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An asymptotic lower bound for the number of irrational even beta values

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Abstract

For even s, let $I(s)$ count the irrational values among $\beta(2), \beta(4), \ldots, \beta(s)$. We prove

$$\liminf_{\substack{s\to\infty \\ 2\mid s}} \frac{I(s)}{\sqrt{s \log\log s/\log s}} \geq 2\sqrt{\frac{\zeta(2)\zeta(3)}{3\zeta(6)}} \approx 1.6098018250.$$

Here β denotes the Dirichlet beta function and ζ the Riemann zeta function. The bound improves the growth rate in our earlier counting result by a factor $\sqrt{\log\log s}$. It has the same growth rate as the recent theorem of Fischler, Lai, Sprang and Zudilin for irrational odd zeta values, with $1/\sqrt{3}$ of their leading constant. The conclusion remains valid after any fixed initial set of even beta values is omitted.

1 The counting problem and the character twist

The Dirichlet beta function is

$$\beta(u) = \sum_{v=0}^{\infty} \frac{(-1)^v}{(2v+1)^u} = L(u, \chi_4), \qquad \chi_4(j) = \begin{cases} 0, & 2 \mid j, \\ 1, & j \equiv 1 \pmod 4, \\ -1, & j \equiv 3 \pmod 4. \end{cases}$$

Its positive odd integer values are nonzero rational multiples of powers of π. The even values have a different arithmetic character. Our aim is to count how many of them must be irrational, without specifying which ones.

Write

$$I(s) = \#\{i : 2 \leq i \leq s,\ 2 \mid i,\ \beta(i) \notin \mathbb{Q}\}, \qquad \kappa = \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \qquad Y(s) = \sqrt{\frac{s \log\log s}{\log s}}.$$

Here and throughout, s denotes the largest allowed index, rather than the number of entries in the list.

Theorem 1.1. *For every $\varepsilon > 0$, all sufficiently large even integers s satisfy*

$$I(s) \geq \left(2\sqrt{\frac{\kappa}{3}} - \varepsilon\right) Y(s).$$

Deleting finitely many indices changes $I(s)$ by a bounded amount, whereas $Y(s) \to \infty$. Thus the same lower limit holds if, for example, the list starts with $\beta(4)$. The theorem gives neither a numerical threshold for s nor a lower bound for the dimension of the span of these values.

Our earlier explicit counting theorem [14, Theorem 1.1] gives

$$I(s) \geq \left\lfloor \pi \sqrt{\frac{2}{135}} \sqrt{\frac{s}{\log s}} \right\rfloor \qquad (s \geq 160, \quad 2 \mid s).$$

Theorem 1.1 improves the growth rate by a factor $\sqrt{\log\log s}$. Its starting endpoint depends on ε and is not computed here, whereas the earlier bound has the stated explicit range. That proof used the rational grids of Lai and Yu [12, Section 2] in the real-character setting of Kumar and Tosi [10], together with an auxiliary-prime test for nonvanishing from Lai and Sprang [11, Lemma 2.3]. The endpoint certificates are in the `even-beta-counts/checks` directory of the project repository [15].

For odd zeta values, Zudilin [17, pp. 1–2] combined two shifted sums to eliminate $\zeta(3)$. Fischler, Sprang and Zudilin [8, Section 5] used families of sums to eliminate several coefficients at once. Lai and Yu [12, Section 2] chose the shifts using Euler's totient function. Fischler, Lai, Sprang and Zudilin [7, Theorem 1.1] took Hankel determinants of the resulting moments and obtained the odd-zeta counting bound with leading constant $2\sqrt{\kappa}$. Kumar and Tosi [10, Theorem 1] state a counting bound of order $\sqrt{s/\log s}$ for the noncritical values of a real Dirichlet character, including χ_4. Their stated constant at conductor four is

$$c(4) = \left(\frac{\kappa}{2} \log \frac{\sqrt{4e^2+16}+4}{\sqrt{4e^2+16}-4} \right)^{1/2}.$$

The proof below does not use that theorem or its denominator estimates.

Our construction is a character twist of [7, Sections 2–6]. On a grid whose denominator is divisible by four, the signs $\chi_4(j)$ combine Hurwitz zeta values into beta values. Only odd j contribute. We therefore put zeros in the numerator only at the fractions used by the character and their integer translates. This retains every elimination weight and reduces the leading logarithmic cost of the numerator by a factor of three. We can consequently use a grid cutoff larger by a factor of $\sqrt{3}$. Together with the density $1/3$ of usable denominators, this explains the ratio $1/\sqrt{3}$ between the two leading constants.

An even pole order selects the even indices under reflection. The combined grid weights are odd, so we also check that the reflected measure has the opposite residues needed for nonvanishing. The arithmetic normalization, analytic test polynomials and signed-measure criterion follow [7] with the restricted numerator.

The argument assumes familiarity with elementary prime valuations, polynomial linear algebra and residues of meromorphic functions. We give the signed-measure criterion in full before applying it to the character weights.

2 The available grids

The grid family is chosen to supply many elimination weights while keeping the rational function's numerator cost under control. For $B > 1$, let

$$\begin{aligned} \Psi_B &= \{b \in \mathbb{Z}_{>0} : \varphi(b) \leq B\}, \\ \mathcal{Q}_B &= \{q \in \Psi_B : 4 \mid q\}, \\ \Theta_B &= \left\{ \frac{a}{q} : q \in \mathcal{Q}_B,\ 1 \leq a < q,\ (a,q) = 1 \right\}, \end{aligned}$$

where φ is Euler's totient function, and put $h_B = |\Theta_B|$. If $q \in \mathcal{Q}_B$, $1 \leq j < q$ and $\chi_4(j) \neq 0$, reducing j/q cancels only an odd divisor. Its reduced denominator b is still divisible by four, and

$\varphi(b) \le \varphi(q)$ because $b \mid q$. Thus j/q and $1 - j/q$ both belong to Θ_B. Conversely, each reduced fraction in Θ_B occurs with nonzero character weight on its own grid. For example, on the $q = 12$ grid, $3/12$ and its reflection reduce to $1/4$ and $3/4$, while $2/12$ carries weight zero.

Using these same denominators, put

$$A_1(B) = \prod_{q \in \mathcal{Q}_B} q^{\varphi(q)}, \qquad A_2(B) = \prod_{q \in \mathcal{Q}_B} \prod_{p \mid q} p^{\varphi(q)/(p-1)}, \qquad A(B) = A_1(B)A_2(B).$$

These are integers since $p - 1$ divides $\varphi(q)$ for $p \mid q$. Erdős [5, pp. 540, 543] proved that $|\Psi_B|/B$ has a limit, crediting joint work with Turán. Dressler [4, Theorem 1.1] evaluated the limit as κ, and Bateman [2, (1.1)–(1.4)] gave error bounds. We use the asymptotic in the same form as Lai and Yu [12, Proposition 2.2],

$$|\Psi_B| = (\kappa + o(1))B. \tag{2.1}$$

Lemma 2.1. *The usable denominators satisfy*

$$|\mathcal{Q}_B| = \left(\frac{\kappa}{3} + o(1)\right) B \qquad (B \to \infty).$$

Proof. For real $x > 0$, let $F(x)$ count all positive integers b with $\varphi(b) \le x$, let $g(x)$ count the odd such integers, and let $T(x)$ count those divisible by four. Separating the power of two in b gives

$$F(x) = 2g(x) + T(x), \qquad T(x) = \sum_{j \ge 1} g(x/2^j).$$

The two copies of g correspond to b odd and b twice an odd integer; in both cases the totient is unchanged. It follows that

$$2T(x) = F(x/2) + T(x/2), \qquad T(x) = \sum_{j \ge 1} 2^{-j} F(x/2^j).$$

These sums are finite for fixed x, since $F(y) = 0$ for $y < 1$. By (2.1), $F(y)/y \to \kappa$ and $F(y) \le Cy$ for $y \ge 1$, with an absolute constant C. Dividing the last sum by x and applying dominated convergence with majorant $C4^{-j}$ yields $T(x)/x \to \kappa \sum_{j \ge 1} 4^{-j} = \kappa/3$. □

The same count determines how many numerator factors we pay for. Each usable denominator contributes exactly $\varphi(q)$ reduced fractions. Writing $T(t)$ for the count in Lemma 2.1, partial summation in the totient variable gives

$$h_B = \sum_{q \in \mathcal{Q}_B} \varphi(q) = BT(B) - \int_0^B T(t)\,\mathrm{d}t = \left(\frac{\kappa}{6} + o(1)\right) B^2. \tag{2.2}$$

In particular $h_B + 1 \le B^2$ for all sufficiently large B, since $\kappa < 6$. Every $q \in \Psi_B$ satisfies $q \ll B \log\log B$ [12, (2.2)]. Hence $\log A_1(B) \le h_B(1 + o(1)) \log B$. For the reverse bound discard the denominators with $\varphi(q) < B/\log B$. By (2.2) they contribute $o(B^2)$ to h_B, and every remaining q satisfies $\log q \ge \log B - \log\log B$. Thus

$$\log A_1(B) = \left(\frac{\kappa}{6} + o(1)\right) B^2 \log B. \tag{2.3}$$

The corresponding coefficient for the full product is $\kappa/2$ [7, Lemma 2.2]. For A_2 the full product is an upper bound, so [7, Lemma 2.2] gives, for all sufficiently large B,

$$\log A_2(B) \le 10B^2(\log\log B)^2. \tag{2.4}$$

3 The twisted functional

Fix a sufficiently large even integer s, and take B sufficiently large with $B^2 < s/100$. We shall eventually choose B in terms of s. Until Section 7, these two parameters remain fixed and n tends to infinity through positive even integers. Write $\Theta = \Theta_B$, $h = h_B$ and $A = A(B)$.

For $(t)_k = t(t+1)\cdots(t+k-1)$, define

$$R_n(t) = A^n (n!)^{s-h} \frac{\prod_{\theta\in\Theta}(t+\theta)_n}{(t)_{n+1}^s}. \tag{3.1}$$

This uses the kernel of [7, (2.1)] with even s and the restricted numerator grid Θ. It is positive for $t > 0$, has degree $nh - (n+1)s$, and satisfies

$$R_n(-n-t) = R_n(t). \tag{3.2}$$

Indeed, reflection pairs θ with $1-\theta$ in the numerator; n is even, and the sign from the denominator disappears because s is even.

Suppose, for a contradiction, that $I(s) < |\mathcal{Q}_B|$. There are then integers w_q, not all zero, such that

$$\sum_{q\in\mathcal{Q}_B} w_q q^i = 0 \quad \text{whenever } 2 \le i \le s, \quad 2 \mid i, \quad \beta(i) \notin \mathbb{Q}. \tag{3.3}$$

This homogeneous rational system has fewer equations than unknowns; clearing denominators gives integral weights. Choose a positive integer A_0 clearing the denominators of the remaining, rational, beta values in the same range. Both A_0 and the w_q are fixed as n varies.

Writing $\zeta(i,\theta) = \sum_{v\ge 0}(v+\theta)^{-i}$ for the Hurwitz zeta function, the character identity underlying the construction is, for $i > 1$ and $4 \mid q$,

$$\sum_{j=1}^{q} \chi_4(j)\zeta(i, j/q) = q^i \beta(i). \tag{3.4}$$

For example, the grid $q = 4$ gives $\zeta(i,1/4) - \zeta(i,3/4) = 4^i\beta(i)$. In general, the integers in the Hurwitz sums are $qv + j$, and $\chi_4(qv+j) = \chi_4(j)$; this proves (3.4) by absolutely convergent summation.

For a polynomial P, set

$$\mathcal{L}_n(P) = A_0 \sum_{q\in\mathcal{Q}_B} w_q \sum_{j=1}^{q} \chi_4(j) \sum_{v=0}^{\infty} R_n(v + j/q)\left[P(v + j/q + n/2) + P(-v - j/q - n/2)\right]. \tag{3.5}$$

We call P admissible if

$$\deg P \le (n+1)s - nh - 2. \tag{3.6}$$

This condition makes the summand $O(v^{-2})$ and guarantees absolute convergence. The proof uses the same moment matrix in two bases. The first makes its scaled determinant an integer. The second bounds the absolute value of that integer by a number less than one. To obtain a contradiction, we allow the size m to vary and prove that at least one permitted size has nonzero determinant. We write

$$\Delta_{n,m} = \det\left(\mathcal{L}_n(t^{i+j})\right)_{0\le i,j<m}, \qquad 2m \le (n+1)s - nh. \tag{3.7}$$

4 Arithmetic normalization

Let $d_n = \operatorname{lcm}(1,\dots,n)$. The role of the numerator zeros is more than analytic: they cancel the possible extra denominators in the finite Hurwitz tails. We spell out this step because twisting the grids must not silently change the integerizer.

4.1 The elementary derivative bounds

The following facts are the building blocks used in [7, Section 3]. A direct proof also shows why their arithmetic does not depend on the parity of s.

Lemma 4.1. *For a fixed positive integer n, each of the following functions F satisfies $d_n^\ell F^{(\ell)}(k)/\ell! \in \mathbb{Z}$ for all $\ell \geq 0$ at the indicated integer evaluation points:*

1. $(t+a)_r/r!$, *with $a \in \mathbb{Z}$ and $0 \leq r \leq n$, at every $k \in \mathbb{Z}$;*
2. $b^n \prod_{p|b} p^{\lfloor n/(p-1) \rfloor} (t+a/b)_n/n!$, *with $(a,b)=1$ and $b>1$, at every $k \in \mathbb{Z}$;*
3. $n!(t+j)/(t)_{n+1}$, *with $0 \leq j \leq n$, at $k=-j$.*

Proof. The case $r=0$ and Taylor orders above the degree are immediate. For the first assertion with $r \geq 1$, a Taylor coefficient of order $\ell \leq r$ is a sum of products obtained by deleting ℓ factors from r consecutive integers, divided by $r!$. For any prime p and integer $v \geq 1$, the original progression has at least $\lfloor r/p^v \rfloor$ factors divisible by p^v. Deleting ℓ factors loses at most ℓ of them. Hence each product has valuation at least

$$v_p(r!) - \ell \lfloor \log_p r \rfloor \geq v_p(r!) - \ell v_p(d_n).$$

This proves integrality after multiplication by d_n^ℓ.

For the second assertion, write its numerator as $\prod_{i=0}^{n-1}(bt+a+bi)$. If $p \nmid b$, this arithmetic progression has at least $\lfloor n/p^v \rfloor$ multiples of each p^v; the same deletion argument applies, and the extra factors b^ℓ are integral. If $p \mid b$, the prefactor $p^{\lfloor n/(p-1) \rfloor}$ alone clears $n!$, since $v_p(n!) \leq \lfloor n/(p-1) \rfloor$. All the numerator coefficients are integers, proving the assertion at these primes too.

For the last assertion set $t=-j+u$. The constant coefficient is $(-1)^j \binom{n}{j}$, and the remaining factor is

$$\prod_{\substack{0 \leq k \leq n \\ k \neq j}} \left(1 + \frac{u}{k-j}\right)^{-1}.$$

Each Taylor coefficient of order ℓ is a sum of products of ℓ reciprocals of nonzero integers of absolute value at most n. Multiplication by d_n^ℓ clears them. □

4.2 The coefficients and the finite tails

Lemma 4.2. *If an admissible $U \in \mathbb{Q}[t]$ satisfies*

$$\frac{d_n^\ell}{\ell!} U^{(\ell)}(k) \in \mathbb{Z} \qquad (k \in \mathbb{Z},\ \ell \geq 0), \tag{4.1}$$

then $d_n^s \mathcal{L}_n(U) \in \mathbb{Z}$.

Proof. Write

$$\widetilde{R}_n(t) = R_n(t)\,[U(t+n/2) + U(-t-n/2)] = \sum_{k=0}^{n} \sum_{i=1}^{s} \frac{a_{i,k}}{(t+k)^i}.$$

The decomposition has no polynomial part by admissibility. The arithmetic building-block argument of [7, (3.5)–(3.6)] gives

$$d_n^{s-i} a_{i,k} \in \mathbb{Z} \qquad (1 \leq i \leq s,\ 0 \leq k \leq n). \tag{4.2}$$

For clarity, $(t+k)^s\widetilde{R}_n(t)$ factors into s copies of $n!(t+k)/(t)_{n+1}$, the blocks

$$b^n \prod_{p\mid b} p^{\lfloor n/(p-1)\rfloor} \frac{(t+a/b)_n}{n!} \quad (b\in\mathcal{Q}_B,\ 1\le a<b,\ (a,b)=1),$$

and $U(t+n/2)+U(-t-n/2)$, together with an integer factor. Lemma 4.1 and the hypothesis on U give $d_n^\ell F^{(\ell)}(-k)/\ell!\in\mathbb{Z}$ for each factor. The leftover constant is integral because $n\varphi(b)/(p-1)\ge \varphi(b)\lfloor n/(p-1)\rfloor$ for every $p\mid b$. Applying the product rule to the Taylor coefficient of order $s-i$ proves (4.2). Neither this calculation nor its building-block estimates require s to be odd. The shifts of U are integral because n is even.

Reflection now has the opposite parity to the odd-zeta construction:

$$a_{i,n-k}=(-1)^i a_{i,k}.$$

Consequently $\rho_i=\sum_{k=0}^n a_{i,k}$ vanishes for every odd i. For $0<\theta<1$, summing the partial fractions therefore gives

$$\sum_{v\ge 0}\widetilde{R}_n(v+\theta)=\rho_{0,\theta}+\sum_{\substack{2\le i\le s\\ 2\mid i}}\rho_i\zeta(i,\theta),\qquad \rho_{0,\theta}=-\sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{i=1}^{s}\frac{a_{i,k}}{(l+\theta)^i}. \tag{4.3}$$

The $i=1$ terms are combined in finite sums before taking their limit. Their common harmonic divergence has coefficient $\rho_1=0$, and the remaining shifted endpoint terms tend to zero. Thus no value $\zeta(1,\theta)$ is introduced.

We claim $d_n^s\rho_{0,\theta}\in\mathbb{Z}$ for each $\theta=j/q$ with $q\in\mathcal{Q}_B$ and $\chi_4(j)\ne 0$. Such θ and $1-\theta$ belong to Θ. Fix $0\le l<k\le n$ and a prime p, and put $e=v_p(d_n)$. If $v_p(l+\theta)\le e$, each term $d_n^s a_{i,k}/(l+\theta)^i$ is p-integral by (4.2). Otherwise, following Sprang [13, Lemma 1.4] and Fischler, Sprang and Zudilin [8, Lemma 2], we evaluate at a numerator zero to replace the terms with a bad denominator by terms integral at p. Here the zero is $t_0=-k+l+\theta$. The block $(t+1-\theta)_n$ vanishes there, since $1\le k-l\le n$. Hence

$$\sum_{i=1}^{s}\frac{d_n^s a_{i,k}}{(l+\theta)^i}=-\sum_{\substack{0\le k'\le n\\ k'\ne k}}\sum_{i=1}^{s}\frac{d_n^s a_{i,k'}}{(l+\theta+k'-k)^i}.$$

Since $v_p(k'-k)\le e<v_p(l+\theta)$, the denominator $(l+\theta+k'-k)^i$ has valuation $i\,v_p(k'-k)\le ie$. Equation (4.2) gives $v_p(d_n^s a_{i,k'})\ge ie$, so every term on the right is p-integral. This proves the claim prime by prime, after summing the finite tails.

Finally apply (3.4) to (4.3) and sum with weights w_q. The irrational beta terms disappear by (3.3); the rational ones are cleared by A_0, while (4.2) clears their coefficients after multiplication by d_n^s. The finite tails have just been cleared by the same multiplier. □

The command `python3 checks/verify_count_examples.py` reproduces the finite-tail cancellations for $B=2$, $s=8$ and $n=2,4$. Run it from `asymptotic-even-beta-counts/` in the project repository [15].

Proposition 4.3. *For $i\ge 0$, define*

$$E_{n,i}=(n!)^{\lfloor i/n\rfloor}(i-n\lfloor i/n\rfloor)!,\qquad H_{n,m}=\frac{d_n^{sm}}{\prod_{i=0}^{m-1}E_{n,i}^2}.$$

For every m allowed in (3.7), $H_{n,m}\Delta_{n,m}\in\mathbb{Z}$.

Proof. Use $U_i(t) = (t)_i/E_{n,i}$ as a basis of polynomials of degree below m. It is a product of normalized Pochhammer polynomials of degrees at most n: divide the i factors into blocks of n and a final shorter block. Lemma 4.1 and the product rule, as in [7, Proposition 3.2], give (4.1) for each product U_iU_j. Lemma 4.2 shows that all entries of $(d_n^s\mathcal{L}_n(U_iU_j))$ are integers. The triangular change of basis has determinant $\prod_{i<m} E_{n,i}^{-1}$. Applying it to the rows and columns gives

$$\det\bigl(d_n^s\mathcal{L}_n(U_iU_j)\bigr)_{i,j<m} = d_n^{sm}\left(\prod_{i<m} E_{n,i}^{-1}\right)^2 \Delta_{n,m} = H_{n,m}\Delta_{n,m}.$$

□

5 Smallness with this same normalization

The analytic gain comes from using a different polynomial basis to estimate the determinant. Its first zeros remove the earliest atoms of the measure. The signs of the character do not affect this argument, which bounds absolute weights.

Proposition 5.1. *There is a constant $C_0 > 0$, independent of n and m, such that, whenever*

$$2m \le (s - B^2 - 10)n, \tag{5.1}$$

we have

$$H_{n,m}|\Delta_{n,m}| \le (C_0 m)^m A^{mn} e^{m^2+(2s+10B^2)mn}\left(\frac{10B^2 n}{m}\right)^{(s-B^2)mn-2m^2}. \tag{5.2}$$

The constant may depend on the fixed parameters s, B, A_0 and w_q.

Proof. We give the estimate of [7, Proposition 4.1] with the character weights included. Put $\nu_i = \lfloor i/(2h+2)\rfloor$ and take the monic polynomials

$$V_i(z) = z^{i-(2h+2)\nu_i}\prod_{\theta\in\Theta\cup\{1\}}\prod_{v=0}^{\nu_i-1}\left(z^2 - (v+\theta+n/2)^2\right).$$

They have degree i, so $\Delta_{n,m} = \det(\mathcal{L}_n(V_iV_j))$. At every contributing point $t = v + j/q$ we have $\chi_4(j) \neq 0$ and $j/q \in \Theta$. Thus both $V_i(t+n/2)$ and $V_i(-t-n/2)$ vanish if $v < \nu_i$. For $v \ge \nu_i$, all their roots in absolute value are no larger than $t + n/2$, which gives

$$|V_i(t+n/2)| = |V_i(-t-n/2)| \le (t+n)^i.$$

Let S_i be the sum defining $\mathcal{L}_n(V_i^2)$ with every weight $w_q\chi_4(j)$ replaced by its absolute value. Cauchy–Schwarz and the determinant expansion give

$$|\mathcal{L}_n(V_iV_j)| \le \sqrt{S_iS_j}, \qquad |\Delta_{n,m}| \le m!\prod_{i<m} S_i \le m^m\prod_{i<m} S_i. \tag{5.3}$$

Write $B_0 = 1 + \max\mathcal{Q}_B$. All contributing points satisfy $t \ge 1/(B_0-1)$. The inequalities

$$(t)_{n+1} \ge \frac{n!}{B_0}(1+t/n)^{n+1}, \qquad (t+\theta)_n \le (t+n)^n, \qquad n! \ge (n/e)^n$$

imply

$$R_n(t) \le B_0^s A^n e^{hn} \left(\frac{n}{t+n}\right)^{(n+1)s-nh}.$$

For $e_i = (n+1)s - nh - 2i$, condition (5.1) gives $e_i - 1 > 10n$. An integral comparison therefore yields

$$\sum_{v \ge \nu_i} \left(\frac{n}{v + j/q + n}\right)^{e_i} \le \frac{2}{(1+\nu_i/n)^{e_i-1}}.$$

Since $h+1 \le B^2$ and $n \ge 2$, we also have $1 + \nu_i/n > (i+1)/(2B^2 n)$. Set $E = (s-B^2)n - 2m > 0$. When $2B^2 n/(i+1) < 1$ the preceding inequalities allow the exponent to be reduced to E; when that ratio is at least one, the same bound follows by first using $(1+\nu_i/n)^{-(e_i-1)} \le 1$. Thus

$$S_i \le C_0 A^n e^{B^2 n} n^{2i} \left(\frac{2B^2 n}{i+1}\right)^E, \qquad C_0 = 4A_0 B_0^s \sum_{q \in \mathcal{Q}_B} q|w_q|.$$

We used only $|\chi_4(j)| \le 1$. Combining with (5.3) and $m! \ge (m/e)^m$ gives

$$|\Delta_{n,m}| \le (C_0 m)^m A^{mn} e^{B^2 mn} n^{m(m-1)} \left(\frac{2eB^2 n}{m}\right)^{Em}. \tag{5.4}$$

Finally, the elementary bounds $d_n \le e^{2n}$ and

$$\log E_{n,i} \ge i \log n - i - n$$

give

$$H_{n,m} \le n^{-m(m-1)} e^{m^2 + 2(s+1)mn}.$$

Multiply this by (5.4), use $2e < 10$ and $B^2 + 2 \le 10B^2$, and obtain (5.2). □

6 Nonvanishing for the signed character measure

Zudilin [16, Proposition 2] used Hankel determinants of positive moments in an irrationality criterion. Fischler, Lai, Sprang and Zudilin [7, Section 1] credit Fauzan's positive-moment construction for $\zeta(5)$ [6, Section 2.4] as the prototype for their determinant argument. The weights in (3.5) can have either sign. We use the signed-measure criterion of Fischler, Lai, Sprang and Zudilin [7, Proposition 5.4] and give its proof to identify the hypothesis our character weights must satisfy.

6.1 A signed-measure criterion

Let $X \subset \mathbb{R}$ be closed and discrete, and let $\mu = \sum_{x \in X} a_x \delta_x$ be a real signed measure. Suppose, for some positive integer D, that

$$\sum_{x \in X} |a_x|(1+|x|)^{D-1} < \infty. \tag{6.1}$$

Extend a_x by zero outside X. Its Cauchy transform and moments are

$$f(z) = \sum_{x \in X} \frac{a_x}{z-x}, \qquad \mu_k = \sum_{x \in X} a_x x^k \quad (0 \le k < D).$$

The series for f is locally uniformly convergent off X, and its residue at x is a_x.

Proposition 6.1 (Fischler–Lai–Sprang–Zudilin)**.** *Assume* (6.1) *and*

$$\mathfrak{A}(x)a_{x+1} = \mathfrak{B}(x)a_x \quad (x \in \mathbb{R}), \tag{6.2}$$

where $\mathfrak{A}, \mathfrak{B} \in \mathbb{C}[z]$ *and* $1 \le d = \max(\deg \mathfrak{A}, \deg \mathfrak{B}) \le D$*. Put*

$$\mathfrak{C}(z) = \sum_{x \in X} a_x \left(\frac{\mathfrak{A}(z) - \mathfrak{A}(x-1)}{z - (x-1)} - \frac{\mathfrak{B}(z) - \mathfrak{B}(x)}{z - x} \right). \tag{6.3}$$

If

$$\mathfrak{A}(z)r(z+1) - \mathfrak{B}(z)r(z) = \mathfrak{C}(z) \tag{6.4}$$

has no solution $r \in \mathbb{C}(z)$*, then for every integer* $M \ge 1$ *with* $2(M + d - 1) - 2 < D$ *there is an* $m \in [M, M + d - 1]$ *such that* $\det(\mu_{i+j})_{0 \le i,j < m} \ne 0$*.*

Proof. All summands in (6.3) are polynomials of degree at most $d - 1$, and (6.1) makes their coefficients absolutely summable. Subtract the polynomial divided differences from $\mathfrak{A}(z)f(z+1) - \mathfrak{B}(z)f(z)$. The remaining partial fractions cancel by (6.2). Consequently

$$\mathfrak{A}(z)f(z+1) - \mathfrak{B}(z)f(z) = \mathfrak{C}(z). \tag{6.5}$$

For discrete semiclassical moment functionals, Dominici and Marcellán [3, Theorem 11] derive the analogous difference equation for the formal Stieltjes transform from the recurrence for the weights.

Set $N = M + d - 1$. We first show that the $N \times M$ Hankel matrix $(\mu_{i+j})_{0 \le i < N,\, 0 \le j < M}$ has rank M. Otherwise there is a nonzero polynomial Q of degree at most $M - 1$ such that

$$\sum_{x \in X} a_x Q(x) x^i = 0 \qquad (0 \le i < N).$$

The polynomial

$$P(z) = \sum_{x \in X} a_x \frac{Q(z) - Q(x)}{z - x}$$

has degree at most $M - 2$ (and is zero if $M = 1$). Expanding $1/(z - x)$ to order N and using the vanishing moments gives

$$Q(z)f(z) - P(z) = \frac{1}{z^N} \sum_{x \in X} a_x Q(x) x^{N-1} \frac{x}{z - x}.$$

The absolute sum of $a_x Q(x) x^{N-1}$ is finite, since $\deg Q + N - 1 \le 2N - 2 < D$. On each line $z = iy$ and $z = 1 + iy$ with $y \ge 1$, the last quotient is bounded in absolute value by 2 and tends to zero for each fixed x. Dominated convergence yields

$$Q(z)f(z) - P(z) = o(y^{-N}) \qquad (y \to \infty) \tag{6.6}$$

on both lines. The polynomial

$$\mathfrak{A}(z)P(z+1)Q(z) - \mathfrak{B}(z)P(z)Q(z+1) - \mathfrak{C}(z)Q(z)Q(z+1)$$

is therefore $o(1)$ on the imaginary axis, by (6.5), (6.6) and $N \ge d + \deg Q$. It vanishes identically. Dividing by $Q(z)Q(z+1)$ makes P/Q a solution of (6.4), a contradiction.

It remains to pass from this rectangular rank to a principal determinant. The leading-minor criterion is classical [9, Chapter X, §10, Lemma 1]. As in [7, Lemma 5.3], we apply it in the form

given by Adiceam, Nesharim and Lunnon [1, Lemma 2.3]. In the $N \times N$ Hankel matrix, let m be the largest number of linearly independent initial columns. We have $m \geq M$. If $m = N$, we are done. Otherwise the next column supplies a recurrence

$$\mu_{i+m} + \sum_{j=0}^{m-1} c_j \mu_{i+j} = 0 \qquad (0 \leq i < N).$$

For the row vectors of the first m columns, this recurrence expresses each row from index m onward as a combination of the preceding m rows. The indices needed in each coordinate lie within the displayed recurrence. Thus the first m rows span all these rows. Since their column rank is m, their first $m \times m$ block is invertible. This is the required principal determinant. □

6.2 The two tails of the character measure

Let $X = \mathbb{Z} + (\Theta \cup \{1\})$. On X, define the combined grid weight

$$w(x) = \sum_{\substack{q \in \mathcal{Q}_B \\ qx \in \mathbb{Z}}} w_q \chi_4(qx), \qquad \widehat{R}_n(x) = R_n(x - n/2).$$

Because each q is divisible by four,

$$w(x+1) = w(x), \qquad w(-x) = -w(x), \qquad w(k) = 0 \ (k \in \mathbb{Z}). \tag{6.7}$$

Equation (3.2) says that $\widehat{R}_n$ is even. The moments in (3.5) are those of the signed measure with masses

$$a_x = \begin{cases} A_0 \operatorname{sgn}(x) w(x) \widehat{R}_n(x), & x \in X, \ |x| > n/2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.8}$$

On the positive tail this follows by putting $x = v + j/q + n/2$; $n/2$ is an integer. On the negative tail, the oddness of w and the extra factor $\operatorname{sgn}(x)$ make the reflected mass equal to the positive one, just as the two evaluations of P in (3.5) require. The sign factor cannot be dropped. The measure is even, but its individual masses can have either sign.

The ratio of consecutive kernel values is

$$\frac{\widehat{R}_n(z+1)}{\widehat{R}_n(z)} = \frac{\mathcal{B}_n(z)}{\mathcal{A}_n(z)}, \tag{6.9}$$

where

$$\mathcal{A}_n(z) = (z + 1 + n/2)^{s+1} (z - n/2) \prod_{\theta \in \Theta} (z + \theta - n/2), \tag{6.10}$$

$$\mathcal{B}_n(z) = (z - n/2)^{s+1} (z + 1 + n/2) \prod_{\theta \in \Theta} (z + \theta + n/2). \tag{6.11}$$

These are the coefficients of [7, (5.11)] with the products taken over our Θ. Both have degree

$$d = s + h + 2.$$

Away from the central gap, (6.9) and periodicity give $\mathcal{A}_n(x) a_{x+1} = \mathcal{B}_n(x) a_x$. At a transition into the positive tail, write $x = n/2 - 1 + \theta$, where $0 < \theta < 1$ is a grid fraction. The factor indexed by $1 - \theta$ in $\mathcal{A}_n(x)$ vanishes. At a transition out of the negative tail, $x = -n/2 - \theta$, and the factor

indexed by θ in $\mathcal{B}_n(x)$ vanishes. Integral endpoints carry no mass by (6.7). Points outside X remain outside X after an integer shift. Thus the recurrence holds for the extension by zero at every real x.

Since $\deg R_n = nh - (n+1)s$, condition (6.1) holds with

$$D = (n+1)s - nh - 1. \tag{6.12}$$

Indeed the summands at infinity are $O(|x|^{-2})$, and X is a finite union of translates of $\mathbb{Z}$.

Lemma 6.2. *For this measure and the polynomials (6.10)–(6.11), the rational equation in Proposition 6.1 has no solution.*

Proof. Suppose $r \in \mathbb{C}(z)$ solves it. The Cauchy transform f solves the same equation. Subtracting and using (6.9), we find that

$$g(z) = \frac{f(z) - r(z)}{\widehat{R}_n(z)} \quad \text{satisfies} \quad g(z+1) = g(z)$$

as a meromorphic identity.

Choose the largest denominator with a nonzero elimination weight, $q_* = \max\{q : w_q \neq 0\}$, and set $\theta_* = 1/q_*$. At either point $x = \pm T + \theta_*$, with integral T, a grid denominator q contributes only if $q_* \mid q$. Maximality leaves just $q = q_*$ among the nonzero weights. Moreover $\chi_4(q_*(\pm T) + 1) = 1$, so

$$w(T + \theta_*) = w(-T + \theta_*) = w_{q_*} \neq 0.$$

Reflection pairs $T + \theta_*$ with $-T - \theta_*$, where the masses agree. Periodicity pairs $T + \theta_*$ with $-T + \theta_*$, where the grid weight is the same and the sign of x changes. Choose T large enough that these latter points lie outside the central gap and avoid the finitely many poles of $r/\widehat{R}_n$ and all zeros or poles of $\widehat{R}_n$. From (6.8), the residues of g at these two points are respectively

$$A_0 w_{q_*} \quad \text{and} \quad -A_0 w_{q_*}.$$

But the points differ by the integer $2T$, so a one-periodic meromorphic function must have equal residues there. This is a contradiction. □

Corollary 6.3. *Let $M \in \mathbb{Z}_{\geq 1}$. If $d \leq D$ and $2(M + d - 1) - 2 < D$, then some integer $m \in [M, M + d - 1]$ satisfies $\Delta_{n,m} \neq 0$.*

Proof. Apply Proposition 6.1 to (6.8), using Lemma 6.2. Its moments are exactly those defining $\Delta_{n,m}$. □

7 The asymptotic comparison

We now choose the parameters. The number of usable grids grows linearly with B, while $\log A(B)$ grows like $B^2 \log B$. The determinant estimate permits B to be larger by a factor $\sqrt{\log\log s}$ than the earlier single-form scale. The following calculation makes the gain and the order of limits explicit.

Fix

$$0 < \lambda < \sqrt{\frac{12}{\kappa}}, \qquad B = \lambda Y(s), \qquad C = \frac{s}{\log\log s}.$$

For all sufficiently large even s, the hypotheses on B and s in Sections 3–5 hold, and

$$2C < s - B^2 - 10.$$

Fix one such s. As n tends to infinity through even integers, take $M = \lceil Cn \rceil$. The degree conditions in Corollary 6.3 hold for all sufficiently large n, since $h \le B^2 - 1$ and $2C < s - B^2 - 10$. Consequently we may choose

$$M \le m = m(n) \le M + d - 1, \qquad \Delta_{n,m} \neq 0, \qquad \frac{m}{n} \longrightarrow C.$$

The same strict inequality ensures (5.1) for these m, once n is large. Proposition 4.3 then gives nonzero integers

$$J_n = H_{n,m(n)}\Delta_{n,m(n)}, \qquad |J_n| \ge 1.$$

We will contradict this using Proposition 5.1.

Take logarithms in (5.2), divide by mn, and first let $n \to \infty$ with s, B, C fixed. The term $\log(C_0 m)/n$ disappears, regardless of the size of the fixed elimination weights or A_0. We obtain

$$\limsup_{\substack{n\to\infty \\ 2|n}} \frac{\log|J_n|}{m(n)n} \le \log A + C + 2s + 10B^2 + (s - B^2 - 2C)\log\frac{10B^2}{C}. \tag{7.1}$$

The right-hand side is now an expression in s. As $s \to \infty$, (2.3)–(2.4) and $\log B \sim \frac{1}{2}\log s$ give

$$\log A \le \left(\frac{\kappa\lambda^2}{12} + o(1)\right) s \log\log s. \tag{7.2}$$

For example, the A_2 contribution is $O(s(\log\log s)^3/\log s) = o(s\log\log s)$, so it does not affect the leading constant. On the analytic side,

$$\log\frac{10B^2}{C} = -\log\log s + 2\log\log\log s + \log(10\lambda^2),$$
$$s - B^2 - 2C = s(1 - o(1)).$$

Inserting these into (7.1) gives

$$\limsup_{\substack{n\to\infty \\ 2|n}} \frac{\log|J_n|}{m(n)n} \le \left(\frac{\kappa\lambda^2}{12} - 1 + o(1)\right) s\log\log s.$$

The coefficient is negative for every sufficiently large s, by the choice of λ. For each such fixed s, we therefore obtain $|J_n| < 1$ for all sufficiently large even n, contradicting $J_n \in \mathbb{Z} \setminus \{0\}$.

The contradiction disproves the assumption $I(s) < |\mathcal{Q}_B|$. Lemma 2.1 now implies

$$\liminf_{\substack{s\to\infty \\ 2|s}} \frac{I(s)}{Y(s)} \ge \frac{\lambda\kappa}{3}.$$

Letting λ increase to $\sqrt{12/\kappa}$ gives $(\kappa/3)\sqrt{12/\kappa} = 2\sqrt{\kappa/3}$, proving Theorem 1.1.

Remark 7.1. The proof concerns growing intervals and uses no explicit threshold for s. In particular, it does not by itself give an irrationality statement for a short prescribed list or for an individual even beta value. Both the character-dependent nonvanishing argument and the fixed-s order of limits are essential to the conclusion above.

Verification. Independent AI models audited the mathematics and exposition of the original and revised proofs. The first proposed the restricted grid.

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