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RAMANUJAN PRIMES AND PRIMES OF EXPONENT ONE IN PRODUCTS OF CONSECUTIVE INTEGERS

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ABSTRACT. Let $M(L)$ be the minimum number of primes occurring to exponent one in a product of L consecutive positive integers. Assuming the ordinary *abc* conjecture, we prove that, for all sufficiently large L, $M(L)$ equals the number of Ramanujan primes at most L. Thus the least length threshold guaranteeing k such primes is the kth Ramanujan prime for every sufficiently large k. The identity holds unconditionally if the last entry is at most $\exp(c(\log L)^{3/2}/\sqrt{\log\log L})$, for an absolute $c > 0$. We determine the limiting count at every fixed ratio of the starting offset to the length. We also prove unconditionally that the factorial count can exceed the minimum by an unbounded amount, and that ordinary primes contain arbitrarily long runs of consecutive non-Ramanujan primes.

1. INTRODUCTION

For integers $L \geq 1$ and $s \geq 0$, set

$$P_L(s) = \prod_{i=1}^{L}(s+i), \qquad S_L(s) = \#\{p \text{ prime} : v_p(P_L(s)) = 1\}, \qquad M(L) = \min_{s\geq 0} S_L(s).$$

The minimum exists by well-ordering. Write $\pi(u)$ for the number of primes at most u, and put $D(u) = \pi(u) - \pi(u/2)$. The kth Ramanujan prime R_k is the least positive integer r for which $D(u) \geq k$ for every real $u \geq r$. Let $\pi_{\mathrm{R}}(L) = \#\{k \geq 1 : R_k \leq L\}$. This definition, and the asymptotic $R_k \sim 2k\log k$, are discussed by Sondow [13]. All logarithms below are natural.

Erdős asked in 1982 whether $M(L) \to \infty$ [4, p. 27]. The related powerful-product conjecture asks whether $M(L) \geq 1$ for *every* $L \geq 3$ [4, p. 28]. Conjectures.io lists the long-block question separately under Erdős problem 137 [1]. In its product $m(m+1)\cdots(m+n)$ the width is $L = n+1$ and the offset is $s = m-1$.

Theorem 1.1. *Assume the ordinary abc conjecture over $\mathbb{Q}$. For all sufficiently large integers L,*

$$M(L) = \pi_{\mathrm{R}}(L). \tag{1.1}$$

In particular $M(L) \sim L/(2\log L)$. For every sufficiently large k, the least L_0 such that $S_L(s) \geq k$ for every $L \geq L_0$ and $s \geq 0$ is R_k.

The exact location of the threshold matters even at a small scale. For instance 199! has 21 prime divisors of exponent one, whereas the product of the 199 integers from 20 to 218 has 18. Moving the right endpoint from 199 to 218 adds 211 to its upper half and loses the four primes $101, 103, 107, 109$ at the lower endpoint. We prove in Section 5 that the difference from the factorial count is unbounded.

Our conditional input is Granville's uniform lower bound for the product of simple prime divisors of a binomial coefficient [6, Theorem 1′]. His Corollary 4 also determines extremal products over simple primes for binomial coefficients. The integer-valued minimum in (1.1) ranges over the unnormalized products $P_L(s)$. Earlier work of Erdős counts block entries having a prime factor

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above the block length [3, Theorem 2]. Hughes gives related conditional reductions for powerful products [9]. Tao studies the frequency of integers in intervals with powerful product [15]. We use no assumption on *abc* for the finite-height theorems or the results on prime clusters.

2. Localization and the exact minimum

Lemma 2.1. *If $0 \le s \le L/2$, then $S_L(s) = D(L+s)$.*

Proof. Each prime $p \le L/2$ divides at least two entries. For $p > L/2$ one has $p > s$, so $v_p(P_L(s)) = v_p((L+s)!)$. In $N!$ the primes of exponent one are exactly those in $(N/2, N]$. □

Lemma 2.2. *Assume abc. If $L \to \infty$ and $s/L \to \infty$, then $S_L(s)/(L/\log L) \to \infty$.*

Proof. Write $N = L+s$. The ordinary *abc* conjecture asserts $c \ll_\eta \mathrm{rad}(abc)^{1+\eta}$ for each $\eta > 0$ and coprime positive integers $a + b = c$, where $\mathrm{rad}(n) = \prod_{p \mid n} p$. Granville's theorem gives, uniformly for $3 \le L \le N/2$,

$$\prod_{p \parallel \binom{N}{L}} p \gg_\epsilon \binom{N}{L}^{1-2/L-\epsilon}. \tag{2.1}$$

Here $p \parallel B$ means $v_p(B) = 1$. Since $P_L(s) = L!\binom{N}{L}$, every prime on the left exceeding L contributes to $S_L(s)$. Removing those at most L costs $O(L)$ on the logarithmic scale by Chebyshev's estimate. Every remaining prime is at most N. Now $\binom{N}{L} \ge (N/L)^L$, by comparing the factors $(N-i)/(L-i)$ for $0 \le i < L$. Take $\epsilon = 1/4$ and $L \ge 8$. This yields

$$\frac{S_L(s)}{L/\log L} \ge \frac{\log L}{\log L + \log(N/L)} \left(\tfrac{1}{2} \log(N/L) - O(1) \right).$$

The right side tends to infinity since $uv/(u+v) \ge \min(u,v)/2$ for $u, v > 0$ and both logarithms tend to infinity. □

Proof of Theorem 1.1. The factorial gives $S_L(0) = D(L) \sim L/(2\log L)$ by Lemma 2.1 and the prime number theorem. Consider a sequence with $s/L \to c > 0$ finite. If $s \le L$, every prime in $((L+s)/2, L+s]$ occurs once, so $S_L(s) \ge ((1+c)/2 + o(1))L/\log L$. If $s \ge L$, every prime in $(s, s+L]$ occurs once, giving $S_L(s) \ge (1+o(1))L/\log L$. The prime number theorem applies since in this regime both endpoints have size comparable to L. At $c = 1$ split the sequence at $s = L$.

On any sequence of minimizers with s/L bounded away from zero, take a subsequence on which s/L converges in $[0, \infty]$. The bounded-ratio estimates and Lemma 2.2 each contradict the factorial upper bound. Hence every minimizing offset has $s = o(L)$. More generally, $S_L(s) \sim L/(2\log L)$ if and only if $s = o(L)$, with sufficiency following from Lemma 2.1.

For large L the minimum is therefore $\min_{L \le t \le 3L/2} D(t)$. Since $D(t) \sim t/(2\log t)$ and $t/\log t$ eventually increases, all $t \ge 3L/2$ have $D(t) > D(L)$ when L is sufficiently large. The preceding minimum equals $\min_{t \ge L} D(t) = \pi_{\mathrm{R}}(L)$, by the defining property of the Ramanujan primes. Real and integer minima agree because D is constant between consecutive integers. For large k, width $R_k - 1$ fails the k-prime condition and every width from R_k onward satisfies it. The threshold in the catalog parameter n is consequently $R_k - 1$. □

Remark 2.3 (The fixed-width problem)**.** For a fixed $L \ge 3$, choose $0 < \epsilon < 1 - 2/L$ in (2.1). If $P_L(s)$ were powerful, each simple prime of $\binom{N}{L}$ would be at most L. Their product would be bounded by $\prod_{p \le L} p$, so (2.1) bounds N in terms of L under *abc*. Together with Theorem 1.1 this allows at most finitely many powerful blocks of width at least three. It leaves open whether any exist. For $L = 3$ specifically the product of the simple primes of $\binom{N}{3}$ would be at most 6. At width two the block $8, 9$ already has powerful product.

3. Uniform unconditional estimates

Theorem 3.1. *There is an absolute constant $c_0 > 0$ such that for all sufficiently large L,*

$$\min_{\substack{s\ge 0\\ L+s\le H(L)}} S_L(s) = \pi_{\mathrm{R}}(L), \qquad H(L) = \exp\left(c_0 \frac{(\log L)^{3/2}}{\sqrt{\log\log L}}\right).$$

For fixed $1+\delta \le A_0 < \infty$ with $\delta > 0$, at least $\gg_{\delta,A_0} L$ primes $p > L$ occur exactly once whenever $L^{1+\delta} \le L + s \le L^{A_0}$.

Proof. Put $N = L + s$. We recover Lemma 2.2 in the specified range. Legendre's factorial formula gives $v_p(\binom{N}{L})\log p \le \log N$, since each $p^j \le N$ contributes zero or one to the valuation. For $N \le L^2$, every prime $p > L$ dividing the binomial coefficient occurs exactly once in the block, since it has at most one multiple there and $p^2 > N$. The number of such primes is at least

$$L\log(N/L)/\log N - \pi(L).$$

Dividing by $L/\log L$ gives a quantity tending to infinity if $N/L \to \infty$. If $N \ge L^{1+\delta}$, the same lower bound is $\gg_\delta L$.

For $L^2 < N \le L^3$, discard primes at most $Z = L(\log L)^{1/3}$. Among the remaining primes dividing $\binom{N}{L}$, any nonsimple p has $bp^2 \in (N-L, N]$ for some $b \le N/Z^2 = o(L)$. For fixed b, the interval $(\sqrt{(N-L)/b}, \sqrt{N/b}]$ has length less than one. There are consequently at most $\pi(Z) + N/Z^2 = o(L)$ discarded primes, each costing at most $\log N$ in the logarithm of the binomial coefficient. This gives at least $(1/2 - o(1))L$ simple primes $> L$.

Suppose now $N \ge L^3$, and write $A = \log N/\log L$. We use Sander's distribution estimate in the form of Granville's Lemma 2 [6]. For $j = 1, 2$ and $Y \le N^{1/j}$, the number of primes $p \le Y$ satisfying $\{N/p^i\} < \sigma_i$ for $1 \le i \le j$ equals $\pi(Y)\prod_i \sigma_i$, with error

$$\ll \left(Y^{\,1-b(\log Y/\log N)^2} + Y^{1+j/2+1/10}N^{-1/2}\right)(\log N)^{4j}. \tag{3.1}$$

The braces denote fractional parts. The bound is uniform for $0 \le \sigma_i \le 1$, with an absolute $0 < b \le 1$ valid for both j.

Put $d = b/(4A^2)$. For each $P = 2^i L$, $i \ge 0$, with $2P \le L^{1+d}$, consider the primes in $(P, 2P]$ satisfying

$$\{N/p\} < L/(2P), \qquad \{N/p^2\} \ge L/P^2.$$

Their residues $N \bmod p < L$ and $N \bmod p^2 > L$ show that they occur once. Subtract the two-coordinate count from the one-coordinate count at $Y = P, 2P$ and difference at these endpoints. The main term is

$$\frac{L}{2P}\left(1 - \frac{L}{P^2}\right)(\pi(2P) - \pi(P)) \gg L/\log L.$$

For $Y = P, 2P$ we have $Y \le \sqrt{N}$. The first error term is at most $L\exp(-3b\log L/(4A^2))$. The second is at most $L^{(37/36)(21/10)-3/2} < L^{0.7}$, since $A \ge 3$. For $N \le H(L)$, $(\log N)^8 \le (\log L)^{12}$, and $A \le c_0\sqrt{\log L/\log\log L}$. Choose c_0 so that $3b/(4c_0^2) > 16$. The errors are $o(L/\log L)$ uniformly. There are $\gg \log L/A^2$ disjoint bands, giving

$$\#\{p > L : p \parallel P_L(s)\} \gg L/A^2 \gg L\log\log L/\log L.$$

This is $\gg_{A_0} L$ on the fixed polynomial range. In the full range, the proof of Theorem 1.1 now applies, since the allowed range includes the entire initial interval $0 \le s \le L/2$. □

A weaker lower bound reaches larger endpoints. The following estimate applies Konyagin's fractional-part theorem twice, to the first and second powers of a possible prime divisor. The first application is related to the method of van Doorn and Tang [10, 2].

Proposition 3.2. *Fix $0 < \beta \le 1/4$, let $B = \log L$, and set $c_\beta = \beta/100$. Uniformly for*

$$L^5 \le N \le \exp(c_\beta B^2/\log B),$$

all but $o_\beta(L^{1-\beta}/\log L)$ of the primes in $(L, L+L^{1-\beta}]$ divide $P_L(N-L)$ exactly once. In particular the long-block count tends to infinity unconditionally for all $L \le N \le \exp(c_\beta B^2/\log B)$.

Proof. Write $h = L^{1-\beta}$, $\delta = 2L^{-\beta}$, and let $\|z\|$ denote distance to the nearest integer. The denominator-one case of Konyagin's Theorem 2 gives the following estimate. If $D/2 \le f^{(r)} \le D$, $|f^{(r+1)}| \le E$ on $[0,h]$, $r \ge 2$, then for $\lambda \ge 1$ the number of integers v there with $\|f(v)\| < \delta$ is

$$\ll h\left((D\lambda^r)^{1/(2r-1)} + \left(\frac{\delta}{D\lambda^r}\right)^{1/(r-1)} + \left(\frac{E\lambda}{D}\right)^{1/(2r)}\right) + 2r\lambda.$$

The implied constant is absolute, in particular independent of r [10, Theorem 2].

For $d = 1, 2$, let $D_j = N\, d(d+1)\cdots(d+j-1)/L^{d+j}$. Choose the first $r \ge 2$ with $D_r \le L^{-\beta/2}$. Since $D_2 \ge L$ and $D_j/D_{j-1} = (d+j-1)/L$, we have

$$3 \le r \le 3c_\beta B/\log B, \qquad D_r > L^{-1-\beta/2}$$

for large L. For the upper bound take $j = \lceil 2c_\beta B/\log B\rceil$, for which the logarithm of the rising factorial is $O_\beta(B)$, and use the assumed bound on N.

Apply the estimate with $f(v) = (-1)^r N/(L+v)^d$, $D = D_r$, $E = D_r(d+r)/L$, and $\lambda = (L^{-\beta/2}/D_r)^{1/r}$. Then $1 \le \lambda < L^{1/r}$ and $(1+L^{-\beta})^{d+r} < 2$, as required for the derivative bounds. The three terms in parentheses are bounded respectively by $B^{-\beta/(12c_\beta)}$, $2B^{-\beta/(6c_\beta)}$, and $B^{-1/(12c_\beta)}$. For the third bound use $1/r \le 1/3$ and $\log(r+d) \le B/6$. Also $r\lambda \ll_\beta BL^{1/3} = o(h/B^2)$. We have therefore discarded $O_\beta(h/B^2)$ integers for each d.

The prime number theorem in short intervals gives $\sim h/B$ primes in $(L, L+h]$, since $h \ge L^{3/4}$ [7, Corollary 1.3]. For every prime surviving both tests,

$$N \bmod p \le p(1-\delta) < L, \qquad N \bmod p^2 \ge \delta p^2 > L.$$

It occurs exactly once. The range $N < L^5$ in the last assertion follows from Theorem 3.1. □

4. Counts at proportional offsets

Let $H_n = \sum_{j=1}^n j^{-1}$, $H_0 = 0$, and $J(x) = xH_{\lfloor x\rfloor} - \lfloor x\rfloor$ for $x \ge 0$.

Theorem 4.1. *If $s/L \to c \in [0,\infty)$, then unconditionally*

$$\frac{\log L}{L}\, S_L(s) \longrightarrow W(c) = 1 + 2[J(c+1) - J(c)] - \tfrac{1}{2}[J(2c+2) - J(2c)].$$

The function W is continuous and strictly increasing.

Proof. A prime occurring once exceeds $L/2$, and, on this scale, its square exceeds $L+s$ for large L. For $u = p/L$ and $c = s/L$, the number of its multiples in the block is

$$\Delta_c(u) = \lfloor (c+1)/u\rfloor - \lfloor c/u\rfloor.$$

On $u > 1$ the singleton indicator is $\Delta_c(u)$. On $1/2 < u \le 1$ it is $2 - \Delta_c(u)$. For each fixed limiting c its support is bounded and its set of discontinuities is finite. Away from arbitrarily small fixed neighborhoods of those discontinuities and the endpoints, the indicators stabilize as $s/L \to c$. The prime number theorem bounds the normalized number of discarded primes by the combined lengths of the neighborhoods in the limit. Integration therefore gives the limit, also when $2c$ is an integer.

Now $\int_1^\infty \lfloor x/u \rfloor \, du = \sum_{j \le x} (x/j - 1) = J(x)$. Adding the two indicator integrals gives the asserted expression. J is continuous, and away from half-integers

$$W'(c) = \frac{2}{\lfloor c \rfloor + 1} - \frac{1}{\lfloor 2c \rfloor + 1} - \frac{1}{\lfloor 2c \rfloor + 2} > 0.$$

Continuity gives strict increase on the full half-line. □

Corollary 4.2. *For any fixed $a \ge 0$, under abc the minimum over $s \ge \lceil aL \rceil$ is asymptotic to $W(a)L/\log L$. For $0 \le a < 1/2$, that minimum equals $\pi_{\mathrm{R}}(L + \lceil aL \rceil)$ for all sufficiently large L. Both statements hold unconditionally if $L + s \le H(L)$ as in Theorem 3.1.*

Proof. The offset $\lceil aL \rceil$ supplies the upper bound. Strict increase and the large-offset bound give $s/L \to a$ for minimizing sequences. When $a < 1/2$, Lemma 2.1 applies there. The prime number theorem puts the minimum of $D(t)$ over $t \ge L + \lceil aL \rceil$ below $t = 3L/2$ for large L, because $1 + a < 3/2$. The same tail-minimum identity as in Theorem 1.1 completes the proof. □

For example $W(0) = 1/2$, $W(1/2) = 3/4$, and $W(1) = 4/3$.

5. Displacement of the minimum and runs of primes

For $0 \le s \le L/2$, Lemma 2.1 gives the exact difference

$$S_L(s) - S_L(0) = \{\pi(L + s) - \pi(L)\} - \{\pi((L + s)/2) - \pi(L/2)\}. \tag{5.1}$$

For every fixed $\epsilon > 0$, all global minimizers satisfy $s < L^{17/30+\epsilon}$ for large L under *abc*. The same upper bound holds unconditionally for minimizers restricted to $L + s \le H(L)$. Indeed those minimizers have $s = o(L)$. For $L^{17/30+\epsilon} \le s \le L/2$, the difference in (5.1) is uniformly asymptotic to $s/(2 \log L) > 0$ by Guth and Maynard [7, Corollary 1.3]. One can decrease ϵ before applying their estimate at $L/2$. Intervals exceeding $x^{0.99}$ may be partitioned into pieces of comparable length $\asymp x^{0.9}$.

Theorem 5.1. *For every pair of positive integers r, B, there is an $H > B$ and infinitely many even L such that*

$$S_L(s) = S_L(0) \quad (0 \le s \le B), \qquad S_L(2H) \le S_L(0) - r.$$

Consequently $\sup_L (S_L(0) - M(L)) = \infty$.

Proof. Choose k large enough in Maynard's admissible linear forms theorem to obtain at least r simultaneous primes [12, Theorem 3.4, arXiv version]. Take $B < h_1 < \cdots < h_k = H$ all divisible by the product of the primes up to k. These shifts form an admissible tuple. For each odd $1 \le d \le 2H$, take a different prime $q_d > 2H$ and require $2t + d \equiv 0 \bmod q_d$. For $1 \le j \le B$ use a further different prime $q'_j > 2H$ to require $t + j \equiv 0 \bmod q'_j$. The Chinese remainder theorem gives a class $t \equiv b \bmod Q$.

The forms $Qn + b + h_i$ remain admissible. At primes dividing Q none vanishes, since $2h_i - d$ and $h_i - j$ are nonzero and smaller in absolute value than their respective moduli. At other primes the change of variable is invertible. Maynard's theorem gives arbitrarily large t in this progression with at least r of the $t + h_i$ prime. All $2t + d$, $1 \le d \le 2H$, are composite by parity or the prescribed factor, as are $t + 1, \ldots, t + B$. Put $L = 2t$. The first pair of braces in (5.1) vanishes throughout $0 \le s \le 2H$. The second pair vanishes for $s \le B$ and is at least r at $s = 2H$. □

Corollary 5.2. *For every r, the ordinary primes contain infinitely many runs of r consecutive non-Ramanujan primes.*

Proof. The construction gives $D(L + 2H) \le D(L) - r$. For each of the last r primes $p \le L$, $D(L) - D(p) \le \pi(L) - \pi(p) \le r - 1$. Thus $D(L + 2H) < D(p)$. At $p = R_j$, one has $D(p) = j$ and all subsequent values are at least j, since D can rise by at most one at any integer. All r primes fail this condition. Their locations tend to infinity with L. □

This proves the non-Ramanujan clause of Sondow's qualitative Conjecture 2 [13]. The conjecture also asks about runs of Ramanujan primes. Sondow, Nicholson and Noe give a quantitative conjecture on the longest runs [14].

Write $\log_j$ for the j-fold iterated logarithm and $s_*(L)$ for the first offset attaining $M(L)$.

Proposition 5.3. *Unconditionally, infinitely many L satisfy*

$$s_*(L) \gg \frac{\log L \log_2 L \log_4 L}{\log_3 L}.$$

For each fixed r, infinitely many L also satisfy $S_L(s) \geq M(L) + r$ simultaneously for $0 \leq s \leq c \log_2 L$, with an absolute $c > 0$.

Proof. Let $Q = \frac{1}{2}\prod_{p \leq y} p$. The primorial covering theorem of Ford, Green, Konyagin, Maynard and Tao [5] gives an interval of length $2h$ each of whose entries has a prime factor at most y, with even $h \asymp y \log y \log_3 y / \log_2 y$. The odd entries, multiplied by $2^{-1} \bmod Q$, give h consecutive nonunits modulo Q. Continue this run to its first following unit e. Then $(e, Q) = 1$ and $(e - i, Q) > 1$ for $1 \leq i \leq h$.

Take a prime $8h < r' < 16h$. Linnik's theorem [8] supplies a prime $p \ll (Qr')^A$ for an absolute A, satisfying $2p \equiv e \bmod Q$ and $p \equiv 4h + 1 \bmod r'$. In particular $p > 4h$. Each of $2p - i$ for $1 \leq i \leq h$ and $p - j$ for $1 \leq j \leq h/2$ is composite, as it shares a prime factor at most y with the odd modulus Q. Put $L = 2p - h$. Then $D(L + s)$ is constant for $0 \leq s < h$ and falls by one at $s = h$. Lemma 2.1 yields $s_*(L) \geq h$. Since $\log L \ll y$ and $L \to \infty$, the first estimate follows.

For the second take $k = k(r)$ from Maynard's theorem and $J = \lfloor (\log_2 X)/10 \rfloor$. Choose distinct primes $J < h_1 < \cdots < h_k = H < 2J$. Let Q_0 be the product of the primes up to $4J$ except the h_i, and impose $t \equiv 0 \bmod Q_0$. This makes $t + u$ composite for $2 \leq u \leq J$. It also makes $2t + d$ composite for $2 \leq d \leq 2H$, except possibly $d = h_i$. A product of two omitted primes exceeds $4J$. Force $t + 1$ and $2t + d$ for $d = 1, h_1, \ldots, h_k$ composite using distinct primes between $8J$ and $16J$. The resulting modulus Q has $\log Q \leq (4 + o(1))J + O_k(\log J)$, so $Q \leq \log X$. The forms $Qn + b + h_i$ are admissible by the same residue check as above. The coefficient bounds of Maynard's uniform Theorem 3.4 apply for $n \in [X, 2X]$, taking the interval length X. For a resulting $t = Qn + b$, put $L = 2t$. Equation (5.1) is zero up to $s = 2J$, and at most $-r$ at $s = 2H$. Now $\log_2 L = \log_2 X + o(1)$, as required. □

6. Finite verification

For $1 \leq L \leq 256$ and $0 \leq s \leq 8192$, direct factorization gives

$$\min_{0 \leq s \leq 8192} S_L(s) = \pi_{\mathrm{R}}(L) \qquad (L \neq 2).$$

At $L = 2$, the observed minimum is zero, attained for example at $8, 9$, while $\pi_{\mathrm{R}}(2) = 1$. Across all 2,097,408 windows the largest excess of the factorial count above this finite minimum is three. It first occurs at length 199, with the first minimizing offset 19.

The ancillary program `erdos137_check.py` sieves the factors of each entry and updates the prime exponents as a window moves one place. It computes the Ramanujan counts independently as suffix minima of D. To terminate that calculation rigorously, set $j = \pi(256) + 1$ and write p_i for the ith prime. Laishram's bound $R_j < p_{3j}$ guarantees that $D(t) \geq j$ for all $t \geq p_{3j}$ [11]. Thus no later value changes a computed suffix minimum for $L \leq 256$. The experiment tests finite ranges of starting offsets. The unbounded-height assertion for long blocks in Theorem 1.1 uses *abc*, and the powerful-product question for every $L \geq 3$ remains open.

AI disclosure

OpenAI's GPT-6.2 Astra carried out the mathematical research, developed the arguments and computations, and wrote this manuscript. Sunay Venkat, the author, takes full responsibility for its contents, including any errors.

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