\ell_2)$ over H lattice points; the choice $A\log(DE) \approx B\log(UV)$ makes this work exactly when $\log M$ exceeds $\frac{9}{8}\log(DE)\log(UV)/\log x$. A kernel vector of the rank-deficient integral matrix gives C_I, with the given representations d_j/e_j, v_j/u_j as denominators, so no coprimality is used.

5.2. **Components of the auxiliary curve.** Let $G_I(d,e;v,u) := e^A u^B C_I(d/e, v/u)$, where $A, B \le K_0$ bound the degrees of C_I. It is a nonzero bihomogeneous form of bidegree (A,B) with coefficients at most x^κ, and it vanishes at every solution with $d/e \in I$. Factor G_I into irreducible elements of $\mathbb{Z}[d,e;v,u]$. There are at most $A+B \le 2K_0$ non-constant irreducible factors; each is bihomogeneous, and by Theorem 3.5 its coefficients are at most $e^{c_4(A+B)}x^\kappa \le x^{\kappa+1}$. Every solution is a zero of one of them. By Gauss's lemma a factor G of bidegree (α,β) is irreducible over $\mathbb{Q}$, and it is of one of the following types.

(a) $\beta = 0$ and $\alpha \ge 2$, or $\alpha = 0$ and $\beta \ge 2$. A rational zero $(w_0, w_1) \ne 0$ of a form $G(d,e)$ would give the linear factor $w_1 d - w_0 e$; so G has no integral zero with $(d,e) \ne 0$ (respectively $(v,u) \ne 0$), and carries no solution.

(b1) A line $pd + qe = 0$ or $pv + qu = 0$.

(b2) $\alpha, \beta \ge 1$ and $\min(\alpha,\beta) = 1$. Then G is irreducible over $\mathbb{C}$. Say $\beta = 1$ and write $G = vP(d,e) + uQ(d,e)$. A common factor of P and Q over $\mathbb{Q}$ would split G, so P and Q are coprime in $\mathbb{Q}[d,e]$; a Bézout identity for their dehomogenisations (in either chart) shows that they have no common zero in $\mathbb{P}^1(\mathbb{C})$. If $G = G_1 G_2$ over $\mathbb{C}$, bihomogeneity makes one factor, say G_1, of bidegree $(a_1, 0)$; if $a_1 \ge 1$ then G_1 vanishes at some $[r:s]$, so $P(r,s) = Q(r,s) = 0$, which is impossible; hence G_1 is a constant. The case $\alpha = 1$ is symmetric.

(b3) $\alpha, \beta \ge 2$, so that $2\alpha + 3\beta \ge 10$.

5.3. **Rational components.**

Lemma 5.2. *Let $\alpha, \beta \ge 1$. Let $A, B \in \mathbb{C}[r,s]$ be coprime binary forms of degree β and $C, E \in \mathbb{C}[r,s]$ coprime binary forms of degree α, and put $R := B^3C^2 - A^3E^2$. Suppose $R \ne 0$ and that β is odd or $\alpha < 3\beta/2$. Then R has at least three distinct zeros in $\mathbb{P}^1(\mathbb{C})$. In particular this holds whenever $\min(\alpha,\beta) = 1$.*

Proof. Coprime forms of positive degree have no common zero in $\mathbb{P}^1(\mathbb{C})$, and none of A, B, C, E is zero. Suppose that R has at most two distinct zeros. A linear change of the variables (r,s) preserves all hypotheses, so we may assume that every zero of R lies in $\{[0:1],[1:0]\}$. Put $a(t) = A(t,1)$, and b, c, e likewise, so that $r_0(t) := R(t,1) = b^3c^2 - a^3e^2$ is a nonzero polynomial whose only root is 0; thus $r_0 = \kappa t^i$ with $\kappa \ne 0$. For a nonzero polynomial p write $p = t^{o(p)}p_1$ with $p_1(0) \ne 0$.

Claim: a and b are monomials, and $3o(b) + 2o(c) = 3o(a) + 2o(e)$. Put $f = b^3c^2$ and $g = a^3e^2$, so $f - g = \kappa t^i$, and let $W := tf' - if$. Since $t(\kappa t^i)' = i\kappa t^i$, also $W = tg' - ig$, and $\deg W \le \min(\deg f, \deg g)$. If $W = 0$ then $tf' = if$ forces f to be a multiple of t^i, and likewise g; then a, b, c, e are monomials and $o(f) = o(g) = i$. If $W \ne 0$, the identity $W = b^2c\,(t(3b'c + 2bc') - ibc)$ shows $u_1 := b_1^2 c_1 \mid W$, and symmetrically $u_2 := a_1^2 e_1 \mid W$. Moreover $t^{o(f)}$ and $t^{o(g)}$ divide W, hence so does t^{M_0} with $M_0 = \max(o(f), o(g))$. A common root $\tau \ne 0$ of u_1 and u_2 would give $f(\tau) = g(\tau) = 0$ and so $\kappa\tau^i = 0$; thus u_1, u_2 and t are pairwise coprime and $u_1 u_2 t^{M_0} \mid W$. Writing $\deg f = o(f) + 3\deg b_1 + 2\deg c_1$ and $\deg g = o(g) + 3\deg a_1 + 2\deg e_1$, the two inequalities $\deg u_1 + \deg u_2 + M_0 \le \deg f$ and $\le \deg g$ add up to $\deg a_1 + \deg b_1 + (2M_0 - o(f) - o(g)) \le 0$. Hence a_1, b_1 are constants and $o(f) = o(g)$, which is the claim.

Conclusion. Let $p = \deg a = o(a)$ and $q = \deg b = o(b)$. As A, B have no common zero at $[0:1]$, $p = 0$ or $q = 0$; as they have no common zero at $[1:0]$, $p = \beta$ or $q = \beta$. Similarly $o(c) = 0$ or $o(e) = 0$. If $p = 0$ and $q = \beta$, the claim gives $3\beta + 2o(c) = 2o(e)$, so $o(c) = 0$ and $2o(e) = 3\beta$; if $p = \beta$ and $q = 0$, it gives $2o(c) = 3\beta$. In both cases β is even and $3\beta \le 2\alpha$, since $o(c), o(e) \le \alpha$, contrary to the hypothesis. Finally, if $\min(\alpha,\beta) = 1$ then either $\beta = 1$ is odd or $\alpha = 1 < 3\beta/2$. □

Proposition 5.3. *Let $\alpha, \beta \ge 1$ with $\min(\alpha,\beta) = 1$, $K \ge 0$ and $\varepsilon > 0$. There is C, depending only on h, $\mathcal{G}$, K, α, β and ε, such that for every $x \ge 1$ and every $G \in \mathbb{Z}[d,e;v,u]$ bihomogeneous of bidegree (α,β), irreducible over $\mathbb{C}$, with coefficients at most x^K, the number of $(d,e,u,v) \in \mathbb{N}^4$ with*

$G(d,e,v,u)=0$, $e^3v^2-d^3u^2=h$ and $\gcd(d,e),\gcd(u,v)\in\mathcal{G}$ is at most Cx^{ε}. The same holds with C independent of h, uniformly in $0<|h|\le x$.

By type (b2) above, this applies to every factor of G_I of that type.

Proof. Let $\beta=1$. (If $\alpha=1$, write $G=dP(v,u)+eQ(v,u)$, let $(r,s)=(v,u)/\gcd(u,v)$ and put $A=-Q$, $B=P$, $C=r$, $E=s$; then $dB(r,s)=eA(r,s)$ and $vE(r,s)=uC(r,s)$, and the argument below applies verbatim.) Write $G=vP(d,e)+uQ(d,e)$ with forms P,Q of degree α, coprime over $\mathbb{C}$ by irreducibility. Let (d,e,u,v) be counted. Then $(d,e)\neq 0$ because $h\neq 0$; let $\gamma=\gcd(d,e)$ and $(r,s)=(d,e)/\gamma$. Put $A=r$, $B=s$ (as forms of degree $\beta=1$), $C=-Q$ and $E=P$. Then $dB(r,s)=eA(r,s)$, and dividing $G(d,e,v,u)=0$ by γ^{α} gives $vE(r,s)=uC(r,s)$.

Resultants. For coprime integral forms P_1,Q_1 of degree $n\ge 1$ with coefficients at most H, let $\mathrm{Res}(P_1,Q_1)$ be the product of the resultants of $P_1(t,1),Q_1(t,1)$ and of $P_1(1,t),Q_1(1,t)$. It is a nonzero integer, $|\mathrm{Res}|\le((2n)!)^2H^{4n}$ by the Leibniz expansion of the Sylvester determinants, and for coprime (r,s) the integer $\gcd(P_1(r,s),Q_1(r,s))$ divides it. Let $g_1=\gcd(A(r,s),B(r,s))$ and $g_2=\gcd(C(r,s),E(r,s))$; these divide $\mathrm{Res}_1:=\mathrm{Res}(A,B)$ and $\mathrm{Res}_2:=\mathrm{Res}(C,E)$. Write $(A,B)(r,s)=g_1(a',b')$ and $(C,E)(r,s)=g_2(c',e')$ with coprime pairs. From $dB=eA$ and $vE=uC$ we get $(d,e)=l(a',b')$ and $(v,u)=\mu(c',e')$ with $|l|=\gcd(d,e)$ and $|\mu|=\gcd(u,v)$. Substituting into $e^3v^2-d^3u^2=h$ gives

$$h\,g_1^3g_2^2=R(r,s)\,l^3\mu^2,\qquad R:=B^3C^2-A^3E^2.$$

Hence $R(r,s)$ divides the nonzero integer $M_0:=h\,\mathrm{Res}_1^3\mathrm{Res}_2^2$, and

$$|M_0|\le|h|\,((2\beta)!)^6((2\alpha)!)^4x^{K(12\beta+8\alpha)}.$$

In particular $R\neq 0$, so by Lemma 5.2 R has at least three distinct zeros in $\mathbb{P}^1(\mathbb{C})$.

Squarefree kernel. Let $g\in\mathbb{Z}[t]$ be the primitive polynomial proportional to the product of the distinct monic irreducible factors of $R(t,1)$ over $\mathbb{Q}$, and let $Q_R(r,s):=s^{e_0+\deg g}g(r/s)$, where $e_0=1$ if R vanishes at $[1:0]$ and $e_0=0$ otherwise. Then Q_R is an integral binary form of content 1, squarefree over $\mathbb{Q}$, of degree q with $3\le q\le 3\beta+2\alpha=:Q_0$ (the lower bound counts the zeros of R), and Q_R divides R in $\mathbb{Z}[r,s]$ by Gauss's lemma. So $Q_R(r,s)$ divides M_0.

Counting. The solution is determined by (r,s), $\gcd(d,e)$ and $\gcd(u,v)$: the pairs (a',b') and (c',e') are determined by (r,s), and the signs of l and μ by the positivity of d,e,u,v. For each of the $2d(|M_0|)$ values $m'=\pm k$, $k\mid M_0$, Theorem 3.3 gives at most $2800q^{1+\omega(m')}\le 2800\,Q_0\,d(|M_0|)^{Q_0}$ coprime pairs with $Q_R(r,s)=m'$, using $q^{\omega(m')}\le 2^{Q_0\omega(M_0)}\le d(|M_0|)^{Q_0}$. Therefore the count is at most $5600\,Q_0\,(\#\mathcal{G})^2\,d(|M_0|)^{Q_0+1}$, and the divisor bound $d(n)\ll_\eta n^\eta$ with $|M_0|\ll x^{O(1)}$ gives Cx^{ε}. The constant depends on h only through $|h|^\eta$; if $|h|\le x$ this factor is absorbed into x^{ε}. □

5.4. **Reduced coordinates.**

Lemma 5.4. *Let $I=(\rho_0,\rho_0(1+M^{-1})]$ with $D/(2E)\le\rho_0\le 2D/E$, and put $\omega_0=\rho_0^{3/2}$.*

(i) *If $(d,e,u,v)\in\mathcal{S}_W$ and $d/e\in I$, then $0<d-\rho_0e\le 4D/M$ and $|v-\omega_0u|\le 17V/M$ (for $x\ge 3|h|M$).*

(ii) *Let $\Lambda_I=T_I(\mathbb{Z}^2)$ and $\Lambda_I'=T_I'(\mathbb{Z}^2)$, where $T_I(x_1,x_2)=(\frac{M}{4D}(x_1-\rho_0x_2),\frac{x_2}{2E})$ and $T_I'(y_1,y_2)=(\frac{M}{17V}(y_1-\omega_0y_2),\frac{y_2}{2U})$. With Lagrange–Gauss reduced bases $g^{(i)}$, $g'^{(i)}$ put $L_i=2/|g^{(i)}|$ and $T_i=2/|g'^{(i)}|$. Then $L_1\ge L_2$, $T_1\ge T_2$, $L_1L_2\asymp DE/M$, $T_1T_2\asymp UV/M$, $L_1\gg(DE/M)^{1/2}$ and $T_1\gg(UV/M)^{1/2}$.*

(iii) *Every $(d',e')\in\mathbb{Z}^2$ with $|d'-\rho_0e'|\le 4D/M$ and $|e'|\le 2E$ has coordinates $\lambda\in\mathbb{Z}^2$, $|\lambda_i|\le L_i$, in the basis $T_I^{-1}g^{(i)}$ of $\mathbb{Z}^2$, and every (v',u') with $|v'-\omega_0u'|\le 17V/M$ and $|u'|\le 2U$ has coordinates τ, $|\tau_i|\le T_i$, in the basis $T_I'^{-1}g'^{(i)}$. These coordinate changes lie in $\mathrm{GL}_2(\mathbb{Z})$.*

This is Reuss's lattice step [14, §2.2] with explicit constants: (i) follows from $d-\rho_0e\le\rho_0e/M$ and from $\omega_0(1-1/M)u\le v\le\omega_0(1+3/M)u$, which uses $|h|\le d^3u^2/(3M)$; (ii) and (iii) follow from

the Gram inequalities of a reduced basis, which give $|\lambda_1 g^{(1)} + \lambda_2 g^{(2)}|^2 \geq \frac{1}{2}(\lambda_1^2|g^{(1)}|^2 + \lambda_2^2|g^{(2)}|^2)$. In the formal proof the constants are $27DE/M \leq L_1L_2 \leq 32DE/M$, $L_1 \geq 5(DE/M)^{1/2}$, $117UV/M \leq T_1T_2 \leq 136UV/M$ and $T_1 \geq 10(UV/M)^{1/2}$.

5.5. **The per-interval CCDN step.**

Lemma 5.5 (Lemma P)**.** *Let $F \in \mathbb{Z}[\lambda_1, \lambda_2; \tau_1, \tau_2]$ be bihomogeneous of bidegree $(3, 2)$, let $h \in \mathbb{Z}\setminus\{0\}$, and let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve over $\mathbb{Q}$ of bidegree $(\alpha, \beta) \neq (0, 0)$. Let $L_1 \geq L_2 > 0$ and $T_1 \geq T_2 > 0$. Then the number of $(\lambda, \tau) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with λ and τ primitive, $|\lambda_i| \leq L_i$, $|\tau_i| \leq T_i$, $F(\lambda; \tau) = h$ and $([\lambda], [\tau]) \in \Gamma$ is at most*

$$2\,c(12)\,d^3B^{1/d}(\log B + d), \qquad d := 2\alpha + 3\beta, \quad B := \max(1, L_1^3T_1^2),$$

with $c(12)$ from Theorem 3.2, uniformly in F, Γ and h.

Geometrically, the twelve monomials σ_k of bidegree $(3, 2)$ embed $\mathbb{P}^1 \times \mathbb{P}^1$ by $\mathcal{O}(3, 2)$, and the counted points are integral points on the affine slice $\{F = h\}$ of the cone over the image of Γ, a curve of degree $2\alpha + 3\beta$. The proof works with the coordinate ring of this slice directly.

Proof. Let $R = \mathbb{Q}[y_0, y_1; y_2, y_3]$, bigraded with y_0, y_1 of bidegree $(1, 0)$ and y_2, y_3 of bidegree $(0, 1)$; let $\sigma_k = y_0^i y_1^{3-i} y_2^j y_3^{2-j}$ for $k = 3i + j$, $0 \leq i \leq 3$, $0 \leq j \leq 2$, so that $\sigma(\lambda, \tau) \in \mathbb{Z}^{12}$ denotes the vector of these monomials. Let G be an equation of Γ, irreducible in R, so that (G) is prime. If $G \mid F$, then F vanishes on Γ and, as $h \neq 0$, nothing is counted. Otherwise the image $\bar{F}$ of F in the domain $R/(G)$ is nonzero; let $S = (R/(G))[\bar{F}^{-1}]$ and let $\psi\colon \mathbb{Q}[x_1, \ldots, x_{12}] \to S$ send x_k to $h\sigma_k/\bar{F}$. Its kernel I is a prime ideal.

Hilbert function. The image under ψ of the polynomials of total degree at most s is $\{\bar{N}\bar{F}^{-s} : N \in R_{(3s,2s)}\}$. Indeed, a monomial x^m with $|m| \leq s$ maps to $h^{|m|}\sigma^m \bar{F}^{\,s-|m|}/\bar{F}^{\,s}$, whose numerator has bidegree $(3s, 2s)$; conversely every monomial of bidegree $(3s, 2s)$ is a product of s of the σ_k, and $h \neq 0$. Multiplication by $\bar{F}^{-s}$ is injective, so this image has dimension $\dim(R/(G))_{(3s,2s)}$. The exact sequence $0 \to R_{(3s-\alpha,2s-\beta)} \xrightarrow{\cdot G} R_{(3s,2s)} \to (R/(G))_{(3s,2s)} \to 0$ and $\dim R_{(a,b)} = (a+1)(b+1)$ give, for $s \geq \alpha + \beta$,

$$(3s + 1)(2s + 1) - (3s - \alpha + 1)(2s - \beta + 1) = (2\alpha + 3\beta)\,s + (\alpha + \beta - \alpha\beta).$$

So I has degree $d = 2\alpha + 3\beta$ in the sense of Theorem 3.2.

Dimension one. Since the Hilbert function grows linearly, any two elements of $\mathbb{Q}[x]/I$ are algebraically dependent: the $(k+1)^2$ products $a^i b^j$ with $i, j \leq k$ lie in a filtration piece of dimension $O(k)$. If $\mathfrak{p} \ni p \neq 0$ is a prime and $\Phi(p, b) = 0$ with $\Phi \neq 0$, cancelling the largest power of the first variable dividing Φ (as $\mathbb{Q}[x]/I$ is a domain) and reducing modulo $\mathfrak{p}$ shows that b is algebraic over $\mathbb{Q}$ modulo $\mathfrak{p}$. Hence the quotient by any nonzero prime is algebraic over $\mathbb{Q}$ and therefore a field, so every nonzero prime is maximal; and $\mathbb{Q}[x]/I$, which is infinite-dimensional and finitely generated, is not a field. So $\dim \mathbb{Q}[x]/I = 1$.

The counted points. Let (λ, τ) be counted. Evaluation at (λ, τ) defines $R/(G) \to \mathbb{Q}$, which sends $\bar{F}$ to $h \neq 0$ and so extends to S; composed with ψ it sends x_k to $\sigma_k(\lambda, \tau)$. Hence every element of I vanishes at $w = \sigma(\lambda, \tau) \in \mathbb{Z}^{12}$, and $|w_k| \leq L_1^i L_2^{3-i} T_1^j T_2^{2-j} \leq L_1^3 T_1^2 \leq B$.

Fibres. Suppose $\sigma(\lambda, \tau) = \sigma(\lambda', \tau')$ with all four vectors primitive. Some $\tau_1^j \tau_2^{2-j}$ is nonzero, and the coordinates with that j show that the cubic monomial vectors of λ and λ' are proportional with a nonzero factor; hence λ and λ' are proportional, and being primitive, $\lambda' = \kappa\lambda$ with $\kappa = \pm 1$. Likewise $\tau' = \mu\tau$ with $\mu = \pm 1$. A nonzero coordinate $\lambda_1^i \lambda_2^{3-i} \tau_1^j \tau_2^{2-j}$ is multiplied by $\kappa^3\mu^2 = \kappa$, so $\kappa = 1$. Thus each point w comes from at most the two pairs $(\lambda, \pm\tau)$.

Therefore the count is at most twice the number of integral points of height at most B on the curve defined by I, and Theorem 3.2 with $N = 12$ gives the bound. □

5.6. **The per-interval bound.**

Proposition 5.6. *In the setting of this section, for each interval I the number of $(d,e,u,v) \in \mathcal{S}_W$ with $d/e \in I$ and $\gcd(d,e), \gcd(u,v) \in \mathcal{G}$ is*

$$\ll_{h,c,\delta,\mathcal{G},\varepsilon} x^{\varepsilon}\Big(1 + \big(L_1(I)^3 T_1(I)^2\big)^{1/10}\Big).$$

Proof. By Section 5.2 every counted solution is a zero of one of at most $2K_0$ irreducible factors G of G_I, each with coefficients at most $x^{\kappa+1}$. We bound the solutions on each factor. (The formal statement gives the bound with L_1 and T_1 computed from some nonzero vectors of Λ_I and Λ_I'; this implies the form stated, since the reduced vectors $g^{(1)}$, $g'^{(1)}$ are shortest.)

Type (a) carries none. For a line $pd+qe=0$, (d,e) is γ times a fixed primitive vector with $\gamma \in \mathcal{G}$, and for each such (d,e) Lemma 4.1(i) gives $\ll x^{\varepsilon}$ pairs (u,v); for a line $pv+qu=0$, (v,u) takes at most $\#\mathcal{G}$ values and Lemma 4.1(ii) gives $O(1)$ pairs (d,e) for each. Type (b2) carries $\ll x^{\varepsilon}$ solutions by Proposition 5.3 with $K=\kappa+1$.

Let G be of type (b3), and fix $\gamma=\gcd(d,e)$ and $\gamma'=\gcd(u,v)$ in $\mathcal{G}$. Put $(d_0,e_0)=(d,e)/\gamma$ and $(v_0,u_0)=(v,u)/\gamma'$; these are primitive, $e_0^3v_0^2-d_0^3u_0^2 = h' := h/(\gamma^3\gamma'^2)$ is a nonzero integer, and $G(d_0,e_0,v_0,u_0)=0$ by bihomogeneity. Since $M \le x^{1/2}$ and x is large, $x \ge 3|h|M$, so Lemma 5.4(i) gives $|d_0-\rho_0 e_0| \le 4D/M$, $|e_0| \le 2E$, $|v_0-\omega_0 u_0| \le 17V/M$ and $|u_0| \le 2U$, and by Lemma 5.4(iii) the coordinates λ, τ of (d_0,e_0) and (v_0,u_0) are primitive with $|\lambda_i| \le L_i$ and $|\tau_i| \le T_i$. In these coordinates $e^3v^2-d^3u^2$ becomes a bihomogeneous form of bidegree $(3,2)$ and G becomes an irreducible form of the same bidegree, since the changes of variables lie in $\mathrm{GL}_2(\mathbb{Z})$; the map to (λ,τ) is injective. Lemma 5.5 with h' and $d=2\alpha+3\beta \ge 10$ bounds the solutions by $2c(12)d^3B^{1/d}(\log B+d)$. Here $B^{1/d} \le 1+(L_1^3T_1^2)^{1/10}$, while $L_1 \ll x$ and $T_1 \ll x$ give $\log B \ll \log x$, and $d \le 5K_0$; so this is $\ll x^{\varepsilon}(1+(L_1^3T_1^2)^{1/10})$. Summing over $(\gamma,\gamma') \in \mathcal{G}^2$ and over the at most $2K_0$ factors proves the proposition. □

6. Summation over intervals

Put $L_* := \max((DE/M)^{1/2}, 32(D+E)/M)$ and $T_* := \max((UV/M)^{1/2}, 194(U+V)/M)$.

Lemma 6.1 (Tail counts)**.** *For $L \ge 32(D+E)/M$, $\#\{I:\ L_1(I) \ge L\} \le 720\,DE/L^2$, and for $T \ge 194(U+V)/M$, $\#\{I:\ T_1(I) \ge T\} \le 7000\,UV/T^2$.*

A short lattice vector is a rational approximation to ρ_0 or ω_0 with bounded numerator; the points ρ_0 are spaced at least $D/(2EM)$ apart, and counting the pairs (denominator, interval) gives the bounds.

Lemma 6.2. *For $0<\theta<2$, $\sum_I L_1(I)^{\theta} \ll_{\theta} ML_*^{\theta}$ and $\sum_I T_1(I)^{\theta} \ll_{\theta} MT_*^{\theta}$.*

The intervals with $L_1(I) \le L_*$ contribute at most $(3M+1)L_*^{\theta}$, and by Lemma 6.1 and $DE \le ML_*^2$ the dyadic shell $2^kL_* \le L_1(I) < 2^{k+1}L_*$ contributes $\ll ML_*^{\theta}(2^{\theta}/4)^k$; the series converges because $\theta<2$.

Theorem 6.3 (The reduced per-box bound)**.** *Let $h \ne 0$, $c \in (0,1)$ and $\mathcal{G}$ be fixed, and let W be a window with $s,t \ge c$. Define*

$$B_{\mathrm{red}}(a,b) := m + \frac{3}{10}\,\ell^* + \frac{1}{5}\,t^*, \tag{6.1}$$

where $\ell^* := \max(\frac{s-m}{2}, a-m, b-m)$ and $t^* := \max(\frac{t-m}{2}, \bar{u}-m, \bar{v}-m)$. Then for every $\varepsilon>0$ the number of $(d,e,u,v) \in \mathcal{S}_W$ with $\gcd(d,e), \gcd(u,v) \in \mathcal{G}$ is $\ll_{h,c,\mathcal{G},\varepsilon} x^{\max(m, B_{\mathrm{red}}(a,b))+\varepsilon}$.

Proof. Put $\delta = \min(\varepsilon/2, 1/2)$ and $M = \lceil x^{(1+\delta)m} \rceil$. Each solution lies in one of the at most $3M+1$ intervals I, so by Proposition 5.6 (with $\varepsilon/2$) the count is at most $\sum_I A\,(1+L_1(I)^{3/10}T_1(I)^{1/5})$ with

$A \ll x^{\varepsilon/2}$. For $L, T \geq 0$, applying $XY \leq X^2 + Y^2$ to $X = (L/L_*)^{3/10}$ and $Y = (T/T_*)^{1/5}$ gives

$$L^{3/10}T^{1/5} \leq L_*^{3/10}T_*^{1/5}\Big(\frac{L^{3/5}}{L_*^{3/5}} + \frac{T^{2/5}}{T_*^{2/5}}\Big).$$

Summing over I with Lemma 6.2 for $\theta = 3/5$ and $\theta = 2/5$, the count is $\ll A\,(M + ML_*^{3/10}T_*^{1/5})$. Now $x^{(1+\delta)m} \leq M \leq 2x^{(1+\delta)m}$, $0 \leq m \leq 3/16$, $L_* \ll x^{\ell^* - \delta m/2}$ and $T_* \ll x^{t^* - \delta m/2}$, so $M \ll x^{m+\delta}$ and $ML_*^{3/10}T_*^{1/5} \ll x^{B_{\mathrm{red}} + \frac{3}{4}\delta m} \leq x^{B_{\mathrm{red}}+\delta}$. Since $\delta \leq \varepsilon/2$, the count is $\ll x^{\max(m, B_{\mathrm{red}})+\varepsilon}$. □

Lemma 6.4. *If $0 \leq a, b \leq \frac{1}{3}$ and $s > 2/27$, then $B_{\mathrm{red}}(a, b) \geq m$.*

Proof. Here $s \leq \frac{2}{3}$, so $t \geq 0$. Then $s - m = s(1 - \frac{9}{8}t) \geq 0$ because $s \geq 2/27$ is equivalent to $t \leq \frac{8}{9}$, and $t - m = t(1 - \frac{9}{8}s) \geq 0$ since $s \leq \frac{2}{3} < \frac{8}{9}$. So $\ell^*, t^* \geq 0$. □

Lemma 6.5 (Balanced form)**.** *If $|a - b| \leq \frac{2}{3}m$, then $B_{\mathrm{red}}(a, b) = \frac{3}{4}m + \frac{1}{10}$.*

Proof. $a - m \leq \frac{s-m}{2}$ if and only if $a - b \leq m$, and $\bar{u} - m \leq \frac{t-m}{2}$ if and only if $\frac{3}{2}(b - a) \leq m$; with the symmetric conditions this gives $\ell^* = \frac{s-m}{2}$ and $t^* = \frac{t-m}{2}$, and then $B_{\mathrm{red}} = \frac{3}{4}m + \frac{3s+2t}{20} = \frac{3}{4}m + \frac{1}{10}$ because $3s + 2t = 2$. □

At $a = b = 1/6$ one has $s = 1/3$, $m = 3/16$ and $B_{\mathrm{red}} = 77/320$.

7. The optimisation and its certificates

Let the gate G be the set of $(a, b) \in [0, 1/3]^2$ with $s, t \geq 1/10$ and $a, b \geq 1/100$, and define

$$\Phi_3(a, b) := \min\{s,\ t,\ B_{\mathrm{red}}(a, b) \text{ if } (a, b) \in \mathsf{G}\},$$
$$\Phi_2^\sharp(a, b) := \min\{s,\ t,\ H_1,\ H_2,\ R \text{ if } (a, b) \in \mathsf{G}_0,\ B_{\mathrm{red}} \text{ if } (a, b) \in \mathsf{G}\},$$

with $\mathsf{G}_0 = \{(a, b) \in [0, 1/3]^2 : s, t \geq 1/10\}$.

Proposition 7.1. (i) *For all $(a_1, a_2, a_3) \in [0, 1/3]^3$, $\min\{\Phi_3(a_1, a_2), \Phi_3(a_1, a_3), \Phi_3(a_2, a_3)\} \leq 77/320$, with equality at $a_1 = a_2 = a_3 = 1/6$.*

(ii) *For all $(a, b) \in [0, 1/3]^2$, $\Phi_2^\sharp(a, b) \leq \frac{1}{4}$, with equality exactly at $(0, \frac{1}{4})$ and $(\frac{1}{4}, 0)$.*

(iii) *If $a, a' \in [19/100, 21/100]$, then $(a, a') \in \mathsf{G}$ and $B_{\mathrm{red}}(a, a') \leq 38059/160000$, with an $O(\eta)$ error on the η-neighbourhood for $0 \leq \eta \leq 1/1000$.*

These are finite statements about explicit piecewise-quadratic functions, proved in Lean by case analysis on the active terms of each maximum and minimum and by exact polynomial inequalities; part (ii) includes the strip inequality $\min\{s, H_2\} \leq \frac{1}{4} + \frac{a}{4}$ for $a \leq 1/100$. The same bounds are certified independently by exact-rational branch and bound (Python `fractions`) in the supplementary scripts: each box bound uses the concavity of m, $(s + m)/2$ and $(t + m)/2$ in s and the linearity of the remaining terms. Table 1 lists the runs; negative controls just below each target fail. The point $(a, b) = (1/200, 197/800)$ shows that R is needed outside G: there $\min\{s, H_2\} = 201/800 > 1/4$.

Proposition 7.2 (Central small-gap exclusion)**.** *Let $n \geq 1$, $H = n+3$, and suppose $(n+1)(n+2)(n+3)$ is powerful. Suppose, for $r = 1, 2, 3$, that $n + r = f_r d_r^3 u_r^2$ with positive integers u_r, $f_r \in \{1, 2\}$ and odd squarefree d_r, and that $H^{19} < d_r^{100} < H^{21}$. Then $\max\{|d_1 - d_2|, |d_3 - d_2|\} > H^{32/1000}$.*

It is proved in Lean without hypotheses. It excludes only the small-gap part of the central box.

Run	functional	region	target T	result (nodes)
K3_MIN	$\min_{\text{pairs}} \Phi_3$	$[0,\frac{1}{3}]^3$	77/320	certified (633)
K3_MIN_CTL	same	same	0.2406	fails near $(\frac{1}{6},\frac{1}{6},0.171)$
K2_FULL	$\Phi_2^\sharp$	$[0,\frac{1}{3}]^2$	1/4	certified (93)
K2_FULL_CTL1	same	same	$\frac{1}{4}-10^{-6}$	fails near $(0,\frac{1}{4})$
CENTRAL	$\min_{\text{pairs}} B_{\text{red}}$	$[\frac{19}{100},\frac{21}{100}]^3$	38059/160000	certified (1)
THETA	$\min_{\text{pairs}} \min(s,t,B_{1/12})$	$[0,\frac{1}{3}]^3$	89/384	certified (633)

TABLE 1. Exact-rational certificates from `certificates.py` and `endpoint_cert.py` (K2_FULL); the outputs, with further runs and controls, are in the supplementary directory. $B_{1/12}$ is defined in Remark 9.3.

8. Proofs of the theorems

8.1. **Fixed width.** *Theorem 1.2.* Split $n+1 \in (X/2, X]$ dyadically and sort the n by (f_1, f_2, f_3) and by the dyadic scales D_i of d_1, d_2, d_3. For each of the three pairs, Lemmas 2.2 and 2.3 map the n of a box injectively into $O(1)$ window sets. With $\hat{a}_i = \max(0, \min(\log D_i/\log X, 1/3))$, Proposition 7.1(i) selects a pair with $s \le 77/320$, or $t \le 77/320$, or $(\hat{a}_i, \hat{a}_j) \in \mathsf{G}$ and $B_{\text{red}} \le 77/320$; Lemma 4.1, or Theorem 6.3 with Lemma 6.4, bounds the box, after a Lipschitz transfer of the exponents by $O(1/\log X)$. Summing over $O((\log x)^4)$ boxes gives the bound for N_k, and Proposition 7.1(iii) gives the central subcount. This proof does not use Theorems 3.1, 3.4 or 3.6.

Theorem 1.1. Here $k = 2$, $h = 1$, and $d = d_1$, $e = d_2$ are squarefree. On boxes with $\hat{a}, \hat{b} \ge \eta$ for a small fixed $\eta > 0$, Proposition 7.1(ii) selects one of $s, t, H_1, H_2, R, B_{\text{red}}$ at most $\frac{1}{4}$, and Lemmas 4.1 and 4.2, Theorem 3.1 (on windows with $D, E \ge x^c$) or Theorem 6.3 give the bound. On thin boxes $\hat{a} < \eta$, the strip inequality $\min\{s, H_2\} \le \frac{1}{4} + \frac{\hat{a}}{4}$ with Lemmas 4.1 and 4.2 gives $x^{1/4+\eta/4+\varepsilon}$, and η is arbitrary; the case $\hat{b} < \eta$ is symmetric. So Theorem 3.1 is needed only on windows with both cube-part scales at least a fixed power of x.

8.2. **Uniformity over all widths.** Put $\theta = 77/320$. Summing Theorem 1.2 over k would not suffice; the argument instead selects three consecutive factors with small coefficients and makes the pair bounds uniform in growing shifts.

Lemma 8.1 (Three consecutive small kernels)**.** *For $n \ge 0$, $k \ge 3$ and $1 \le j \le k$, let $f_j = \prod_{p<k,\ v_p(n+j)\ odd} p$. There is $1 \le i \le k-2$ with $f_i, f_{i+1}, f_{i+2} \le B(k)$, where $B(k) = 30030$ for $k \le 14$ and $B(k) = k^8$ for $k \ge 15$.*

Proof. If $k \le 14$, take $i = 1$: every f_j divides $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. Let $k \ge 15$. If a prime $p < k$ divides f_j, then $p \mid n+j$; there are at most $\lfloor k/p \rfloor + 1 \le 2\lfloor k/p \rfloor$ such j, and $p^{\lfloor k/p \rfloor} \mid k!$. Hence $\prod_{j=1}^k f_j$ divides $(k!)^2$, so $\prod_j f_j \le k^{2k} \le k^{8\lfloor k/3 \rfloor}$, using $2k \le 8\lfloor k/3 \rfloor$ for $k \ge 15$. If no admissible i existed, each of the $\lfloor k/3 \rfloor$ disjoint blocks $\{3l+1, 3l+2, 3l+3\}$ would contain some $f_j \ge k^8 + 1$, and since all $f_j \ge 1$ we would get $(k^8+1)^{\lfloor k/3 \rfloor} \le \prod_j f_j \le (k^8)^{\lfloor k/3 \rfloor}$, a contradiction. □

For a window W and an integer $h \ne 0$ let $A_W(h)$ be the number of $(d, e, u, v) \in \mathcal{S}_W$, defined with shift h, such that $\gcd(d, e) = \gcd(u, v) = 1$.

Lemma 8.2 (Uniform primitive windows)**.** *Fix $0 < c \le 1/10$, $0 < \delta \le 1/100$ and $\varepsilon > 0$. For sufficiently large x, uniformly in the integer shift $1 \le |h| \le x^{\delta/4}$ and the window W, the number $A_W(h)$ of solutions in $\mathcal{S}_W$ with $\gcd(d, e) = \gcd(u, v) = 1$ satisfies $A_W(h) \ll x^{s+\varepsilon}$ and $A_W(h) \ll x^{t+\varepsilon}$, and if also $s, t \ge c$, then $A_W(h) \ll_{c,\delta,\varepsilon} x^{\max(m, B_{\text{red}}(a,b))+\varepsilon+2\delta}$.*

Proof. The first two bounds hold for every $x > 1$ uniformly in $0 < |h| \le x$. Every solution has $d \le 2D$, $e \le 2E$ and $u, v \le 32x$. For coprime (d, e) the number of coprime (u, v) with $u, v \le 32x$

and $e^3v^2 - d^3u^2 = h$ is at most $2\tau(|h|)(4\log_2((d^3e^3+1)(32x)^2)+2)$ (the primitive count behind Lemma 4.1(i)), and for each (u,v) Lemma 4.1(ii) with $G_0 = 1$ gives at most $2800 \cdot 3^{1+\omega(|h|)}$ coprime pairs (d,e). Summing over the at most $4DE$ pairs (d,e), respectively $4UV$ pairs (u,v), with $\tau(|h|), 3^{\omega(|h|)} \ll_\varepsilon x^\varepsilon$, gives the two bounds. For the third bound, $|h| \le x^{\delta/4}$ controls every place where the shift enters Sections 5 and 6: it gives $3|h|M \le x$, the hypothesis of Lemma 5.4(i); it gives $|h| \le x$, the range of the second form of Proposition 5.3 and of the primitive counts of the previous sentences, which replace Lemma 4.1 on the lines of type (b1); and it gives $3 + 8|h| \le x^{\delta/3}$, so that in Lemma 5.1 the approximation of v/u by $(d/e)^{3/2}$ loses at most a factor $x^{\delta/3}$ and the determinant estimate holds with degrees and constants depending only on c and δ. Lemma 5.5 does not involve h, and since the solutions are primitive no division by $\gcd(d,e)$ or $\gcd(u,v)$ is needed. Hence Proposition 5.6 holds with $\mathcal{G} = \{1\}$ and an implied constant independent of h in this range, and summing it over the intervals as in the proof of Theorem 6.3 gives $A_W(h) \ll x^{\max(m, B_{\mathrm{red}}(a,b))+\varepsilon+\delta}$, which implies the stated bound. □

Lemma 8.3 (Three-term count with small coefficients)**.** *Let $\mathcal{Q}$ be the positive powerful integers. For every $\varepsilon > 0$ there are $\eta > 0$ and C_ε such that, for $y \ge 2$ and integers $1 \le f_1, f_2, f_3 \le y^\eta$,*

$$\#\{0 \le N \le y : N + r \in f_r\mathcal{Q} \ (r = 1,2,3)\} \le C_\varepsilon y^{\theta+\varepsilon}.$$

Proof. Take δ and ε' small in terms of ε, and η small in terms of δ and ε. The $N \le y^\theta$ contribute at most $y^\theta + 1$. For $y^\theta < N \le y$ write $N + r = f_r d_r^3 u_r^2$ $(r = 1,2,3)$ and $F = \max_r f_r \le y^\eta$.

The primitive equation. For a pair (i,j) the identity of Lemma 2.2 gives $e^3v^2 - d^3u^2 = h$ with $d = f_i d_i$, $e = f_j d_j$, $u = f_j u_i$, $v = f_i u_j$ and $1 \le h = f_i^2 f_j^2 (j-i) \le 2F^4$; only this identity is used, since the f_r are arbitrary. Dividing by $g = \gcd(d,e)$ and $q = \gcd(u,v)$ gives a primitive solution (d', e', u', v') with shift $h' = h/(g^3q^2)$, an integer because $g^3q^2 \mid h$; so $g, q, h' \le 2F^4$, and its size $d'^3u'^2$ lies between $(N+1)/2$ and $F^4(N+3)$.

The window and the shift. For y large, $h' \le 2F^4 \le (N+1)/200 \le d'^3u'^2/100$, so Lemma 2.3 places the primitive solution in a window $W' = (D', E', x')$ from the grid $\{2^{k/8}\}$ with $N + 1 \ll x' \ll F^4(N+1)$. As $N > y^\theta$ and $F \le y^\eta$, with η small in terms of δ we get $h' \le 2F^4 \le x'^{\delta/4}$, so Lemma 8.2 applies to W' uniformly in the shift.

The exponents and the choice of pair. Let $\hat{a}_r$ be $\log d_r / \log(N+1)$ clipped to $[0, 1/3]$. The exponents (a,b) of W' differ from $(\hat{a}_i, \hat{a}_j)$ by $O((1 + \log F)/\log N)$, the clipping included, and s, t, m, ℓ^*, t^* and B_{red} are Lipschitz in (a,b); this perturbation costs a factor $O(F^{O(1)})$. Proposition 7.1(i) at $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ selects a pair with $s \le \theta$, or $t \le \theta$, or $(\hat{a}_i, \hat{a}_j) \in \mathsf{G}$ and $B_{\mathrm{red}} \le \theta$; in the last case $s, t \ge c = 1/20$ for y large, and Lemma 6.4 at $(\hat{a}_i, \hat{a}_j)$, where $s \ge 1/10 > 2/27$, gives $m \le B_{\mathrm{red}}$ there. In each case Lemma 8.2 bounds the primitive solutions in W' with shift h' by $\ll_\varepsilon F^{O(1)}(N+1)^{\theta+\varepsilon'+2\delta}$.

Summation. The key consisting of the pair, g, q and the window W' determines h', and together with the primitive solution it determines N. There are $O(F^8(\log y)^3)$ keys. The total is therefore $y^\theta + 1 + F^{O(1)}(\log y)^3 y^{\theta+\varepsilon'+2\delta}$, and with $F \le y^\eta$ the choice of parameters makes the exponent less than $\theta + \varepsilon$. □

Proof of Theorem 1.3. Theorem 3.6 applies to every powerful $P(N,k)$ with $N \ge N_0$ and $k \ge 3$: the condition $k < N$ is automatic, because if $N \le k$, Bertrand's postulate gives a prime p with $N < p \le N + k < 2p$, which divides $P(N,k)$ exactly once. So for $n_1 = \max(N_0, 1) \le n \le x$ every powerful $P(n,k)$ with $k \ge 3$ has $k \le K_x = \exp((\log x)^{7/10}) = x^{o(1)}$.

Width at least three. Let x be large, so that $14 \le K_x \le x$ and $K_x^8 \le x^\eta$. Lemma 8.1 gives an index $i \le K_x$ with $f_i, f_{i+1}, f_{i+2} \le K_x^8$, and $(n+l)/f_l$ is powerful: a prime $p \ge k$ dividing $n + l$ divides no other factor, so $p^2 \mid n + l$, and a prime $p < k$ has even exponent in $(n+l)/f_l$. So $N = n + i - 1 \le 2x$ satisfies $N + r \in f_{i+r-1}\mathcal{Q}$ for $r = 1,2,3$, and Lemma 8.3 with $\varepsilon/2$ and $y = 2x$

applies to each of the at most K_x indices and K_x^{24} coefficient triples, and the count is at most $n_1 + K_x^{25} C(2x)^{\theta+\varepsilon/2} \ll_\varepsilon x^{\theta+\varepsilon}$.

Width two. If $(n+1)(n+2)$ is powerful, both factors are powerful since they are coprime, and Theorem 1.1 with $x+1$ applies to $n+1$; this gives the bound $x^{1/4+\varepsilon}$ for $k \geq 2$.

Non-powerful members. A non-powerful z with $N < z \leq N+K$ and $P(N,K)$ powerful forces $K \geq 3$, and then $K < N$ by the Bertrand argument. Starts $N < n_1$ give $z < 2n_1$; every other start is counted by the first bound with $\varepsilon/2$ and carries at most $K \leq K_x \leq x^{\varepsilon/2}$ such z.

The explicit choices of δ, ε', η, c and of the thresholds in x and y, and the resulting constants, are those of the formal proofs of Lemmas 8.2 and 8.3 and Theorem 1.3 (Appendix A). All implied constants depend only on ε and on the threshold N_0 of Theorem 3.6. □

9. Remarks

Remark 9.1 (1/4 is the limit of this method for $k=2$). At $(a,b) = (0, \frac{1}{4})$ every bound used here is at least $\frac{1}{4}$: there $m = \frac{45}{256}$, $s = H_2 = R = \frac{1}{4}$, $t = \frac{5}{8}$, $H_1 = \frac{3}{4}$ and $B_{\mathrm{red}} = \frac{673}{2560}$. At this corner d_1 is bounded, $d_2 \approx x^{1/4}$, $u_1 \approx x^{1/2}$ and $u_2 \approx x^{1/8}$. Going below $\frac{1}{4}$ at this corner requires a further bound, uniform in d_1 and u_2, for the integral points on the genus-one curves $Y^2 = d_1 u_2^2 X^3 - d_1$ (a bound $x^{o(1)}$ would suffice); this paper uses none.

Remark 9.2 (Comparison with Reuss). Reuss's per-box exponent $R(a,b)$ equals $m + \min(\ell^*, t^*)$. Our exponent is $m + \frac{3}{10}\ell^* + \frac{1}{5}t^*$. At Reuss's binding point $s = t = 2/5$ ($m = 9/50$, $\ell^* = t^* = 11/100$) these are 0.29 and 0.235. The difference is entirely the per-interval count: Reuss counts rational points of the auxiliary curve with one coordinate pair at a time [14, §2.2, Lemma 8], whereas Lemma 5.5 counts integral points of the curve cut out jointly by the auxiliary equation and by (1.1).

Remark 9.3 (A genus-one saving would help). If every component of bidegree $(2,2)$ in Proposition 5.6 carried $\ll x^\varepsilon \max(1, L_1^3 T_1^2)^{\theta'}$ solutions for some $\theta' \in [\frac{1}{12}, \frac{1}{10}]$, the per-box exponent would become $B_{\theta'} := m + 3\theta'\ell^* + 2\theta' t^*$. For $\theta' = 1/12$ the run THETA certifies $\sup_{[0,1/3]^3} \min_{\text{pairs}} \min(s, t, B_{1/12}) = 89/384 \approx 0.2318$. This would need a uniform saving over the exponent $1/10$ of Theorem 3.2 for integral points on the degree-10 curves of Lemma 5.5. Nothing in this remark is claimed; the value $89/384$ is an exact-rational certificate, not a Lean theorem.

Remark 9.4 (Relation to Tao's model problem). Tao [17, Remark 2.12] proposes to show that the variety $n_1^2 n_2^3 + 1 = m_1^2 m_2^3$ [17, (2.9)] has $o(H^2)$ integral points of height at most H. The pair equation (1.1) is this variety with $h = 1$, and the geometry of Lemma 5.5 is related to it; we have not examined what our method gives for the model problem as posed, where the cube variables need not be squarefree. Theorem 1.3 improves the quantity in Tao's Theorem 1.8 through the uniform triple count instead.

Remark 9.5 (The constant 9/8). The exponents depend on the determinant constant through $m = c_0 st$ with $c_0 = 9/8$; in the balanced case at $s = 1/3$ the bound is $c_0/8 + 1/10$. Within the determinant estimate of Lemma 5.1 (a rectangular monomial set and a generalised Vandermonde bound), the AM–GM inequality shows that no choice of A and B lowers 9/8. This remark is a written argument; it is not formalized.

10. Verification and disclosure

Trust base. The formal proofs are checked by the kernel of Lean 4.33.1 with Mathlib at commit `0df444a360eaa60ab8c11dca51a86af692955474`. Every declaration of Table 2 depends exactly on the axioms `propext`, `Classical.choice` and `Quot.sound`; there is no `sorry`, no `native_decide` and no added axiom. The results from the literature enter only as the named hypotheses of Section 3. What must be trusted beyond the kernel is therefore: those six results; the agreement

of each Lean hypothesis with its source, discussed in Section 3; and the agreement of the Lean statements of Table 2 with Theorems 1.1–1.3.

Statements and independent checks. Each statement in Table 2 is stated independently of its proof, and the proofs were checked against it by the Comparator tool, which compares the compiled statement with an independently compiled copy, together with an export of the proof checked by the independent kernel Nanoda with only the three standard axioms permitted. The declarations are in the module `CountingClosure` of the accompanying repository, compiled with zero warnings and audited for their exact axiom sets. The replay recorded in `replay/receipts/powerful-counting.json` checks every release declaration of this paper, the nine of Table 2 and the supporting declarations of Table 3, with Nanoda (standard axioms only), and compares each against its independent statement file in `replay/statements/powerful-counting/`; a negative control, the statement of Theorem 1.2 with 77/320 replaced by 76/320, is rejected. The Lean names of intermediate steps locate their proofs; only the declarations of Table 2 are compared with stated paper results.

Hypotheses from the literature. The statements of Theorems 3.1, 3.2 and 3.6 were compared with the arXiv texts [14, 4, 17], and Theorem 3.3 with the journal text [16, pp. 795–796] and Theorem 3.5 with [3, Lemma 1.6.11], by an AI reviewer. Theorem 3.4 was compared with Theorem 15 in the author's manuscript of [12] and with its restatement in [15, Theorem 2]; the published text of [12] has not been compared.

Use of AI systems. All results, proofs (written and formal), computation scripts and the text of this manuscript were produced by AI systems. The results and their proofs come from large language models running as agents in the Jocelyn research harness, together with Lean and exact-arithmetic tools; much of the writing and revision of the manuscript, its reviews and the resulting corrections were done with AI coding assistants from several providers, operated by the author outside the harness. The author chose the problem, operated the harness, set its objectives, decided the content of the manuscript and commissioned the reviews, and takes responsibility for the manuscript. All reviews were carried out by AI systems; no human has reviewed the results.

Status. The novelty of the results has been checked only by bounded literature searches (arXiv, zbMATH in part, the papers citing Reuss, and Tao's paper and its references), not certified.

Reproduction. The Lean sources, statement files and replay receipt are in release `v1.0.1` of the accompanying repository, https://github.com/dannyply/math-frontier; the receipt `replay/receipts/powerful-counting.json` records the commit at which it was produced. There `scripts/build.sh` compiles the library and fails on any warning, `scripts/audit.sh` checks the axioms and the statement hash of every release declaration against `release/manifest.json`, and `scripts/replay.sh` runs the independent replay. The certificate scripts are reproduced by `python3 certificates.py && python3 endpoint_cert.py` in the supplementary directory, in about a minute on a laptop; all arithmetic is exact.

Acknowledgements

The author thanks the Mathlib community for the formal library used in the accompanying repository.

Appendix A. Lean declarations

Table 2 lists the release declarations of the counting closure; each has an independent statement file and passed the replay of Section 10. Table 3 gives the Lean declaration behind every statement of the paper. Names are relative to `JocelynWorkspace.Formalization.PowerfulCounting`, except those beginning with `Erdos137`, which are relative to `JocelynWorkspace`; `CC` is `CountingClosure` and `CC.Final` is `CountingClosure.Erdos137Final`. A module name followed by eight hexadecimal digits denotes its declaration `JocelynClaims.claim_`⟨hash⟩ whose hash begins with those digits; the release declarations are listed in full in `release/manifest.json` of the repository, and a hash prefix locates any other declaration in the Lean sources. Intermediate declarations are checked as part of the dependency closure of the release declarations; only the release declarations are compared with stated results.

Result	Lean declaration	Named hypotheses
Theorem 1.2	`counting_thm_k3`	Stewart, CCDN, Gelfond
Theorem 1.1	`counting_thm_k2_reuss`	Stewart, CCDN, Gelfond, Reuss, Heath-Brown
Theorem 1.3	`counting_thm_uniform_reuss`	the five above and Tao
Proposition 5.6	`counting_prop_interval`	Stewart, CCDN, Gelfond
Theorem 6.3	`counting_thm_Bred`	Stewart, CCDN, Gelfond
Proposition 5.6, uniform in h	`counting_UPI`	Stewart, CCDN, Gelfond
Lemma 8.2, third bound	`counting_lem_uniformwindows_TB`	Stewart, CCDN, Gelfond
Lemma 8.3	`counting_lem_uniformtriple`	Stewart, CCDN, Gelfond
Reuss window convention	`reuss_window_bridge`	none

Table 2. Release declarations of the counting closure.

Statement	Lean declarations
Theorem 1.1	`CC.Final.counting_thm_k2_reuss`; `CC.C12 a9994a76`, `CC 6ee9f26a`
Theorem 1.2	`CC.Final.counting_thm_k3`; `CC.C11 d974288e`
Theorem 1.3	`CC.Final.counting_thm_uniform_reuss`; `CC 3103e62f`, `e09c0d72`, `CC.C23 8e090865`, `AllWidthFromUniformTriple 2be59d9f`, `AllWidthFromPairAndUnion fdb026ef`, `NonpowerfulMembers ceda7068`
Lemmas 2.1, 2.2	`DecompositionAndEnvelope 9673eb5d`
Lemma 2.3	`StrictWindows f0cfdddd` (fixed h); growing shifts: `CC.C21.PcPellWin.growing_cover`
Reuss's window convention	`CC.Final.reuss_window_bridge`
Lemma 4.1	`CC.C08 ca19e5d2`
Lemma 4.2	`CC.C13 a92c7c0a`
Lemma 5.1	`Paper3OriginalActualAuxiliaryPolynomial 1d348eff`
Section 5.2	`CC.C02 7327564c`
Lemma 5.2	`CC.C01.PcCurves.three_roots_gen`
Proposition 5.3	`CC.C01 6be17294`; uniform in h: `CC.C16 b07245cc`
Lemma 5.4	`CC.C09 cc873556`, `CC.C04 f5611586`
Lemma 5.5	`CC.C03 3ede4992`; type (b3): `CC.C03 f159a569`
Proposition 5.6	`CC.Final.counting_prop_interval`; `CC.C05 d9675218`, `CC.C06 40f41d3d`, `CC.C07 71200785`
Lemmas 6.1, 6.2	`LatticeTails d31e9b77`, `LatticeMoments de91d4e3`
Theorem 6.3	`CC.Final.counting_thm_Bred`; `CC.C10 664a2185`, `8d943e1c`
Lemma 6.4	`DecompositionAndEnvelope 928cf894`
Lemma 6.5	`DecompositionAndEnvelope.JocelynExpAlg.balanced_aux`
Proposition 7.1	(i) `TripleExponentCertificate d17527ab`; (ii) `PairExponentCertificate c433f729`; (iii) `DecompositionAndEnvelope 9673eb5d`, `f1b2bf39`
Proposition 7.2	`Erdos137.WidthThree.CentralSet.SmallGapExclusion.small_gap_exclusion`
Lemma 8.1	`AllWidthFromUniformTriple.JocelynAllWidth.goodtriple` (same statement: `GoodTriple 6ba68896`); $k \geq 15$: `Erdos137.FixedWidth.GoodIndex.three_consecutive_small_kernels`
Lemma 8.2	`CC.C08.PcPell.uniform_s`, `CC.C08.PcPell.uniform_t` (first two bounds); `CC.Final.counting_lem_uniformwindows_TB` (third bound), via `CC.C19 62fb44f0` (`counting_UPI`), `CC.C10 8d943e1c`, `CC.C20 88a200ce`
Lemma 8.3	`CC.Final.counting_lem_uniformtriple`; `CC.C21 b14c9f52`

TABLE 3. Concordance between the statements of the paper and the Lean development.

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