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EQUAL PRIME DIVISORS IN DISJOINT BLOCKS OF CONSECUTIVE INTEGERS

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Abstract. We construct infinitely many disjoint pairs of consecutive blocks, of lengths three and four in that order, whose products have identical prime divisors. Their starting positions disprove the finiteness and separation predictions in Erdős's 1976 formulation. The modern Erdős problem 931 requires the earlier block to be at least as long as the later one and remains open. For any fixed lengths $k, \ell \geq 2$, the number of equal-support pairs with increasing first terms at most X, even allowing overlap, is $O_{k,\ell,\varepsilon}(X^{2/3+\varepsilon})$. In the modern length order, we prove finiteness on fixed conics, on fixed equations in separated rational functions, and for pairs admitting two independent multiplicative relations. We also give exact counts in three strips of bounded separation, without a bound on either starting term.

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1. The historical and ordered problems

The products $34 \cdot 35 \cdot 36$ and $48 \cdot 49 \cdot 50 \cdot 51$ have exactly the same five prime divisors $2, 3, 5, 7, 17$. They are the first pair in an infinite family that disproves a conjecture of Erdős from 1976 and Conjecture 11 of Lebowitz-Lockard from 2025. The modern Erdős problem #931 imposes a further condition on which block comes first. It remains open [13, 18, 15].

1.1. **The order of the two blocks.** Write $P_k(n) = \prod_{i=1}^{k}(n+i)$ and $\mathrm{rad}(a) = \prod_{p|a} p$, the product over distinct prime divisors. On page 29 of his 1976 paper, Erdős conjectured that $P_{k_1}(n_1)$ and $P_{k_2}(n_2)$ could have equal radicals only finitely often when $\min(k_1, k_2) \geq 3$ and $n_2 \geq n_1 + k_1$. He also suggested that equality might imply $n_2 > 2(n_1 + k_1)$. His printed display (10) permits the

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shorter block to come first [13]. The catalog asks whether, for each fixed $k_1 \geq k_2 \geq 3$, there are finitely many pairs (n_1, n_2) with $n_2 \geq n_1 + k_1$ and equal radicals. Thus it requires the earlier block to be at least as long as the later one. In 1980 Erdős also proposed finiteness for two blocks of the same fixed length $k > 2$, using least common multiples, which have the same prime divisors as the products. Both questions remain open [11, 15, 14]. Lebowitz-Lockard's Conjecture 11 predicts finiteness for fixed lengths $k, \ell > 1$ other than $(2, 2)$, with starting values $m < n$. When a two-term block is allowed, the earlier triple $v - 1, v, v + 1$ and the later pair $v^2 - 1, v^2$ already give a counterexample for every integer $v \geq 3$. Their products differ by a factor of v. The Pell construction gives infinitely many pairs with both lengths at least three, as required by Erdős in 1976 [18].

1.2. **The Pell construction.** For four consecutive integers centered at $U/2$, the inner pair has product $(U^2 - 1)/4$ and the outer pair has product $(U^2 - 9)/4$. The difference is two. The two numbers $2v^2$ and $2(v - 1)(v + 1)$ also differ by two, and together they contain exactly the primes in the three-term block centered at v. Equating the inner product with $2v^2$ gives the Pell equation $U^2 - 8v^2 = 1$ and automatically matches the outer product as well.

Theorem 1.1. *For every $j \geq 3$, write $(3 + \sqrt{8})^j = U_j + v_j\sqrt{8}$. Put $v = v_j$ and $U = U_j$. The earlier block $[v - 1, v + 1]$ and the later block $[(U - 3)/2, (U + 3)/2]$ are disjoint integer intervals whose products have the same radical. In Erdős's notation their positions satisfy*

$$\frac{n_2}{n_1 + k_1} = \frac{U - 5}{2(v + 1)} < 2, \qquad \lim_{j\to\infty} \frac{n_2}{n_1 + k_1} = \sqrt{2}.$$

Proof. Multiplication by $3 + \sqrt{8}$ sends (U, v) to $(3U + 8v, U + 3v)$. Starting at $(1, 0)$ gives positive, strictly increasing U_j, v_j for $j \geq 1$, with every U_j odd. Taking norms gives $U^2 - 8v^2 = 1$. Since $v \geq v_3 = 35$, we have $U > \sqrt{8}\, v > 2v + 5$, so the later block starts beyond $v + 1$.

Pair the two outer terms of the four-term block and the two inner terms. Its product is

$$\frac{(U^2 - 9)(U^2 - 1)}{16} = 4v^2(v^2 - 1) = 4v\,(v - 1)v(v + 1).$$

The multiplier $4v$ contributes only primes already in the earlier block. Every prime dividing v occurs there, and one of three consecutive integers is even. This proves the radical equality. The choices $n_1 = v - 2$, $k_1 = 3$, $n_2 = (U - 5)/2$, $k_2 = 4$ give the stated ratio. From $U^2 = 8v^2 + 1$ we have $U < 3v$ and $U/v \to \sqrt{8}$. The separation inequality therefore fails for infinitely many distinct pairs, even if one allows finitely many exceptions. □

j	(U_j, v_j)	Earlier block	Later block	Product ratio
3	$(99, 35)$	$34, 35, 36$	$48, 49, 50, 51$	140
4	$(577, 204)$	$203, 204, 205$	$287, 288, 289, 290$	816
5	$(3363, 1189)$	$1188, 1189, 1190$	$1680, 1681, 1682, 1683$	4756

The construction also gives a lower bound for Lebowitz-Lockard's counting function $F_{3,4}(X)$, which counts pairs whose later starting value is at most X. Put $\eta = 3 + \sqrt{8}$. Taking conjugates gives $U_j = (\eta^j + \eta^{-j})/2$, so the later block starts at $\eta^j/4 + O(1)$. Consequently this family alone contributes $(\log X)/(\log \eta) + O(1)$ distinct pairs as $X \to \infty$ [18].

1.3. **Removing an endpoint.** At $j = 3$, dropping 48 preserves all five primes. For every $j \geq 4$, each endpoint has a prime divisor greater than three. Such a prime divides no other term of the block, so dropping either endpoint loses it.

To see this, the recurrence above gives $v_h \equiv h \pmod 2$. Advancing by two steps multiplies $U_h + v_h\sqrt{8}$ by $17 + 6\sqrt{8}$, so $(U_{h+2}, v_{h+2}) \equiv -(U_h, v_h) \pmod 3$. Thus U_h is coprime to 6 for even h, and v_h is coprime to 6 for odd h.

For $j = 2m \geq 4$, the endpoints are $U_m^2 - 2$ and $U_m^2 + 1$, since $U_{2m} = 2U_m^2 - 1$. Neither is divisible by three. The first is odd and greater than one. The second is 2 (mod 8) and greater than two. For $j = 2m + 1 \geq 5$, multiplication of the powers with exponents $m + 1$ and $\pm m$ gives $(U_j - 3)/2 = 8v_m v_{m+1}$ and $(U_j + 3)/2 = U_m U_{m+1}$. Among $m, m + 1$, the odd index $h \geq 3$ contributes $v_h > 1$ coprime to 6 to the first product, and the even index $h \geq 2$ contributes $U_h > 1$ coprime to 6 to the second. This proves the endpoint claim.

1.4. **Results for the catalog's length order.** For any fixed lengths $k, \ell \geq 2$ and every $\varepsilon > 0$, the number of equal-radical pairs with starts $1 \leq x < y \leq X$ is $O_{k,\ell,\varepsilon}(X^{2/3+\varepsilon})$. This gives the power saving sought in the discussion at the end of the 2025 preprint. The estimate counts all admissible pairs in the catalog's order, including those on any exceptional curves. See Section 2 and [18].

For fixed $k_1 \geq k_2 \geq 3$, every fixed affine conic and every fixed polynomial graph contains only finitely many catalog pairs. Assuming Vojta's conjecture with multiplicity truncated at one, all but finitely many such pairs lie on finitely many algebraic curves. A higher-degree rational curve can still survive this reduction, and the global finiteness question remains open. See Section 3.

For a fixed curve with infinitely many ordered pairs we obtain an additional necessary condition in Section 5. At least two geometric branches must meet at one marked grid point. Section 6 analyzes the gap between the blocks, and Section 7 proves that only finitely many possible image curves have prescribed projection degrees. The appendices examine particular ramification patterns. These restrictions leave open the existence of an infinite ordered family.

2. A power-saving counting bound

For positive integers m and h, put $B_h(m) = m(m + 1) \cdots (m + h - 1)$. Write $\operatorname{rad}(n)$ for the product of the distinct primes dividing n. Let $F_{k,\ell}(X)$ count the pairs $1 \leq x < y \leq X$ with $\operatorname{rad} B_k(x) = \operatorname{rad} B_\ell(y)$. Overlapping blocks are allowed. This is the counting function in Lebowitz-Lockard's 2025 preprint. Theorem 5 there gives an upper bound $X^{1+o(1)}$, and the author raises the question of a bound $O(X^{1-\delta})$ for some $\delta > 0$ [18].

Theorem 2.1. *For every fixed $k, \ell \geq 2$ and every $\varepsilon > 0$,*

$$F_{k,\ell}(X) \ll_{k,\ell,\varepsilon} X^{2/3+\varepsilon}.$$

In particular the bound applies to the disjoint pairs in Erdős problem 931. The proof bounds the number of pairs at each scale. It permits an infinite sequence of increasingly sparse examples [11].

2.1. **Two estimates for the prime factors. Partners of one fixed block.** If one start is fixed, the number of possible partners up to X is $X^{o(1)}$, uniformly for starts up to X. In fact the common radical R satisfies $R \leq (X + \max(k, \ell))^{\max(k,\ell)}$. If s distinct primes divide R, then $s! \leq R$, whence $s = O_{k,\ell}(\log X / \log\log X)$. Every prime in the partner's first term divides R. For any fixed $\eta > 0$, the number of possible first terms is at most

$$X^\eta \prod_{p \mid R} (1 - p^{-\eta})^{-1} \leq X^\eta (1 - 2^{-\eta})^{-s} = X^{\eta + o(1)}.$$

Since η is arbitrary this proves the estimate. We use it in both directions.

Large repeated prime powers. Define $E_h(u) = B_h(u)/\operatorname{rad} B_h(u)$. For $U \geq 2$, $1 \leq T \leq U$, and every $\eta > 0$,

$$\#\{1 \leq u \leq 2U : E_h(u) > T\} \ll_{h,\eta} U^\eta (U/T + T^2). \tag{2.1}$$

To prove this, write $d_i = (u+i)/\operatorname{rad}(u+i)$, $0 \le i < h$. The ratio $E_h(u)/\prod_i d_i$ is bounded by a constant C_h because a prime dividing two block terms divides their nonzero difference, which is less than h. Put $L = T/C_h$. Bounded L gives (2.1) directly, so suppose $L \ge 2$.

If some prime $p > L$ divides d_i, then $p^2 \mid u+i$. Summing $(2U+h)/p^2$ over $p > L$ and the h positions gives $O_h(U/L)$ possibilities. In the remaining case, $E_h(u) > T$ implies $\prod_i d_i > L$ and all prime factors are at most L. Multiply these factors one at a time until their product first reaches L, retaining their positions. This gives divisors $a_i \mid d_i$ with $L \le A = \prod_i a_i \le L^2$.

Set $m_i = a_i \operatorname{rad}(a_i)$, so that $m_i \mid u+i$. For any compatible tuple, $\gcd(m_i, m_j) \mid j-i$ for $i < j$. Checking prime exponents gives

$$\operatorname{lcm}(m_0, \dots, m_{h-1}) \;\ge\; \frac{\prod_i m_i}{\prod_{i<j} \gcd(m_i, m_j)} \;\gg_h\; \prod_i m_i.$$

The Chinese remainder theorem consequently bounds the number of u per tuple by $O_h(U/\prod_i m_i + 1)$. The number of ordered tuples with $A \le L^2$ is $O_h(L^2(1+\log L)^{h-1})$, by summing over the first $h-1$ entries and bounding the last by $L^2/\prod_{i<h-1} a_i$. For $0 < \eta < 1$, the sum of the reciprocal products is at most

$$\sum_{\prod a_i \ge L} \frac{1}{\prod_i a_i \operatorname{rad}(a_i)} \le L^{-1+\eta} \left(\sum_{a \ge 1} \frac{a^{-\eta}}{\operatorname{rad}(a)} \right)^h.$$

The last series converges. Its Euler factor at p is $1 + p^{-1-\eta}/(1-p^{-\eta})$. These estimates prove (2.1), after choosing the intermediate exponents small enough to absorb powers of $\log U$.

2.2. **When the later block is longer.** Suppose $k < \ell$ and $Y \le y < 2Y$. Equal radicals and $x < y$ imply

$$E_\ell(y) \ge B_\ell(y)/B_k(x) \gg_{k,\ell} Y^{\ell-k}.$$

For large Y this exceeds $Y^{1/3}$. Applying (2.1) at that threshold leaves $O(Y^{2/3+\eta})$ choices for y, each with $Y^{o(1)}$ partners. Summing over intervals with Y a power of two proves the theorem in this case.

2.3. **When the lengths are equal.** Write $h = k = \ell$ and $T = X^{1/3}$. By (2.1) and the partner estimate, pairs with either $E_h(x) > T$ or $E_h(y) > T$ contribute $O_{h,\varepsilon}(X^{2/3+\varepsilon})$. For every remaining pair the product ratio equals $a/b = E_h(x)/E_h(y) < 1$ in lowest terms, with $a, b \le T$.

For $u = 2x+h-1$, $v = 2y+h-1$, put

$$H(u) = \prod_{i=0}^{h-1} (u + 2i - h + 1) = 2^h B_h(x).$$

The shifts sum to zero. The coefficient of u^{h-2} is $-\frac{1}{2}\sum_{i=0}^{h-1}(2i-h+1)^2 < 0$. It follows that $H(u)^{1/h} = u + O_h(1/u)$ for our positive arguments, and $H(u)^{1/h} \asymp_h u$. For $\theta = (a/b)^{1/h}$, the ratio identity gives $u \asymp_h \theta v$ and

$$|u/v - \theta| \ll_h 1/(\theta v^2).$$

When $V \le v < 2V$, distinct fractions u/v are separated by at least $1/(4V^2)$. Only $O_h(1+\theta^{-1})$ slopes occur on this scale. On a fixed reduced slope s/t, write $u = sw,\ v = tw$. The polynomial $bH(sw) - aH(tw)$ has at most h zeros. If it vanished identically, its coefficients of degrees h and $h-2$ would force $s = t$ and $a = b$. This contradicts $x < y$.

There are $O_h(\log(2X))$ scales. Summing the per-ratio bound uses $\sum_{1 \le a < b \le T} (b/a)^{1/h} \ll_h T^2$, since $1/h < 1$. The contribution is $O_h(T^2 \log(2X))$, as required.

2.4. **When the earlier block is longer.** Suppose $k > \ell$. Restrict the starts to $K \le x < 2K$, $Y \le y < 2Y$, with K, Y powers of two and $K \le Y$. Write $N = Y^{2/3}$. If $K \le N$, the partner estimate bounds the number of pairs by $O(NY^{o(1)})$. Otherwise put $A = K/N$ and $B = Y/N$. Discarding starts with $E_k(x) > A$ or $E_\ell(y) > B$ costs $O_{k,\ell,\eta}(Y^{2/3+\eta+o(1)})$ pairs by (2.1), since $K/A = Y/B = N$ and $A^2, B^2 \le N$.

For the remaining pairs the reduced ratio $a/b = B_k(x)/B_\ell(y) = E_k(x)/E_\ell(y)$ has $a \le A$, $b \le B$, and $a/b \asymp_{k,\ell} R = K^k/Y^\ell$. The number of available ratios is at most

$$O_{k,\ell}\Big(\min(B^2R, A^2/R)\Big).$$

Indeed, for each b there are at most $O(Rb)$ positive integers a, and for each a at most $O(a/R)$ positive integers b. As functions of K, the two bounds vary as K^k and K^{2-k}, respectively. Since $k \ge 3$, their minimum is at most their common value at $K = Y^{(\ell-1)/(k-1)}$. This gives

$$\#\{\text{ratios}\} \ll_{k,\ell} Y^{(\ell-1)/(k-1)-1/3}. \tag{2.2}$$

For each ratio, the pairs lie on the curve $bB_k(x) - aB_\ell(y) = 0$. Put $g = \gcd(k, \ell)$ and give x, y weights $\ell/g, k/g$. The terms of greatest weight are $bx^k - ay^\ell$. Over $\mathbb{C}$ this factors into g distinct binomials $x^{k/g} - c_j y^{\ell/g}$ up to a constant. Each binomial is irreducible because the two exponents are coprime. Indeed, in the Laurent polynomial ring a unimodular monomial change of variables sends $x^{k/g}y^{-\ell/g} - c_j$ to $z - c_j$, and the original binomial has no monomial factor. The highest-weight forms multiply under factorization. The highest-weight form of every nonconstant factor thus contains a nonempty product of those binomials, with a pure x term of degree at least k/g. Every component has ordinary degree at least k/g.

Pila's lattice-point theorem for an absolutely irreducible plane curve of degree $d \ge 2$ bounds its integer points in a square of side $2Y$ by $O_{d,\eta}(Y^{1/d+\eta})$, uniformly in its coefficients. A complex component distinct from its conjugate contributes only $O_{k,\ell}(1)$ real points by Bézout. Hence a ratio contributes $O_{k,\ell,\eta}(Y^{g/k+\eta})$ pairs [23]. Combining this with (2.2) gives an exponent at most $2/3 + \eta$, since $g \mid k - \ell$ implies $g/k \le (k-\ell)/(k-1)$. There are $O((1 + \log X)^2)$ pairs of scales. Taking each intermediate η sufficiently small in terms of the stated ε completes the proof. □

The theorem gives a power saving over the cited 2025 estimate. The optimal exponent remains open.

3. Finiteness on polynomial graphs and conics

Fix integers $k_1 \ge k_2 \ge 3$. Put

$$F(x) = \prod_{i=1}^{k_1}(x + i), \qquad G(y) = \prod_{j=1}^{k_2}(y + j).$$

Call a pair of integers with $x \ge 0$ admissible if

$$y \ge x + k_1, \qquad \operatorname{rad} F(x) = \operatorname{rad} G(y).$$

Here $\operatorname{rad}(n)$ is the product of the distinct prime divisors of n. Problem 931 asks whether there are finitely many admissible pairs for each choice of k_1, k_2 [9, 15].

Theorem 3.1. *Each fixed affine plane curve of degree at most two contains only finitely many admissible pairs. Each fixed graph $y = P(x)$ or $x = P(y)$, for any $P \in \mathbb{Q}[X]$, also contains only finitely many. The result applies to each curve separately. Finiteness in the whole plane remains open.*

The boundary $x = 0$ gives the fixed support of $F(0) = k_1!$. The consecutive integers $y+1, y+2$ then have all their prime factors in this fixed set. The S-unit equation for their difference has finitely many solutions, by Siegel on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus the remainder can take $x \ge 1$ [2].

3.1. **Poles and unmatched marked points.** Write $V = \{-k_1, \dots, -1\}$, $H = \{-k_2, \dots, -1\}$. On the smooth projective normalization $\widetilde{C}$ of a geometrically irreducible curve defined over $\mathbb{Q}$, regard its nonconstant coordinates x, y as morphisms to $\mathbb{P}^1$. Let I be the union of their poles. Outside I put $E = x^{-1}(V)$, $D = y^{-1}(H)$, and take these as sets of distinct geometric points. If C has infinitely many admissible points, all but finitely many are smooth, hence lift to $\widetilde{C}(\mathbb{Q})$. We claim

$$|I| + |E \triangle D| \leq 2. \tag{3.1}$$

Pass to a number field K over which the finitely many marked points and their images are rational. Enlarge a finite set S of places to include the archimedean places, primes of bad reduction of the curve or the coordinate maps, and primes at which distinct marked points or distinct designated values coalesce. Such an S exists by spreading out the smooth projective curve, the two morphisms and the marked sections. In particular, at each $T \in E \setminus D$ the finite value $y(T)$ reduces outside H for every place outside S. Include the analogous exclusions for $x(T)$ at $T \in D \setminus E$.

An admissible lift cannot reduce to a pole at $\mathfrak{p} \notin S$ because its two coordinates are rational integers. Suppose it reduces to $T \in E \setminus D$. Its x-coordinate then reduces to some $v \in V$, so the rational prime p below $\mathfrak{p}$ divides $F(x)$. The equality of radicals gives $p \mid G(y)$. Thus y reduces to a member of H at $\mathfrak{p}$, contradicting the prescribed reduction of $y(T)$. The same reasoning interchanging x, y excludes $D \setminus E$. A prime at which both coordinates hit marked values is allowed. The argument excludes it only from reductions to the specified unmatched points on the normalization.

The lifts are consequently S-integral on the complement of $I \cup (E \triangle D)$. By Siegel's theorem, a curve with at least three geometric points removed from its smooth projective completion has finitely many S-integral points. This proves (3.1) [2]. In the special case $y = P(x)$, the same reduction is visible directly over $\mathbb{Q}$. For each distinct irreducible factor A of F whose roots are absent from $G(P(X))$, clear denominators in the second polynomial. The resultant of A with that polynomial is a fixed nonzero integer after integral scaling. A prime of $A(x)$ at an admissible integer x divides that resultant, up to the finitely many denominator primes. Apply the same argument to every factor exclusive to $G(P(X))$. Over a splitting field, outside finitely many additional primes, this excludes reductions to each unmatched root. Together with the pole at infinity two distinct unmatched roots would contradict Siegel. In particular, conjugate unmatched roots count separately in (3.1).

3.2. **The inverse image of consecutive integers.**

Lemma 3.2. *Let V_0, H_0 be finite intervals of consecutive integers, with $|H_0| = h \geq 3$. If $P \in \mathbb{Q}[X]$ has degree $d \geq 2$, then $W = P^{-1}(H_0)$, taken in $\mathbb{C}$ without multiplicity, satisfies*

$$|V_0 \triangle W| \geq 2.$$

Suppose the difference has size at most one. A nonintegral member of W would be the only member of $W \setminus V_0$, so Galois invariance makes it a rational number z. Write b for its multiplicity in its fibre. The monic polynomials for different fibres differ only in their constant terms. Newton's identities therefore give identical power sums of orders $1, \dots, d-1$, with multiplicity, in each fibre. The other members of this fibre and every member of every other fibre are integers. Hence bz^{d-1} is an integer. If $q \geq 2$ is the reduced denominator, then $q^{d-1} \mid b$, impossible for $d \geq 3$ since $2^{d-1} > d \geq b$. When $d = 2$, it forces $b = 2$, making z the vertex. The vertex value must be at one end of H_0 since the other fibres have real roots. The distances from z to preimages of the nearest two other levels have ratio $\sqrt{2}$. These distances are nonzero rational numbers, a contradiction. Thus every member of W is integral.

Let $r = hd - |W|$, the sum of each root's multiplicity minus one over these fibres. These multiplicities contribute r zeros of P', counted with multiplicity, so $r \le d-1$. There are at least $|W|-2$ pairs $u, u+1$ in $W \cap V_0$. If W is missing one point of V_0, the remaining points form at most two runs. If W has one additional point, the whole of V_0 remains. This also gives the bound under exact equality.

For each such pair the intermediate value theorem gives

$$\Delta P(u) = P(u+1) - P(u) \in \{-1, 0, 1\}.$$

A larger jump in either direction would cross an intermediate level of H_0 at a nonintegral member of W. Every zero difference gives by Rolle's theorem a critical point in $(u, u+1)$, away from W. These intervals are disjoint. At most $d-1-r$ differences are zero, so at least

$$(hd - r - 2) - (d - 1 - r) = (h-1)d - 1 \ge 2d - 1$$

differences are 1 or -1. Each of the two equations $\Delta P = \pm 1$ has at most $d-1$ roots, since ΔP has degree $d-1$. This proves the lemma.

For $y = P(x)$ of degree at least two, (3.1) and the lemma are incompatible. For $x = P(y)$ of degree at least two, positive y sufficiently large gives either $P(y) < 0$ or $P(y) > y$, which violates admissibility.

Consider a line with both coordinates nonconstant. If infinitely many admissible points lie on it then its slope and intercept are rational. Write $y = ax+b$. The required positive unbounded branch gives $a > 0$. Equation (3.1) gives $|V \triangle W| \le 1$ for $W = (H-b)/a$. Since $|V| = k_1 \ge k_2 = |W|$, it follows that $W \subseteq V$ and $k_1 - k_2 \le 1$. The common difference of the ordered elements of W is a positive integer. A difference of at least two would give span $2(k_2 - 1) > k_1 - 1$, so $a = 1$. The only possibilities are $b = 0$ or $b = 1$, which violate $y \ge x + k_1$. A horizontal line bounds x. On a vertical line with x fixed, the prime support is fixed, so $y+1$ and $y+2$ are units outside a fixed finite set of primes. The S-unit equation with their difference equal to one has finitely many solutions, also by Siegel on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

3.3. **Conics with two places at infinity.** For a conic containing infinitely many admissible points it suffices to consider a smooth geometrically irreducible projective conic defined over $\mathbb{Q}$. Indeed a reducible conic is a union of lines, and a line through two rational points is defined over $\mathbb{Q}$. Any geometrically irreducible curve of degree two containing infinitely many rational points is itself defined over $\mathbb{Q}$, as follows either by projective interpolation or by Bézout applied to its field conjugates. A smooth conic with a rational point has normalization $\mathbb{P}^1$.

If either coordinate projection has degree one, the other coordinate is a rational function $R(t) \in \mathbb{Q}(t)$ of the first. Write $R = P + Q$ with $P \in \mathbb{Q}[t]$ and Q proper. At an unbounded sequence of integral inputs with $R(t)$ integral, a fixed integer multiple of $Q(t)$ is integral and tends to zero. This forces $Q = 0$ since a nonzero rational function has only finitely many zeros. Lemma 3.2 covers the resulting polynomial graph in either orientation.

Assume both projections have degree two. If the line at infinity meets the conic in two distinct points, these must both be real. An unbounded real branch has a real limit point at infinity and complex conjugation preserves the two-point set. Both coordinates have simple poles at both points. Indeed the denominator in the projective expressions $x = X/Z$, $y = Y/Z$ has simple zeros there, and cancellation in either coordinate would lower its degree.

Use a real projective parameter u taking the two poles to $0, \infty$. There are real constants with $abcd \ne 0$ such that

$$x = A + au + b/u, \qquad y = B + cu + d/u.$$

Equation (3.1) gives the exact equality $E = D$. This is a finite nonempty subset of $\mathbb{C}^\times$. The two fibre involutions $u \mapsto b/(au)$ and $u \mapsto d/(cu)$ preserve it, so their composition multiplies it by a

root of unity. The multiplier $bc/(ad)$ is real, hence is 1 or -1. For $bc = ad$ the two coordinate functions are affinely dependent. For $bc = -ad$ they satisfy

$$(x - A)^2 - L(y - B)^2 = 4ab, \qquad L = (a/c)^2 > 0.$$

The first involution fixes x and sends y to $2B - y$. The second fixes y and sends x to $2A - x$. The exact equality of the fibres implies $2B - H = H$ and $2A - V = V$. Thus A, B are their respective block midpoints. Moreover

$$\{(v - A)^2 : v \in V\} = L\{(h - B)^2 : h \in H\} + 4ab.$$

The span of such a square set for k successive integers is $R(k) = \lfloor (k-1)^2/4 \rfloor$, strictly increasing for $k \geq 2$. The equality gives $L = R(k_1)/R(k_2)$. With $k_1 = k_2$ the square sets are identical, whence $L = 1$ and $4ab = 0$, impossible. With $k_1 > k_2$ it gives $L > 1$. On an unbounded positive branch the conic equation gives $x/y \to \sqrt{L} > 1$, contrary to the required order.

3.4. **Conics with one place at infinity.** With both projections of degree two and a single place at infinity, the coordinates on the normalization are quadratic polynomials $f(u), g(u)$. Their fibre involutions are reflections. If their centres agree the quadratics are affinely dependent. Otherwise normalize the parameter by an affine change to make these involutions $R(u) = -u$ and $S(u) = 1 - u$.

The sets $E = f^{-1}(V)$, $D = g^{-1}(H)$ are finite, nonempty, invariant under R, S respectively, and satisfy $|E \triangle D| \leq 1$ by (3.1). If they are equal their common set is invariant under the nonzero translation $S \circ R$, which is impossible. Up to interchanging the coordinates and renormalizing the two reflections, write $D = E \cup \{z\}$. For any complex u, invariance of the two indicator functions gives

$$\mathbf{1}_E(u + 1) - \mathbf{1}_E(u) = \mathbf{1}_{\{-z\}}(u) - \mathbf{1}_{\{z-1\}}(u).$$

The jumps are distinct, since otherwise the nonempty finite set E would be translation invariant. On each coset of $\mathbb{Z}$ the left side sums to zero by finite support. The two jumps must therefore lie on the same coset. On every other coset the indicator is zero. On that one coset it rises once and falls once. Hence E is a full finite arithmetic progression with step one.

Translate the parameter so that E becomes a consecutive integer interval. The corresponding quadratic takes integer values on all of E, and its entire inverse image of its designated output block equals E. There are at least three distinct input points since the output block has at least three levels. Interpolating three of them shows that the translated quadratic has rational coefficients. This contradicts the lemma of Lemma 3.2. The same proof applies after the coordinate interchange because each output block has at least three levels.

This exhausts the possibilities for a fixed conic and proves the theorem. □

3.5. **A conditional reduction to rational curves.** Assume Vojta's conjecture with counting truncated at multiplicity one for rational points on $X = \mathbb{P}^1 \times \mathbb{P}^1$. Take the reduced divisor $\mathcal{D}$ consisting of the vertical fibres at $V \cup \{\infty\}$ and the horizontal fibres at $H \cup \{\infty\}$. This divisor has simple normal crossings, and $K_X + \mathcal{D}$ has bidegree $(k_1 - 1, k_2 - 1)$. Take $S = \{\infty\}$ and the ample line bundle $\mathcal{O}(1, 1)$. For every $\epsilon > 0$ the conjecture then gives, outside a proper closed subset and finitely many further points,

$$(k_1 - 1)\log x + (k_2 - 1)\log y \leq N_S^{(1)}(\mathcal{D}, (x, y)) + \epsilon(\log x + \log y) + O(1)$$

at positive integral x, y [28]. For such points a finite prime contributes once exactly when it divides $F(x)G(y)$, even when it divides both factors. Thus the counting function is $\log \operatorname{rad}(F(x)G(y)) + O(1)$. At an admissible point this equals $\log \operatorname{rad} F(x) + O(1) \leq k_1 \log x + O(1)$, with constants allowed to depend on the fixed block sizes. Substituting gives

$$(k_2 - 1 - \epsilon)\log y \leq (1 + \epsilon)\log x + O(1).$$

Since $y \geq x$ and $k_2 \geq 3$, any fixed $0 < \epsilon < 1/2$ bounds y outside that proper closed subset.

Consequently the conjecture places all but finitely many admissible pairs on a finite union of curves. A curve carrying infinitely many such points must have genus zero by Siegel, and its poles and unmatched marked preimages must satisfy (3.1). The preceding lemmas exclude conics and polynomial graphs. Proving finiteness on the remaining higher-degree rational curves would complete this conditional route.

4. Fixed equations and the ordered problem

Theorem 4.1. *Fix integers $k \geq \ell \geq 3$, positive integer vectors $A = (A_1, \dots, A_k)$, $B = (B_1, \dots, B_\ell)$, and $c \in \mathbb{Q}_{>0}$. There are only finitely many pairs of integers $x \geq 0$, $y \geq x + k$ satisfying*

$$\prod_{i=1}^{k} (x+i)^{A_i} = c \prod_{j=1}^{\ell} (y+j)^{B_j}. \tag{4.1}$$

The catalog asks for finiteness assuming only that the two blocks have the same prime divisors. Equation (4.1) requires a fixed equality of multiplicities after assigning weights to the terms. The ramification criterion below gives a condition for such weights to exist on a proposed algebraic family [11, 15].

4.1. A polynomial with several fibres on one arithmetic progression.

Lemma 4.2. *Let P be a nonconstant complex polynomial. If the full inverse image of a finite set T of at least two values is an arithmetic progression, then* $\deg P \leq 2$. *Inverse images here consist of distinct points.*

An affine change of input makes the progression a block of consecutive integers. At any $t \in T$, the monic polynomial obtained by dividing $P - t$ by its leading coefficient has only real roots. Making this affine change also in the output gives a real polynomial whose target values are all real.

Put $d = \deg P$. Rolle's theorem and the multiplicities of the roots of one real-rooted fibre account for all $d - 1$ derivative roots. Between consecutive distinct roots there is exactly one derivative root, of multiplicity one. A multiple derivative root must therefore be a root of that fibre. Applying this to two disjoint fibres shows that every critical point is simple. The resulting d monotone branches each attain each marked value. Indeed, a root at a critical endpoint counts on both adjacent branches, and the d roots counted with multiplicity must occupy all d branches.

Choose adjacent targets $a < b$ in sorted order. On each branch their preimages are adjacent grid points $i_j, i_j + 1$. An intervening integer would have image strictly between a and b. The lower sites $i_1 < \cdots < i_d$ are strictly increasing, and the orientation alternates over the branches. For odd $d \geq 3$, the first power sums of these two fibres thus differ by an alternating sum of d ones, which is nonzero. For even $d \geq 4$, their second power sums differ, up to sign, by

$$(2i_2 + 1) - (2i_1 + 1) + \cdots + (2i_d + 1) - (2i_{d-1} + 1) > 0.$$

Newton's identities give equal power sums of all orders below d, because the two monic fibre polynomials differ only by a constant. This proves the lemma.

4.2. Chebyshev fibres in arithmetic progression.

Write $T_n(\cos\theta) = \cos(n\theta)$. Suppose the complete real-rooted fibre $T_n^{-1}(w)$ is a progression of at least three distinct points. Then $n \geq 3$, w is real, and $|w| \leq 1$, since a real level outside that interval has at most two real preimages.

For $-1 < w < 1$ all n roots are simple and their mean is zero. For $n \geq 5$ their second and fourth moments are $1/2$ and $3/8$. For instance, write $w = \cos\theta$ and list the roots as $\cos((\theta + 2\pi j)/n)$, $0 \leq j < n$. Averaging the double- and quadruple-angle formulas gives those moments. The ratio of the fourth moment to the square of the second for a centered progression of n points is

$3(3n^2-7)/(5(n^2-1))$, by summing second and fourth powers. Equality with $3/2$ would give $n^2=9$. For $n=3$ the middle point is zero, so $w=0$. For $n=4$ the roots have the form $-3h,-h,h,3h$. Their mean square gives $h^2=1/10$, hence $w=T_4(h)=7/25$.

For $w=\pm 1$, the distinct roots in decreasing order are cosines of angles in $[0,\pi]$ with common difference $2\pi/n$. Three consecutive such cosines have zero second difference only when the middle cosine is zero. Thus a progression of at least three points has exactly three, including zero. Since $T_n(0)=0$ for odd n, the critical level requires even n. This parity statement suffices below.

4.3. **Proof of Theorem 4.1.** If d is the greatest common divisor of all the entries of A, B, any solution makes c a d-th power in $\mathbb{Q}_{>0}$. Taking positive d-th roots reduces to the case $d=1$. Write $F(X)=\prod(X+i)^{A_i}$ and $G(Y)=c\prod(Y+j)^{B_j}$ after this reduction. An infinite set of solutions contains a sequence on which both coordinates tend to infinity, since each fixed value of either coordinate gives finitely many solutions of (4.1).

By the Bilu–Tichy theorem over $\mathbb{Q}$, an infinite sequence of integral solutions entails a decomposition $F=\phi\circ U,\ G=\phi\circ W$. Here $U=U_0\circ L_1,\ W=W_0\circ L_2$ for invertible rational affine maps L_i, and (U_0,W_0), up to order, is one of their five standard pairs. All these polynomials have rational coefficients [8]. **At least two distinct zeros of ϕ.** The inverse images of its full zero set under U, W are exactly $\{-k,\dots,-1\}$ and $\{-\ell,\dots,-1\}$, respectively. The progression lemma gives $\deg U, \deg W \le 2$. At an integer solution the rational numbers $u=U(x), w=W(y)$ have a fixed common denominator bound. Write $m=\deg\phi$ and let a_m, a_{m-1} be its two highest coefficients. On either real tail ϕ is strictly monotone. Thus large u, w of the same sign with $\phi(u)=\phi(w)$ are equal. For opposite signs, m must be even. Translating the input by $a_{m-1}/(ma_m)$ removes the term of degree $m-1$. Equality of the centered polynomial values first gives $|w|/|u|\to 1$, then, by factoring the difference of the m-th powers of their absolute values, gives

$$u+w=-2a_{m-1}/(ma_m)+O(1/|u|).$$

On a lattice of bounded denominator this forces eventual equality with the constant. The original solutions therefore lie, up to finitely many, on at most two fixed conics $U(x)=W(y)$ or $U(x)+W(y)=-2a_{m-1}/(ma_m)$. This step also follows from a published refinement [7]. At primes outside the finite set dividing the numerator or denominator of c, (4.1) implies that the two interval products have the same prime divisors. The fixed-conic finiteness proof applies with precisely this extension. On a normalized component, points can reduce at an unmatched marked preimage only at one of the finitely many excluded primes or primes of bad reduction. Enlarging this finite set by the primes of c leaves the application of Siegel's theorem intact. The rest of that proof concerns the marked places and the real order $y\ge x+k$. Each conic therefore contributes finitely many solutions. (Section 3)

One distinct zero of ϕ. In this case $\phi(Z)=a(Z-b)^m$. Every root multiplicity of both F and G is divisible by m. Their joint gcd is one, so $m=1$. Thus $U(x)=W(y)$ at every solution, and both standard polynomials must have their entire fibre over the same real value b on a real arithmetic progression of at least three distinct points. The affine input changes preserve this property.

- The first standard kind contains a monomial, and the second contains a quadratic. A real monomial takes a prescribed value at at most two real points. A quadratic has at most two distinct preimages.
- The fifth kind contains $(aZ^2-1)^3$. Its zero fibre has at most two distinct points. Every other real fibre contains a nonreal point, since a nonreal cube root of the level has preimages under the quadratic but cannot be attained at a real input.
- In the third kind the pair is $D_p(Z,a^q), D_q(Z,a^p)$, with $\gcd(p,q)=1$ and $a\neq 0$. For a Dickson polynomial to have a complete real fibre of at least three points its parameter must be positive. Indeed, $D_n(Z,s)=2s^{n/2}T_n(Z/(2\sqrt{s}))$, so for $s<0$ its derivative has a

nonreal root when $n \geq 3$. The derivative of a real-rooted fibre polynomial has only real roots. Positivity of both parameters and coprimality imply $a > 0$. Both Chebyshev fibres therefore have the same normalized level $b/(2a^{pq/2})$. At an interior level the computation above permits only degree 3 at zero or degree 4 at $7/25$. The degrees would have to agree, contrary to coprimality. An endpoint level would make both degrees even.

- The fourth kind is $a^{-p/2}D_p(Z, a), -b_0^{-q/2}D_q(Z, b_0)$ with $\gcd(p, q) = 2$. Again both parameters must be positive, and both degrees are even. The leading coefficients consequently have opposite signs. This persists under the real affine input changes, so the two polynomials cannot take equal values along a sequence with both inputs tending to positive infinity.

This exhausts the possibilities and proves (4.1) has finitely many ordered solutions. Hajdu and Tijdeman treat polynomials with simple rational roots [17]. The argument here allows arbitrary prescribed positive multiplicities on both consecutive blocks.

4.4. **A constraint on prime exponents.** Fix a positive integer M. Consider ordered pairs of blocks with equal prime supports in which no individual term is divisible by a prime to exponent greater than M. For each common prime $s > k$, draw an edge between the positions i, j that it divides, and label it with $\nu_s(y+j)/\nu_s(x+i)$. Each position is unique on its side because the block lengths are at most $k < s$.

Only finitely many such pairs admit positive vertex weights with A_i/B_j equal to the label on every edge. In particular, only finitely many have a graph with no cycles, allowing parallel edges to form two-edge cycles. To prove this, the labels range over the finite set $\{u/v : 1 \leq u, v \leq M\}$. On each component choose weight one at a vertex and propagate along a spanning tree. All the resulting rational weights belong to a fixed finite set because each tree path has at most $k + \ell - 1$ edges. Clear denominators. Whenever the labels are consistent, the quotient of the resulting weighted products is supported only on primes at most k. Their exponents are bounded because both the weights and all the term valuations are bounded. Thus there are only finitely many possible A, B, c, and (4.1) gives finiteness. Consequently, an infinite sequence with uniformly bounded prime exponents must eventually display incompatible ratios around a cycle.

The obstruction occurs in a known genuine example. Jud McCranie's triples beginning at 2650 and 58563 have shared support $\{2, 3, 5, 11, 13, 17, 53, 241\}$. The terms $2650 = 2 \cdot 5^2 \cdot 53$ and $58565 = 5 \cdot 13 \cdot 17 \cdot 53$ give two edges between the same positions. The prime 5 requires $2A_1 = B_3$, while 53 requires $A_1 = B_3$. Thus any equality of the form (4.1) with positive weights must introduce at least one of these two primes into c [21]. An exhaustive computation with both first terms at most 5,000,000 and $3 \leq \ell \leq k \leq 12$ found 257 ordered pairs of blocks, including 16 pairs of triples. For each first term and length, the program sums fixed 64-bit labels of the distinct primes in the interval, sorts these sums, and compares the exact prime sets within each group. Equal prime sets always receive the same sum. Hash collisions between different sets can add comparisons but cannot change the answer. The program also checks its result against a direct all-pairs search on a small range. The 16 pairs of triples reproduce the previously published list [21]. The largest later first term in the full computation is 893024, paired with the seven-term block beginning at 172. The ancillary files contain the census program and its full output.

4.5. **Two triple pairs from a single formula.** The square $58564 = 242^2$ sits at the center of McCranie's later triple, and $242 = 2 \cdot 11^2$. Its earlier center is $2651 = 11(242 - 1)$. These identities suggest the centers

$$u = s(2s^2 - 1), \qquad v = 4s^4, \qquad s \geq 2.$$

Exactly two pairs of triples in this entire family have equal prime support. Their first terms are $(13, 63)$ and $(2650, 58563)$.

Indeed $v-1=(2s^2-1)(2s^2+1)$, while Sophie Germain's factorization gives

$$u^2-1=(s^2-1)(2s^2-2s+1)(2s^2+2s+1)=(s^2-1)(4s^4+1).$$

Every prime dividing $2s^2+1$ must therefore divide $s(2s^2-1)(s^2-1)(4s^4+1)$. Such a prime is odd, divides neither s nor $2s^2-1$, and cannot divide $4s^4+1$, which is congruent to 2 modulo the prime. It must divide s^2-1, hence it is 3. Thus $2s^2+1=3^m$. The case $m=2$ gives $s=2$. For $m>2$, Ljunggren's theorem on square repunits applied to $(3^m-1)/(3-1)=s^2$ gives $m=5,\ s=11$. Direct factorization verifies equal supports for both pairs. This classifies the proposed family at every scale [20].

4.6. **Five ordered examples from one cubic identity.** The largest pair in the computation has a seven-term block centered at $t=175$. Its later triple has center $s=893025=(t^3-7t)/6$. For any integer $t\geq 4$, this formula makes s integral and gives

$$6(s-1)=(t-3)(t+1)(t+2),\qquad 6(s+1)=(t-2)(t-1)(t+3).$$

The seven-term block centered at t and the triple centered at s have equal prime support exactly when every prime factor of t^2-7 belongs to $\{2,3,7\}$. Indeed, each prime greater than three in the earlier six outer terms appears in the later outer terms, and every such prime dividing t divides s. The sole possible additional primes on the later side divide t^2-7. If such a prime divides an earlier term $t+i$, $|i|\leq 3$, it divides i^2-7, whose possible absolute values are $7,6,3,2$. Conversely 2 and 3 occur in both blocks, and if $7\mid t^2-7$ then $7\mid t$.

This criterion can be solved completely. The valuation at 2 of t^2-7 is one for odd t and zero otherwise. Its valuation at 7 is one if $7\mid t$, zero otherwise. We therefore need $t^2=7+2^a7^c3^b$, with $a,c\in\{0,1\}$. Write $b=3j+r$, $0\leq r\leq 2$, and set $C=2^a7^c3^r$. The integers $X=C3^j$, $Y=Ct$ satisfy $Y^2=X^3+7C^2$, with $0<7C^2\leq 111132$. Bennett and Ghadermarzi solved all Mordell equations with nonzero constant of absolute value at most 10^7. Their Table 6 gives $X<10^{16}$ throughout that range. As $3^{34}>10^{16}$, it is enough to test $0\leq b\leq 101$ [5]. Taking the integer square root of each of the $4\cdot 102$ values $7+2^a7^c3^b$ and squaring it back gives precisely $t=4,5,7,13,14,175$ for $t\geq 4$. The respective exponent triples (a,b,c) are

$$(0,2,0),\quad (1,2,0),\quad (1,1,1),\quad (1,4,0),\quad (0,3,1),\quad (1,7,1).$$

The blocks overlap for $t=4$. Thus the complete list of disjoint first-term pairs for this family is

$$(2,14),\quad (4,48),\quad (10,350),\quad (11,440),\quad (172,893024).$$

4.7. **A ramification criterion.** Suppose a smooth geometrically integral curve over $\mathbb{Q}$ has rational coordinate functions f,g, both nonconstant. Suppose their poles form a single Galois orbit, common to both maps, and that their marked finite preimages agree exactly,

$$E=f^{-1}(\{-k,\dots,-1\})=g^{-1}(\{-\ell,\dots,-1\}).$$

At any $t\in E$ write $e_f(t),e_g(t)$ for the local ramification indices. If positive integer weights can be chosen so that

$$A_ie_f(t)=B_je_g(t)\quad\text{whenever } f(t)=-i,\quad g(t)=-j, \tag{4.2}$$

then the quotient of the two weighted products in (4.1), evaluated at f,g and omitting c, has zeros and poles only on the common pole orbit. The quotient is rational over $\mathbb{Q}$, so its valuations there are equal. A principal divisor has degree zero, hence they all vanish. The quotient is a rational constant, positive if the curve contains an ordered nonnegative integral pair. The theorem now gives finiteness on this curve.

This applies in particular if $P(f)=Q(g)$ for any nonconstant rational functions P,Q over $\mathbb{C}$. Set $A_i=e_P(-i)$, $B_j=e_Q(-j)$. Multiplicativity of ramification indices gives (4.2), including at poles of the outer maps. It also applies whenever the bipartite multigraph with vertices $-i,-j$ and one edge per $t\in E$ is a forest. To see the general criterion, choose a positive weight at one

vertex of each connected component and propagate the forced ratios along its edges. Consistent propagation is equivalent to product one around every cycle, including two-edge cycles formed by parallel edges. Clearing denominators gives (4.2). This is the ramification-label criterion for polynomial common composites in the setting of Beals, Wetherell and Zieve. Their example with two incompatible indices illustrates why equality of supports alone does not supply these weights [3]. As one application, let $f, g \in \mathbb{Q}[t]$ be nonconstant polynomials of degrees m, n with the exact marked-preimage equality above. Pakovich's theorem supplies a common composite if $|E| \geq \operatorname{lcm}(m, n)$. Since the total finite ramification of a polynomial of degree m is $m-1$, we have $|E| \geq (k-1)m + 1$ and also $|E| \geq (\ell - 1)n + 1$. An infinite ordered family of this type would therefore require, simultaneously,

$$k \leq n/\gcd(m,n), \qquad \ell \leq m/\gcd(m,n), \qquad |E| < \operatorname{lcm}(m,n).$$

Even in this range it must have a cycle on which the index ratios are inconsistent. The argument permits no conclusion about an arbitrary admissible pair whose multiplicities vary independently, or about a curve with a single unmatched marked point [22].

4.8. **A two-pole criterion from averaging over fibres.** Consider a rational curve over $\mathbb{Q}$ with two distinct real conjugate coordinate poles. Suppose f has order d at each pole, g has order e at each, and the finite preimages E above agree exactly. Put

$$h = \gcd(d, e), \qquad p = e/h, \qquad q = d/h.$$

If $\ell \geq 2q + 1$, the curve contains only finitely many ordered integer solutions. In particular this excludes every such family with e an integer multiple of d, including all equal-order families. Here the coordinate functions can have arbitrary lower-order terms.

Let S send a polynomial U in the second coordinate to its average $U(g(t))$ over $f(t) = x$, with ramification multiplicities and normalization factor $1/(2d)$. Let T be the reverse average over $g(t) = y$, with normalization factor $1/(2e)$. These traces are polynomials, since their only possible pole is at infinity. Define the nonzero real numbers r_+, r_- to be the limits of g^q/f^p at the two poles. For each $j \geq 1$, the trace of g^{qj} has degree at most pj, with coefficient of x^{pj} equal to $(r_+^j + r_-^j)/2$. Exactly half of the sheets approach each pole, and $g^{qj}/x^{pj} \to r_\pm^j$ on these sheets. Lower powers give strictly lower degree by the local pole orders. The reversed assertion gives coefficient $(r_+^{-j} + r_-^{-j})/2$ for the trace of f^{pj}.

On the marked grids V, H, S, T are stochastic averages. Every point over a marked value contributes a positive weight in each direction. By Riemann–Hurwitz the total ramification of g away from the two poles is $2e$. Therefore $|E| \geq 2e(\ell - 1)$, while $|E| \leq 2dk$, giving $k \geq 2p$. If $k = 2p$, apply T to $\prod_{v \in V}(X - v)$. Its trace vanishes at every $y \in H$ but is of degree $2q$ with positive leading coefficient $(r_+^{-2} + r_-^{-2})/2$. It cannot vanish at the $\ell \geq 2q + 1$ distinct points of H.

Now $k \geq 2p + 1$. Restrictions to V retain all coefficients of polynomials of degree at most $2p$. The two-step averaging operator $M = ST$ preserves this polynomial space and its subspace of degrees below $2p$. On their one-dimensional quotient its eigenvalue is

$$\lambda = \frac{(r_+^2 + r_-^2)(r_+^{-2} + r_-^{-2})}{4}.$$

A stochastic matrix has all eigenvalues of modulus at most one, while Cauchy–Schwarz gives $\lambda \geq 1$. Hence $r_+^2 = r_-^2$. There is a monic polynomial U of degree $2p$ fixed by M on V. Indeed, take a convergent subsequence of the Cesàro averages of $M^j X^{2p}$ on the finite grid. All values are bounded, the leading coefficient remains one, and the difference between an average and its image under M tends to zero.

The support of the two-step chain is symmetric inside each connected component of the grid incidence graph. The maximum principle for finite stochastic matrices makes U constant on each component. The polynomial $W = TU$ thus takes that same value on its vertices in H, giving

$U(f(t)) = W(g(t))$ at every $t \in E$. The leading coefficient of W is $1/r_+^2 = 1/r_-^2$, so the leading terms cancel at both poles. The difference has total pole order at most $4pd - 2$. On the other hand $|E| \geq 2d(k-1) \geq 4pd$, by Riemann–Hurwitz applied to f. A nonzero rational function has equal total zero and pole orders, so $U(f) = W(g)$ identically. The common-composite criterion and the fixed-weight theorem now apply.

4.9. **Every fixed equation in separated variables.**

Theorem 4.3. *Let $P, Q \in \mathbb{Q}(Z)$ be any two nonconstant rational functions. Among ordered pairs with equal prime support in problem 931, only finitely many satisfy*

$$P(x) = Q(y).$$

Here one can allow the common value to be infinity. In particular an infinite family must eventually leave every prescribed finite collection of such equations.

Suppose a geometrically irreducible component contains infinitely many admissible points. It is defined over $\mathbb{Q}$, since otherwise a distinct conjugate would intersect it infinitely often. Denote its coordinate functions on the normalization by f, g. The integral-point argument for fixed curves gives genus zero. There is one rational coordinate pole or a pair of real quadratic conjugate poles. In the two-pole case, each coordinate has poles at both, and the finite preimages of the two marked grids agree exactly. In the one-pole case their symmetric difference contains at most one geometric point. These statements follow by good reduction and Siegel's theorem as in the fixed-conic proof. With two poles, or with exact agreement and one pole, the relation $P(f) = Q(g)$ gives the fixed-weight criterion above. (Section 3) [2]

There remains one pole and exactly one unmatched point. Choose a rational parameter t with its pole there. The functions f, g are rational-coefficient polynomials of degrees m, n, and $\mathbb{C}(f, g) = \mathbb{C}(t)$ by birationality. An unbounded ordered sequence gives $m \leq n$. Their rational function fields intersect nontrivially since they contain $P(f) = Q(g)$. In characteristic zero the common-composite theorems supply a common polynomial composite of degree $\operatorname{lcm}(m, n)$ and a common right polynomial factor of degree $\gcd(m, n)$. Birationality forces $\gcd(m, n) = 1$. We may therefore write

$$U(f) = W(g), \qquad \deg U = n, \quad \deg W = m$$

with polynomials U, W over $\mathbb{C}$. The image is the entire curve $U(X) = W(Y)$. Indeed its irreducible equation has X-degree n and Y-degree m, the degrees of the opposite coordinate projections. It divides $U(X) - W(Y)$, so the quotient is constant. Properness of the normalization now shows that for every finite w, every pair in $U^{-1}(w) \times W^{-1}(w)$ occurs at some parameter [3]. Write $V = \{-k, \ldots, -1\}$ and $H = \{-\ell, \ldots, -1\}$. In each fibre product $U^{-1}(w) \times W^{-1}(w)$, call a pair mismatched if exactly one coordinate is marked. There can be just one such pair over all w, since each one requires a different unmatched parameter. Suppose first that its x-coordinate is marked and its y-coordinate is unmarked, and let w_0 be their common image. There is exactly one marked x and exactly one unmarked y above w_0. At all other w where any marked value occurs, both complete fibres are marked.

If there is a marked y above w_0, every x above it must be marked. Hence $U^{-1}(w_0)$ is a singleton and $V = U^{-1}(U(V))$. Since $|V| \geq 3$, there are at least two levels in $U(V)$, and the progression lemma gives $n \leq 2$. If there is no marked y above w_0, then $W^{-1}(w_0)$ is a singleton and $H = W^{-1}(W(H))$. A polynomial with a singleton fibre is an affine change of a monomial. Any other single fibre with three or more points is a regular polygon, so it cannot be precisely the collinear set H. Thus $W(H)$ contains at least two levels and the progression lemma gives $m \leq 2$. Interchanging sides treats a mismatched pair of the opposite kind. In all cases $\min(m, n) \leq 2$, hence $m \leq 2$.

For $m = 1$ the image is a polynomial graph, already excluded. For $m = 2$, coprimality makes n odd and the ordering gives $n \geq 3$. Translate t so that $f = at^2 + c_0$ with a, c_0 real. Since W

is quadratic, $W(g(-t)) = W(g(t))$ implies either $g(-t) = g(t)$ or $g(-t) = 2b - g(t)$. The first alternative conflicts with birationality. In the second, $b = g(0)$ is real and $g(t) = b + tR(t^2)$ with R a real polynomial.

The set $E = f^{-1}(V)$ is invariant under $t \mapsto -t$. The set $D = g^{-1}(H)$ thus differs from its reflection by at most two points. If $H \neq 2b - H$, the two different reflected target sets each contain an extra level, and their two disjoint full g-fibres would together have at most two points. This would give two different singleton finite fibres of g, impossible since its total finite ramification is $n - 1$. Thus $H = 2b - H$, D is reflection-invariant too, and its single discrepancy with E is the fixed point $t = 0$.

Put $z = t^2$, $J(z) = zR(z)^2$ of odd degree n, and $T = \{(h - b)^2 : h \in H\}$. The set T has at least two values. The entire preimage $J^{-1}(T)$ differs from the real arithmetic progression $V' = (V - c_0)/a$ precisely at zero. If $0 \in V' \setminus J^{-1}(T)$, all marked targets of J are positive. All their preimages are real since they belong to V', and are positive since $J(z) \leq 0$ for real $z \leq 0$. Removing zero therefore removes an endpoint of V'. The remaining full progression contradicts the progression lemma.

Finally suppose $J^{-1}(T) = V' \cup \{0\}$ with $0 \notin V'$. Then $0 \in T$. All preimages of a positive marked value are real and positive, so every critical point is positive by Rolle's theorem with multiplicities. Every zero of J other than 0 is a root of R, hence a critical point. Thus the sorted preimage set consists of zero followed by the progression V' of positive points. Apply the monotone-branch argument in the progression lemma to the targets zero and the least positive member of T. Each branch joins consecutive points in this sorted set. The first gap has some length $d_0 > 0$ and all remaining branch gaps have the common lattice step. The orientations alternate. Because the degree n is odd, the last $n - 1$ signed gaps cancel in pairs. The first power sums of the two fibres differ by $\pm d_0$, contradicting Newton's identity. This exhausts the one-pole case and proves the theorem.

5. Marked fibres and multiplicative rank

In this section $k \geq \ell \geq 3$ are fixed, $V = \{-k, \ldots, -1\}$, and $H = \{-\ell, \ldots, -1\}$. A multiplicative relation among the $k + \ell$ entries is an integer vector of exponents whose corresponding product is one. Independence of relations means linear independence of their exponent vectors over $\mathbb{Q}$.

Theorem 5.1. *If a fixed irreducible plane curve contains infinitely many admissible pairs, at least one point in $V \times H$ has two or more distinct geometric preimages on its normalization. In particular every plane curve smooth at the grid contains only finitely many admissible pairs.*

Theorem 5.2. *On* every *fixed plane curve, only finitely many admissible pairs have a multiplicative dependence among their $k + \ell$ block entries. The integer exponents in the dependence may vary freely from pair to pair.*

Theorem 5.3. *Across the entire plane, only finitely many admissible pairs have two linearly independent multiplicative relations among their entries. In particular only finitely many have at most $k + \ell - 2$ distinct common prime divisors. This conclusion is unconditional.*

A rational curve can have a multibranch singularity. Even two triples can share their prime divisors while all six entries are multiplicatively independent. An example follows the proof of Theorem 5.2.

5.1. **Marked points on the parameter line.** Suppose a curve contains infinitely many admissible pairs. A constant coordinate gives only finitely many pairs by the S-unit equation applied to two consecutive entries of the other block. Infinitely many rational points force the irreducible component containing them to be defined over $\mathbb{Q}$. Removing finitely many singular image points, lift the pairs to the normalization. The integral-point theorem gives a rational parameter line

with either one rational pole for both coordinates or two common real quadratic conjugate poles. The poles of an individual coordinate constitute a nonempty Galois-stable subset of those poles. In the two-pole case its two pole orders agree [2]. Write f, g for the two coordinate functions. Put $E_f = f^{-1}(V)$, $E_g = g^{-1}(H)$, and $S = E_f \cap E_g$, all as geometric sets. By (3.1), the number of poles plus $|E_f \triangle E_g|$ is at most two. Indeed, away from a fixed finite set of primes, an integral point cannot reduce to an unmatched marked point. At such a reduction one block would have a prime divisor absent from the other. Siegel's theorem gives the stated bound on the number of punctures. In what follows assume each occupied cell of $V \times H$ has just one geometric lift. Each common marked point is then rational by Galois invariance. The sole unmatched point, when present, is also rational. In particular all points in both marked sets are real.

5.2. **Traces for polynomial coordinates of equal degree.**

Lemma 5.4. *Let $f, g \in \mathbb{Q}[t]$ be polynomials of the same degree d parametrizing a nonlinear curve with infinitely many admissible pairs. Then $k = \ell$, and the ratio of leading coefficients $a = \operatorname{lc}(g)/\operatorname{lc}(f) = u/v$ in lowest terms satisfies*

$$1 < \frac{u}{v} < \frac{k-1}{k-2}, \qquad u^{\,k-1} v^{\,k-2} \mid d.$$

In particular d cannot be a prime power.

Proof. The order at an unbounded positive branch gives $a \geq 1$. The average of g over $f = X$, counted with multiplicity, is $aX + b$. Indeed $g - af$ has degree less than d, and the requisite Newton sums do not depend on the constant coefficient of $f - X$. By (3.1), at most one geometric point belongs to just one marked inverse image. The entirely matched f-levels span at least $k-2$ and their averages lie in the interval spanned by H. Thus $a(k-2) \leq \ell - 1$. If equality holds, the two extreme matched fibres have means at the two extremes of H, so every value in each equals its mean. The two fibres have at least $2d-(d-1) = d+1$ distinct roots in total, by the derivative bound. Hence $g - af - b$ vanishes identically. This excludes $k > \ell$ on a nonlinear image. For $k = \ell$, $a = 1$ likewise forces an affine relation by using the direction in which all levels are matched, with leading ratio one in either direction.

Now $a > 1$. A missing point over an endpoint of V is necessary, since otherwise the matched levels span $k - 1$. The other $k - 1$ levels are consecutive and wholly matched, and all $\ell = k$ reverse levels are matched. For $j \geq 1$ the trace of $\binom{g(t)}{j}$ over $f = X$ is a polynomial of degree j with leading coefficient $da^j/j!$. Polynomiality follows by symmetry in the inverse roots, and the leading coefficient follows from $g/f \to a$ at infinity. At integer matched levels the trace is integral. Taking jth finite differences gives $da^j \in \mathbb{Z}$ for $j \leq k-2$. The reverse direction gives $d/a^j \in \mathbb{Z}$ for $j \leq k-1$. Coprimality of u, v gives the divisibility claimed. The strict upper bound on a forces both u and v to exceed one. □

5.3. **One pole.** The coordinate functions are real rational-coefficient polynomials of degrees m, n. On an infinite positive branch the order $y \geq x + k$ gives $m \leq n$. An affine relation is covered by the fixed-line result, so the four derivatives of $f, g, g + f, g - f$ are nonzero.

Let $N = |S|$, $a = |E_f \setminus S|$, $b = |E_g \setminus S|$, and $s = a + b \leq 1$. On the parameter line, at least $N - 1 - s$ gaps between successive common marked points contain no unmatched point. On such a gap each coordinate changes between the endpoints by $0, 1$, or -1. A larger change would cross an intervening marked level. Rolle's theorem therefore gives a zero of one of the four derivatives in each open gap. At the marked points themselves, f' and g' have respectively $km - N - a$ and $\ell n - N - b$ zeros, counting multiplicity. The sum of degrees of the four derivatives is at most $m + 3n - 4$. Hence

$$N - 1 - s \leq m + 3n - 4 - (km - N - a) - (\ell n - N - b).$$

Using $N \le km - a$ gives

$$(\ell - 3)n \le m - 3 + 2s - a \le m - 1.$$

Thus $\ell = 3$. This argument for $\ell \ge 4$ used only the reality of the marked points.

Translate the later levels to $\{-1, 0, 1\}$. For a real polynomial whose fibres over two distinct real values consist entirely of real roots, every intermediate value has as many distinct real preimages as the degree. To see this, Rolle's theorem applied to each real-rooted fibre shows that all critical points are real. A repeated critical point would have to be a root in both fibres, since the logarithmic derivative of a real-rooted polynomial has strictly negative derivative away from its roots. Thus the critical points are simple. The polynomial has one monotone branch between successive critical points. Each of the two endpoint fibres fills every branch when roots are counted with multiplicity, and intermediate values lie strictly inside every branch's range.

In particular, the central fibre of g has n distinct points. At most b miss the earlier grid. Uniqueness of the lift in each cell gives $n \le k + b$. Each of the $k - 2$ interior earlier levels has m distinct points and at most three of these can occupy its three cells. If $m \ge 4$, each interior row needs an unmatched point, so $k = 3, a = 1, b = 0$, giving $n \le 3 < m$. Thus $m \le 3$.

The case $m = 1$ is the polynomial-graph theorem. Equal degrees $m = n = 2$ or 3 are covered by Lemma 5.4, which excludes all prime-power degrees.

For $m = 2$, complete the square in f. Since every marked lift is rational, the k earlier levels give k distinct rational squares in arithmetic progression. Fermat's four-squares theorem forces $k = 3$. None of these squares is zero, since the next two squares would then have ratio two. Thus f has six distinct marked lifts, $|E_g| \le 7$, and $2n + 1 \le 7$. For $n = 3$ equality is required. Both simple critical points of g occur at the two outer marked values and are rational. Write them $t_0 \pm h$ with $h \ne 0$. After shifting the argument and subtracting its central value, g is proportional to $z^3 - 3h^2 z$, with $z = t - t_0$. The central marked fibre would contain $t_0 \pm \sqrt{3}h$, a contradiction [12]. It remains to consider $m = 3, n > 3$. At least $k - 2 - a$ earlier interior rows have three distinct lifts taking all three values $-1, 0, 1$. The field trace $T(X)$ of g over $f = X$ is a polynomial of degree at most $\lfloor n/3 \rfloor$. Polynomiality follows from the monic equation for the inverse roots, and its growth at infinity gives the degree bound. On each of these rows T vanishes. For $k \ge 5$, if $a = 1$ then $n \le k$ and $k - 3 > \lfloor k/3 \rfloor$. If $a = 0$ then $n \le k + 1$ and $k - 2 > \lfloor (k + 1)/3 \rfloor$. Thus $T = 0$.

For $k = 3$, $n > 3$ requires $n = 4, b = 1, a = 0$. Since $N \ge 2n + 1 - b = 8$, at least two rows have three distinct lifts. Their traces vanish, so the affine polynomial T vanishes. For $k = 4$ one has $n \le 5$, so two such rows again suffice. If there are fewer than two, the unmatched point must lie in one interior row, since the other interior rows each take all three levels. Both outer rows are then full but have fewer than three lifts, placing the two simple rational critical points of f at the extreme levels. Using their rational midpoint and half-distance to normalize the parameter puts f, up to an affine output change, in the form $z^3 - 3z$ with critical values ± 2. The two interior levels are $\pm 2/3$. Neither has a rational inverse image. If z is integral at 3 then $z^3 - 3z$ is integral there, and otherwise its valuation is $3v_3(z)$, never -1. This contradicts rationality of all marked lifts.

A full row with a double critical root would consequently give $2u + v = 0$ at two distinct values $u, v \in \{-1, 0, 1\}$, which is impossible. Triple critical roots were excluded by the real-fibre argument. Every full row therefore takes the three distinct values and the trace of g^2 equals two there. That trace has degree at most $\lfloor 2n/3 \rfloor < k - a$, so it is identically two. Since the parametrization is birational, the minimal equation of g over $\mathbb{Q}(f)$ has degree three. Its first two coefficients are $0, -1$ by Newton's identities. The curve consequently satisfies $Y^3 - Y = P(X)$ with $P \in \mathbb{Q}[X]$, and Theorem 4.3 proves finiteness.

5.4. **Two poles.** Use a real parameter placing the two poles at zero and infinity. The coordinates are Laurent polynomials with exponents from $-d$ to d and $-e$ to e, respectively, with both extreme

coefficients nonzero. The order of the blocks forces $d \le e$. The marked sets agree exactly, of size N, and uniqueness of lifts gives $N \le k\ell$.

The two real arcs contain at least $N-2$ gaps between successive marked parameters. Each gives a distinct zero of the derivative of one of $f, g, g+f, g-f$. Using the logarithmic derivative operator $t\,d/dt$, their respective numbers of zeros away from the poles, with multiplicity, are at most $2d, 2e, 2e, 2e$. Subtract the $2kd-N$ and $2\ell e-N$ zeros of the first two at the marked points. This yields

$$N-2 \le 2d+6e-(2kd-N)-(2\ell e-N), \qquad (\ell-3)e \le d+1$$

using $N \le 2kd$ in the second inequality. The separate ramification bounds give $2(k-1)d \le N$ and $2(\ell-1)e \le N$.

If $\ell \ge 5$ this forces $d = e = 1$ or a contradiction. For $\ell = 4$, either $e = d$, or $e = d+1$ and every inequality needed for the last upper bound is an equality. In that case $N = 2kd \le 4k$, giving $d \le 2$. For $\ell = 3$, $2(k-1)d \le 3k$ gives $d \le 2$. When $d = 2$ it further gives $k \le 4$, and $4e \le 3k$ gives $e \le 3$.

If $d = 1$ or $e = d$, the trace criterion of Section 4.8 applies. For the remaining $d = 2, e = 3$, the trace of f over g is constant, that of f^2 over g is affine, and that of g over f is affine. Indeed these functions have no finite poles in the base and their growth exponents at infinity are at most $2/3, 4/3, 3/2$, respectively.

If $\ell = 4$, there are $N = 4k$ lifts and every grid cell is occupied. Since $18 \le N \le 24$, $k = 5$ or 6. Each earlier multiplicity is one. For $k = 6$ every later multiplicity is also one. For $k = 5$, each later column has exactly one double lift and four simple ones. Constancy of the trace of f makes the doubled row the same in all four columns.

If $\ell = 3$, the bounds $12 \le N \le 3k$ and $k \le 4$ force $k = 4, N = 12$. Each row has one double lift and two simple lifts. Its trace of g is the sum of the three distinct later values plus the repeated value. An affine function taking integer values in a span of two on four consecutive integers must be constant, so the repeated column is the same in every row. Each later column has the four earlier labels once, plus two further labels counting multiplicity. Translate the earlier labels to $0, 1, 2, 3$. The sum of the two extra labels is constant by the trace of f. In this four-element set, at most two unordered pairs realize any fixed sum. The affine trace of f^2 thus takes at most two values at three points and is constant. The sum and sum of squares determine the unordered pair of extra labels. Consequently later multiplicities depend only on the row.

In both cases write A_i for the later multiplicity in row i, and B_j for the earlier multiplicity in column j. At each lift the two vanishing orders of the products $\prod_i (f-v_i)^{A_i}$ and $\prod_j (g-h_j)^{B_j}$ agree. Their total finite zero orders are equal, and each has its total pole order divided equally between the two poles. Their quotient is constant, yielding a fixed separated equation and finiteness. This completes Theorem 5.1. The inequality $(\ell-3)e \le d+1$ also applies when all marked lifts are real and several lifts occupy a cell.

5.5. **Varying multiplicative relations.** Map the open part of the given curve into the algebraic torus by

$$(x, y) \longmapsto (x+1, \ldots, x+k, y+1, \ldots, y+\ell).$$

If its image lies in a translate of a proper algebraic subgroup, there is an integer character, with nonzero exponent vector, that is constant on the curve. A point with positive integer coordinates makes this constant rational. If the character involves just one block, its coordinate is constant because the distinct linear factors have distinct roots. If it involves both blocks, rearrangement gives a fixed relation $P(x) = Q(y)$ between two nonconstant rational functions. In either case there are only finitely many admissible pairs.

Otherwise the curve is contained in no proper coset. The bounded-height theorem of Bombieri, Masser and Zannier bounds the height of all points on it lying in any proper algebraic subgroup,

uniformly over the subgroups. Every multiplicatively dependent tuple lies in such a subgroup. These tuples are rational, so bounded height gives finiteness, proving Theorem 5.2. Combining this with the reduction in Section 3.5 would give global finiteness of all dependent admissible tuples, conditional on Vojta's truncated-counting conjecture [10, 16]. The triples starting at 2650 and 58563 illustrate the other possibility. The factorizations are

2650	$2 \cdot 5^2 \cdot 53$	58563	$3^5 \cdot 241$
2651	$11 \cdot 241$	58564	$2^2 \cdot 11^4$
2652	$2^2 \cdot 3 \cdot 13 \cdot 17$	58565	$5 \cdot 13 \cdot 17 \cdot 53$

In a multiplicative relation let a, b, c, d, e, f be the exponents in increasing order of the six entries. The valuations at 5 and 53 give $2a + f = a + f = 0$, so $a = f = 0$. The valuations at $13, 3, 241, 11$, in that order, then give $c = d = b = e = 0$. This verifies independence while the table verifies equality of prime support.

The remaining possibility under Vojta is a curve with a multibranch point on the grid whose general admissible tuples have full multiplicative rank. The arguments here determine neither its existence nor the global finiteness of the ordered pairs.

5.6. **Two independent relations on the whole plane.** Put $N = k + \ell$, $H = \log(y + \ell)$, and $u = (x+1, \dots, x+k, y+1, \dots, y+\ell)$. Suppose there is an infinite sequence of distinct admissible pairs with at least two independent relations. By Theorem 5.2 every fixed proper algebraic subset of the plane contains only finitely many members of the sequence. We can take $H \to \infty$.

First, $x/H^B \to \infty$ for every fixed $B > 0$. Indeed, applying Stewart and Yu to $1+(y+1) = y+2$ gives

$$H \ll_\eta \operatorname{rad}((y+1)(y+2))^\eta \le ((x+k)^k)^\eta$$

for every $\eta > 0$. Choose $\eta < 1/(kB)$ [27]. **The two logarithmic heights are comparable.** In the prime-valuation matrix of u, the entries are integers of size $O(H)$ and there are N columns. A nonzero minor of maximal rank and Cramer's rule give two independent integer kernel vectors with coefficients $O_N(H^N)$. For each vector write A, B for the sums of its earlier and later exponents. Taking real logarithms in the relation gives

$$A \log x + B \log y = O_N(H^N/x).$$

Eliminating $\log y$ from the two equations gives $|A_1 B_2 - A_2 B_1| \log x = O_N(H^{2N}/x)$. The integer determinant is zero for all sufficiently large members. An integer linear combination therefore gives a nonzero kernel vector (a, b) with both block sums zero and coefficients $O_N(H^{2N})$. This also holds if the two original pairs of sums are both zero.

For a nonzero block $a = (a_i)$ of sum zero, some first moment $\sum_i i^r a_i$ is nonzero for $1 \le r \le k-1$, by the Vandermonde determinant. Expanding $\sum_i a_i \log(x+i)$ to this first nonvanishing order bounds its absolute value below by $c_k x^{-r}$ and above by $C_k H^{2N} x^{-r}$. The first nonzero numerator is an integer and the remainder is $O_k(H^{2N} x^{-r-1})$. The same bounds hold on the other side with y, t, ℓ in place of x, r, k. Neither block can vanish because its nonzero partner would then have logarithmic sum zero. Since the two sums cancel, $y^t \ll_N H^{2N} x^r$. Consequently $\log y \le k \log x$ for all sufficiently large members. Here $\log H = o(\log x)$ follows from the preceding Stewart–Yu bound.

The relations stay independent in both coordinate blocks. For every real vector $v = (v_i)$ and integer x tending to infinity,

$$\sum_p \left| \sum_i v_i \, v_p(x+i) \right| \log p = \|v\|_1 \log x + O_k(\|v\|_1). \tag{5.1}$$

Without cancellation, the left side would be $\sum_i |v_i| \log(x+i)$. Cancellation occurs only for $p \le k$. For each such prime, all terms but one have valuation at most $\max_{1 \le h \le k} v_p(h)$, since p^j

dividing two entries divides their difference. The triangle inequality bounds the error by twice the contribution of these lower valuations. This proves (5.1), uniformly in real v.

Choose two orthonormal rows spanning a real two-dimensional subspace of the relation space at each point, and pass to a convergent subsequence of these $2 \times N$ matrices, with limit M. For every (a, b) in a relation plane the two prime sums on the left of (5.1) are identical, one evaluated on each block. The comparability of $\log x, \log y$ now gives $\|a\|_1 \asymp_N \|b\|_1$, uniformly on the plane. Hence both coordinate projections of the limiting plane are injective. In particular there are an earlier column i and a later column $k + j$ in M whose two-by-two determinant is nonzero.

The height contradiction. For a large positive integer Q, round QM entrywise to an integer matrix Φ. Its two rows are characters of the N-dimensional torus. They give a map from the fixed block surface X with coordinates u to the two-dimensional torus. Its generic degree Δ is at least the absolute determinant of the two selected columns of Φ, hence $\Delta \gg_M Q^2$.

Here is a local verification of the degree assertion. At the grid cell $(-i, -j)$ take $z = x + i,\ w = y + j$. The two characters are monomials in z, w times analytic units with nonzero values at the origin. Since the exponent minor is nonsingular, taking local logarithms of the two units absorbs them into a coordinate change $(z, w) \mapsto (zU, wV)$ with U, V analytic units. In the new coordinates, the monomial map has a kernel of order the absolute determinant, consisting of pairs of roots of unity. All inverse images of a sufficiently small generic source point's image under this local monomial map stay in the coordinate bidisc and are distinct. Thus the original rational map is dominant and its generic degree has the claimed lower bound.

On a dense Zariski-open subset of the fixed surface, Habegger's height inequality for a rank-two character map is

$$h(\Phi(u)) \geq c_X \, \frac{\Delta}{\|\Phi\|} \, h(u) - C_\Phi. \tag{5.2}$$

The open subset and additive constant may depend on the fixed matrix, and $c_X > 0$ does not. This is the open-subset inequality for an individual map. It applies on our product surface even though its vertical and horizontal curves are anomalous. We use the projective sup height, for which $h(u) = H$. Habegger's Euclidean archimedean convention differs by a bounded additive term absorbed in C_Φ. By Theorem 5.2 only finitely many sequence points fall outside the open set for this fixed Q [16]. For each fixed Q the rows of Φ differ from Q times the orthonormal rows at sufficiently late sequence points by at most one in each entry. Those real rows have zero sum against every prime-valuation vector and against the archimedean logarithm vector at their point. In computing the height of either rational output character, one may therefore replace its exponents place by place with the bounded errors. The sum over all places of the absolute logarithms of one entry u_i is $2 \log u_i$. Consequently $h(\Phi(u)) \leq 4NH$ on this tail, with the constant independent of Q.

The lower bound in (5.2) is at least $c_{X,M} QH - C_\Phi$. Fixing Q large enough, then sending H to infinity, gives a contradiction. This proves Theorem 5.3. If the common radical has at most $N - 2$ prime divisors, the valuation matrix has at most $N - 2$ rows, giving the stated corollary.

6. Bounds and exact enumerations by separation

Fix $k \geq l \geq 3$. Write $F_k(x) = \prod_{i=1}^{k} (x + i)$. We study $x \geq 0,\ y \geq x + k$ with $\operatorname{rad} F_k(x) = \operatorname{rad} F_l(y) = R$, and put $d = y - x$. The finiteness of all such pairs remains open.

6.1. **Consecutive differences contain every common prime.** Every $p \mid R$ divides some $x + i$ and some $y + j$. Hence it divides $d + j - i$. All of these differences are positive because $d \geq k$. Consequently

$$R \mid \prod_{h=1-k}^{l-1} (d + h).$$

For the particular later terms $y+j, y+j+1$, only the $k+1$ differences $d+j-k,\dots,d+j$ are needed. For the early terms $x+i, x+i+1$, only $d-i,\dots,d-i+l$ are needed.

For each $A>0$, only finitely many admissible pairs have $d \le (\log y)^A$. Indeed, Stewart and Yu's second theorem applied to $1+(y+1)=y+2$ gives $\log(y+2) \ll_\eta \operatorname{rad}((y+1)(y+2))^\eta$, for every $\eta>0$. Their factor p' equals one because one summand is one. The radical is at most $(d+1)^{k+1}$. Choosing $\eta < 1/(A(k+1))$ proves the assertion [27]. **Uniform version for rational slopes.** For any fixed $B>0$, only finitely many admissible pairs allow positive coprime integers p,q satisfying

$$\max(p,q,|qy-px|) \le \big(\log(y+l)\big)^B.$$

Put $L=\log(y+l)$, $z=qy-px$. A sequence of distinct pairs has $L\to\infty$. Stewart and Yu applied to the later consecutive pair gives $L \ll_\eta (x+k)^{k\eta}$. Hence x eventually exceeds every fixed power of L. The assumed inequality forces $p \ge q$, since otherwise $qy-px \ge x$, and it also gives $\log x \sim \log y$. If a prime divides $x+i$ and $y+j$, it divides

$$D_{ij} = q(y+j)-p(x+i) = z+qj-pi.$$

The product of the nonzero $|D_{ij}|$ is $L^{O_{k,l,B}(1)}$. Call a row or column unmatched when it contains no zero D_{ij}. All prime divisors of its block entry divide this product. If two columns are unmatched, divide the two entries by their gcd and apply Stewart and Yu to their fixed nonzero difference divided by the same gcd. The least greatest prime factor among the three terms is at most l, and their common radical product bound is still a fixed power of L. This gives $\log y \ll_\eta L^{O_{k,l,B}(1)\eta}$, a contradiction for small η. Two unmatched rows give the same contradiction using $\log x \sim \log y$.

It remains that at most one row and one column are unmatched. If $p=q=1$, disjointness makes every D_{ij} nonzero. Otherwise $p\ge 2$. The matched column indices belong to one residue class modulo p, so there are at most $1+\lfloor (l-1)/p \rfloor$. To cover at least $l-1$ columns and $k-1$ rows requires $k=l=3$, $p=2, q=1$. A zero now forces the integer $z=2i-j$ into a fixed finite set. The single unmatched row and column provide $2(x+i_0)$ and $y+j_0$ with a fixed nonzero difference after passing to one choice of z, i_0, j_0. Both have prime divisors in a fixed finite set since the nonzero D_{ij} are bounded. The S-unit equation gives only finitely many such pairs. This proves the uniform assertion.

Assuming the classical *abc* conjecture, for every $\epsilon>0$,

$$d \gg_{k,l,\epsilon} \max\Big(x^{(k-1)/(k+l-1)-\epsilon}, y^{(l-1)/(k+l-1)-\epsilon}\Big).$$

Shorey and Tijdeman's Theorem 8.1(b) gives $R \gg x^{k-1-\eta}$ and $R \gg y^{l-1-\eta}$. The displayed divisibility gives $R \ll (d+1)^{k+l-1}$. The fixed $x=0$ boundary has only finitely many partners [26]. In fact, for any fixed integer polynomial $H(X,Y)$, set

$$\Delta_H = \{H(-i,-j) : 1\le i\le k,\ 1\le j\le l\}, \qquad s=|\Delta_H|.$$

Then $R \mid \prod_{\delta\in\Delta_H}(H(x,y)-\delta)$. Outside the finitely many *level curves* $H=\delta$, the same arguments show that $|H(x,y)| > (\log y)^A$ for all sufficiently large admissible y, and that under *abc* it is $\gg \max(x^{(k-1)/s-\epsilon}, y^{(l-1)/s-\epsilon})$. For $H(X,Y)=U(X)-V(Y)$ with U,V nonconstant, only finitely many admissible pairs lie on those levels by Theorem 4.3.

6.2. **Counting by gap, with no height cutoff.** For every $\epsilon>0$, a prescribed gap d contributes $O_{k,l,\epsilon}(d^\epsilon)$ pairs even when x,y are unbounded. In fact the total over $k\le d\le D$ is $O_{k,l}(D(1+\log D)^{C_l})$ for a constant C_l.

Use the first two early terms. Let s_d count the distinct prime factors of $M_d=\prod_{h=-1}^{l-1}(d+h)$. The pair $(-(x+1), x+2)$ is a solution to $u+v=1$ in a multiplicative group of rank $2s_d$. The Beukers–Schlickewei bound gives at most $2^{8(2s_d+1)}$ possibilities. Since $M_d \ll_l d^{l+1}$, the estimate $s_d! \le M_d$ gives $s_d = O_l(\log d/\log\log d)$ and proves the pointwise claim [6]. For the sum set

$z = 2^{16}$, $s = l + 1$. Bound $z^{\omega(M_d)}$ by $\prod_{h=-1}^{l-1} z^{\omega(d+h)}$. Hölder's inequality with all exponents equal to s reduces the mean to the mean of $(z^s)^{\omega(n)}$. For the integer $q = z^s$, this is at most the q-fold divisor function, whose sum up to X is at most $X(1 + \log X)^{q-1}$ by summing its defining factorizations. This proves the logarithmic version.

More generally, on any level $H(x, y) = b$ with $b \notin \Delta_H$, the same unit-equation argument bounds the number of admissible pairs by $O_{H,k,l,\epsilon}((1+|b|)^{\epsilon})$, with no height cutoff. Use all prime divisors of $\prod_{\delta\in\Delta_H}(b-\delta)$ and the two consecutive early terms. For each candidate x from that unit equation, the two consecutive late terms give at most the same unit-equation bound for y. Multiplying the bounds doubles the absolute constant in the exponent of the number of primes, which preserves the stated estimate. Summing over $|b| \le D$, $b \notin \Delta_H$, gives $O_{H,k,l}(D(1 + \log D)^{C_H})$ by the same shifted Hölder argument. Thus the finite list of exceptional translates of any prescribed plane curve is explicitly indexed by its values on the marked grid. The estimate leaves the behaviour on those translates open.

6.3. **Entire gap strips by positive Pell equations.** The following counts are exhaustive in the indicated gap ranges. The starting positions are unrestricted.

Earlier length	Later length	Maximum gap	Number of pairs
3	3	1000	15
4	3	500	23
4	4	250	5

For two triples the complete list of preceding integers (x, y) is

$$(2, 7),\ (3, 7),\ (4, 13),\ (4, 47),\ (11, 25),$$
$$(12, 62),\ (13, 47),\ (19, 53),\ (19, 97),\ (32, 118),$$
$$(33, 48),\ (53, 97),\ (54, 74),\ (88, 622),\ (168, 322).$$

For each d choose an adjacent pair in either block and take the prime divisors of the applicable differences above. The program chooses the pair whose allowed primes have the smallest product. For each squarefree product Δ of these primes, including 1 and excluding 2, solve $U^2 - 2\Delta V^2 = 1$. If P is the largest allowed prime, Lehmer [19] proved that every pair of consecutive positive integers supported on those primes occurs as $((U_h - 1)/2, (U_h + 1)/2)$ for a multiple $1 \le h \le \max(3, (P+1)/2)$ of the fundamental solution. Only V_h supported on the allowed primes need be used. The consequence of Lehmer's theorem is an exact search in each gap. The final step tests *every* term in both original blocks. The self-check recovers all 23 consecutive pairs supported on $2, 3, 5, 7$. On the common range, converting first terms to preceding integers recovers the corresponding records of the independent sieve through five million.

6.4. **If a fixed algebraic curve contains infinitely many admissible points.** In that case there is an explicit one-parameter subfamily on which **every sufficiently large index** gives an admissible pair. The parameter can be either an arithmetic progression, one rational prime raised to successive powers with fixed scale and translation, or successive powers of a unit in a real quadratic field. In particular the counting function of admissible pairs on the curve up to height X is $\gg_C \log X$ for all sufficiently large X.

An active irreducible component is defined over $\mathbb{Q}$ and has smooth normalization $\mathbb{P}^1$. By the integral-point and unmatched-fibre argument, the common pole set is either a single rational point or two real quadratic conjugate points. If it consists of one point there is at most one unmatched finite geometric point between the inverse images of the two marked grids. For two poles there are none. Horizontal or vertical components are already finite. See Section 3 and [2].

One pole. Choose a rational parameter t with that pole at infinity, so $x = f(t), y = g(t)$ for $f, g \in \mathbb{Q}[t]$. The denominators of the rational parameters at integral points are bounded divisors of one fixed integer. Indeed, clear the coefficient denominators in $f(t) - x$ and use the rational-root

theorem at the resulting integer polynomial, whose leading coefficient is independent of x. Rescale to an integer parameter T and retain the notation f, g.

Let a finite set S of rational primes contain coefficient denominators and contents in the primitive factorizations of $F_k(f(T)), F_l(g(T))$, and the primes dividing resultants between distinct factors. Outside S, reductions of distinct factors have no common root. The only possible exclusive irreducible factor is a single primitive linear polynomial $L(T) = aT + b$ with $a \neq 0$, since a unique unmatched geometric root is Galois-fixed. In its absence both prime supports agree outside S for every integer parameter yielding integer coordinate values. In its presence they agree at such parameters precisely when $L(T) \neq 0$ is supported on S. The parameter $L(T) = 0$ is excluded from the admissible positive blocks.

Choose an unbounded real parameter ray containing infinitely many admissible parameters. On this ray $f \geq 0$ and $g - f - k \geq 0$ eventually, since these are polynomial inequalities. If D_0 clears the coefficient denominators of both f, g, take $M = D_0 \prod_{p \in S} p$. For $T \equiv T_* \pmod{M}$ with T_* admissible, both coordinate changes are integers divisible by every $p \in S$. Hence the support tests at these primes keep the same outcome.

With no unmatched factor, take the whole residue class on the chosen ray. The heights grow polynomially in the index. With L present, $|L(T)|$ is unbounded along that ray and is supported on the fixed set S, so some $p \in S$ has unbounded valuation on a subsequence. Choose a witness T_* on the ray so far out that $\operatorname{sgn}(L(T_*)/a)$ is the ray's sign, and with $v_p(L(T_*)) \geq v_p(|a|M)$. Choose $e > 0$ so that p^e is one modulo the prime-to-p part of $|a|M$. For $j \geq 0$ define

$$T_j = \frac{L(T_*)p^{ej} - b}{a}.$$

The integer $L(T_*)(p^{ej} - 1)$ is divisible by $|a|M$, also at p by choice of the witness. Thus T_j is integral and congruent to T_* modulo M. The parameters escape on the correct ray, $L(T_j)$ remains supported on S, and their coordinate heights grow at most exponentially in j.

Two poles. Write K for their real quadratic field and σ for its nonidentity automorphism. Choose a degree-one parameter u over K vanishing at the first pole and infinite at the second. Scale it to value one at a rational point away from the poles. Then $u^\sigma = 1/u$ as rational functions, since their divisors coincide and their values agree there. A finite nonzero $u \in K$ gives a rational point of the source exactly when $u^\sigma = 1/u$. Both $x = f(u)$ and $y = g(u)$ are Laurent polynomials over K.

Choose a real end approached by infinitely many admissible points. It specifies a sign of u and either $|u| \to 0$ or $|u| \to \infty$. Pick an admissible witness u_* on that sign component and a totally positive norm-one unit $\eta \in \mathcal{O}_K^\times$ bigger than one in the chosen real embedding. Let S include all rational primes below places where u_* is not a unit, where a coefficient denominator occurs, or where primitive factorizations of $F_k(f(u)), F_l(g(u))$ have bad contents. The two Laurent expressions have the same finite nonzero roots, so over K they have exactly the same distinct Laurent irreducible factors. At places away from S, all evaluations at $u_*\eta^j$ are integral and their prime supports coincide.

At every place above a prime $p \in S$, choose a sufficiently large power of the prime ideal that congruence of a unit v to one modulo this power makes both $f(u_*v) - f(u_*)$ and $g(u_*v) - g(u_*)$ lie in $p\mathcal{O}_{K,\mathfrak{p}}$. This follows term by term for the finitely many Laurent monomials and also holds with v^{-1}. The residue unit groups are finite, so one exponent $e > 0$ works for all primes and all $v = \eta^{ej}$, $j \in \mathbb{Z}$. Now $u_j = u_*\eta^{ej}$ is norm one and has the witness's sign. Let $j \to +\infty$ or $j \to -\infty$ according to the chosen end. The coordinates are rational, integral everywhere, and retain their divisibilities by $p \in S$. On the chosen end the real inequalities hold eventually because they are Laurent polynomial inequalities valid along an unbounded subsequence. The heights are at most exponential in $|j|$.

In all three cases there are only finitely many repeated images from different parameters, since the normalization is one-to-one off the finitely many singularities of the plane curve. The claimed logarithmic lower bound follows. Under the conditional reduction in Section 3.5, an infinite set of ordered admissible pairs would contain one of these families and would therefore satisfy the same lower bound globally. No such family for the catalog order has been produced.

7. Rigidity at fixed projection degrees

Theorem 7.1. *Fix the two target grids V, H, with $k = |V| \geq \ell = |H| \geq 3$. Fix degrees $D \leq E$. There are only finitely many birational image curves (f, g) of these degrees on a rational parameter line with the following properties.*

- *With one common pole, the symmetric difference between $f^{-1}(V)$ and $g^{-1}(H)$ has at most one geometric point.*
- *With two common poles, the two marked sets coincide, and each coordinate has equal orders at the two poles.*

This assertion is geometric over $\mathbb{C}$. Section 3 supplies its hypotheses for a curve carrying infinitely many admissible points.

Proof. For fixed degrees the maps and poles belong to a parameter space of finite type. Label the marked parameters and specify their target values and ramification indices. There are finitely many possible lists. If one list gave infinitely many distinct image curves, there would be a one-parameter algebraic family with nonconstant image in the projective space of implicit equations. After a finite base change the poles and all individual marked parameters can be followed separately. Normalize the pole to infinity in the polynomial case, and the poles to $0, \infty$ in the Laurent case. Work at a general smooth base parameter s.

In the polynomial case let

$$W = (\partial_s f)(\partial_t g) - (\partial_s g)(\partial_t f).$$

At a common marked point $t_i(s)$ of local degrees u_i, v_i, the functions $f - f(t_i)$ and $g - g(t_i)$ have factors $(t - t_i)^{u_i}$ and $(t - t_i)^{v_i}$, times local units. Their target values are fixed. Differentiating the factors shows that the two terms in W agree in their order-$(u_i + v_i - 2)$ leading contributions, so W vanishes there to order at least $u_i + v_i - 1$.

Let N be the number of common marks and let u_0, v_0 be the total ramification indices at the unmatched points on the two sides, respectively, taking zero if absent. The lower bound for the number of zeros is $kD + \ell E - u_0 - v_0 - N$. Since $N \leq kD - u_0$ and $v_0 \leq E$, this is at least $(\ell - 1)E \geq D + E$, greater than $\deg W \leq D + E - 1$.

In the Laurent case use $t\partial_t$ for the parameter derivative. The new W has total pole degree at most $D + E$, since the pole orders are fixed throughout the family. All marks match, giving at least $\ell E > D + E$ zeros by the same calculation. In both cases $W = 0$. Over characteristic zero this Jacobian identity implies that the map from the parameter surface (s, t) to (f, g) has image of dimension at most one. Every fibre already parametrizes a curve, so the image curve is fixed. This contradicts the choice of family. □

The proof gives a finite algorithm over the algebraic numbers for fixed degrees. Enumerate the ramification lists and enforce the polynomial identities specifying the zeros of all $f - v, g - h$. Require different marked parameters to be distinct and exclude common compositional factors by retaining only maps whose image has both prescribed projection degrees. Use the normalized pole or poles and enough distinct marked points to fix three points in all and remove source automorphisms. The resulting reduced parameter spaces are zero-dimensional. This does not give a bound on the degrees themselves.

7.1. **A grid bound without a reality hypothesis.** Suppose a nonlinear birational one-pole image is parametrized by polynomials of degrees $m \le n$, and put $g_0 = \gcd(m, n)$. Write S for the set of shared marked parameters. Consecutive real members of S on an interval free of unmatched points differ in each coordinate by 0 or ± 1. By Rolle's theorem such an interval contains a zero of a derivative of $f, g, f + g$, or $f - g$. None of these derivatives is identically zero. At most one interval has to be omitted, hence $|S \cap \mathbb{R}| \le m + 3n - 2$.

For each nonreal shared parameter t, the distinct ordered pair $(t, \bar{t})$ annihilates the two divided differences of f and g. These polynomials in two variables have no common curve component, since the parametrization is birational, and the diagonal is not a component in characteristic zero. Their degrees are $m-1, n-1$. At infinity they already intersect at the $g_0 - 1$ points $[\zeta : 1 : 0]$ with $\zeta^{g_0} = 1$, $\zeta \ne 1$. Bézout bounds the number of remaining intersections by $(m-1)(n-1) - (g_0 - 1)$. Finally the derivative of each coordinate bounds its total finite ramification, with at most one unmatched geometric point removed. Combining the two counts gives

$$\max\{(k-1)m, (\ell-1)n\} \le |S| \le mn + 2n - g_0.$$

In particular $\ell \le m + 2$, independently of the degree of the later projection.

7.2. **An explicit degree bound for a cubic first coordinate.**

Proposition 7.2. *If a birational one-pole curve with earlier projection of degree three carries infinitely many admissible points, the later block length is* $3, 4$, *or* 5. *The corresponding later projection degree* n *is at most* $114,192$, *or* 216, *respectively. In particular only finitely many image curves can arise in this cubic case, even when the block lengths vary.*

Proof. The preceding grid bound gives $\ell \le 5$. We may take $n > 3$ by Lemma 5.4. Translate the earlier grid to $\{0, \dots, k-1\}$. Since the earlier map has at most two units of finite ramification and at most one marked point can be unmatched, $3(k-1) \le \ell n$.

For each marked level of the later coordinate, form the multiset of n earlier values on that fibre, counted with multiplicity. Its power sums through order $L = \lceil n/3 \rceil - 1$ are independent of the chosen level. Indeed, the trace of the jth power of the early coordinate is polynomial in the late coordinate, and has degree at most $\lfloor 3j/n \rfloor$. All entries of these multisets lie on the earlier grid apart from at most one distinct value c in one multiset. If such a value occurs it is rational by Galois invariance of the sole unmatched parameter. With multiplicity $e \le n$, it satisfies $ec^L \in \mathbb{Z}$ by comparison with a wholly matched fibre. Thus $2^L > n$ forces $c \in \mathbb{Z}$.

Take $h = 2\lfloor L/2 \rfloor \ge 2$. Comparing the hth moments about $(k-1)/2$ gives $|c - (k-1)/2| \le n^{1/h}(k-1)/2$. After translating the possible integer values to start at zero, the difference of any two multiset histograms is a polynomial $W(z)$ with

$$\deg W \le D = \tfrac{1}{2}\lfloor \ell n/3 \rfloor (1 + n^{1/h}), \qquad \|W\|_1 \le 2n, \qquad (z-1)^{L+1} \mid W(z).$$

Here $\|W\|_1$ sums absolute coefficients. The last divisibility follows by writing derivatives at 1 as falling factorial moments. For a prime power p^a, the norm of $1 - \zeta_{p^a}$ to $\mathbb{Q}$ equals p. On all conjugates the absolute value of W is at most $2n$. Consequently the cyclotomic polynomial Φ_{p^a} divides W whenever $p^{L+1} > (2n)^{\varphi(p^a)}$. If the degrees of these coprime divisors, including $(z-1)^{L+1}$, exceed D, then $W = 0$.

In this event all column polynomials of the implicit equation $F(X, Y) = 0$ are identical. Indeed F may be taken monic in X by taking the resultant of the two polynomial parametrizations. Birationality gives a single copy of the irreducible implicit equation. The degree in Y is three with constant cubic coefficient. For $\ell > 3$, four identical specializations in Y force F to be independent of Y, a contradiction. For $\ell = 3$, translating the later levels to $-1, 0, 1$ gives $F(X, Y) = Q(X) + c'(Y^3 - Y)$. Theorem 4.3 excludes an infinite admissible family on this curve.

The finite part of the degree comparison uses integer powers. On a range $a \le n \le b$, set $L = \lceil a/3 \rceil - 1$, $h = 2\lfloor L/2 \rfloor$, $K = \lfloor \ell b/3 \rfloor$. When $2^L > b$ and $h \ge 2$, let D_0 be $L+1$ plus the degrees for the prime powers $p^j \le 2b$ passing $p^{L+1} > (2b)^{\varphi(p^j)}$. The two tests $2D_0 > K$ and $(2D_0 - K)^h > bK^h$ imply $D_0 > D$, with the bounds valid uniformly across the range. The ancillary program `verify_cubic_degree.py` checks $4 \le n < 600$ separately and $600 \le n < 6000$ in intervals of 100. The largest individual degrees left by the test are exactly those in the statement, and all the intervals pass.

For $n \ge 6000$ write $z = n/6 \ge 1000$. Rosser and Schoenfeld's estimates [25, Corollary 1] give

$$\pi(z) - \pi(z/2) > \frac{z}{\log z} - \frac{1.25506z}{2\log(z/2)} > 40.$$

For the last inequality, use $\log(z/2)/\log z \ge 1 - \log 2/\log 1000$ and $z/\log z$ increases here. Each prime exceeds $\sqrt{2n}$, and $p - 1 \le n/6$, so every one passes the norm test. Their degrees total more than $40(n/12 - 1) > 3n$, whereas $2^h > n$ and $D \le (5n/3)(1+2)/2 < 3n$. This finishes the degree bound. Finally $3(k-1) \le \ell n$ bounds the early length as well, so Theorem 7.1 gives the assertion about image curves. □

7.3. **A finite check for two triples and one quartic projection.**

Proposition 7.3. *For two triples no ordered infinite family has a birational one-pole parametrization with earlier degree $D = 4$.*

Center both grids at $\{-1, 0, 1\}$. Riemann–Hurwitz gives $2E \le 12$ even with a single unmatched point. Equal degree four is excluded by Lemma 5.4, so $E = 5$ or 6.

The image curve is defined over $\mathbb{Q}$, as it contains infinitely many rational points. The birational implicit polynomial $F(X, Y)$ is monic in X of degree E, with constant coefficient $a \ne 0$ on Y^4. Its monomials satisfy $4i + Ej \le 4E$ for X^iY^j, as follows from the pole parametrization or the resultant. For $E = 5$ its highest weighted part is $X^5 + aY^4$ and positivity gives $a < 0$. For $E = 6$ its highest part is $(X^3 - hY^2)^2$ for $h > 0$. Indeed the inverse branches over a large earlier coordinate have the same positive value of X^3/Y^2 to leading order, since f^3/g^2 has a single limit at the common pole.

The three column specializations $F(X, j)$ are monic split polynomials of degree E. Their root sums are constant in j and their second power sums are affine, by the traces of f, f^2 over $\mathbb{C}(g)$. The row specializations are a times monic quartics with root sums affine in the row index, by the reverse trace. All roots are drawn from the opposite three marks except at most one root value in a single row or a single column. For a possible exceptional value, specify its multiplicity and the other roots. The first-moment condition then determines it, including when the exceptional row is an endpoint.

For every row and column multiset satisfying these conditions, evaluate the specializations on all nine cells. Their indicated values must agree after the common factor a is applied to the row data. This either rejects the data, determines a, or leaves it free. The polynomial F is uniquely determined by the six specializations for fixed a. In fact two possible reconstructions differ by a multiple of $(X^3 - X)(Y^3 - Y)$, whose leading weight exceeds $4E$ for $E = 5, 6$. Explicitly interpolate the three columns quadratically in Y, then add the quadratic interpolation in X of the missing row data. In the $E = 6$ case the coefficient of X^3Y^2 additionally determines a by the preceding square identity.

For $E = 5$ this gives 46 fixed equations and 102 pencils. For $E = 6$ it gives 15 fixed equations. Every irreducible fixed candidate is excluded because the odd part of its Y-discriminant has degree greater than three in X. A degree-four polynomial cover can have at most three finite branch values of odd discriminant order. Its monic minimal-polynomial discriminant differs from the function-field discriminant by a square.

Separated pencils are excluded by the fixed separated-equation theorem. For every other pencil compute the product $G(X,a)$ of distinct factors of the Y-discriminant having odd multiplicity and positive X-degree. Its generic degree exceeds three. To find exceptional a, take the rational negative roots of the leading X-coefficient of G times its X-discriminant, and recompute the odd part of the original discriminant at each such value. None has degree at most three. The ancillary program `verify_triple_quartic.py` performs these steps using exact rational arithmetic.

Appendix A. Galois coordinate projections

Theorem A.1. *Fix $k \geq \ell \geq 3$. Let an irreducible curve over $\mathbb{Q}$ have rational normalization and two nonconstant coordinate functions whose poles form a single pair of real quadratic conjugates. If either coordinate projection is Galois over $\mathbb{C}$, the curve contains only finitely many ordered pairs in Erdős problem 931.*

For an infinite family, both coordinates have both poles, with the same local order at the two poles for each coordinate separately. Their full inverse images of the corresponding negative-integer blocks must agree. Indeed, away from finitely many primes of bad reduction, an integral solution cannot reduce to a point belonging to just one marked inverse image. On a number field splitting the marked points, such a point would give a third omitted point in addition to the two poles, contradicting Siegel's theorem. See Section 3.

Call the Galois coordinate f, the other g, and their marked sets V, H, respectively. Write $E = f^{-1}(V) = g^{-1}(H)$ and $s = |H| \geq 3$. We prove that f, g have a common nonconstant composite.

A.1. **The group and the fibre averages.** Choose a real parameter t sending the poles to $0, \infty$. The deck group of f acts transitively on these two points. Its subgroup fixing both consists of rotations $t \mapsto \zeta_d^j t$, where $\zeta_d = \exp(2\pi i/d)$. This subgroup has index two, so f has degree $2d$ and order d at each pole. Rotation invariance gives

$$f(t) = At^d + Bt^{-d} + C, \qquad A, B, C \in \mathbb{R}, \quad AB \neq 0.$$

The full group G is generated by $R(t) = \zeta_d t$ and a reflection $S(t) = r/t$ with $r^d = B/A$. Write $g = \sum_{i=-e}^{e} c_i t^i$ with real coefficients and $c_e c_{-e} \neq 0$. Every $\gamma \in G$ preserves E.

For a polynomial ψ, define

$$T_\gamma \psi(y) = \frac{1}{2e} \sum_{g(t)=y} e_g(t)\, \psi(g(\gamma t)),$$

where $e_g(t)$ is the local degree. The field trace is regular at finite y because $g \circ \gamma$ has poles only above infinity. It therefore gives a polynomial of degree at most $\deg \psi$. Let L_0, L_∞ denote the limits of $g(\gamma t)/g(t)$ at the poles. There are e inverse branches near each pole, so the coefficient of y^j in $T_\gamma(y^j)$ is $(L_0^j + L_\infty^j)/2$. On H, the operator is a stochastic matrix with rational entries. The trace method for algebraic correspondences also appears in Bellaïche's Section 4 [4]. **Averaging lemma.** Let $G_0 \leq G$. Suppose $1 \leq m < s$ and a character $\chi : G_0 \to \{\pm 1\}$ satisfies $L_0(\gamma)^m = L_\infty(\gamma)^m = \chi(\gamma)$ for every $\gamma \in G_0$. There is a monic real polynomial p of degree m with $p(g(\gamma t)) = \chi(\gamma) p(g(t))$ identically.

Indeed, $M = |G_0|^{-1} \sum_\gamma \chi(\gamma) T_\gamma$ contracts the supremum norm on H and preserves the leading coefficient of a degree-m polynomial. Evaluation on H is injective in degree at most m. A convergent subsequence of the Cesàro averages of $M^j(y^m)$ gives a monic p with $Mp = p$. The unsigned average of the same matrices has reciprocal transition support by inversion in G_0, so it has a stationary distribution positive on every state. Jensen's inequality for the square, integrated against this distribution, is an equality because $Mp = p$. Every transition thus satisfies $\chi(\gamma) p(g(\gamma t)) = p(g(t))$ at its marked parameter. At each pole the leading terms cancel, bounding

the total pole order of the difference by $2em-2$. Riemann–Hurwitz bounds the total finite ramification of g by $2e$, since the poles already contribute $2(e-1)$. Hence $|E| \geq 2e(s-1) > 2em-2$, proving the identity.

A.2. **The possible rotations and reflections.** Put $e/d = u/q$ in lowest terms. The slope of $T_\gamma(y)$ is rational by evaluation at two rational points of H, and belongs to $[-1,1]$ as a real eigenvalue of a stochastic matrix. For R this slope is $\cos(2\pi u/q)$. Twice the cosine is an algebraic integer, so rationality forces $q \in \{1,2,3,4,6\}$.

For a reflection the multipliers are $L = c_{-e}r^{-e}/c_e$ and L^{-1}. Their mean belongs to $[-1,1]$, hence $|L| = 1$. When $B/A > 0$ choose r real, giving $L = \pm 1$. When $B/A < 0$, the choices of r have arguments $(2j+1)\pi/d$. For odd q, choosing $q \mid 2j+1$ again gives $L = \pm 1$. For even q, the choice $j = 0$ gives $L = \pm\exp(-i\pi u/q)$, of order $2q$. The rational cosine condition excludes $q = 4, 6$ in this case. For $q = 2$ take $m = 2$. The squared multipliers are $+1$ for the rotation and -1 for this reflection.

In all cases with a reflection multiplier ± 1, take $m = q/\gcd(q,2)$. At both poles the mth powers of the rotation and reflection multipliers are signs. Since the sign for each generator is independent of the pole, these signs multiply under composition and define a character on G. The same holds in the separate $q = 2, m = 2$ case. Whenever $m < s$, the averaging lemma gives $p(g)^2 \in \mathbb{C}(t)^G = \mathbb{C}(f)$.

A.3. **Three marked values.** It remains that $s = 3$ and $q = 3$ or 6. Put $U = R$ or R^2, respectively. The pole multipliers of U are primitive cube roots. On the three-point set H, the functions $1, y, y^2$ form a basis. The degree filtration shows that T_U and T_{U^2} each have trace $1 - \frac{1}{2} - \frac{1}{2} = 0$. All diagonal entries of both stochastic matrices are therefore zero.

On every marked U-orbit, values of g differing by one or two steps are distinct. Three available values force $g(U^3 t) = g(t)$ on E, hence identically since $|E| \geq 4e > 2e$. The first two elementary symmetric functions of $g(t), g(Ut), g(U^2 t)$ agree on E with those of H. The leading terms cancel at both poles, because the leading factors are proportional to the three cube roots of unity. The total pole orders are at most $2e-2$ and $4e-2$, respectively. Both symmetric functions are consequently constant. It follows that $P_H(g)$, for $P_H(Y) = \prod_{h\in H}(Y-h)$, is the third symmetric function minus $\prod_{h\in H} h$, and is invariant under U.

There is a reflection with multipliers ± 1 by the preceding analysis. The averaging lemma with $m = 1$ shows that it acts on g by an affine isometry preserving H, and thus fixes $P_H(g)^2$. For $q = 6$ the same argument applies to R^3, whose two multipliers are -1. Since R^2, R^3 generate R, in both cases $P_H(g)^2 \in \mathbb{C}(f)$.

A.4. **Recovering the multiplicities.** We have obtained a nonconstant identity $P(f) = Q(g)$, with Q polynomial. The rational function P is polynomial as well because a finite pole of P would give a pole away from the common pole pair. For $v \in V, h \in H$ let $a_v = e_P(v)$, $b_h = e_Q(h)$. At each $t \in E$, the chain rule for local degrees gives $a_{f(t)} e_f(t) = b_{g(t)} e_g(t)$. Hence

$$\frac{\prod_{v\in V}(f-v)^{a_v}}{\prod_{h\in H}(g-h)^{b_h}}$$

has divisor supported at the two poles. The two orders there are equal because each coordinate has equal pole orders at the two points. Their sum is zero, so the ratio is constant. In the original rational coordinates this constant is rational, and evaluation at a positive integral pair makes it positive. Theorem 4.1 now proves finiteness.

Appendix B. Two diagonal factors for equal block lengths

For $k \geq 3$, let $V = \{j - (k-1)/2 : 0 \leq j < k\}$ and put $P(X) = \prod_{v\in V}(X - v)$. Thus $V = -V$, and $P(x)$ is the product of the length-k block centered at x.

Theorem B.1. *Fix positive integers a, b, c, d and $\lambda \in \mathbb{Q}^\times$. The curve*

$$P(x)^a = \lambda P(y)^b (y-x)^c (y+x)^d \tag{B.1}$$

contains only finitely many pairs x, y that are centers of disjoint positive integer blocks, with y the later center.

Suppose an irreducible component has infinitely many such points. On its smooth projective normalization let $f = x,\ g = y$ have degrees D, E. Both coordinates are unbounded on a subsequence, since (B.1) has only finitely many y for fixed x. Siegel's theorem and the classification of affine rational curves with infinitely many integral points give genus zero and one rational pole or two real conjugate poles. The poles are common to both functions, since each function is unbounded on that subsequence and the poles form a single Galois orbit. Write $s \le 2$ for their number. The ordering implies $D \le E$. At a finite parameter, (B.1) gives $f \in V$ if and only if $g \in V$, including on either diagonal. Let N be the number of these marked parameters. Equal degrees are already excluded by the polynomial common-composite criterion at one pole and the fibre-averaging criterion at two poles. Both produce fixed positive multiplicities on the terms of the two blocks. The fixed-multiplicity finiteness theorem applies after the rational affine change from centers to endpoints by Section 4 and [2].

Assume $D < E$. At each pole g dominates f, since their respective local pole orders are $E/s, D/s$. Both diagonals $g - f, g + f$ have degree E. Comparing pole orders in (B.1) gives $kaD = (kb + c + d)E$, in particular $a > b$. For t marked, write $u_t = e_f(t), v_t = e_g(t)$. Away from the two diagonals $au_t = bv_t$, so v_t is divisible by $m = a/\gcd(a, b) \ge 2$. There are at most $2E$ marked parameters on the diagonals. Riemann–Hurwitz for g, followed by counting its k marked fibres with multiplicities, gives

$$(k-1)E + 2 - s \le N \le 2E + (k-2)E/m.$$

This excludes $k \ge 5$.

For $k = 4$, equality forces $m = 2,\ a = 2b$, and each diagonal has E distinct simple zeros, all with $v_t = 1$. At a zero of the first diagonal, $2bu_t = b + c$, so c/b is a positive odd integer. The analogous statement holds for d/b. The pole identity gives $(c+d)/b < 4$, hence $c = d = b$. Rescale the grid to $\{-3, -1, 1, 3\}$ and put $X = (x^2 - 5)/4,\ Y = (y^2 - 5)/4$. The component covers

$$(X^2 - 1)^2 = \mu(Y^2 - 1)(Y - X).$$

Evaluation at a nonzero rational solution gives $\mu \in \mathbb{Q}^\times$. The weighted highest part $X^4 - \mu Y^3$ is irreducible, so the curve is irreducible. The coprime exponents give one point at infinity, with X of pole order three. Equivalently, the degrees of X, Y are $3, 4$ and their local pole orders have ratio $3 : 4$. At $X = -5/4$, the cubic $(Y^2 - 1)(Y + 5/4)$ has critical points $(-5 \pm \sqrt{73})/12$. Their critical values are distinct quadratic conjugates, so the rational nonzero level has three simple points. Adjoining $x = \sqrt{4X + 5}$ gives a double cover ramified at these three points and infinity. Riemann–Hurwitz gives positive genus, a contradiction.

B.1. **The cubic and its logarithmic differential.** For $k = 3$ rescale to $P(x) = x^3 - x$. The finite grid lines, both diagonals and the line at infinity form the nine projective lines of the B_3 reflection arrangement. Blow up their points of multiplicity at least three, obtaining a smooth projective surface S and a simple normal crossing boundary B, the reduced total transform. The rational function

$$\Phi = \frac{P(x)^a}{P(y)^b (y-x)^c (y+x)^d}$$

has logarithmic differential $\theta = d\Phi/\Phi$, a section of the rank-two bundle $\Omega^1_S(\log B)$.

Its zero scheme is finite. The line residues are nonzero, including the residue at infinity, since $T = 3a - 3b - c - d > 0$. An exceptional line with zero residue has nonconstant restriction of Φ,

as the distinct incident directions have nonzero weights. Thus no boundary component lies in the zero scheme. At a critical point off the lines, applying $x\partial_x + y\partial_y$ to $\log \Phi$ gives

$$T + 2a/(x^2 - 1) - 2b/(y^2 - 1) = 0.$$

A critical curve could not be vertical. On a branch where x tends to infinity this identity gives $y^2 \to 1 + 2b/T > 1$. But $\partial_y \log \Phi$ then tends to $-b(3y^2 - 1)/(y(y^2 - 1)) \neq 0$.

The local colengths of the zero scheme sum to $c_2(\Omega^1_S(\log B)) = \chi(S \setminus B) = 8$. Indeed the complement is the square of $\mathbb{C} \setminus \{-1, 0, 1\}$ with two disjoint copies of that punctured line removed. This gives $(-2)^2 - 2(-2) = 8$ [1]. Let $\nu : C \to S$ normalize the chosen component and $\Delta = \nu^{-1}(B)_{\mathrm{red}}$. Its degree is $N + s$. At each branch put z_t equal to the minimum of the vanishing orders of the two coefficients of $\nu^*\theta$ in a local logarithmic frame. The section generates the line $\mathcal{O}_C(\sum z_t[t])$ after saturation. The pullback map to $\Omega^1_C(\log \Delta)$ kills this line, since Φ is constant on C, and is nonzero on the quotient. Taking determinants gives

$$\deg \nu^*(K_S + B) \leq N + s - 2 + \sum_t z_t. \tag{B.2}$$

The surface maps to $\mathbb{P}^1_x \times \mathbb{P}^1_y$, since the two centers resolving the projection $[x : y : z] \mapsto ([x : z], [y : z])$ were blown up. The pullback of the logarithmic two-forms for the four marked vertical and four marked horizontal fibres, including infinity, maps to $\mathcal{O}_S(K_S + B)$. The divisor of this map is supported on B. Consequently the left side of (B.2) is at least $2D + 2E$.

We show $\sum z_t \leq 16$. At a zero of θ all incident residues vanish, so Φ is locally a unit and $F = \log(\Phi/\Phi(p))$ is holomorphic. Its level $F = 0$ is reduced and has no boundary component, since either exception would give a divisorial logarithmic zero. In local coordinates X, Y, write $\mu = i(F_X, F_Y)$ and $h = \mathrm{mult}_p F$. The polar formula $i(F, F_Y) = \mu + i(F, X) - 1$ follows by differentiating on each branch of $F_Y = 0$, where $dF = F_X\, dX$, and using $i(X, F_Y) = i(X, F) - 1$. Away from the boundary, use a generic direction to obtain total branch order $\mu + h - 1$ for the gradient ideal. This is at most 2μ, since the derivatives belong to the $(h - 1)$-st power of the maximal ideal. At a boundary axis $X = 0$ the ideal is (XF_X, F_Y). Along every branch its minimum order is $\mathrm{ord}_t F_Y$, by differentiating $F = 0$. Summing gives $i(F, F_Y) = \mu + i(F, X) - 1$, the colength of that same ideal. At two boundary axes the orders of XF_X, YF_Y agree branchwise. Their total is $i(F, XF_X) = \mu + i(F, X) + i(F, Y) - 1$, again the logarithmic colength. Summing over the branches of our component only decreases these bounds [24]. It follows that $N \geq 2D + 2E - 16$. Together with $2E \leq N \leq 3D$, this gives $2E - D \leq 16$. Only 28 positive degree pairs $D < E$ satisfy these inequalities.

B.2. **The finite valuation check.** For $i, j \in \{-1, 0, 1\}$ let $L_{ij} = \sum u_t$, $M_{ij} = \sum v_t$ over the branches with $(f(t), g(t)) = (i, j)$. The row sums of L are D, the column sums of M are E, and their supports agree. The trace of f^h over $\mathbb{C}(g)$ is polynomial of degree at most $\lfloor hD/E \rfloor$, by its growth at the common poles. For $h = 1, 2$, the column first moments of M are constant and the second moments are affine in j. Thus, for integers l, r, z, w,

$$M_{-,j} = l + wj, \quad M_{+,j} = r + wj, \quad M_{0,j} = z - 2wj, \qquad l + r + z = E.$$

The reverse trace of g is affine in f, since $E/D \leq 3/2$. Hence $L_{i,+} - L_{i,-}$ is affine in i.

Write $a/b = A/B_0$ in lowest terms. The four off-diagonal cells $(0, \pm 1), (\pm 1, 0)$ have $AL_{ij} = B_0 M_{ij}$. At least one is occupied. Otherwise all N points would be on the two diagonals, and at least one above $f = 0$ would be counted twice, giving $N < 2E$. Thus A divides the nonzero gcd of the four M_{ij}. Given M, A, B_0, the four off-diagonal entries of L, its row sums and its first-moment progression determine L after one corner entry is chosen.

At a marked branch let w, z be the vanishing orders of $g - f, g + f$, with order zero for a nonvanishing diagonal. Then

$$au = bv + cw + dz, \qquad \sum w = \sum z = E.$$

On an incident diagonal its order is $\min(u, v)$ when $u \neq v$, and lies between u and E when $u = v$. At the center at most one of w, z exceeds that minimum. Some noncentral diagonal cell is occupied, since otherwise every preimage of $g = \pm 1$ would lie above $f = 0$. These two fibres have at least $E + 2 - s > D$ distinct preimages. With a, b scaled to A, B_0, any branch in such a corner fixes one diagonal weight as $(Au - B_0 v)/w$ or $(Au - B_0 v)/z$. The pole identity fixes their sum. Negating one coordinate allows $c \leq d$ in these geometric tests.

The verifier enumerates the displayed M, L tables, the possible corner weights, and the branch allocations for each cell. For each pair of cumulative diagonal contacts it retains the maximum possible count of marked parameters. It tests contacts (E, E) and $N \geq \max(2E, 2D + 2E - 16)$. The sole result, with exponents divided by their joint gcd, is

$$(D, E; a, b, c, d) = (5, 6; 2, 1, 1, 1).$$

Every entry of the surviving matrix M is two. The rows of L are $(2, 1, 2)$, $(1, 3, 1)$, and $(2, 1, 2)$. The ancillary program `verify_diagonal_arrangement.py` checks the list using exact integer and rational arithmetic.

For the remaining equation put $X = x^2$. Its quotient is

$$X(X - 1)^2 = \lambda' y(y^2 - 1)(y^2 - X).$$

The weighted highest part $X^3 - \lambda' y^5$ is irreducible. There is a single point at infinity with X of pole order five, by the coprime pole orders as above. At $X = 0$ the two points $y = \pm 1$ give simple zeros. At $y = 0$ the derivative in X is one, and the equation gives a zero of order three. Thus $x^2 = X$ ramifies at these three points and infinity. Its normalization has positive genus, excluding the last case.

Appendix C. A degree-two projection with two poles

Proposition C.1. *Fix $k \geq l \geq 3$. Let $C/\mathbb{Q}$ be a fixed irreducible plane curve whose smooth projective normalization has genus zero and two real conjugate poles, both simple for its early coordinate x. Then C contains only finitely many integer pairs*

$$0 \leq x, \qquad y \geq x + k, \qquad \operatorname{rad}\left(\prod_{i=1}^{k}(x + i)\right) = \operatorname{rad}\left(\prod_{j=1}^{l}(y + j)\right).$$

The degree of the later projection is unrestricted. This covers every two-pole rational curve with x-projection of degree two. Higher-degree early projections and general one-pole rational curves remain outside this theorem.

C.1. **The norm and trace of the later projection.** Suppose there are infinitely many ordered pairs. The Siegel argument in Section 3 shows that the poles of both coordinates coincide and that their marked finite fibres are equal as geometric sets. Translate x, y separately so the two marked sets are $V = \{0, \ldots, k - 1\}$ and $H = \{0, \ldots, l - 1\}$. Write X, Y for the translated coordinates. Both have real expansions at the two poles. Galois conjugacy gives Y equal pole orders $n \geq 1$, hence degree $2n$. For $n = 1$ the equation below has total degree at most two, and the conic theorem applies.

The quadratic extension generated by the projection X gives the equation

$$Y^2 - S(X)Y + T(X) = 0.$$

Its trace and norm have no finite poles, $\deg S \le n$, and $\deg T = 2n$. At each marked integer $v \in V$ the two Y-values indexed by the geometric X-fibre, with multiplicity, belong to H. In particular $S(v) \in \{0, \dots, 2(l-1)\}$. For every $h \in H$ write

$$h^2 - hS(X) + T(X) = LQ_h(X), \qquad L = \mathrm{lc}(T).$$

Each Q_h is monic of degree $2n$, and all its roots belong to V. Thus $Q_h \in \mathbb{Z}[X]$. The common marked preimage contains at least $2n(l-1)$ points by Riemann–Hurwitz for Y, which has ramification $2n-2$ at the poles. It contains at most $2k$ points via the X-projection. Conversely it contains at least $2k-2$ points via that degree-two projection, and at most $2nl$ via Y. Consequently

$$n(l-1) \le k \le nl + 1. \tag{C.1}$$

Suppose S is constant. The quadratic involution preserves the common marked preimage and sends Y to $S - Y$, so the set H is symmetric about $S/2$. The polynomial $S^2/4 - T(X)$ pulls back the set $\{(h - S/2)^2 : h \in H\}$ to exactly V. The set has at least two values. Lemma 4.2 below gives $2n \le 2$.

If $n \ge 2$ and S is affine nonconstant, its degree is less than n. The two leading real coefficients of Y/X^n at the poles therefore have opposite signs, and $L < 0$. At either endpoint v of V, all $Q_h(v)$ are nonnegative, since all their roots lie in the grid and their degree is even. Hence $(h - Y_1(v))(h - Y_2(v)) \le 0$ for every $h \in H$. The two branch values at each endpoint must be $0, l-1$. This forces $S(0) = S(k-1) = l-1$, ruling out affine nonconstant S. We may therefore assume $2 \le d = \deg S \le n$.

C.2. **A discrete norm bound.** An integer-valued polynomial S on k consecutive integers with values in $[0, 2(l-1)]$ satisfies

$$\frac{\binom{k+d}{2d+1}}{\binom{2d}{d}} \le k(l-1)^2. \tag{C.2}$$

To verify this, let $\Delta f(x) = f(x+1) - f(x)$, $R(x) = \binom{x}{d}\binom{x-k}{d}$, and $G = \Delta^d R$. Its leading difference is $\Delta^d G = \binom{2d}{d}$. Since R vanishes on $0, \dots, d-1$ and $k, \dots, k+d-1$, summation by parts gives

$$\sum_{x=0}^{k-1} U(x)G(x) = (-1)^d \sum_{x=d}^{k-1} (\nabla^d U)(x) R(x), \qquad \nabla U(x) = U(x) - U(x-1).$$

Thus G is orthogonal to polynomials of degree less than d. Taking $U = G$ and using Vandermonde's identity gives $\sum G^2 = \binom{2d}{d}\binom{k+d}{2d+1}$. Project $S - (l-1)$ on G. Its leading difference $\Delta^d S$ is a nonzero integer. This yields (C.2).

Write $A(k,d)$ for the left side of (C.2), divided by k. Its consecutive ratios in d equal

$$\frac{A(k,d+1)}{A(k,d)} = \frac{k^2 - (d+1)^2}{4(2d+1)(2d+3)}.$$

They decrease in d, so the minimum on $2 \le d \le n$ is attained at 2 or n. From the product expression $A(k,d) = (d!)^2 \prod_{j=1}^{d} (k^2 - j^2)/((2d)!(2d+1)!)$ we have

$$A(k,2) = \frac{(k^2-1)(k^2-4)}{720}, \qquad A(k,n) \ge \left(\frac{l-1}{2}\right)^{2n} B_n, \qquad B_n = \frac{(3n)!\, n!}{2(2n+1)(2n)!^2}.$$

Here $k \ge (l-1)n$ was used. The ratio $B_{n+1}/B_n = 3(3n+1)(3n+2)/(4(2n+1)(2n+3))$ exceeds one for $n \ge 2$.

- For $l = 3$ and $n \ge 11$, $k \ge 22$ and $B_{11} = 13485/2261 > 4$, so both endpoints contradict (C.2).
- For $l = 4$ and $n \ge 4$, $k \ge 12$, and $B_4(3/2)^8 = (11/28)(3/2)^8 > 9$.

- For $l \geq 5$ and $n \geq 3$, put $s = l - 1 \geq 4$. Then $k \geq 3s$, so $A(k, 2) > s^2$. Also $B_3(s/2)^6 = (3/10)(s/2)^6 > s^2$.
- For $n = 2$, necessarily $d = 2$. Set $r = \lfloor (k-1)/2 \rfloor \geq l - 2$. The integer $\Delta^2 S \neq 0$ gives $r^2 \leq |S(2r) - 2S(r) + S(0)| \leq 4(l-1)$, impossible for $l \geq 7$.

C.3. **Exact finite calculation.** The program `verify_quadratic_projection.py` uses integer arithmetic and rational fractions from the Python standard library. When $l = 3$ it counts all nonconstant traces of degree at most n on the first $2n$ integers. Each successive $(n+1)$-st difference must vanish, so the first $n + 1$ grid values determine the rest. For $n = 2, 3, \ldots, 10$, the counts are

$$36,\ 22,\ 8,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0.$$

This leaves $(n, l) = (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (4, 3)$, with k in (C.1).

For each such triple, enumerate the monic polynomials of degree $2n$ with roots in V by enumerating their multisets of roots. The Q_h all have the same coefficients above degree n. The vector of coefficients of degrees 1 through n is affine in h. If c_0, c_1, c_2 are the constant terms at the first three levels, then $c = c_0 + c_2 - 2c_1 = 2/L \neq 0$. All remaining constants must be $c_0 + h(c_1 - c_0) + \binom{h}{2}c$. These identities enumerate every possible tuple. Reversing the late labels allows the first two low-coefficient vectors to be ordered lexicographically. Equal vectors would give constant trace.

Recover $S = 1 + L(Q_0 - Q_1)$. For every v, the list of distinct indices h with $Q_h(v) = 0$ must contain one or two values. Their sum, with repetition when there is only one, must be $S(v)$. After this check only $(n, l, k) = (2, 3, 4)$ survives, up to late-label reflection. Its three root multisets are

$$\{1, 1, 2, 2\}, \qquad \{0, 1, 2, 3\}, \qquad \{0, 0, 3, 3\}.$$

They give $S = X^2 - 3X + 3$, $T = (X^2 - 3X + 2)^2/2$. The discriminant $S^2 - 4T$ has leading coefficient -1, precluding any unbounded real branch. This excludes the final case.

Appendix D. Unramified coverings of punctured spheres

Let V, H be blocks of $k_1, k_2 \geq 3$ consecutive integers in their respective affine coordinates. Write $Y_1 = \mathbb{P}^1 \setminus (V \cup \{\infty\})$ and $Y_2 = \mathbb{P}^1 \setminus (H \cup \{\infty\})$.

Proposition D.1. *Let $U = \overline{C} \setminus E$ be a connected complex curve of finite type with two finite holomorphic unramified coverings $f_i : U \to Y_i$. Suppose the inverse images on $\overline{C}$ of the two distinguished punctures ∞ are the same set I. For each i, suppose the ramification index of the compactified f_i has one value m_i throughout I. Then the affine image of (f_1, f_2) lies on a plane curve of degree at most two.*

For an infinite sequence of ordinary rational integer pairs on a birational parameter curve, the hypotheses on I follow from integrality. The normalization has genus zero and one rational pole or two conjugate real quadratic poles. This last classification is for ordinary integral points over $\mathbb{Q}$, with no finite primes inverted. A nonconstant rational function has a pole somewhere, so both coordinate functions then have these same poles. In the two-pole case Galois conjugation also equates the local orders for each coordinate. Thus the fixed-conic theorem excludes a solution of the ordered Erdős 931 from any pair of genuine unramified covers of the indicated punctured spheres with exactly matching marked preimages [2].

D.1. **A common quotient from the distinguished cusp.** The universal cover of U is the hyperbolic upper half-plane $\mathbb{H}$. Composing it with the covering maps to Y_i gives finite-area torsion-free groups $\Gamma_U \subseteq \Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$, with $U = \mathbb{H}/\Gamma_U$ and $Y_i = \mathbb{H}/\Gamma_i$. At each puncture, a canonical local coordinate q is defined up to a factor of modulus one. Explicitly, send a lift of the cusp to infinity, write the positive stabilizer translation as $z \mapsto z + w$, and put $q = \exp(2\pi i z/w)$. Covering a cusp with ramification m gives $|q_{\text{target}}| = |q_{\text{source}}|^m$ exactly. This follows either from the index m between cusp stabilizers or from the uniquely normalized cusp coordinates [29]. For

sufficiently small $\delta > 0$, take the target neighbourhood $|q_i| < \delta^{m_i}$ of the distinguished cusp of each Y_i. Its pullback to U is the union of $|q_p| < \delta$ for $p \in I$, the same union for the two maps. Pulling this set back to $\mathbb{H}$ gives a common disjoint collection $\mathcal{B}$ of open horoballs. The group Γ_i preserves $\mathcal{B}$ and acts transitively on it, since its target neighbourhood is attached to one cusp.

The full orientation-preserving stabilizer G of $\mathcal{B}$ is discrete. There are two balls with different ideal centres, since a cofinite hyperbolic surface group is non-elementary. An element sufficiently close to the identity fixes each of these balls. Indeed a chosen interior point remains in its ball and distinct balls are disjoint. An isometry fixing two such balls is the identity. Indeed upon moving their centres to zero and infinity it is a positive dilation, and preserving the height of the ball at infinity forces its dilation factor to be one. This proves discreteness. Each Γ_i has finite index in G by the hyperbolic area formula. Concretely, an embedded disk away from stabilizers in $\mathbb{H}/G$ would have infinitely many disjoint lifts of identical positive area in the finite-area quotient $\mathbb{H}/\Gamma_i$ if the index were infinite.

The quotient $W = \mathbb{H}/G$ is therefore a finite-area hyperbolic orbifold. The Y_i cover it as orbifolds. Adjoining their cusps extends these maps to finite holomorphic maps of compact surfaces. Each $\overline{Y}_i$ is a sphere, so $\overline{W}$ is a sphere by Riemann–Hurwitz. By transitivity, the ideal centres of the packed balls form a single cusp of W whose preimage on each $\overline{Y}_i$ consists exactly of its distinguished cusp. Use a target coordinate taking this cusp of W to infinity. Each quotient map $P_i : \overline{Y}_i \to \overline{W}$ is a polynomial since its only pole is infinity. The common covering gives

$$P_1(f_1) = P_2(f_2).$$

Let T be the set of all other cusps of W. Since the preimages of cusps under an orbifold cover are precisely cusps, $P_1^{-1}(T) = V$ and $P_2^{-1}(T) = H$.

If $|T| \geq 2$, Lemma 4.2 gives $\deg P_1, \deg P_2 \leq 2$, as desired. The set T cannot be empty since V is nonempty. Suppose $T = \{a\}$. A hyperbolic finite-area genus-zero orbifold with exactly two cusps has an elliptic point b. Without an elliptic point its orbifold Euler characteristic would be zero. Write $e \geq 2$ for its stabilizer order. Both Γ_i are torsion-free, so every point over b has ramification exactly e. Consequently $P_i = b + A_i^e$ for polynomials A_i over $\mathbb{C}$. The finite set $T' = \{z : z^e = a - b\}$ has e distinct points, and the two inverse images under A_i are precisely V, H. Lemma 4.2 gives $\deg A_i \leq 2$. The identity of composites implies

$$A_1(f_1) = \zeta A_2(f_2)$$

for some e-th root of unity ζ, because the meromorphic functions on U form a field. This is a plane equation of degree at most two and finishes the proof.

This addresses modular correspondences that are unramified outside their matching marked values, including covers defined on a common noncongruence surface. A rational map with additional critical values need not be a covering of the indicated punctured sphere.

D.2. **A cyclotomic construction ruled out in all degrees.** Let D_m be the monic polynomial satisfying $D_m(t + t^{-1}) = t^m + t^{-m}$. Consider a norm-one real quadratic unit t, an early coordinate $f = A + a(t^m + t^{-m})$ with $A, a \in \mathbb{Q}$, $a \neq 0$, and a later Laurent polynomial $g(t)$ with real coefficients. Suppose every point of the common marked inverse set $E = f^{-1}(V) = g^{-1}(H)$ is a root of unity. Assume g is rational-valued at infinitely many powers of the unit and its coefficients belong to the real quadratic field. Then the later coordinate either is a polynomial in the early coordinate or is a single harmonic $B + b(t^n + t^{-n})$. In the graph case use Theorem 3.1. In the remaining case the early coordinate has dihedral Galois projection on the Laurent parameter line. Its marked fibres can also be treated directly using the common composition of the two Dickson polynomials, after rational affine changes, and Theorem 4.3. Indeed the affine shifts and scales of the later harmonic are rational by interpolation at rational values of the unit traces.

Write n for the common pole order of g at zero and infinity. The Laurent polynomial $g(t)-g(1/t)$ vanishes on every point of E, since g takes real values there. Riemann–Hurwitz for the degree-$2n$ map g, with ramification contribution $2n-2$ already at the two poles, gives $|E| \geq 2n(k_2-1) > 2n$. Thus $g(t) = g(1/t)$, so it is a polynomial $R(z)$ in $z = t + t^{-1}$. For the norm-one unit, infinitely many distinct z's are rational, and so are their images. Interpolation gives $R \in \mathbb{Q}[z]$.

The normalized early levels $(V-A)/a$ form an arithmetic progression of at least three members of $\{-2,-1,0,1,2\}$. Indeed, every value $\tau + \tau^{-1}$ with $\tau = t^m$ is both rational and an algebraic integer of modulus at most two. Let S be the resulting set of phase values τ, and let $b = |S|$. The allowable phases above $2, -2, 0, -1, 1$ are respectively $1, -1, \pm i$, the primitive third roots, and the primitive sixth roots. Thus $|E| = bm$ and

$$2n \leq \frac{bm}{k_2 - 1}.$$

For any marked later value h_0, the monic polynomial $P = (R - h_0)/\operatorname{lc}(R)$ has algebraic-integer roots and rational coefficients, hence integral coefficients. Applying this at $h_0 + 1$, after choosing adjacent levels, gives $q = |1/\operatorname{lc}(R)| \in \mathbb{Z}_{>0}$. The normalized marked levels of P span $q(k_2-1)$. The complete fibres above the smallest and largest levels lie in $[-2, 2]$. This gives $n+1$ successively alternating points in that interval for the two levels. To see this even with multiple roots, all intermediate levels have n distinct real roots by Rolle and the monotone branch decomposition. The intervening extrema and the two outer branches each reach the appropriate boundary level.

The span is at most four. Subtract the midpoint of the two levels from P and compare with the monic D_n, whose magnitude is at most two on $[-2, 2]$. A larger span would force a polynomial of degree at most $n-1$ to change sign at least n times. At span four the signs are weak. For alternating nodes $z_0 < \cdots < z_n$, the last divided difference of that polynomial is zero, and its terms have the same weak sign because the denominators $\prod_{r \neq j}(z_j - z_r)$ alternate. All terms vanish. Hence at span four P is a translate of D_n.

At span three we must have $k_2 = 4$ and $q = 1$. Every eligible initial progression with at least four levels contains either the phases $\tau^3 = 1$ or the phases $\tau^3 = -1$. Hence E contains an entire coset $t^{3m} = \pm 1$. Since $2n \leq 8m/3 < 3m$, Fourier orthogonality on that coset gives

$$\operatorname{Var}(P(t + t^{-1})) = 2\sum_{j=1}^{n} a_j^2, \qquad P(t + t^{-1}) = a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j}).$$

Here $a_j \in \mathbb{Z}$ and $a_n = 1$. Values in an interval of length three have variance at most $9/4$. Thus $a_j = 0$ for $1 \leq j < n$ and P is again a translate of D_n.

At span two we have $k_2 = 3$ and $q = 1$. Put $a_{-j} = a_j$. On each coset $t^m = \tau$ the variance is at most one. Choose $t_0^m = \tau$ with $|t_0| = 1$. The Fourier coefficient in residue class $r \pmod m$ is

$$t_0^r C_r(\tau), \qquad C_r(T) = \sum_{q:\, |r+qm| \leq n} a_{r+qm} T^q.$$

The squared magnitude is an integer, since $C_r(\tau)$ is an algebraic integer in $\mathbb{Q}$, $\mathbb{Q}(i)$, or $\mathbb{Q}(\sqrt{-3})$. For real-valued samples the coefficients at r and $-r$ are conjugates. Every pair with $r \not\equiv -r \pmod m$ must vanish by the variance bound. The only nonconstant self-conjugate class is $r = m/2$ for even m. Pairing positive and negative exponents shows that its Fourier coefficient is an integer combination of $2\cos((j + 1/2)\arg\tau)$. For $\tau = 1, -1, \pm i$, or a primitive sixth root, these belong respectively to $2\mathbb{Z}$, $\{0\}$, $\sqrt{2}\mathbb{Z}$, or $\sqrt{3}\mathbb{Z}$. It too must vanish, except possibly above a primitive third root.

Each C_r, $r \neq 0$, now vanishes on at least $b-2$ distinct phases, while the difference of its extreme exponents is at most $\lfloor 2n/m \rfloor \leq \lfloor b/2 \rfloor$. This forces $C_r = 0$ for $b \geq 5$. At $b = 4$ the unique eligible progression is $\{-2, 0, 2\}$, which contains no primitive third phase, so all four phases give zeros and

the same conclusion follows. The remaining frequencies are multiples of m. Inversion symmetry gives $P \in \mathbb{Z}[t^m + t^{-m}] = \mathbb{Z}[D_m(z)]$, as claimed.

Computational materials

The ancillary directory contains the programs and coordinate lists used in this paper. For the bounded-height census, run `block_census.py` with arguments 5000000 and 12 using Python with NumPy. The columns of `block_census_5m.txt` are the two lengths and the two *first terms*. For the bounded-gap searches, run `verify_gap_strip.py` with the two lengths and the gap bound in Section 6. Its listed coordinates are the integers *preceding* the blocks. The complete gap lists appear in `gap_strip_certificate.txt`. The quartic calculation uses Python with SymPy. The quadratic-projection and diagonal calculations use the Python standard library.

AI disclosure

OpenAI's GPT-6.2 Astra carried out the mathematical research, developed the arguments and computations, and wrote this manuscript. Sunay Venkat, the author, takes full responsibility for its contents, including any errors.

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