Two shorter intervals of even Dirichlet beta values

twinheterodox

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Abstract

We prove that each of the sets $\{\beta(6), \beta(8), \ldots, \beta(66)\}$ and $\{\beta(8), \beta(10), \ldots, \beta(98)\}$ contains an irrational number. These sets contain 31 and 46 values, respectively, improving the previous intervals with the same lower endpoints, which contained 35 and 52 values.

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1 The results and their source

For real $s > 0$, the Dirichlet beta function is

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}.$$

An irrationality statement about a finite set of its even values locates an irrational value without identifying which member it is. The lower endpoint matters: a result for a set containing $\beta(4)$ does not locate an irrational value beyond $\beta(4)$.

Theorem 1. *Each of the following sets contains an irrational number:*

$$\{\beta(6), \beta(8), \ldots, \beta(66)\}, \qquad \{\beta(8), \beta(10), \ldots, \beta(98)\}.$$

The powered-kernel construction of [2, Theorem 1] gives the corresponding upper endpoints 74 and 110. The comparison is

Lower endpoint	Previous interval	Present interval
$\beta(6)$	$6, 8, \ldots, 74$: 35 values	$6, 8, \ldots, 66$: 31 values
$\beta(8)$	$8, 10, \ldots, 110$: 52 values	$8, 10, \ldots, 98$: 46 values

Here the displayed integers are the arguments of β. Neither conclusion follows just by deleting the first value from the fifteen-value result for $\beta(4), \ldots, \beta(32)$ in [1]. Similarly, the first conclusion of Theorem 1 does not imply the second: its irrational member could be $\beta(6)$.

The proof combines two existing constructions. We use the order-31 rational kernel and its isolated-pole denominator estimate from [1, Sections 2–4 and Appendix A], in place of the order-35 kernel in [2]. We then apply the powering and differentiation procedure of [2, Sections 2–4]. Squaring the kernel and differentiating four times leaves the even indices $6, \ldots, 66$; cubing it and differentiating six times leaves $8, \ldots, 98$.

Two points must be checked for this combination. First, the denominator saving at an isolated pole must remain available after powering. Second, the finite rational constant used to prove nonvanishing must be nonzero for each new power. We prove the first point by retaining the identities at the integer zeros, and certify the second by exact arithmetic at the fixed base index 96. The prime-density and

phase bounds come from [1]. Their certificates are in the `fifteen-beta-values/checks` directory of the project repository [10].

The half-integer poles belong to a family of alternating hypergeometric forms that Rivoal and Zudilin used for even beta values [5, Lemma 2], following the odd-zeta construction of Ball and Rivoal [4]. Zudilin placed zeros throughout the pole interval [6, proof of Lemma 1 and Section 3]. With different block parameters and normalization, Lai and Zhou proved that at least one of $\beta(2), \dots, \beta(10)$ is irrational [7, Theorem 6.1]. To remove $\beta(2)$, Krisman [3, Sections 1.2 and 4] combined this geometry with the second derivative that Rivoal had used to remove $\zeta(3)$ [9, Section 2, Lemma 1]. Krisman's triple zeros survive two differentiations. The powers in [2] let us retain those zeros while removing further low indices. The intended prerequisites are partial fractions, elementary p-adic valuations, and asymptotic estimates for factorials.

2 Why powers move the interval

The integer zeros of the kernel will do two jobs: they suppress the early terms of the analytic sum and allow cancellation in its rational constant. Its half-integer poles produce beta values. Increasing the order of differentiation raises the first possible beta index, while the pole orders determine the last.

Fix

$$N = 96, \qquad r = \frac{3}{4}, \qquad \delta_j = \frac{j}{96} \quad (0 \le j \le 30),$$

and let n range over positive multiples of N. Set

$$I_j(n) = [\delta_j n, (1-\delta_j)n] \cap \mathbb{Z}, \qquad B_a(n) = \#\{j : a \in I_j(n)\} \quad (0 \le a \le n).$$

We usually write B_a for $B_a(n)$. At $n = 96$, the blocks are $[0, 96], [1, 95], \dots, [30, 66]$, and $B_a(96) = 1 + \min(a, 96 - a, 30)$. Thus the pole orders rise from 1 at either end to 31 in the middle. Define

$$\Lambda_n = [-(1+r)n - 1, rn] \cap \mathbb{Z}, \qquad R_n(t) = \frac{(2t+n+1)\prod_{\ell \in \Lambda_n}(t-\ell)^3}{\prod_{a=0}^{n}(t+a+\frac{1}{2})^{B_a}}. \tag{2.1}$$

The two degree parameters we need are

$$\gamma = \sum_{j=0}^{30}(1-2\delta_j) = \frac{341}{16}, \qquad G = \gamma - 3(1+2r) = \frac{221}{16}. \tag{2.2}$$

There are $\gamma n + 31$ denominator factors and $3\#\Lambda_n = 15n/2 + 6$ integer-zero factors. Consequently $\deg R_n = -Gn - 24$. The linear prefactor cancels one order at the central pole; keeping the nominal orders B_a simply gives a zero leading partial-fraction coefficient there.

The kernel satisfies

$$R_n(-n-1-t) = R_n(t). \tag{2.3}$$

Indeed, reflection permutes the integer zeros and the pole blocks. The numerator product has even degree, whereas the prefactor and the denominator both change sign.

Fix a positive integer q, and put

$$L_{q,n} = \sum_{t=0}^{\infty}(-1)^t \left(\frac{d}{dt}\right)^{2q} R_n(t)^q. \tag{2.4}$$

Each integer zero now has order $3q > 2q$. At integer arguments,

$$(R_n^q)^{(2q)}(t) = 0 \quad (0 \le t \le rn), \qquad (R_n^q)^{(2q)}(-\ell) = 0 \quad (1 \le \ell \le n) \tag{2.5}$$

Write

$$R_n(t)^q = \sum_{a=0}^{n} \sum_{i=1}^{qB_a} \frac{a_{q,n;a,i}}{(t+a+\frac{1}{2})^i}, \qquad \sigma_a(s) = \sum_{h=0}^{a-1} \frac{(-1)^h}{(2h+1)^s}.$$

Absent coefficients are set to zero, and $(i)_d = i(i+1)\cdots(i+d-1)$ denotes the rising factorial. Termwise differentiation and a shift of the beta series give

$$L_{q,n} = \rho_{q,n} + \sum_{i=1}^{31q} \tau_{q,n;i}\beta(i+2q), \tag{2.6}$$

$$\tau_{q,n;i} = 2^{i+2q}(i)_{2q} \sum_{a=0}^{n} (-1)^a a_{q,n;a,i}, \tag{2.7}$$

$$\rho_{q,n} = -\sum_{a,i} (-1)^a 2^{i+2q}(i)_{2q} a_{q,n;a,i}\sigma_a(i+2q). \tag{2.8}$$

All series converge absolutely. Reflection (2.3) gives $a_{q,n;n-a,i} = (-1)^i a_{q,n;a,i}$. Since n is even, the terms with odd i cancel in (2.7). Thus

$$L_{q,n} \in \mathbb{Q} + \sum_{\substack{2q+2 \le s \le 33q \\ s \text{ even}}} \mathbb{Q}\beta(s). \tag{2.9}$$

For $q = 2$ the upper endpoint is 66; for $q = 3$ the nominal endpoint 99 is odd and disappears, leaving 98.

3 The same clearing factor works after powering

The denominator estimate in [1] discards an isolated pole whenever it is the only source of a bad evaluation denominator. We show that powering preserves this choice. This is essential: a larger denominator estimate would need a new comparison with analytic decay.

3.1 The inherited integer

We specify the integer D_n of [1, Section 3 and Appendix A], so that the arithmetic and analytic bounds refer to the same normalization. Write $v_p(p) = 1$ and $v_p(0) = +\infty$. For an odd prime set

$$\lambda_p = \lfloor \log_p(7n) \rfloor, \qquad \nu_p(a) = 3 \sum_{\ell \in \Lambda_n} v_p(2a+1+2\ell) - \sum_{b \ne a} B_b v_p(b-a).$$

At 2, put $Z = 3\#\Lambda_n$ and

$$\lambda_2 = \lfloor \log_2 n \rfloor, \qquad \nu_2(a) = 1 - Z - \sum_{b \ne a} B_b v_2(b-a).$$

The small-prime exponents are

$$e_p = \max\{0, \max_a(-\nu_p(a) + 33\lambda_p)\} \qquad (p \text{ odd},\ p^2 \le 7n), \tag{3.1}$$

$$e_2 = \max\{0, \max_a(-\nu_2(a) + 31\lambda_2)\}. \tag{3.2}$$

For an odd prime $p \le n$ with $p^2 > 7n$, define

$$\begin{aligned} N_{p,j}(a) &= \#\{b \in I_j(n) : b \equiv a \pmod p\}, \\ c_p(a) &= 3\#\{\ell \in \Lambda_n : p \mid 2a+1+2\ell\} + \mathbf{1}_{p \mid n-2a}, \\ F_p(a) &= \sum_{j=0}^{30} N_{p,j}(a) - c_p(a). \end{aligned}$$

A pole-address class is *repeated* if it contains at least two addresses in $\{0,\dots,n\}$, and *isolated* otherwise. Let M_p be the maximum of $F_p(a)$ over repeated classes, and put

$$e_p = \max(0, M_p + 2), \qquad D_n = \prod_{p \le n} p^{e_p}. \tag{3.3}$$

For $p > n$ the exponent is zero.

Proposition 2. *For each fixed positive integer q, all the coefficients $D_n^q \rho_{q,n}$ and $D_n^q \tau_{q,n;i}$ are integers.*

3.2 Local series and the small primes

A convenient way to keep track of powering is the following elementary rule. Say that a series has bound (v, λ) at p if its degree-h coefficient has valuation at least $v - \lambda h$. Products add the first parameters and preserve a common slope λ. Hence its qth power has bound (qv, λ): every summand in a coefficient of total degree h has valuation at least $qv - \lambda h$.

At $t = -a - \frac{1}{2} + u$, remove the nominal pole of R_n. The remaining series is

$$H_a(u) = (n - 2a + 2u) \prod_{\ell \in \Lambda_n} (-a - \tfrac{1}{2} - \ell + u)^3 \prod_{b \ne a} (b - a + u)^{-B_b}. \tag{3.4}$$

The coefficient $a_{q,n;a,i}$ is $[u^{qB_a - i}] H_a(u)^q$. The local estimates of [1, Appendix A], obtained by expanding the reciprocal linear factors, give the bounds $(\nu_p(a), \lambda_p)$ and $(\nu_2(a), \lambda_2)$ for H_a. Therefore

$$v_p(a_{q,n;a,i}) \ge q\nu_p(a) - \lambda_p(qB_a - i). \tag{3.5}$$

At an odd prime, a finite harmonic sum of exponent $i + 2q$ costs at most $(i + 2q)\lambda_p$. The combined lower bound is

$$q\nu_p(a) - \lambda_p(qB_a - i) - (i + 2q)\lambda_p = q\{\nu_p(a) - (B_a + 2)\lambda_p\} \ge -qe_p.$$

At 2, the harmonic denominators are odd, so no such cost occurs; $B_a \le 31$ and (3.2) suffice. The powers of two and the integer rising factorial in (2.7)–(2.8) only improve these bounds.

3.3 Why an isolated pole remains harmless

For odd p with $p^2 > 7n$, the local calculation in [1, Section 3, local-coefficient lemma] gives H_a bound $(B_a - F_p(a), 1)$. Indeed, each nonunit linear factor has valuation one, every nonzero difference is smaller than p^2, and the central factor $2u$ also has bound $(1, 1)$. Consequently

$$v_p(a_{q,n;a,i}) \ge i - qF_p(a). \tag{3.6}$$

If a is isolated, every denominator factor in (3.4) has unit constant term, and the whole local series is integral at p. This last assertion holds for every odd prime, regardless of its size.

It remains to handle the rational correction. For $1 \le \ell \le n$ set

$$T_q(\ell, a) = \sum_i (i)_{2q} a_{q,n;a,i} (a - \ell + \tfrac{1}{2})^{-i-2q}.$$

Reindexing the finite harmonic sums in (2.8), and then using (2.5), gives

$$\rho_{q,n} = -\sum_{\ell=1}^{n} (-1)^\ell \sum_{a \ge \ell} T_q(\ell, a), \qquad \sum_{a \ge \ell} T_q(\ell, a) = -\sum_{a < \ell} T_q(\ell, a). \tag{3.7}$$

The second identity lets us choose either side of $-\ell$. Zudilin used zeros in the pole interval to shift the summation limit in the rational term [6, proof of Lemma 1]. Krisman's derivative version [3, Lemma 4.6] gives this choice at $q = 1$. For a fixed ℓ, an evaluation denominator is divisible by p only in the single class $2(a - \ell) + 1 \equiv 0 \pmod p$. If this class is isolated, choose the side of (3.7) that does not contain its address.

For example, among addresses $0, \dots, 12$ modulo 11 the class of 8 is isolated. At $\ell = 3$ it is the only class with a bad evaluation denominator. The sum over $a < 3$ avoids it. The identity making this possible still holds after powering because the integer zero has order $3q$, greater than the derivative order $2q$.

If the bad class is repeated and $p^2 > 7n$, its evaluation denominators have valuation one. Equation (3.6) then gives

$$v_p\big((i)_{2q} a_{q,n;a,i} (a - \ell + \tfrac{1}{2})^{-i-2q}\big) \ge i - qF_p(a) - (i + 2q) \ge -q(M_p + 2) \ge -qe_p.$$

Terms with unit evaluation denominators, and the beta coefficients, are covered by the same exponent; isolated coefficients are integral. For $p > n$ all classes are isolated, so the complementary choice avoids the bad class entirely, even if its denominators have higher valuation. Together with the small-prime argument this proves Proposition 2.

4 Decay with this normalization

The importance of Proposition 2 is that its integer is the qth power of the original one. Both the main arithmetic cost and the analytic decay are therefore multiplied by q.

In $\log D_n - G \log(n!)$, the large-prime contribution is at most $(8.10809 + o(1))n$ by the certificate [1, (4.5)] for the isolated-pole exponents (3.3). The small odd primes contribute $o(n)$ and the prime 2 contributes at most $15n \log 2 + o(n)$ [1, Section 4 and Appendix A]. Hence

$$\log D_n \le G \log(n!) + (8.10809 + 15 \log 2 + o(1))\, n. \tag{4.1}$$

Put $H(u) = u \log u$ for $u > 0$. For $x > r$, define

$$\Phi(x) = 3\{H(x + 1 + r) - H(x - r)\} - \sum_{j=0}^{30} \{H(x + 1 - \delta_j) - H(x + \delta_j)\}.$$

The uniform Stirling estimate of [1, Section 4] gives, for $t \ge rn + 1$,

$$\log R_n(t) = -G \log(n!) + n\Phi(t/n) + O(\log(n + t)). \tag{4.2}$$

The global maximum of Φ is bounded in [1, (4.9)], and the derivative estimate in the same proof controls the tail. Thus

$$\Phi(x) \le -18.83511 \quad (x > r), \qquad \Phi(x) \le \Phi(3) - 3 \log(x/3) \quad (x \ge 3). \tag{4.3}$$

Proposition 3. *For each fixed positive integer q,*

$$\limsup_{\substack{n\to\infty\\ 96\mid n}} \frac{1}{n}\log|D_n^q L_{q,n}| \le q(8.10809 + 15\log 2 - 18.83511) < -0.3298q,$$

where $\log 0 = -\infty$.

Proof. For $t \ge rn+1$ every linear factor is bounded away from zero and there are $O(n)$ factors, counted with multiplicity. Each logarithmic derivative of fixed positive order of R_n is therefore $O_q(n)$ on this range. The formula for a derivative of an exponential gives

$$|(R_n^q)^{(2q)}(t)| \le C_q n^{2q} R_n(t)^q.$$

The first identity in (2.5) removes all preceding integer terms.

For $rn+1 \le t \le 3n$, equations (4.2)–(4.3) bound the sum by a polynomial in n times $(n!)^{-qG}\exp(-18.83511qn)$. For $t > 3n$, the second bound in (4.3) supplies $(3n/t)^{3qn}$. Its sum absorbs the fixed polynomial in $n+t$ coming from the Stirling error and again costs only a polynomial in n. Combining with (4.1) cancels the factorial terms and proves the assertion. The strict numerical inequality follows from $\log 2 < 0.6931472$, since $\sum_{j=0}^{9}(0.6931472)^j/j! > 2$. □

5 A seed that prevents vanishing

Small integral forms give a contradiction only when they are nonzero. We use the auxiliary-prime criterion of Lai and Sprang [8, Lemma 2.3], in the prime-dilation form developed in [2, Section 4] and [1, Section 5]. After dilation, all beta coefficients will vanish modulo that prime, while a fixed rational seed constant survives. This permits us to avoid the finitely many primes dividing any hypothetical common denominator of the beta values.

5.1 Centering and the trimmed seed

The change of variable $x = 2t+n+1$ makes the poles integers. Put

$$F_n(x) = \frac{x\prod_{\substack{v \text{ odd}\\ |v|\le(5/2)n+1}}(x-v)^3}{\prod_{a=0}^{n}(x-n+2a)^{B_a}}, \qquad F_n(x)^q = \sum_{a,i}\frac{c_{q,n;a,i}}{(x-n+2a)^i}.$$

Counting powers of two gives

$$R_n(t) = 2^{Gn+25}F_n(2t+n+1), \qquad L_{q,n} = 2^{q(Gn+25)+2q}\ell_{q,n}, \tag{5.1}$$

where the centered form has expansion

$$\ell_{q,n} = \sum_{a,i}(-1)^a(i)_{2q}c_{q,n;a,i}\{\beta(i+2q) - \sigma_a(i+2q)\}. \tag{5.2}$$

Denote its rational constant by $\kappa_{q,n}$.

Take $n = Np = 96p$, where $p > 337$ is prime. The zeros divisible by p in $F_{Np}(pu)$ are pv with v odd and $|pv| \le 240p+1$. Thus $|v| \le 239$. The selected pole addresses are pa, $0 \le a \le 96$, and $B_{pa}(Np) = B_a(96)$. They give the fixed rational function

$$f(u) = \frac{u\prod_{\substack{v \text{ odd}\\ |v|\le 239}}(u-v)^3}{\prod_{a=0}^{96}(u-96+2a)^{B_a(96)}}. \tag{5.3}$$

The endpoint 239 is important: this is not F_{96}, whose last positive zero is 241.

For clarity, the factor counts are

$$W = \gamma N = 2046, \qquad GN = 1326, \qquad A_0 = (W + 31) - 1 - 3 \cdot 240 = 1356.$$

The last count says that $p^{A_0} F_{Np}(pu)$ has a nonzero reduction after the selected factors have been extracted.

Lemma 4. *Write*

$$f(u)^q = \sum_{a=0}^{96} \sum_{i=1}^{qB_a(96)} \frac{\widetilde{c}_{q;a,i}}{(u - 96 + 2a)^i}.$$

For every prime $p > 337$,

$$p^{1356q-i} c_{q,Np;pa,i} \equiv (-1)^q \widetilde{c}_{q;a,i} \pmod{p}, \tag{5.4}$$

$$v_p(c_{q,Np;a,i}) \geq i - 1326q \qquad (p \nmid a). \tag{5.5}$$

The quantities reduced in (5.4) are integral at p.

Proof. We include the counts from [1, Section 5] to explain why they pass to powers. In each block $[pj, p(96 - j)]$, every nonzero address class occurs $96 - 2j$ times and the zero class occurs once more. Hence the weighted pole counts are W for each nonzero class and $W + 31$ for zero.

After extracting the selected factors from $F_{Np}(pu)$, all remaining factors reduce to constants. Their combined unit is -1. Indeed, the $240p$ odd numbers from $-240p + 1$ to $240p - 1$ contain each residue 240 times. Their nonzero residue product is 1 by Wilson's theorem. The two extra endpoints $\pm(240p + 1)$ contribute -1, also after cubing. In the denominator, each complete nonzero residue block contributes -1, while powers of 2^{p-1} are 1; the total sign is $(-1)^W = 1$. Thus, locally away from the other selected poles,

$$p^{1356} F_{Np}(pu) \equiv -f(u) \pmod{p}. \tag{5.6}$$

At the pole $u = 96 - 2a$, multiply by its nominal pole-removing power. The other selected differences $2(b - a)$ are units, as are the constant terms of unselected denominator factors after their factors of p have been separated. The resulting local series is in $\mathbb{Z}_{(p)}[[w]]$. Raising it to the qth power and extracting the coefficient of degree $qB_a(96) - i$ proves (5.4).

At a nonselected address the prefactor is a unit, the numerator has at least 240 zeros in each residue class before cubing, and its pole-address class has weighted count W. Removing the nominal pole gives a local bound $(B_a + 720 - W, 1) = (B_a - 1326, 1)$. All nonzero differences have valuation at most one: their absolute values are below $337p < p^2$. The power rule and extraction of degree $qB_a - i$ now give (5.5). □

5.2 Separating the constant from the beta coefficients

Define the rational seed constant

$$k_q = -\sum_{a=0}^{96} \sum_{i} (-1)^a (i)_{2q} \widetilde{c}_{q;a,i} \sigma_a(i + 2q), \qquad \tau_q = (1356 + 2)q = 1358q. \tag{5.7}$$

The two parts of τ_q match the pole and the derivative. At a selected address pa, multiplying the term of order i by p^{τ_q} gives p^{i+2q} times the p-integral quantity on the left of (5.4). A harmonic denominator of exponent $i + 2q$ can consume exactly that power. At an unselected address, (5.5) leaves at least $(1356 - 1326)q = 30q$ further powers of p after the same loss. This is why only the selected addresses can contribute to the seed.

Proposition 5. *Every beta coefficient of $p^{\tau_q}\ell_{q,Np}$ is divisible by p, and all its coefficients are integral at p. Its rational constant satisfies*

$$p^{\tau_q}\kappa_{q,Np} \equiv (-1)^q \chi_4(p) k_q \pmod{p}, \qquad \chi_4(p) = (-1)^{(p-1)/2}. \tag{5.8}$$

Proof. Put $s = i + 2q$. The scaled beta coefficients have valuation at least s by the preceding bounds and the integrality of $(i)_{2q}$. Every $2h+1$ with $h < Np$ is below $2Np < p^2$, so a harmonic term costs at most s powers of p. This proves integrality of the scaled constant and makes its unselected terms zero modulo p. For a selected address pa, only denominators divisible by p survive in $p^s\sigma_{pa}(s)$. Writing $2h+1 = p(2v+1)$ gives

$$h = pv + \frac{p-1}{2}, \quad 0 \le v < a, \qquad p^s\sigma_{pa}(s) \equiv \chi_4(p)\sigma_a(s) \pmod{p}.$$

There is no further p in $2v+1$, since $p > 337 > 191$. Combining this identity, $(-1)^{pa} = (-1)^a$, and (5.4) with the constant part of (5.2) proves (5.8). □

6 Exact finite certificates and the conclusion

It remains to prove $k_2 \neq 0$ and $k_3 \neq 0$. This is a calculation on the fixed function (5.3), not on an increasing sequence of kernels. At $u_a = 96 - 2a$, form

$$J_a(w) = (u_a + w) \prod_{\substack{v \text{ odd} \\ |v| \le 239}} (u_a - v + w)^3 \prod_{\substack{0 \le b \le 96 \\ b \neq a}} (2(b-a) + w)^{-B_b(96)}.$$

Then

$$\widetilde{c}_{q;a,i} = [w^{qB_a(96)-i}] J_a(w)^q.$$

The exact reductions of (5.7) are

q	k_q mod 10007	k_q mod 10009	k_q mod 10037	Coefficients per prime
1	3738	5333	938	2077
2	8279	5481	5660	4154
3	9978	6289	4500	6231

The first two residues at $q = 1$ reproduce [1, Section 5]. The column for 10037 comes from an independent reconstruction. Each of the last two rows contains a nonzero residue, proving $k_2 \neq 0$ and $k_3 \neq 0$, respectively.

Here is a reproducible specification of the calculation. One route multiplies the truncated factors in J_a^q directly. The other writes its unit part as $g(w) = \prod_\nu (v_\nu + w)^{e_\nu}$, with integer exponents, and uses

$$\frac{g'(w)}{g(w)} = \sum_{j \ge 0} \left(\sum_\nu e_\nu \frac{(-1)^j}{v_\nu^{j+1}} \right) w^j.$$

If the coefficient in parentheses is d_j and $g(w) = \sum_{k \ge 0} g_k w^k$, then

$$g_0 = \prod_\nu v_\nu^{e_\nu}, \qquad g_k = \frac{1}{k} \sum_{j=0}^{k-1} d_j g_{k-1-j}.$$

At the central pole, remove the factor w^q before this recurrence and restore it afterwards. All three moduli are prime by trial division. Pole differences are at most 192, nonzero numerator differences

at most 335, harmonic denominators at most 191, and the recurrence indices are below 93. All inversions used above are therefore legitimate.

The two methods agree on every coefficient. The seed checker also verifies reflection and the vanishing odd beta coefficients and computes the constant in two summation orders. From the `beta-intervals/` directory of the project repository [10], run

```
python3 -B checks/interval_seeds.py
python3 -B checks/interval_seeds.py --cross-check
```

The second command requires `python-flint`; the first uses only the Python standard library.

Proof of Theorem 1. Fix $q = 2$ or $q = 3$, and suppose that all beta values in (2.9) are rational. One positive integer T clears their denominators. Proposition 2 gives

$$TD_n^q L_{q,n} \in \mathbb{Z} \qquad (96 \mid n),$$

and Proposition 3 makes these integers tend to zero.

To show that infinitely many of these integers are nonzero, we use p^{τ_q} at the single prime p on the subsequence $n = 96p$. The corresponding seed k_q is a nonzero rational number. For every sufficiently large prime outside the finite set dividing T or the numerator or denominator of k_q, all the beta values are integral at p. Proposition 5 makes their coefficients in $p^{\tau_q}\ell_{q,96p}$ divisible by p and its constant a unit. Thus $\ell_{q,96p} \neq 0$, and (5.1) gives $L_{q,96p} \neq 0$. Infinitely many nonzero integers $TD_{96p}^q L_{q,96p}$ would tend to zero, a contradiction. □

The argument up to the finite calculation applies to every fixed q for which $k_q \neq 0$. The two assertions above use only the separately certified cases $q = 2, 3$.

Verification. A second AI model independently audited the original proof and exposition of Theorem 1 and reproduced the seeds and decay bounds.

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