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The dimension–degree boundary for Gaussian moment vanishing

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Abstract

We classify complex polynomials of degree at most three in three independent standard real Gaussian variables whose positive moments all vanish. Up to a complex orthogonal linear change of coordinates and a nonzero scalar, every such polynomial has one-sided circular support or belongs to a single exceptional orbit. For every fixed polynomial Q, its mixed moments $\mathbb{E}[QP^m]$ vanish whenever $m > 3 \deg Q$, and the coefficient three is optimal. The proof extracts a Gaussian coordinate in which the polynomial is affine linear, then classifies that family using formal stationary frames and analytic period estimates.

Combining this ternary-cubic result with Wilson's two-variable theorem, Liu's quadratic theorem and Long's counterexamples gives the exact dimension–degree boundary for the Gaussian Moments Conjecture: the implication from pure-moment vanishing to eventual mixed-moment vanishing holds universally precisely when there are at most two variables, the degree is at most two, or there are three variables and the degree is at most three. We reproduce the all-order moment identities for the two minimal failures.

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1 What vanishing moments tell us

Let $X_1, \dots, X_n$ be independent standard real Gaussian random variables. We allow a polynomial $P(X)$ to have complex coefficients. Suppose that

$$\mathbb{E}[P(X)^m] = 0 \qquad \text{for every integer } m \geq 1. \tag{1}$$

The Gaussian Moments Conjecture asks whether this forces

$$\mathbb{E}[Q(X)P(X)^m] = 0 \qquad \text{for all sufficiently large } m \tag{2}$$

for every fixed polynomial Q. The threshold may depend on both P and Q.

The complex coefficients matter. For real P, the second moment in (1) forces $P = 0$. For complex P, cancellation can make every positive moment vanish even when P is far from zero. For example, $(X_1 + iX_2)/\sqrt{2}$ has this property. The question is whether a fixed polynomial multiplier can keep exposing a nonzero moment at arbitrarily large powers.

We call (1) *purity* and (2), for every Q, *safety*. These names distinguish the hypothesis from the conclusion; safety without purity is a separate question. In the original formulation of Derksen, van den Essen and Zhao [1], the assertion that every pure polynomial is safe was a conjecture in every dimension. Long's examples show why the dimension cannot be ignored [8].

Here the question is more precise. For integers $n \geq 1$ and $d \geq 0$, let $\mathsf{G}(n, d)$ mean that every pure complex polynomial of total degree at most d in n standard Gaussian variables is safe.

Theorem 1.1 (Dimension–degree boundary)**.**

$$\boxed{\mathsf{G}(n,d) \iff n \leq 2 \text{ or } d \leq 2 \text{ or } (n,d) = (3,3).}$$

The entries below describe a universal implication for all polynomials in each dimension–degree range. Individual safe polynomials also exist outside those ranges.

	$d \leq 2$	$d = 3$	$d \geq 4$
$n = 1, 2$	purity implies safety	purity implies safety	purity implies safety
$n = 3$	purity implies safety	purity implies safety	failures exist
$n \geq 4$	purity implies safety	failures exist	failures exist

The only positive case needed beyond the existing binary and quadratic theorems is a cubic in three variables. Its quantitative form is useful:

Theorem 1.2 (Ternary cubics)**.** *Let $P \in \mathbb{C}[X_1, X_2, X_3]$ have total degree at most three and satisfy (1). Then, for every nonzero polynomial Q,*

$$\mathbb{E}[QP^m] = 0 \qquad \textit{whenever } m > 3 \deg Q. \tag{3}$$

For $Q = 0$ the conclusion is immediate. No smaller constant can replace three uniformly over this class.

The proof gives a classification as well as a vanishing bound. Put

$$Z = \frac{X_1 + iX_2}{\sqrt{2}}, \qquad W = \frac{X_1 - iX_2}{\sqrt{2}}, \qquad T = X_3. \tag{4}$$

Call a phase *one-sided* if all its monomials have circular weight $\deg_Z - \deg_W$ of one strict sign; T has weight zero.

Theorem 1.3 (Normal forms for pure ternary cubics)**.** *A complex polynomial of degree at most three in three Gaussian variables is pure if and only if, after a complex orthogonal linear change of coordinates and multiplication by a nonzero scalar, it is one-sided or is the exceptional phase*

$$P_* = W + ZT - \frac{1}{2}Z^3.$$

The zero polynomial is included in the one-sided case: its empty support satisfies either strict-sign condition. Thus the classification consists of one-sided support and one exceptional orbit. One-sided phases have cutoff $m > \deg Q$; the exceptional orbit accounts for the sharp coefficient three.

Notation and reading guide

The coordinate and parameter conventions used throughout are:

Symbol	Meaning
n, d	Number of real Gaussian variables and upper bound on $\deg P$
Z, W, T	Circular pair with $\mathbb{E}[Z^a W^b] = \delta_{ab} a!$, and an independent standard Gaussian
t, r	Formal moment parameter; $r = 1/t$ near infinity, with a finite root cover when stated
Φ_Q	Marked series $\sum_{m\geq 0} \mathbb{E}[QP^m]t^m/m!$
$q(x)$	Gaussian exponent after a coordinate change, including its linear and constant terms

All polynomial degrees are total degrees. An expectation is the algebraic Gaussian moment functional, extended to formal series coefficient by coefficient. Complex coordinate changes preserve this functional when they preserve its bilinear covariance form. Analytic periods enter only after the comparison hypotheses are checked.

For the boundary statement and its examples, read Sections 1–7 and Section 12. The full ternary-cubic proof occupies Sections 8–11; the proof map below locates its two main reductions.

Proof map

Write $P = F_3 + F_2 + F_1 + F_0$ in homogeneous pieces. The logical route is:

Step	Statement used	Proof location
Find a Gaussian direction	Purity gives a nonzero v with $v^T v = 0$, $\nabla F_3(v) = 0$, $F_2(v) = 0$	Section 11
Make one coordinate affine linear	The frame test gives $P = Wh(Z,T) + G(Z,T)$	Lemma 7.1
Classify the affine family	One-sided support or the exceptional orbit	Section 9
Transfer the mixed bound	Weight cancellation or the exact shear source identity	Section 6

Formal tools in Section 8 and analytic comparison in Section 10 support the two classification steps. Section 12 assembles the conclusions.

The leading cubic has nine geometric types. The abbreviated table below groups types with the same exclusion argument; the full table appears in Section 11. In every singular row, the task is to prove that purity supplies a direction passing the frame test.

Leading cubic	Argument when the frame test fails
Smooth	Gradient criterion, Corollary 11.3
Nodal; transverse line–conic; nonconcurrent lines	Ordinary-node analysis, Corollary 11.16
Cuspidal; tangent line–conic	Proposition 11.21
Three concurrent distinct lines	Proposition 11.22
Double line plus a line; triple line	Quadratic elimination, Proposition 11.25

Section 6 proves sharpness with a three-term example. The difficult step is reducing an arbitrary pure ternary cubic to a family with a direct moment calculation; Section 7 states the two reductions.

Attribution and scope

We prove the ternary-cubic normal-form classification and its sharp mixed cutoff, and deduce the dimension–degree boundary. Wilson's Theorem 1 supplies the unrestricted two-variable result and the classification by dimension alone [13]. The quadratic theorem is due to Liu [7, Theorems 2.1 and 2.2]; we include a short proof of the form used here. Long's cubic four-variable and quartic three-variable counterexamples are adopted from [8, Proposition 4.1 and Theorem 5.1], with their all-order moment identities reproduced below.

For formal exponential decomposition and local Fourier–Laplace background, see Sabbah [11]. The polynomial stationary-frame argument below allows nonconvergent Laurent coefficients over a valued field. The analytic period comparisons are applied to rational parameter families, whose local Puiseux expansions converge.

2 A few Gaussian identities

All our generating functions are formal power series. We never need to assume that the integral of $e^{tP(X)}$ converges. Define

$$\Phi_Q(t) = \sum_{m\geq 0} \mathbb{E}[QP^m]\frac{t^m}{m!}. \tag{5}$$

Then P is pure exactly when $\Phi_1 = 1$, and is safe exactly when every Φ_Q is a polynomial. Thus a degree bound on Φ_Q gives an explicit vanishing threshold.

In the circular coordinates of (4), Z, W are polynomials in the independent real Gaussians X_1, X_2, but are not independent random variables: $W = \overline{Z}$. Their moments are

$$\mathbb{E}[Z^a W^b] = \delta_{ab} a!. \tag{6}$$

Rotating (X_1, X_2) proves the vanishing when $a \neq b$; when $a = b$, $ZW = (X_1^2 + X_2^2)/2$ has moments $a!$. An additional independent standard Gaussian T has $\mathbb{E}[T^{2j}] = (2j-1)!!$ and zero odd moments.

Lemma 2.1 (One sign of weight)**.** *Give Z, W, T weights $1, -1, 0$, respectively. If every monomial of P has strictly positive weight, or every monomial has strictly negative weight, then P is pure and*

$$\mathbb{E}[QP^m] = 0 \qquad (m > \deg Q).$$

Proof. In the positive case, every monomial of P^m has weight at least m. A monomial of Q has weight at least $-\deg Q$. For $m > \deg Q$ their product therefore has nonzero weight, and (6) kills it. The negative case follows by exchanging Z and W. □

Complex orthogonal changes of polynomial coordinates preserve the Gaussian moment functional. This follows from its source identity $\mathbb{E}[e^{u^T X}] = e^{u^T u/2}$. We use this algebraic invariance; we do not interpret a complex coordinate change as a real change of integration contour.

Lemma 2.2 (Linear circular contraction)**.** *Let a, b be constants and let $F(t, z)$ be a formal power series in t whose coefficients are polynomials in z. Then*

$$\mathbb{E}\Big[F(t, Z)e^{t(a+bZ)W}\Big] = \frac{1}{1-bt}\, F\left(t, \frac{at}{1-bt}\right). \tag{7}$$

Proof. It suffices, coefficient by coefficient in t, to take $F = Z^j$. Using (6), the left side is

$$\sum_{k\geq j} t^k [z^k] z^j (a+bz)^k = \sum_{r\geq 0} \binom{j+r}{j} a^j b^r t^{j+r} = \frac{(at)^j}{(1-bt)^{j+1}}.$$

Every fixed coefficient uses only finitely many terms, which also justifies the statement for t-dependent F. □

3 Quadratics in any number of variables

We include a short proof of the positive quadratic result. It explains why degree two is safe in every dimension without introducing the finite-moment thresholds that are unnecessary for the boundary. The classification and sharper surrounding results are due to Liu [7, Theorem 2.1].

Proposition 3.1. *Every pure complex quadratic in n standard Gaussian variables is safe. In fact*

$$\mathbb{E}[QP^m] = 0 \qquad (m > \lfloor n/2 \rfloor \deg Q)\,.$$

Proof. Write $P = X^T AX + \ell^T X + c$, with $A = A^T$, and put $\Sigma(t) = (I - 2tA)^{-1}$ and $D(t) = \det(I - 2tA)$. Completing the square formally gives

$$\Phi_1(t) = D(t)^{-1/2} \exp\left(ct + \frac{t^2}{2}\ell^T \Sigma(t)\ell\right). \tag{8}$$

If $\Phi_1 = 1$, then $D = e^S$ for a rational function S regular at zero. Hence $D'/D = S'$. A rational derivative has zero residue at every finite pole, whereas D'/D has the positive multiplicity as residue at every zero of D. Thus D has no zero, so $D = 1$. It follows that A is nilpotent and

$$c = 0, \qquad \ell^T A^j \ell = 0 \quad (j \geq 0).$$

Conversely these identities make (8) equal to one.

With an additional source u, the same calculation now gives

$$\mathbb{E}[e^{tP+u^T X}] = \exp\big(u^T \mu(t) + \tfrac{1}{2} u^T \Sigma(t) u\big)\,, \qquad \mu(t) = t\Sigma(t)\ell. \tag{9}$$

Both μ and Σ are polynomials in t. Let s be the dimension of $\operatorname{span}\{\ell, A\ell, A^2\ell, \ldots\}$, with $s = 0$ if $\ell = 0$. This space is totally isotropic, since $(A^i\ell)^T A^j \ell = \ell^T A^{i+j}\ell = 0$. Nondegeneracy of the ambient form gives $s \leq h := \lfloor n/2 \rfloor$. Nilpotence gives $A^s\ell = 0$, so $\deg_t \mu \leq s$. Also $\deg_t \Sigma \leq n - 1 \leq 2h$.

For a monomial of degree q in X, a term in (9) uses p covariance factors and $q - 2p$ mean factors. Its t-degree is at most $2hp + h(q - 2p) = hq$. Source differentiation and linearity prove the result. □

Corollary 3.2 (Ternary quadratic normal form)**.** *Every pure quadratic in three variables has, in Gaussian coordinates, the form*

$$P = Z(\alpha Z + \beta T + \gamma).$$

In particular it is one-sided.

Proof. The preceding proof gives $c = 0$, nilpotent symmetric A, and $\ell^T A^j \ell = 0$ for every $j \geq 0$. If $A \neq 0$, let k be largest with $A^k \neq 0$. In dimension three this last power has rank one, so $A^k = \lambda v v^T$ for a nonzero null vector v, and $Av = 0$. The identity $\ell^T A^k \ell = 0$ gives $v^T \ell = 0$. Since ℓ is also null, the one-dimensional nondegenerate quotient $v^\perp / \mathbb{C}v$ forces $\ell \in \mathbb{C}v$. Choose $Z = v^T X$ and a circular partner W. The equation $Av = 0$ makes P independent of W, and $\operatorname{tr} A = 0$ removes its T^2 coefficient. This is the displayed form. If $A = 0$, the null linear form $\ell^T X$ can itself be chosen as a multiple of Z; the zero polynomial is immediate. □

4 Two variables, with no degree restriction

The needed input is Wilson's support theorem [13, Theorem 1]: if a polynomial in two Gaussian variables is pure, then in the coordinates Z, W every monomial has weight $\deg_Z - \deg_W$ of one strict sign. Lemma 2.1 immediately gives the threshold $m > \deg Q$. This proves $\mathsf{G}(2, d)$ for every d.

The one-variable conclusion follows as well. Embed a one-variable polynomial into two variables by ignoring the second. A multiplier in the larger space can be averaged in that unused variable. In particular, safety in two variables implies safety in one. No extension of Wilson's theorem to three variables is being assumed here.

5 The two failures that determine the boundary

Long's examples [8] have the same marked generating function. Writing them side by side makes both the threshold and the failure of eventual vanishing transparent.

5.1 A quartic in three variables

Use Z, W, T as above and set

$$P_3 = (1 + Z)\left(W - \frac{2 + Z}{2} T^2\right), \qquad Q_3 = Z.$$

The total degree is four. For any polynomial A, first average over T, then apply Lemma 2.2 with $a = b = 1$. Put $z_* = t/(1 - t)$. The result is

$$\begin{aligned} \mathbb{E}[A(Z) e^{tP_3}] &= \frac{A(z_*)}{1 - t} \left(1 + t(1 + z_*)(2 + z_*)\right)^{-1/2} \\ &= A\left(\frac{t}{1 - t}\right), \end{aligned}$$

because the expression inside the square root is $(1 - t)^{-2}$. The root with constant term one cancels the factor $(1 - t)^{-1}$.

5.2 A cubic in four variables

Use two independent circular pairs (Z_1, W_1) and (Z_2, W_2) and set

$$P_4 = (1 + Z_2)\big(W_1(1 - Z_1) + W_2\big), \qquad Q_4 = Z_2.$$

The total degree is three. For a formal parameter s, circular contraction gives

$$\mathbb{E}[e^{sW_1(1-Z_1)}] = \frac{1}{1+s}.$$

Substitute $s = t(1 + Z_2)$, then contract the second pair. Again

$$\mathbb{E}[A(Z_2)e^{tP_4}] = \frac{A(z_*)}{1-t}\frac{1}{1+t(1+z_*)} = A\left(\frac{t}{1-t}\right).$$

In either example, $A = 1$ gives purity, while $A = z$ gives

$$\Phi_Q(t) = \frac{t}{1-t}, \qquad \mathbb{E}[QP_j^m] = m! \neq 0 \quad (m \geq 1).$$

Thus one linear multiplier sees a nonzero moment at every positive power. These examples disprove $\mathsf{G}(3, 4)$ and $\mathsf{G}(4, 3)$.

6 Why the cubic cutoff is three

The factor three in Theorem 1.2 has a concrete source. Consider

$$P = aW + (c + dZ)T - \frac{Z(c + dZ)^2}{2a}, \qquad a \neq 0. \tag{10}$$

For sources α, β, γ, first contract W and then average over T. If $u(z) = c + dz$, the exact identity is

$$\begin{aligned}\mathbb{E}[e^{tP+\alpha Z+\beta W+\gamma T}] = \exp\Bigg(&\alpha(at+\beta) + \frac{\gamma^2}{2} + \gamma t\, u(at+\beta) \\ &- \frac{\beta t}{2a}u(at+\beta)^2\Bigg).\end{aligned} \tag{11}$$

Setting the sources to zero proves purity. Give the sources α, β, γ weights $1, 3, 2$, respectively. Every term in the exponent has t-degree at most its source weight. The same is true after exponentiation. It follows that

$$\deg_t \Phi_{Z^i W^j T^k} \leq i + 3j + 2k \leq 3(i + j + k).$$

Thus every member of (10) satisfies (3).

Now take the simplest member,

$$P_* = W + ZT - \frac{1}{2}Z^3.$$

With only the W source left, (11) becomes

$$\mathbb{E}[e^{tP_*+\beta W}] = \exp\left(-\frac{\beta t(t+\beta)^2}{2}\right).$$

The highest t-degree in its β^D coefficient comes only from $(-\beta t^3/2)^D/D!$. Therefore, for every $D \geq 1$,

$$\mathbb{E}[W^D P_*^{3D}] = (3D)! \left(-\frac{1}{2}\right)^D \neq 0. \tag{12}$$

Since $\deg W^D = D$, any proposed uniform coefficient smaller than three would already fail on this family. This establishes the sharpness part of Theorem 1.2 independently of the general reduction.

Lemma 6.1 (The exceptional orbit)**.** *If $d \neq 0$, every phase in (10) is equivalent to P_* under a complex orthogonal linear change and nonzero scalar multiplication. If $d = 0$, it is one-sided in suitable Gaussian coordinates.*

Proof. Apply the null rotation

$$Z = Z', \qquad T = T' + \lambda Z', \qquad W = W' - \lambda T' - \lambda^2 Z'/2, \qquad \lambda = c/a.$$

It preserves $2ZW + T^2$ and changes the phase to $aW' + dZ'T' - d^2 Z'^3/(2a)$. For $d \neq 0$, choose $\kappa^2 = a/d$, set $Z' = \kappa Z''$, $W' = W''/\kappa$, and multiply by κ/a. The result is P_*. When $d = 0$, the original phase is a pure linear form, so Corollary 3.2 applies. □

7 The two reductions for a ternary cubic

To prove the ternary-cubic theorem, we must handle a polynomial with no preferred coordinate. Write it in homogeneous pieces as

$$P = F_3 + F_2 + F_1 + F_0.$$

We want a direction in which P is only affine linear. The direction must also be isotropic for the Gaussian bilinear form; an arbitrary linear change would lose the moment identities.

Lemma 7.1 (The algebraic frame test)**.** *There are Gaussian coordinates (Z, W, T) in which P is affine linear in W if and only if there is a nonzero vector $v \in \mathbb{C}^3$ such that*

$$v^T v = 0, \qquad \nabla F_3(v) = 0, \qquad F_2(v) = 0. \tag{13}$$

Proof. Polarization gives

$$D_v^2 P(x) = 2\nabla F_3(v)^T x + 2F_2(v).$$

The last two conditions in (13) say exactly that this polynomial vanishes. Complete the nonzero null vector v to a Witt basis: choose a null partner with pairing one, and a unit vector in their orthogonal complement. In these coordinates D_v is the W derivative. Hence $D_v^2 P = 0$ says that no second or higher power of W occurs. The converse reads off v from the W direction. □

The hard reduction is an existence statement, not this algebraic test.

Proposition 7.2 (The required Gaussian direction)**.** *If a pure ternary polynomial has degree exactly three, then a vector satisfying (13) exists.*

The proof divides according to the projective cubic $F_3 = 0$. Over $\mathbb{C}$, the list is short: a smooth cubic, an irreducible nodal or cuspidal cubic, a line with a smooth conic, three distinct lines, or a repeated line. The first group has no leading singular direction at all. In the singular cases the issue is whether the singular direction is null and whether F_2 vanishes there. Cases failing this

test are excluded by their period behavior. Once a direction passes the test, Lemma 7.1 supplies the desired coordinates.

Section 11 proves this statement for all nine leading types. Its formal frames, regular-rank accounting and uniform cut estimates include critical degeneracies and contributions at infinity.

Proposition 7.3 (Classification with an affine-linear coordinate)**.** *Let*

$$P = W(a + bZ + cT + dZ^2 + eZT + fT^2) + G(Z, T), \qquad \deg G \le 3.$$

If P is pure, then, up to a complex orthogonal linear change of coordinates and a nonzero scalar multiple, it is one-sided or equal to $P_* = W + ZT - Z^3/2$. In particular it is safe, with $\mathbb{E}[QP^m] = 0$ whenever $m > 3 \deg Q$.

The proof below reduces the pure cases to the one-sign argument, the quadratic calculation, and the shear mechanism of Section 6. Some excluded cases first reduce to a one-variable polynomial exponential period. The proof must retain the determinant factor from eliminating a Gaussian variable; omitting it can turn a decaying period into an apparent constant. It must also use the actual period selected by the Gaussian moment functional, rather than every solution of an enlarged differential equation.

The next four sections separate the tools from the two reductions: formal arguments in Section 8, the affine-coordinate classification in Section 9, analytic comparison in Section 10, and geometric frame extraction in Section 11.

8 Formal tools

The first tools are algebraic: an obstruction for a cubic Gaussian series, a stationary-ball criterion over a valued field, and the corresponding criterion with logarithmic volume. They are used in both reductions. The analytic interpretation is developed separately in Section 10.

Here a *differential module* over a parameter field K, such as $\mathbb{C}(t)$ or $\mathbb{C}((r))$, is a finite-dimensional K-vector space M with a connection ∇ satisfying $\nabla(am) = a'm + a\nabla m$. A K-linear functional $J : M \to L$ into a differential extension field is *horizontal* if $J(\nabla m) = (J(m))'$. The distinguished amplitude class [1] is called the *seed*. In a Gaussian application $J([Q]) = \Phi_Q$, so purity means $J([1]) = 1$. The obstructions below exclude a horizontal quotient from the submodule generated by this seed to the scalar module K, with connection given by ordinary differentiation, taking [1] to one. Lemma 10.1 explains why purity would supply such a quotient.

8.1 Removing the cubic height without asymptotics

The extra term T^3 can be dealt with algebraically. This avoids using an asymptotic formula for a cubic Gaussian integral.

Lemma 8.1 (An algebraic Riccati obstruction)**.** *The equation $f_x = x - f^2$ has no solution algebraic over $\mathbb{C}(x)$.*

Proof. At a finite point, a finite branch of f satisfies a regular analytic first-order equation and cannot branch. At a pole, put $g = 1/f$. Then $g_x = 1 - xg^2$, again a regular equation; a zero of g is simple. Thus the normalized algebraic curve of $\mathbb{C}(x, f)$ is unramified over every finite x. If its degree is d and there are r points over infinity, Riemann–Hurwitz gives $2g - 2 = -d - r$. This forces $d = r = 1$. So f would be rational. An integer-order pole of f at infinity makes f^2 dominate both x and f_x; if f is bounded there, the term x cannot cancel. Both possibilities are impossible. □

Lemma 8.2 (Cubic Gaussian obstruction)**.** *Let $A, B, C \in \mathbb{C}(z)$ be regular at zero, with $A(0) = C(0) = 0$, $B(0) = -1/2$ and $A \neq 0$. Define the formal unit*

$$I = \mathbb{E}_T \exp\bigl(AT^3 + (B + 1/2)T^2 + CT\bigr).$$

Then I'/I is not rational in z.

Proof. Write $M_j = \mathbb{E}[T^j \exp(AT^3 + (B + 1/2)T^2 + CT)]$ and $J = M_1$. Gaussian integration by parts gives, for $j \geq 0$,

$$3AM_{j+2} + 2BM_{j+1} + CM_j + jM_{j-1} = 0, \tag{14}$$

where the last term is zero for $j = 0$. Also $I' = A'M_3 + B'M_2 + C'J$ and $J' = A'M_4 + B'M_3 + C'M_2$. For clarity, the first two eliminations are

$$M_2 = -\frac{2BJ + CI}{3A}, \qquad M_3 = \frac{(4B^2 - 3AC)J + (2BC - 3A)I}{9A^2}.$$

The case $j = 2$ of (14) supplies M_4.

In an algebraic extension choose $a^3 = 3A$ and put

$$X = \frac{B^2 - 3AC}{a^4}, \qquad F = \frac{J}{I}, \qquad f = -aF - \frac{B}{a^2}.$$

Substitution in these three recurrences, using $a' = A'/a^2$, gives

$$\frac{I'}{I} = U - aX'F, \qquad U = \frac{A'(2BC - 3A)}{9A^2} - \frac{B'C}{3A}, \tag{15}$$

$$f' = X'(X - f^2). \tag{16}$$

These identities follow by differentiation and polynomial substitution; they do not involve an integration contour or a limit of a Gaussian integral. They can also be checked by substituting $A = a^3/3$ in (14) before differentiating.

If A has order $q > 0$, then X has a pole of order $4q/3$, because $B^2 - 3AC$ is a unit. Thus $X' \neq 0$ and aX' is a nonzero rational function. If I'/I were rational, (15) would make F rational and f algebraic over $\mathbb{C}(z)$. The function field $\mathbb{C}(z, a)$ is finite over $\mathbb{C}(X)$, so f would be algebraic over $\mathbb{C}(X)$. Equation (16) contradicts Lemma 8.1. □

8.2 A formal leading-coefficient criterion

For a polynomial phase, the remaining shrinking argument can be replaced by a valuation calculation. The construction below includes critical points that collide, remain multiple, or escape to infinity. The key is to group them before making a Gaussian approximation.

Write v for the valuation of a formal parameter r, normalized by $v(r) = 1$. The valuation of a polynomial is the least valuation of its coefficients. We use the formal Newton–Puiseux theorem: finitely many algebraic elements over $\mathbb{C}((r))$ belong to one field $\mathbb{C}((r^{1/N}))$ [12, Corollary 1.10]. Characteristic zero makes each minimal polynomial separable; no convergence hypothesis is needed.

Lemma 8.3 (Polynomial separation of formal balls)**.** *Let finitely many disjoint balls have centers c_B in a Puiseux field and radii specified by $v(x - c_B) \geq \rho_B$. After one common ramification, put $\delta_B = r^{\rho_B}$. For any one ball B and integer $j \geq 0$, there is a polynomial $q_{B,j}(x)$ over the Laurent field whose substitution $x = c_C + \delta_C y$ has integral coefficients for every ball C, and whose reduction is y^j on B and zero on every other ball.*

Proof. The finitely many valuations $v(c_B - c_C)$ give a finite tree of clusters of centers. At a node with scale r^d, the coordinate $u = (x - c)/r^d$ has distinct constant reductions $a_1, \dots, a_s$ on its children. The Lagrange polynomial

$$e_i(u) = \prod_{k \neq i} \frac{u - a_k}{a_i - a_k}$$

reduces to one on child i and zero on every other child.

Start at the chosen leaf with $((x - c_B)/\delta_B)^j$. Induct upward. Suppose a polynomial h already has the required reductions on the chosen child. On each of the other child balls its coefficients have some finite lower valuation. The restriction of e_i there has strictly positive valuation. Consequently $e_i^M h$, for a sufficiently large fixed integer M, has positive coefficient valuation on all those children. On the chosen child $e_i^M = 1+$ positive-order terms, so the existing reductions are unchanged. There are finitely many nodes, and this constructs the required polynomial. A node with just one leaf requires no filter. This also explains why independently normalized Lagrange polynomials, with no control of their growth on other scales, would not suffice. □

Theorem 8.4 (Formal stationary balls)**.** *Let $\varphi(r, x) \in \mathbb{C}((r))[x]$ have degree $D \geq 2$, and let $h(r) \in \mathbb{C}((r)) \setminus \{0\}$. Put*

$$M = \mathbb{C}((r))[x]/(\partial_x + \varphi_x)\mathbb{C}((r))[x], \qquad \nabla[q] = [q_r + (\varphi_r + h'/h)q].$$

After a finite ramification, this module has a full frame of rank-one formal exponential functionals. If a_D is the leading coefficient of φ, every zero-exponential functional in this frame takes the seed $[1]$ to a series of valuation at least

$$v(h) - \frac{v(a_D)}{D}. \tag{17}$$

In particular, if $Dv(h) > v(a_D)$, the differential submodule generated by $[1]$ has no horizontal quotient taking $[1]$ to the constant one.

Proof. The two twisted derivatives commute, so the parameter connection is well defined on the quotient. A nonzero polynomial of degree m has spatial twisted derivative of degree $m + D - 1$. Reduction of successive leading terms therefore gives a basis $1, x, \dots, x^{D-2}$ and dimension $D - 1$.

Split φ_x, retaining root multiplicities. For a critical root ξ, expand

$$\varphi(\xi + u) = c_\xi + \sum_{j=2}^{D} a_{\xi,j} u^j, \qquad \rho_\xi = \max_{2 \leq j \leq D,\ a_{\xi,j} \neq 0} \frac{-v(a_{\xi,j})}{j}.$$

Enlarge the ramification so that all these scales are integral, and set $\delta_\xi = r^{\rho_\xi}$. The polynomial

$$\varphi(\xi + \delta_\xi y) - c_\xi$$

has integral coefficients, no linear term, and at least one unit coefficient. Its reduction $F_\xi(y)$ has degree $d_\xi \geq 2$. Here and below we may replace r by a root of r and renormalize the valuation; every displayed strict inequality is unchanged. As $a_{\xi,D} = a_D$,

$$\rho_\xi \geq -v(a_D)/D. \tag{18}$$

The balls partition all critical roots. If another critical root ζ satisfies $v(\zeta - \xi) \geq \rho_\xi$, translate the integral scaled polynomial by $(\zeta - \xi)/\delta_\xi$. All its coefficients remain integral, and the coefficient

of degree d_ξ remains a unit: higher-degree terms have positive valuation. The linear term vanishes because ζ is critical. It follows from the definition of the maximum that $\rho_\zeta = \rho_\xi$. Thus their balls are identical. If two of the balls intersect, one contains the other; the same argument makes them identical. Otherwise they are disjoint.

The scaled derivative has reduction F'_ξ of degree $d_\xi - 1$. It therefore has exactly $d_\xi - 1$ integral roots, with multiplicity. Here is a direct check of that count: factor it in a Puiseux field. An integral root contributes a linear factor to the reduction; for a root of negative valuation, factor out that root and the remaining factor reduces to a constant. Multiplicativity of the coefficient valuation leaves a nonzero constant scalar. Hence the degree of the reduction counts exactly the integral roots. The ball B thus contains $d_B - 1$ critical roots. Since $\deg \varphi_x = D - 1$,

$$\sum_B (d_B - 1) = D - 1. \tag{19}$$

A functional for every dimension in a ball. The fixed polynomial F_B has twisted quotient

$$H_B = \mathbb{C}[y]/(\partial_y + F'_B)\mathbb{C}[y]$$

with basis $1, y, \dots, y^{d_B - 2}$. Let $\ell^0_{B,j}$ be its dual basis. Write

$$\varphi(\xi_B + \delta_B y) = c_B + F_B(y) + E_B(r, y), \qquad E_B \in r\mathbb{C}[[r]][y].$$

For a polynomial amplitude q define

$$\ell_{B,j}(q) = h\delta_B e^{c_B} \ell^0_{B,j}\left(e^{E_B(r,y)} q(\xi_B + \delta_B y)\right). \tag{20}$$

Each coefficient uses finitely many polynomial terms. The exponential in this formula is purely formal. The product rule conjugates the spatial twisted derivative to $\partial_y + F'_B$, so the functional annihilates the defining relations of M.

We spell out the parameter check because it retains the moving volume. At fixed y, put $w = \xi'_B + \delta'_B y$. Differentiating (20), then subtracting its value on $q_r + (\varphi_r + h'/h)q$, leaves, inside the fixed functional,

$$e^{E_B}\left(wq_x + \frac{\delta'_B}{\delta_B} q + \varphi_x wq\right) = \frac{1}{\delta_B}(\partial_y + F'_B)\left(e^{E_B} wq\right).$$

It is zero. Thus the functionals are horizontal, including the Jacobian δ_B and normalization h.

Completeness and the diagonal connection. Choose the polynomials $q_{B,j}$ of Lemma 8.3, for $0 \le j < d_B - 1$. Index rows by functionals (B, j) and columns by amplitudes (C, k), and form the normalized evaluation matrix

$$A_{(B,j),(C,k)} = \frac{\ell_{B,j}(q_{C,k})}{h\delta_B e^{c_B}}.$$

Its entries are integral, and its reduction is the identity: E_B has positive order and the amplitudes reduce to the chosen monomials on their own ball and zero on every other ball. Thus A is invertible over the ramified Laurent field. There are $D - 1$ rows by (19), equal to $\dim M$, so the amplitude classes are a basis and these functionals account for the entire module.

Remove from c_B its finite negative-power part $q_B(r)$. The scalar

$$g_B = h\delta_B \exp(c_B - q_B)$$

is an invertible Laurent series: $\exp(c_B - q_B)$ is a formal power-series unit, with a nonzero constant term. Consequently the matrix of the functionals $e^{-q_B}\ell_{B,j}$ in the amplitude basis is $\mathrm{diag}(g_B)A$, with g_B repeated for every row on ball B. It is invertible over the same Laurent field. The horizontality already proved says precisely that these rows give an isomorphism to the direct sum of rank-one connections $\partial_r + q'_B$, one for each pair (B, j). This supplies the diagonal gauge explicitly; it does not infer semisimplicity from a rank count. If $q_B = 0$, formula (20) takes the seed to a series of valuation at least $v(h) + \rho_B$, proving (17).

Every rank-one differential module is simple, so this constructed direct sum is semisimple. A quotient on the submodule generated by the seed therefore extends to the whole module by choosing a differential complement. A nonzero exponential factor has no horizontal map to the Laurent field: the logarithmic derivative of a nonzero Laurent series has at most a simple pole, whereas q'_B has a pole of order at least two when $q_B \neq 0$. On the zero part, every horizontal functional is a constant combination of those in the full frame. If the seed bound is strictly positive, their values cannot sum to one. □

The argument is a formal stationary-phase construction for polynomial phases over a Laurent field, including nonconvergent coefficients. For the classical local Fourier–Laplace theory of exponential connections, see Sabbah [11].

Corollary 8.5 (Leading-coefficient obstruction)**.** *For a rational polynomial phase $\varphi(t, x)$ of degree $D \geq 2$ and nonzero rational scalar normalization $h(t)$, let v_∞ mean order at $\varepsilon = 1/t = 0$. If*

$$v_\infty(h^D/a_D) > 0, \tag{21}$$

its seed submodule cannot have a rational horizontal quotient with seed value one. In particular any horizontal formal functional at another point cannot take that seed to one.

Proof. Apply the theorem at infinity. A rational horizontal quotient remains a quotient on extension to the formal Laurent field. A formal functional with seed value one restricts to such a rational quotient: its values on successive seed derivatives are $1, 0, 0, \ldots$. □

The inequality must be strict. For example $\varphi = -x^2/(2t^2)$ and $h = 1/t$ have $v_\infty(h^2/a_2) = 0$, and their normalized Gaussian period is a nonzero constant. Rescaling that functional gives seed value one, so the conclusion would fail if equality were allowed.

8.3 A polynomial phase with logarithmic volume

The puncture itself can be retained in the formal construction when the phase is polynomial. This gives a stronger bound than the one for dx: logarithmic volume does not change under a scaling of x.

Theorem 8.6. *Let $K = \mathbb{C}((r))$, $\varphi \in K[x]$, and $h \in K \setminus \{0\}$. On*

$$H = K[x, x^{-1}]/(x\partial_x + x\varphi_x)K[x, x^{-1}], \qquad \nabla[q] = [q_r + (\varphi_r + h'/h)q],$$

there is, after finite ramification, a full frame of rank-one formal exponential functionals. Every zero-exponential functional in this frame has seed value of valuation at least $v(h)$. Thus if $v(h) > 0$, the differential seed submodule has no horizontal quotient taking $[1]$ to one. This includes a constant phase.

Proof. Write $D = \deg \varphi$, taking $D = 0$ for a constant phase, including zero. If $D \geq 1$, the quotient has basis $1, x, \ldots, x^{D-1}$. Indeed, on a negative monomial the lowest term of $x\partial_x + x\varphi_x$ is its nonzero integer exponent times that monomial. This eliminates all negative powers without introducing lower ones. On polynomials, successive leading terms eliminate all degrees at least D. Conversely an exact polynomial cannot come from a Laurent polynomial with a negative lowest term, and a nonzero polynomial preimage of degree m has exact degree $m + D$. This proves independence. For $D = 0$, the quotient of $x\partial_x$ has the single basis element 1.

Use the critical balls constructed in Theorem 8.4. At most one contains the origin. Every other ball has $\rho_B > v(\xi_B)$, so $\gamma_B = \delta_B/\xi_B$ has positive valuation. With the same c_B, F_B, E_B and fixed functionals as before, the local logarithmic functionals are

$$q \longmapsto h e^{c_B} \frac{\delta_B}{\xi_B} \ell^0_{B,j}\left(\frac{e^{E_B} q(\xi_B + \delta_B y)}{1 + \gamma_B y} \right). \tag{22}$$

The geometric denominator and all negative powers in q expand coefficientwise because $v(\gamma_B) > 0$. Every coefficient is a polynomial in y. The normalized logarithmic Jacobian is $dx/x = \delta_B\, dy/(\xi_B + \delta_B y)$.

We retain the puncture in a separate central ball. If a critical ball contains zero, write it as $v(x) \geq \rho_0$ and put $\delta_0 = r^{\rho_0}$. Translation within that ball gives

$$\varphi(\delta_0 y) = c_0 + F_0(y) + E_0(r, y), \qquad c_0 = \varphi(0), \quad E_0 \in r\mathbb{C}[[r]][y], \quad F_0(0) = 0.$$

Here $d_0 = \deg F_0 \geq 2$, and the ball contains $d_0 - 1$ critical roots with multiplicity. If no critical ball contains zero, choose ρ_0 sufficiently large that the central ball is disjoint from all critical balls and every nonconstant coefficient of $\varphi(\delta_0 y)$ has positive valuation. Then $F_0 = 0$. Such a choice exists: there are finitely many roots and coefficients. This also handles $D \leq 1$ without any critical balls.

For $d_0 \geq 2$, the fixed punctured quotient

$$H_0 = \mathbb{C}[y, y^{-1}]/(y\partial_y + yF_0')\mathbb{C}[y, y^{-1}]$$

has basis $1, y, \ldots, y^{d_0-1}$, by the same reduction argument. For $F_0 = 0$, it has basis 1 and dual functional the constant term. For each element λ_j of the dual basis define

$$q \longmapsto h e^{c_0} \lambda_j\left(e^{E_0} q(\delta_0 y)\right). \tag{23}$$

There is no factor δ_0: $dx/x = dy/y$. Each coefficient is a Laurent polynomial, so the expression is well defined.

We check both identities, including parameter dependence. Spatial exactness in the central ball is conjugation by e^{E_0} of $y\partial_y + yF_0'$. The derivative discrepancy is δ_0'/δ_0 times the same exact expression applied to $e^{E_0} q$. On an outside ball put $x = \xi_B + \delta_B y$ and $w = \xi_B' + \delta_B' y$. After removing $h e^{c_B}$, the derivative discrepancy in (22) is

$$e^{E_B}\left[\left(\frac{\delta_B'}{x} - \frac{\delta_B w}{x^2}\right) q + \frac{\delta_B w}{x}(q_x + \varphi_x q)\right] = (\partial_y + F_B')\left(e^{E_B}\frac{w}{x} q\right).$$

It is annihilated by $\ell^0_{B,j}$. The identical product rule proves spatial exactness there. Thus every displayed functional is horizontal for the stated connection.

Completeness follows from polynomial separation of these disjoint balls, not merely their number. For the central ball use the monomials of the basis of H_0; for each outside ball use

$1, \dots, y^{d_B-2}$. Lemma 8.3 gives polynomial amplitudes with exactly these reductions on their own ball and zero on the others. Normalize the central columns by he^{c_0} and the outside columns by $he^{c_B}\delta_B/\xi_B$. Their matrix reduces to the identity. If the central ball contains critical roots, it contributes their multiplicity plus one; otherwise it contributes one. Together with all outside roots this gives D dimensions for $D \geq 1$, or one for $D = 0$, the full quotient rank.

As in Theorem 8.4, this frame is a direct sum of rank-one exponential factors after ramification. For a zero factor, (23) has seed valuation at least $v(h)$; (22) has valuation at least $v(h)+\rho_B-v(\xi_B) > v(h)$. Semisimplicity now extends a quotient of the differential seed submodule to the full module. Nonzero exponential factors cannot map horizontally to the Laurent field. A constant combination of the zero-factor seed values cannot be one if $v(h) > 0$. □

Remark 8.7 (Why two irregular ends require another argument). For $\varphi = r(x + x^{-1})$ and $h = 1$, put $\theta = r\partial_r$. Every logarithmic period satisfies

$$(\theta^2 - 4r^2)I = 0.$$

Indeed $y = x + x^{-1}$ and $z = x - x^{-1}$ obey $(x\partial_x + x\varphi_x)z = y + rz^2$ and $y^2 - z^2 = 4$. These identities give the equation by integration by parts. The quotient has rank two, and $[1], [y]$ are independent: its basis is $x^{-1}, 1$ and $[y] = 2[x^{-1}]$.

An explicit formal basis of scalar solutions is

$$I = \sum_{n\geq 0} \frac{r^{2n}}{(n!)^2}, \qquad J = I \log r - \sum_{n\geq 1} \frac{H_n r^{2n}}{(n!)^2}, \quad H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

The coefficient recurrence $n^2 a_n = a_{n-1}$ proves the first equation; $H_n - H_{n-1} = 1/n$ proves the second. Their θ-Wronskian is one: its derivative is zero by the equation and its constant term is one. Thus the logarithm is a full rank-two phenomenon, with nontrivial unipotent monodromy that no finite ramification removes. These are the familiar modified Bessel solutions; see [2, 10.25.1, 10.31.2]. This example does not refute the desired logarithmic bound: multiplication by $h = r$ makes both seed values tend to zero. It does refute an extension of the semisimple polynomial-frame theorem to all Laurent phases. The remaining proof must allow such extensions.

The residue dimension at the puncture is essential. Dropping it would give rank $D - 1$ instead of D and an incomplete frame. This theorem still permits nonconvergent formal coefficients, but is a valued-field statement. It does not cover a phase with both positive and negative powers of x.

Corollary 8.8. *Suppose $\varphi(t, x)$ is rational in t and polynomial either in x or in x^{-1}. For the logarithmic-volume connection normalized by $h = 1/t$, no horizontal formal functional at another point can have seed value one.*

Proof. For a polynomial in x, apply Theorem 8.6 at $r = 1/t = 0$, where $v(h) = 1$. The rational seed restriction is justified as in Corollary 8.5. For a polynomial in x^{-1} use $y = 1/x$; $dx/x = -dy/y$ only changes a constant factor. □

9 Classification with an affine-linear coordinate

We now prove Proposition 7.3. The coefficient of W determines a disjoint set of cases. Most are settled formally; the logarithmic case uses the comparison and decay argument of Section 10.

9.1 An exhaustive split of the affine-linear coordinate

Write $h = a + bZ + cT + dZ^2 + eZT + fT^2$ and $\Delta = e^2 - 4df$. The following split is exhaustive and its branches are disjoint. The last column gives the proposition proving each branch.

Coefficient condition	Reduction	Proof
$\Delta \neq 0$	Laurent phase on $\mathbb{C}^*$ with logarithmic volume	Proposition 10.10
$\Delta = 0$, $f \neq 0$	$h = a+bZ+cT+fT^2$ after a null rotation	Proposition 9.3
$f = e = 0$, $d \neq 0$, $c \neq 0$	$W(a + T + Z^2) + G$ after normalization	Proposition 9.4
$f = e = 0$, $d \neq 0$, $c = 0$	$Wh(Z) + G$	Proposition 9.1
$d = e = f = 0$, $c \neq 0$	$W(a + bZ + cT) + G$	Corollary 9.2
$d = e = f = c = 0$	$Wh(Z) + G$	Proposition 9.1

Here is the algebra behind the reductions. The null rotation

$$Z = Z', \quad T = T' + \lambda Z', \quad W = W' - \lambda T' - \lambda^2 Z'/2$$

preserves $2ZW + T^2$. If $f \neq 0$, choose $\lambda = -e/(2f)$. When $\Delta = 0$ the quadratic part of h becomes just fT'^2. If $f = e = 0$ and $c \neq 0$, choose $\lambda = -b/c$ to remove the linear Z term without changing dZ^2. If also $d \neq 0$, the Gaussian scaling $Z = \kappa Z'$, $W = W'/\kappa$, followed by a nonzero scaling of P, makes the T and Z^2 coefficients both one; take $\kappa^2 = c/d$. All additional terms enter a polynomial G of degree at most three. If $\Delta = 0$ and $f = 0$, then $e = 0$, which proves that the table omits no case.

9.2 A complete algebraic subcase

Suppose

$$P = Wh(Z) + v(Z)T^2 + u(Z)T + r(Z),$$

where the coefficient degrees make P cubic. Thus $\deg h \leq 2$, $\deg v \leq 1$, $\deg u \leq 2$, $\deg r \leq 3$. If $g = th(g)$ and $J = 1 - th'(g)$, formal circular contraction followed by the quadratic Gaussian identity gives

$$\Phi_1 = \frac{\exp\big(tr(g) + t^2 u(g)^2/(2K)\big)}{J\sqrt{K}}, \qquad K = 1 - 2tv(g). \tag{24}$$

The circular identity here is the one-variable formal Lagrange formula; it also follows directly from coefficient extraction. For completeness, for any numerator A, its formula is

$$\mathbb{E}[A(Z)e^{tWh(Z)}] = \frac{A(g)}{1 - th'(g)}.$$

Indeed the left side is $\sum_{m\geq 0} t^m [z^m] A(z)h(z)^m$; the same expression is the formal residue of $A(z)/(z - th(z))$, whose unique small pole is g. The identity remains valid for the t-dependent numerator in (24).

If $h(0) \neq 0$, change variables formally from t to $z = g(t)$ and put

$$A = h(z), \quad B = h(z) - 2zv(z), \quad C = h(z) - zh'(z).$$

The equation $\Phi_1 = 1$, after squaring, equates the exponential of a rational function with a rational function. The residue argument used in Section 3 forces

$$C^2 B = A^3, \qquad 2Br + zu^2 = 0. \tag{25}$$

These identities first force $\deg h \le 1$. To see this, if the distinct nonzero roots of h are α_i with multiplicities m_i, then

$$\frac{C}{h} = 1 - z \sum_i \frac{m_i}{z - \alpha_i}.$$

When $\deg h \ge 2$, the reduced numerator has the same positive degree as its denominator, has leading coefficient $1 - \deg h$, and is nonzero at each α_i. It therefore has a root away from the roots of h, contradicting $C^2 \mid h^3$.

If $h = a \neq 0$, (25) gives $v = 0$ and $r = -zu^2/(2a)$. The cubic degree bound forces $\deg u \le 1$, giving exactly (10). If $h = a + bz$ with $ab \neq 0$, it gives

$$v = \frac{h - h^3/a^2}{2z}.$$

This has a nonzero quadratic coefficient, contradicting $\deg v \le 1$. Thus this second branch cannot be a cubic.

If $h(0) = 0$, the fixed point is $g = 0$. Writing $b = h'(0)$, $v_0 = v(0)$, $u_0 = u(0)$ and $r_0 = r(0)$, purity in (24) forces

$$(1 - bt)^2(1 - 2v_0 t) = 1, \qquad tr_0 + \frac{t^2 u_0^2}{2(1 - 2v_0 t)} = 0.$$

Hence $b = v_0 = u_0 = r_0 = 0$. Every surviving monomial has strictly positive Z-minus-W weight, so Lemma 2.1 applies.

This proves the classification on the displayed subcase. The following arguments remove T^3, handle a linear T coefficient of W, and settle the rank-one and mixed quadratic coefficients. The nondegenerate logarithmic branch is excluded in Section 10.

9.3 A coefficient independent of the Gaussian height

Proposition 9.1 (The thirteen-coefficient case)**.** *Suppose a cubic has the form*

$$P = Wh(Z) + sT^3 + v(Z)T^2 + u(Z)T + r(Z),$$

where s is constant. If it is pure, then $s = 0$; it is one of the pure cases described in Section 9.2, and obeys $m > 3 \deg Q$.

Proof. If $h(0) \neq 0$, use $z = g$, $t = z/h(z)$ in the formal Lagrange identity. The remaining cubic Gaussian series has

$$A = \frac{zs}{h}, \quad B = -\frac{1}{2} + \frac{zv}{h}, \quad C = \frac{zu}{h}.$$

Purity would give $I = He^{-R}$, where $H = (h - zh')/h$ and $R = zr/h$. If $s \neq 0$, this has rational logarithmic derivative, contradicting Lemma 8.2. If $h(0) = 0$, the fixed point is zero. Use instead $A = ts$, $B = -1/2 + tv(0)$, $C = tu(0)$, $H = 1 - th'(0)$ and $R = tr(0)$. The same contradiction applies. Thus $s = 0$ in both cases, and the already proved algebraic subcase finishes the argument. □

Corollary 9.2 (The affine coefficient of W)**.** *If $P = W(a + bZ + cT) + G(Z, T)$ is a pure cubic, it satisfies $m > 3 \deg Q$. If $c \neq 0$, it is in fact quadratic.*

Proof. Put $q = 1 - bt$ and $g = at/q + (ct/q)T$. Circular contraction gives

$$\Phi_1 = q^{-1}\mathbb{E}_T[e^{tG(g,T)}].$$

The coefficient of T^3 in the exponent is $A(t) = tG_3(ct/q, 1)$. If it is nonzero, the cubic Gaussian obstruction applies, whereas purity makes the remaining Gaussian series equal to q, after removing a rational scalar exponential. For $c \neq 0$, the rational function ct/q is nonconstant, so $A = 0$ forces $G_3 = 0$. Proposition 3.1 applies. If $c = 0$, use Proposition 9.1. □

9.4 Closing the full rank-one WT^2 family

Proposition 9.3. *For $f \neq 0$ and $\deg G \leq 3$, the polynomial*

$$P = W(a + bZ + cT + fT^2) + G(Z, T)$$

is pure if and only if $b = 0$ and $G = 0$. Its pure members obey $m > \deg Q$.

Proof. Put $q = 1 - bt$ and $A(T) = a + cT + fT^2$. Exact circular contraction gives

$$J(p) = \frac{1}{q}\mathbb{E}_T[p(T)\exp\{tG(tA/q, T)\}], \qquad J(1) = \Phi_1.$$

Gaussian integration by parts says that J annihilates $\partial_T + \varphi_T$; direct parameter differentiation makes it horizontal for the polynomial phase and normalization

$$\varphi = tG(tA/q, T) - T^2/2, \qquad h = 1/q.$$

The phase has degree at least two, since it equals $-T^2/2$ at $t = 0$.

Suppose first $b = 0$ and write $G = \sum_{j\geq 0} Z^j G_j(T)$. If a term $A^j G_j$ attains the actual degree $D \geq 2$ of the phase, the leading coefficient is a polynomial in t containing a nonzero positive power. Distinct j give distinct powers t^{j+1}, so they cannot cancel identically. Its order at infinity is negative, whereas $v_\infty(h) = 0$. Criterion (21) excludes purity. Because $\deg A = 2$, every nonzero G_j with $j \geq 1$ attains some degree at least two. The only remaining possibility is $G = \alpha T + \beta$. Its exact unit series is $\exp(\beta t + \alpha^2 t^2/2)$, which is one only if $G = 0$.

Suppose now $b \neq 0$. Divide G by $S = bZ + A(T)$ as a polynomial in T:

$$G = R_0(Z) + TR_1(Z) + S(\alpha Z + \beta T + \gamma).$$

The remainder has basis $1, T, Z, ZT, Z^2, Z^2T, Z^3$. At $Z = -A/b$ these monomials have distinct leading degrees $0, 1, 2, 3, 4, 5, 6$. Thus if $V(T) = R(-A/b, T)$ has degree n, then $\deg R(sA, T) \leq n$ for every scalar s; a constant V forces R to be that same constant. Write $g = tA/q$. The phase is exactly

$$tR(g, T) + \alpha g^2 + \beta gT + \gamma g - T^2/2.$$

Since $g = -A/b + O(t^{-1})$, its first term is $tV + O(1)$, with all these remainders of degree at most n.

If $\alpha \neq 0$, the limiting remaining phase has degree four with leading coefficient $\alpha f^2/b^2$. If $\alpha = 0$, $\beta \neq 0$, it has degree three with leading coefficient $-\beta f/b$. Otherwise its quadratic coefficient tends to $-\gamma f/b - 1/2$. A leading term from tV has valuation -1; a higher-degree nonzero limiting term has valuation zero. The only flattened case has $\alpha = \beta = 0$, $\gamma = -b/(2f)$. Its exact quadratic coefficient outside tR is $-1/(2q)$, of valuation one. Thus in every case the actual leading coefficient

a_D has $v_\infty(a_D) \le 1$. This also covers V constant or linear. But $v_\infty(h) = 1$ and $D \ge 2$, so $Dv_\infty(h) - v_\infty(a_D) \ge D - 1 > 0$. Purity is impossible by the same criterion.

Conversely $WA(T)$ has strictly negative circular weight. For a monomial $Z^i W^j T^k$, a mixed moment can be nonzero only when the number of W factors balances the number of Z factors; in particular it vanishes for $m > \deg Q$. □

9.5 A mixed chart with a complete formal proof

One of the previously analytic exclusions admits a direct formal proof. It is the whole family in the next proposition, not a generic choice of its coefficients.

Proposition 9.4. *For every $a \in \mathbb{C}$ and every $G \in \mathbb{C}[Z,T]$ of total degree at most three, the polynomial*

$$P = W(a + T + Z^2) + G(Z,T)$$

has a nonzero positive pure Gaussian moment.

Proof. We construct the exact formal seed functional and apply Corollary 8.5. No analytic integral is needed.

The formal source. The circular fixed point is $g = t(a+T+g^2)$, with determinant $D = 1-2tg$. Make the formal Gaussian change

$$T = Y - t^2(a+Y)^2, \qquad z = t(a+Y).$$

It has $g = z$ and $dT/dY = D$. For any change $T(Y) = Y + O(t)$ with polynomial formal coefficients, the identity

$$\mathbb{E}_T[F(T)] = \mathbb{E}_Y\Big[F(T(Y))T'(Y)e^{(Y^2 - T(Y)^2)/2}\Big]$$

holds coefficientwise. To prove it, use the homotopy $T_u = Y + u(T-Y)$: its derivative is the Gaussian expectation of $(\partial_Y - Y)\{F(T_u)\dot{T}_u e^{(Y^2-T_u^2)/2}\}$, which is zero. Thus the determinant cancels and gives

$$\Phi_1 = \mathbb{E}_Y[e^{\Psi}], \qquad \Psi = tG(z, Y - z^2) + Yz^2 - z^4/2.$$

Define

$$F = z/t - a - z^2, \qquad \varphi(t,z) = tG(z,F) - F^2/2,$$

and let

$$M = K[z]/(\partial_z + \varphi_z)K[z], \qquad \nabla[q] = [q_t + (\varphi_t - 1/t)q], \qquad K = \mathbb{C}(t).$$

These are vector-space quotients. The formal map $J(q) = \mathbb{E}_Y[q(t(a+Y))e^{\Psi}]$ annihilates $(\partial_z + \varphi_z)K[z]$ by Gaussian integration by parts. Differentiating at fixed Y differs from applying ∇ by $J((\partial_z + \varphi_z)(zq/t)) = 0$. So J is horizontal and $J(1) = \Phi_1$. The term $-1/t$ is the normalization $dz/t = dY$; removing it changes the problem. Leading-degree reduction gives $\dim_K M = \deg_z \varphi - 1$.

The exhaustive degree split. Put

$$V(z) = G(z, -a - z^2), \quad L(z) = G_T(z, -a - z^2), \quad B_1 = z^2 G_{TT}(z, -a - z^2)/2 + z(a + z^2).$$

If $s = [T^3]G$, direct Taylor expansion gives

$$\varphi = tV + B_0 + B_1/t + B_2/t^2, \quad B_0 = zL - (a+z^2)^2/2, \quad B_2 = sz^3 - z^2/2. \tag{26}$$

To check every degree drop, write $v_1 = [ZT^2]G$, $v_0 = [T^2]G$ and $u_2 = [Z^2T]G$. Then

$$[z^6]V = -s, \qquad [z^5]V = v_1, \qquad [z^4]V = v_0 - u_2 - 3as.$$

If $n = \deg V \geq 4$, the degree of φ is n and each B_j has degree at most n. If $n \leq 3$, or V is constant, these three identities force $s = v_1 = 0$ and $u_2 = v_0$. Hence $B_0 = -z^4/2 + O(z^3)$, $\deg B_1 \leq 3$ and $B_2 = -z^2/2$. In that case $\deg_z \varphi = 4$. There is no omitted higher-degree term tending to zero at infinity.

If $n \geq 4$, the leading coefficient of φ has a nonzero tV_n term and therefore has order -1 at infinity. If $n \leq 3$ or V is constant, its leading coefficient is exactly $-1/2$ and has order zero. In both cases $h = 1/t$ has order one, so

$$v_\infty(h^D/a_D) > 0.$$

The leading-coefficient obstruction excludes $J(1) = 1$. The degree calculation was exhaustive, so this proves the proposition even at multiple critical points and degree drops. The full formal frame, including every escaping dimension, is supplied by Theorem 8.4 rather than a new case-by-case saddle calculation. □

Proof of Proposition 7.3. Use the disjoint case table in Section 9.1. Proposition 10.10 excludes its first row. Proposition 9.3 treats the second and leaves only a one-sign product. Proposition 9.4 excludes the third. Proposition 9.1 proves the fourth and sixth; its pure cases are one-sign phases or the shear family. Corollary 9.2 treats the fifth and leaves a quadratic. These rows exhaust all coefficients, including their zero faces.

Corollary 3.2 makes every pure ternary quadratic one-sided. The shear family is one-sided when $d = 0$ and is equivalent to P_* otherwise, by Lemma 6.1. This proves the stated classification.

The one-sign and ternary quadratic cases have cutoff $m > \deg Q$, and the shear has cutoff $m > 3 \deg Q$. The null rotations, Gaussian coordinate scalings and nonzero scalar multiples of P used in the reductions preserve the threshold and multiplier degree. Thus every pure member of the full affine-linear family satisfies $m > 3 \deg Q$. □

10 Analytic comparison and logarithmic exclusion

This section supplies the period interpretation and the nondegenerate logarithmic exclusion used above. Comparison identifies the actual period; the estimates retain its normalization, full rank and behavior at every end.

10.1 What purity gives in a differential module

The following lemma is the precise passage from the formal Gaussian functional to an analytic period. Its hypotheses must be verified after each change of variables or localization.

Lemma 10.1 (The seed functional)**.** *Let M be a finite-dimensional differential module over $K = \mathbb{C}(t)$, let $s \in M$, and suppose a K-linear horizontal map $J : M \longrightarrow \mathbb{C}((t))$ satisfies $J(s) = 1$. Then the differential submodule $U = K\langle\partial_t\rangle s$ has a horizontal quotient $L : U \longrightarrow K$ with $L(s) = 1$.*

At an ordinary base point, L extends locally to an analytic horizontal functional on M. For the period conclusion, assume explicitly that M is the generic exponential de Rham system of regular functions f, g on a smooth affine complex variety, where f is the base projection and f, g are algebraically independent. Remove the finite exceptional base set in the Hien–Roucairol comparison

theorem. Then the extension is represented by a flat rapid-decay cycle, and its period evaluated on s is identically one.

Proof. On the generators $s, \nabla s, \nabla^2 s, \dots$, the values of J are $1, 0, 0, \dots$. Therefore its restriction takes values in K. Every rational relation between these generators is respected because J was defined on M, so this restriction is well defined and horizontal. Its value at s makes it surjective.

Choose a point where M, U and L are ordinary, extend the fiber functional from U to M, and solve the dual analytic ODE. Uniqueness preserves its restriction to U. Perfectness of the period pairing is Hien–Roucairol [6, Theorem 2.7]; their Theorem 3.5 represents holomorphic solutions by rapid-decay periods. Together these statements give the asserted cycle. A scalar rational normalization is an invertible gauge away from its zeros and poles and must be retained in the period. □

There are three distinct verifications here. First, integration by parts must make J descend to the exact displayed de Rham quotient. Second, every inverted spatial coordinate must be a valid formal Laurent unit on the selected branch. Third, the comparison theorem requires a smooth affine total space and regular functions after removing the finite set of poles in the parameter. The theorem gives a representation, not a bound at infinity. In particular, it gives no shrinking estimate on its own.

There is also a useful purely formal way to obtain a contradiction.

Lemma 10.2 (A full formal frame and a positive seed)**.** *Assume the hypotheses of the first paragraph of Lemma 10.1. Suppose that, after a finite ramification $r = t^{-1/N}$ at infinity, M has a full formal frame of rank-one exponential factors. For every factor with zero exponential part, assume its horizontal functional takes s into $r\mathbb{C}[[r]]$. Then $J(s) = 1$ is impossible.*

Proof. Enlarge the ramification to include the rational powers in the frame. The module over $\mathbb{C}((r))$ is then a direct sum of rank-one modules, so the quotient on U extends to a horizontal map from the whole module to $\mathbb{C}((r))$. A nonzero exponential factor has no such map: the logarithmic derivative of a Laurent series has at most a simple logarithmic pole and cannot equal the derivative of a nonzero negative-power exponential part. On the zero part, all horizontal functionals are constant linear combinations of the supplied frame. Their values on s have strictly positive r-order and cannot sum to one. □

The word *full* is essential. Counting some critical points or writing a few scalar asymptotic expansions does not supply this lemma's hypothesis. The finite and escaping pieces must together give an invertible period matrix of the full de Rham rank. Also, a fractional power becomes an integer after ramification; the proof uses its strictly positive order, not an unsupported monodromy inference.

10.2 A curve-length estimate

Lemma 10.3. *A real algebraic curve of degree at most D has length at most $\pi D\rho$ in a disk of radius ρ. Consequently its logarithmic length in $a \le |x| \le b$ is at most a constant times $D(1 + \log(b/a))$.*

Proof. Average the number of intersections with lines over directions and apply the one-dimensional coarea formula. Outside exceptional line components the number is at most D; the components themselves satisfy the same estimate. In each dyadic annulus $\rho \le |x| \le 2\rho$, divide the disk estimate by ρ to bound $\int |dx|/|x|$ by $2\pi D$. Sum the dyadic annuli. □

10.3 The logarithmic branch, including its removed component

For $\Delta \neq 0$, $f \neq 0$, choose $\eta^2 = \Delta$ and set

$$\rho = \frac{e+\eta}{2f}, \quad \sigma = \frac{e-\eta}{2f}, \quad u = \frac{\rho c - b + 1/t}{\eta}, \quad v = \frac{b - 1/t - \sigma c}{\eta}, \quad k = uv - a/f.$$

On the circular constraint $z = th(z,T)$, put $x = T + \rho z + u$, so that $T + \sigma z + v = k/x$. Then

$$z = \frac{f}{\eta}(x - k/x - u + v), \quad T = x - u - \rho z, \qquad \frac{dT}{1 - th_Z} = \frac{dx}{t\eta x}.$$

The coefficient of t^{-2} in k is $-1/\eta^2$, so the conic is generically smooth. The exponent $tG(z,T) - T^2/2$ is Laurent polynomial in x. On the formal Gaussian branch, $x = 1/(\eta t) + O(1)$, which makes every Laurent power legitimate.

If $f = 0$, then $e \neq 0$. Put $z_0 = -c/e$ and $x = z - z_0$. With $A(z) = a + bz + dz^2$, the constraint gives

$$T = \frac{z/t - A(z)}{ex}, \qquad \frac{dT}{1 - th_Z} = \frac{dx}{tex}.$$

This localization is valid if $c \neq 0$, or if $c = 0$ and $a \neq 0$. If $a = c = 0$, the chosen fixed point is instead $g = 0$ identically; inverting g would discard the very branch being tested. In that case the exact unit series is

$$\mathbb{E}_T\left[\frac{e^{tG(0,T)}}{1 - t(b + eT)}\right].$$

Use $x = T - (1/t - b)/e$, now a valid Laurent unit. Its volume is $-dx/(tex)$ and its exponent is polynomial in x. These formulas cover both the smooth conic and the removed-component branch. We next connect these coordinates to the formal Gaussian functional. This will close every one-ended phase without an analytic estimate.

Lemma 10.4 (Formal logarithmic transport)**.** *In each of the three charts above, write $X = X(t,T)$ for the formal coordinate, D for the displayed circular denominator, and B for $tG(g,T)$ or $tG(0,T)$ on the removed component. Put $\kappa = \eta, e, -e$, respectively. Then*

$$J(q) = \mathbb{E}_T\left[\frac{q(X)e^B}{D}\right]$$

is horizontal on the logarithmic twisted quotient with phase φ and normalization $h = 1/t$, and $J(1) = \Phi_1$.

Proof. All Laurent powers of X are legitimate by the unit checks above. The coordinate identities give

$$X_T = t\kappa X/D, \qquad \varphi(t,X) = B - T^2/2.$$

Consequently Gaussian integration by parts gives $J((x\partial_x + x\varphi_x)q) = 0$: the corresponding integrand, multiplied by $t\kappa$, is $(\partial_T - T)\{q(X)e^B\}$.

For parameter horizontality set $w = 1/D$ and $R = X_t/X_T$. Differentiating $w = X_T/(t\kappa X)$ gives the exact identity

$$w_t + w/t = \partial_T(wR).$$

Subtracting $J(q_t + (\varphi_t - 1/t)q)$ from $\partial_t J(q)$ now gives the Gaussian expectation of $(\partial_T - T)\{wRq(X)e^B\}$, hence zero. These are formal coefficientwise identities, without an integral-convergence hypothesis. The seed equality is the circular contraction formula. □

Proposition 10.5. *Every chart above whose actual Laurent phase has at most one irregular end is not pure. The removed-component branch $a = c = f = 0$, $e \neq 0$, is therefore excluded for every polynomial G, without a degree bound.*

Proof. At most one irregular end means that φ is polynomial in x or in x^{-1}, including the constant case. Purity would give the horizontal functional in Lemma 10.4 seed value one, contrary to Corollary 8.8. The removed-component phase is always polynomial in x, irrespective of $\deg G$. □

10.4 Resolving a Laurent phase

The logarithmic estimate needs two ingredients: a complete description of the regular part, and a bound for the distinguished differential $d\log x$. The first is a nearby-cycle calculation. The second is a calculation on annuli. Keeping these ingredients separate prevents a perfect period pairing from being mistaken for a growth estimate.

Throughout this subsection the coefficients are rational functions of the parameter. In particular their Puiseux expansions converge. The nonconvergent formal statements proved earlier have a different scope.

The two-ended twisted quotient is finite dimensional without a genericity assumption. If the actual Laurent orders are $p, q \geq 1$, the classes $x^{-q}, \dots, x^{p-1}$ are a basis. The top term of $x\varphi_x$ eliminates every exponent at least p, without crossing below $-q$; the bottom term eliminates every exponent below $-q$, without crossing above $p-1$. An exact expression supported in this interval would require its preimage to have highest exponent at most -1 and lowest exponent at least zero. This proves independence and rank $p+q$. Coefficient specializations use their actual orders.

Lemma 10.6 (A marked model)**.** *Let $\varphi(r,x) \in \mathbb{C}(r)[x, x^{-1}]$ be nonconstant in x. After a finite ramification of r there is a smooth surface proper over a parameter disc, unchanged over its punctured disc, with these properties:*

1. *Its generic fibre is $\mathbb{P}^1$. Its central fibre is a reduced normal-crossing curve, with local equations $r = u$ or $r = uv$.*
2. *The phase extends to a morphism to $\mathbb{P}^1$. Its zero and pole divisor and the divisor of x have normal crossings with the central fibre.*
3. *The horizontal sections $x = 0, \infty$ are disjoint and meet smooth points of the central fibre. Locally $x = u^a v^b U$, where a, b are integers and U is holomorphic and nonvanishing.*

Additional markings used in the construction are not additional punctures of the exponential connection.

Proof. Mark $0, \infty$ and every point of the generic fibres $\varphi^{-1}(0), \varphi^{-1}(1), \varphi^{-1}(\infty)$. Retain multiplicities in the divisors, but identify identical sections. Convergent Newton–Puiseux makes the finitely many marked branches meromorphic sections after one finite ramification. Start with $\mathbb{P}^1$ over that disc and blow up collisions of distinct sections. This terminates: the contact order of each pair is a finite integer and decreases on blowing up their common point.

Every centre is a smooth point of a single central component. Indeed a section cannot pass through a node $uv = r$: two positive integral orders cannot have sum one. Thus each new exceptional component has multiplicity one and all nodes still have the equation $uv = r$. The final marked sections are disjoint and miss the nodes.

The divisor of the meromorphic function φ is supported on the central components and its marked zero and pole sections. If a point were an indeterminacy, its local divisor would have a

zero branch and a pole branch meeting there. After cancelling common factors its expression is a unit times u^a/v^b, $a, b > 0$. The equation $\varphi = 1$ then has a horizontal branch through that point. At a node this contradicts the section property; at a smooth point it contradicts the separation from the zero or pole section. These are all possible intersections. Hence φ is a morphism. The asserted normal crossings and the expression for x follow from its divisor: horizontally x has just the two marked sections. Only those original sections are removed from the generic fibre; marking a zero or a point with phase value one does not remove it. □

The marking of phase value one is necessary. For $\varphi = r/x^2$, the zero and pole sections are already disjoint, but the map is undefined at $r = x = 0$. Its value-one branches $x = \pm\sqrt{r}$ force the cover $r = s^2$ and the chart $x = sy$, where the phase is $1/y^2$. The marked construction therefore detects a transition that merely separating zeros and poles would miss.

We use the following precise comparison facts. For a meromorphic connection on a punctured disc, its moderate nearby cycles describe its formal regular summand, with monodromy. Nearby cycles commute with proper direct image. For a rank-one exponential on a surface with normal crossings, a component on which the phase has a vertical pole contributes no moderate nearby cycles. At a node between a finite-phase component and a pole component, the surviving object is the exponential connection on the finite component, localized at the node; it is not an extra point class. At a node between two finite components, the local complex is the usual nearby annulus complex, including its monodromy. These are standard comparison and normal-crossing facts, not new theorems here; see Roucairol [10, Section 1.2 and Lemma 1.4] and Sabbah [11, Section 1.b, especially Proposition 1.5].

For clarity, their local exponential models are a holomorphic unit (finite phase), $\exp(u^{-p})$ at a finite–pole boundary, and $\exp(u^{-p}v^{-q})$, $p, q > 0$, when both components are poles. An analytic unit in a pole monomial is absorbed in a coordinate. This may multiply the equation of the base by a nonvanishing unit; locally that change does not change the vanishing assertion or the nearby-cycle object up to the usual unit identification. In the f^s description the identification is multiplication by $\exp(s \log U)$, with a local analytic choice of $\log U$. One keeps the actual gluing maps and monodromy, rather than just adding the ranks of the components.

Lemma 10.7 (Lifting the regular part)**.** *For the marked model, the complete nearby-cycle space of the exponential direct image is represented, on a sufficiently small parameter sector, by flat rapid-decay chains made from the following pieces:*

1. *bounded deformations of finitely many paths and cycles in the finite-phase central components, away from their nodes and poles;*
2. *a bounded number of bridges and circles in the necks $uv = r$ between finite-phase components;*
3. *descending phase tails at irregular ends and circles about regular punctures.*

The coefficients of each chosen flat chain are fixed constants. Their nearby-cycle classes exhaust the formal regular part, with its actual gluing maps and monodromy.

Proof. Apply proper nearby-cycle comparison to the compactified surface, with the exponential connection localized at $r = 0$ and at the two original punctures. The normal-crossing rules just stated kill the interiors of the pole components. Along a finite component they leave its ordinary exponential connection, with localization at its pole ends and at any original regular puncture. A finite–pole node gives that localized end. A finite–finite node retains the nearby annulus. Thus

no pole component is silently counted as an additional regular class, and no annulus or puncture residue is discarded.

Here is a chain description of this calculation. Remove small discs about the finitely many nodes and punctures in each finite component. Triangulate the remaining compact surfaces, keeping the boundary circles. At an irregular boundary attach the finitely many descending sectors. At a regular puncture keep the circle. The one-dimensional exponential period comparison identifies the resulting relative chain complexes with the component de Rham complexes; for a constant phase this is ordinary homology. At each finite–finite node glue by the actual annulus chain complex, with both its degree-zero and degree-one terms. Here is the total complex and the degree in which it is used. Let U be the disjoint union of these component pieces with their descending ends, V the disjoint union of the finite–finite necks, and W their boundary-collar overlaps. Write $C_\bullet$ for chains with the stated rapid boundary condition at the ends, and with the exponential local coefficients. Set

$$C_\bullet^{\mathrm{gl}} = \mathrm{Cone}\Big(C_\bullet(W) \xrightarrow{(i_*,-j_*)} C_\bullet(U) \oplus C_\bullet(V) \Big).$$

Thus a class includes its matching boundary chains; taking the separate component homology groups would lose these data. The neck complex retains both C_0 and C_1. Let $\mathcal{K}_\psi$ denote the moderate nearby-cycle de Rham complex on the central curve of the marked model, before proper pushforward. We use the unshifted curve de Rham normalization (functions in degree zero, one-forms in degree one), and take its hypercohomology over that central curve. The integration comparison and proper specialization give

$$H_1(C_\bullet^{\mathrm{gl}}) \xrightarrow[\text{integration}]{\simeq} \mathbb{H}^1(\mathcal{K}_\psi)^\vee \xrightarrow[\text{proper comparison}]{\simeq} \mathrm{Sol}(M_0).$$

Here M_0 is the formal zero-exponential summand, and Sol denotes its horizontal dual on the chosen parameter sector, with nearby-cycle monodromy retained. The first arrow is obtained on U, V, W from the curve period pairing and then on their mapping cones. The second is the moderate-nearby-cycle/proper-image comparison just cited, in fibre cohomological degree one. This also specifies the specialization map used here: lift the glued chains, integrate, and take their zero-exponential asymptotic classes. On a non-Stokes sector the latter forgets possible exponentially decaying terms in a lift. Naturality of integration under the two collar inclusions makes these maps agree. In particular this comparison is of complexes with their gluing maps and monodromy; equality of Euler characteristics or a count of local saddles is not being used to infer surjectivity.

Choose a finite basis of this finite-dimensional homology. Smooth pieces and their boundary points lift by the implicit function theorem; an annulus is lifted by $uv = r$. Descending ends are continued by the inverse phase coordinate. On a fixed small sector these choices are isotopies, so the homology coefficients are constants. No Morse condition on the phase is used. Degenerate critical points stay inside the triangulated pieces, and a constant phase on a component is allowed.

For the tail continuation, choose the ray of r so that no unbounded critical value lies on the negative real phase axis for small r. There are finitely many excluded directions: critical values are convergent Puiseux functions. Bounded critical values fit in a fixed disc. From a sufficiently negative phase value every chosen inverse branch therefore continues to $-\infty$ without meeting a critical value. In the two-ended Laurent case it ends at $x = 0$ or ∞. The same choice can avoid the finitely many limiting Stokes directions of the parameter connection.

This constructs flat lifts whose nearby-cycle classes exhaust the space computed by proper comparison. No quantitative estimate has been used in that completeness calculation. □

Lemma 10.8 (The logarithmic bound on the lifted pieces)**.** *For the chains of Lemma 10.7, the period of $e^{\varphi}d\log x$ is $O(1+|\log r|)$ along the chosen ray. The bound is uniform for a fixed finite collection of generators. With a fixed Laurent amplitude the same construction has at most an additional algebraic power of $|r|^{-1}$.*

Proof. On a compact finite-phase piece the phase and the coefficients of $d\log x$ are bounded, so its integral is bounded. In a finite–finite neck write

$$r = uv, \qquad x = u^a v^b U(u,v).$$

At fixed r,

$$d\log x = \left(a - b + \frac{uU_u - vU_v}{U}\right)\frac{du}{u}.$$

The coefficient is uniformly bounded in a small bidisc. A radial bridge across $|r|/\eta \le |u| \le \eta$, with bounded angular detours, has logarithmic length $O(1+|\log r|)$. Circles have bounded logarithmic length. The phase is holomorphic on this bidisc and is uniformly bounded there. This treats every finite–finite neck.

Near an irregular end of a finite component, use a local root of the phase unit to write $\varphi = w^{-p}$, $p \ge 1$. At a finite–pole node one can simultaneously write $r = wv$ after changing the other coordinate by a unit. On $|v|, |w| < \eta$, the same calculation bounds $d\log x$ by a constant times $|dw|/|w|$. Along a descending phase ray, $s = -\varphi > 0$, its integral is bounded by a constant times

$$\int_{s_0}^{\infty} e^{-s}\frac{ds}{s}.$$

If the node chart ends, truncate it first at $|w| = |r|^{\alpha}$, where $0 < \alpha < 1$. Its phase is already of size $-|r|^{-\alpha p}$. Continue that inverse branch along the negative phase axis as in Lemma 10.7.

We give a uniform bound for that continuation, rather than declaring it negligible from a local saddle formula. Choose M large enough to include all coefficient transition scales in $|r|^{M} \le |x| \le |r|^{-M}$. Such M exists because the coefficients are rational and there are finitely many Laurent powers. In this annulus each descending path lies on $\operatorname{Im}\varphi = 0$, a real algebraic curve of bounded degree after clearing its Laurent denominator. The curve-length bound used in Lemma 10.3 gives $\int |dx|/|x| = O(1+|\log r|)$, by dyadic annuli. Along the entire tail the exponential is at most $\exp(-|r|^{-\alpha p})$. Outside that annulus the leading Laurent monomial dominates uniformly, so $|d\log x/d\varphi| \le C/|\varphi|$ and its descending integral is bounded by the same exponential tail integral. This proves the required uniform tail bound. A fixed Laurent amplitude contributes only a power of $|r|^{-1}$ on the annulus and algebraic growth in s outside; the exponential still dominates it.

At a regular puncture use a small fixed circle in its smooth marked chart. Both its phase and logarithmic length are bounded. There are finitely many pieces and their chain coefficients are fixed. Adding their bounds proves the lemma. □

Theorem 10.9 (Logarithmic seed bound)**.** *Let $\varphi(t,x) \in \mathbb{C}(t)[x,x^{-1}]$. On a generic ray at $t=\infty$, the zero-exponential part of every scalar flat period of $e^{\varphi(t,x)}dx/x$ is $O(1+\log|t|)$. This includes all critical degeneracies, coincident actions and regular punctures. Consequently, for a rational normalization $h(t)$ with $v_\infty(h) > 0$, its differential seed submodule admits no rational horizontal quotient taking the seed to one. The same exclusion holds for a horizontal formal Gaussian functional at another point.*

Proof. If the phase is independent of x, the sole residue period is a constant times $e^{\varphi(t)}$. Its zero-exponential part is bounded when φ is regular at infinity, and absent otherwise. For a nonconstant phase put $r = 1/t$ and use Lemmas 10.6–10.8. The construction includes regular puncture circles when there is just one irregular end. Finite ramification changes $\log|r|$ only by a constant factor. The lifted chains have at most algebraic growth on every Laurent amplitude by Lemma 10.8. On the chosen generic ray, a full period vector with such growth has no exponentially growing factor: after formal exponential decomposition a largest growing factor could not cancel on every basis amplitude. The other nonzero factors are exponentially decaying there. The nearby-cycle basis therefore gives a full basis of regular functionals modulo those decaying terms, with constant coefficients in the flat basis. Every one of their seed periods has the asserted logarithmic bound. Thus their zero-exponential seed expansions have no negative algebraic powers. Multiplication by h makes all of them tend to zero.

For a constant phase the rank-one period already gives the same contradiction directly. For a nonconstant phase, a quotient with normalized seed one would extend locally to a horizontal functional on the full module, by the seed restriction and analytic extension argument of Lemma 10.1. Its comparison hypotheses hold on $(\mathbb{A}^1 \setminus S) \times \mathbb{G}_m$: remove the finitely many parameter poles, zeros of h, and comparison exceptions. The base projection and nonconstant phase are regular and algebraically independent there. Its scalar seed is constant. Distinct formal exponential factors separate, so its zero-exponential seed is also one. This contradicts the bound after normalization. A formal Gaussian functional with unit seed first restricts to the same rational seed quotient, as in the first part of Lemma 10.1. □

The comparison theorems supply completeness here. The bounds come from the logarithmic differential on the marked model and the explicit descending-tail estimate. Neither part follows from a finite symbolic calculation. This proof uses convergence of the rational parameter family; it makes no assertion for arbitrary nonconvergent Laurent families or arbitrary coefficient rings.

Proposition 10.10. *If $e^2 - 4df \neq 0$, no polynomial G, of any degree, makes $P = Wh(Z,T) + G(Z,T)$ pure.*

Proof. The smooth-conic, linear-conic and removed-component charts are exhaustive. Their Laurent-unit checks and Lemma 10.4 give the formal Gaussian functional with seed value Φ_1 and normalization $1/t$. Theorem 10.9 excludes $\Phi_1 = 1$ in every chart, using its actual Laurent orders. No image-kernel calculation is used. □

10.5 Why normalization and full rank matter

The following exact examples prevent several tempting shortcuts. For positive real t,

$$\frac{1}{t}\int_{\mathbb{R}} e^{-x^2/(2t^k)}\,dx = \sqrt{2\pi}\,t^{k/2-1}.$$

The unique critical point is bounded and has zero action for every k, yet the normalized period decays for $k = 1$, is constant for $k = 2$, and grows for $k = 3$. A critical-point count or a zero-action label alone gives no decay estimate.

For the fixed phase tx^k, $k > 2$, the critical point is degenerate and the decay scale is $t^{-1/k}$, not a Gaussian Hessian scale. The fixed quotient in Theorem 8.4 retains its full multiplicity $k - 1$. For $tV_n x^n - x^4/2$, $1 \leq n \leq 3$, the $n - 1$ bounded and $4 - n$ escaping critical dimensions total three. Counting only bounded points loses actual rank; the stationary-ball construction retains all

roots and tests its full matrix using polynomial separation. Finally, $t^{-3/2} = r^3$ under $r = t^{-1/2}$: a claim about a noninteger exponent before ramification must not be reused after ramification. Lemma 10.2 uses positive seed order and an actual full frame instead.

11 Extracting a Gaussian affine-linear frame

We prove Proposition 7.2 by considering the leading projective cubic. Each case either provides the Gaussian direction in (13) or excludes purity. The final table records the entire case split, including lower-degree terms and singular critical sets.

11.1 One exclusion criterion for several geometric cases

The smooth case does not need a list of critical-point formulas. The same argument also treats many singular leading forms. Its hypotheses must hold in every complex direction. A bound only on the original real integration space would not suffice.

We state the criterion for a polynomial exponential amplitude, then specialize it to the Gaussian. Norms are Hermitian norms on complex affine space; the Gaussian quadratic remains holomorphic.

Theorem 11.1 (A gradient criterion)**.** *Let $P, q \in \mathbb{C}[x_1, \dots, x_n]$, with P nonconstant and $\deg q \le J$, where $J \ge 1$. Suppose that outside a compact set*

$$\|\nabla P(x)\| \ge c\|x\|^{J-1}, \qquad \frac{\|\operatorname{Hess} P(x)\|}{\|\nabla P(x)\|^2} \le C \tag{27}$$

for some $c > 0$ and $C < \infty$. For every fixed polynomial A, no nonzero flat period of

$$A(x)e^{tP(x)-q(x)}\,dx_1 \cdots dx_n$$

is a finite exponential-polynomial sum $\sum_\lambda e^{\lambda t} p_\lambda(t)$ with polynomial p_λ. In particular it cannot be a nonzero constant.

Proof. We first justify the full cycle space and the tails. The gradient is nonzero outside a compact set, so the affine critical set is compact and algebraic, hence finite. The gradient bound is stronger than the condition excluding vanishing cycles at infinity. The finite-critical thimble decomposition therefore applies, with the whole local Milnor homology at each isolated critical point, not just its Morse approximation [9, Part I, Sections 1–4]. When several critical points have the same value, take disjoint Milnor balls and apply excision to their complement. This gives the direct sum of their full local homology before any values are grouped.

Here the Gaussian or polynomial amplitude can be kept on these fixed P-cycles. Choose a generic argument θ of t. Outside the local critical neighbourhoods, lift a descending phase ray by

$$P(x) = \lambda - e^{-i\theta}s, \qquad \frac{dx}{ds} = -e^{-i\theta}\frac{\overline{\nabla P(x)}}{\|\nabla P(x)\|^2}.$$

For large s the lift is outside the compact set, because its phase value is unbounded. The first condition in (27) gives $\|x\|^J \le C_1(1+s)$. The real differential of this vector field has norm at most a constant times the second ratio in (27), hence is bounded. Transported $(n-1)$-cycle volumes grow at most as $\exp(C_2 s)$; the radial speed is bounded as well. Also $|A(x)e^{-q(x)}| \le C_3(1+s)^M \exp(C_4 s)$ for some M. Thus $\exp(-|t|s)$ dominates the complete tail, uniformly for large $|t|$ in a smaller sector. The same estimate applies to every fixed polynomial amplitude.

We must also check that these are a full basis for the actual phase, rather than selected admissible cycles. Put

$$F_\sigma = P - \sigma q/t, \qquad 0 \le \sigma \le 1.$$

Since $\|\nabla q\| \le C_5 \|\nabla P\|$ outside the same compact set, all large $|t|$ give

$$\|\nabla F_\sigma\| \ge \tfrac{1}{2}\|\nabla P\|.$$

The ratio involving the Hessian stays uniformly bounded too. Indeed $\|\operatorname{Hess} q\| = O(\|x\|^{J-2})$ for $J \ge 2$, while that Hessian is zero for $J = 1$. All critical points and their values therefore remain in fixed compact sets. On the preimage of a phase half-plane beyond those values, the parameter field

$$W_\sigma = \frac{q}{t}\frac{\overline{\nabla F_\sigma}}{\|\nabla F_\sigma\|^2}$$

satisfies $\partial_\sigma F_\sigma + dF_\sigma(W_\sigma) = 0$ and $\|W_\sigma\| \le C_6(1 + \|x\|)/|t|$. Its flow is complete for $0 \le \sigma \le 1$. Extend it by a cutoff away from that half-plane. It gives an isotopy of the ambient relative pairs defining the full rapid-decay homology. This explicitly prevents an infinity class from appearing during the deformation.

The fixed P-thimbles have uniformly descending tails for every F_σ, by the preceding estimates. Truncate them beyond a common large phase value. Their boundaries stay in the descending half-plane throughout the isotopy, so they define its flat relative classes. They are a full basis at $\sigma = 0$ and remain one at $\sigma = 1$. Finite critical points may merge, and their critical values may coincide; the argument uses the full relative pairs and does not require a Morse family. The same cycles are flat as t varies in the chosen sector.

These improper chains and their homotopies are admissible for the period pairing. One can integrate first on truncated chains and then pass to the limit: the tail bound makes every boundary term tend to zero. Indeed the differential of W_σ grows at most as $C(1 + \|x\|^J)/|t|$; along the controlled tails this is $C'(1 + s)/|t|$. The homotopy volumes have at most this exponential growth, still dominated by the descending factor for all large $|t|$. Thus Stokes and differentiation under the integral hold throughout; the relative-pair calculation does not depend on a convergence claim for the original real Gaussian integral.

It remains to determine the local asymptotics at a finite critical point with value λ. For an analytic amplitude germ, the vanishing-cycle density is a finite sum of terms $s^{\alpha+k}(\log s)^j$, with analytic coefficient series and $\alpha > -1$. Its Laplace asymptotics consequently have the form

$$e^{\lambda t} \sum_{\alpha,k,j} c_{\alpha,k,j}\, t^{-\alpha-k-1}(\log t)^j, \qquad k \ge 0.$$

This is the local analytic result of Pham [9, Part II, Section 1.1, (1.5)–(1.6) and the footnote on printed page 26]; it permits degenerate isolated critical points. Pham credits Maisonobe and Rombaldi for the local solution comparison used there. The amplitude Ae^{-q} is analytic. Every nonzero term has a strictly negative power of t and tends to zero after removing $e^{\lambda t}$. The tails outside the local neighbourhood are exponentially negligible by the uniform bound already proved.

The finite-critical decomposition accounts for the full module. Group equal critical values before separating the distinct formal exponential factors. Within each such factor the displayed local expansions contain only decaying powers and logarithms. A nonzero polynomial prefactor cannot occur. The zero function is allowed when all these contributions vanish. This proves the theorem. □

Corollary 11.2 (Original Gaussian markers)**.** *For the standard Gaussian quadratic $q = \sum x_i^2/2$, suppose (27) holds with $J = 2$. Then every original marked series Φ_A is either zero or is not a finite exponential-polynomial sum. In particular P is neither pure nor safe.*

Proof. The exact top de Rham quotient and connection are

$$M = \mathbb{C}(t)[x] \Big/ \sum_i (\partial_i + tP_i - x_i)\mathbb{C}(t)[x], \qquad \nabla[A] = [A_t + PA].$$

Gaussian integration by parts makes $[A] \mapsto \Phi_A$ horizontal on this vector-space quotient. The spatial and parameter derivatives commute; no ordinary Jacobian-ideal quotient is substituted. If some Φ_A is a nonzero finite exponential-polynomial sum, a constant-coefficient differential operator isolates one exponential and kills its polynomial prefactor, giving $ce^{\beta t}$, $c \neq 0$. Because

$$\partial_t \Phi_A = \Phi_{PA},$$

this is still an original fixed polynomial marker. Replace P by $P - \beta$ and divide the marker by c. Its formal seed value is one. Lemma 10.1 gives an actual constant period for that marker, contradicting Theorem 11.1. Its comparison hypotheses hold with base t and phase $tP - q$ on $\mathbb{A}^1 \times \mathbb{A}^n$, after removing the finite exceptional base set. No convergence of the original Gaussian series is assumed. Finally $\Phi_1(0) = 1$, so it is a nonzero series; purity or safety would make it a polynomial. □

Corollary 11.3 (Smooth leading form, in every degree)**.** *If $\deg P = D \geq 2$ and the homogeneous leading form satisfies $\nabla F_D(v) = 0$ only for $v = 0$, then P is not pure. In fact Corollary 11.2 holds for every original marker.*

Proof. Compactness of the complex unit sphere gives $\|\nabla P(x)\| \geq c\|x\|^{D-1}$ for large $\|x\|$. Also $\|\operatorname{Hess} P\| = O(\|x\|^{D-2})$, so its ratio to the squared gradient is $O(\|x\|^{-D})$. Both conditions for $J = 2$ follow, with all lower terms retained. □

Lemma 11.4 (The ordinary-node test)**.** *Let $P = F_3 + F_2 + F_1 + F_0$ in $n \geq 2$ variables. Suppose its projective singular set is finite and every singular point $[v]$ has $\ker \operatorname{Hess} F_3(v) = \mathbb{C}v$. If $F_2(v) \neq 0$ at every such point, then $\|\nabla P(x)\| \geq c\|x\|$ outside a compact set. Consequently P is not pure under the Gaussian functional.*

Proof. Otherwise choose an unbounded sequence with $\|\nabla P(x)\|/\|x\| \longrightarrow 0$. A projective limit $[v]$ is singular for F_3. Choose a fixed complement to $\mathbb{C}v$ and write $x = sv + y$, where $y = o(s)$. With $H = \operatorname{Hess} F_3(v)$, homogeneity gives the exact expansion

$$\nabla P(sv + y) = sHy + \nabla F_3(y) + s\nabla F_2(v) + \nabla F_2(y) + \nabla F_1.$$

The map H is injective on the chosen complement. Divide by s. The error from $\nabla F_3(y)/s$ is $o(\|y\|)$, so the equation first forces y to stay bounded. A limit y_0 satisfies $Hy_0 + \nabla F_2(v) = 0$. Pair bilinearly with v. Symmetry and $Hv = 0$ kill the first term, whereas Euler's identity gives $v \cdot \nabla F_2(v) = 2F_2(v)$, a contradiction. The Hessian of a cubic is $O(1 + \|x\|)$, so the second condition in (27) follows from the first. Apply Corollary 11.2. □

11.2 A common residual family for ordinary nodes

For a pure cubic with only ordinary leading nodes, the preceding lemma forces F_2 to vanish at some node. An isotropic such node gives the frame already proved in Lemma 7.1. Every nonisotropic such node has the following single normal form.

Lemma 11.5. *Let $[w]$ be an ordinary node of a ternary F_3, with $F_2(w) = 0$ and $w^T w \neq 0$. An invertible affine change of coordinates puts the pair consisting of P and the Gaussian quadratic into*

$$P = T(uv + h) + au^3 + dv^3 + bu^2 + cv^2 + eu + fv + k,$$
$$q = \tfrac{1}{2}(Au^2 + Bv^2 + mT^2 + 2Cuv + 2DuT + 2EvT) + \ell_u u + \ell_v v + \ell_T T + q_0,$$

where $m \neq 0$ and $\det \operatorname{Hess} q \neq 0$. All Gaussian means and cross terms must be retained. Its leading curve is irreducible nodal when $ad \neq 0$, a transverse line and smooth conic when exactly one of a, d vanishes, and three nonconcurrent lines when $a = d = 0$.

Proof. Take T in the direction w and choose a complementary plane. The singular-point equations make F_3 linear in T; its coefficient is a nondegenerate binary quadratic because the node is ordinary. Factor that quadratic and choose its two linear factors as coordinates u, v. Thus $F_3 = Tuv + B_3(u, v)$. Replace T by T plus a linear form in u, v to remove the $u^2 v$ and uv^2 terms of B_3.

There is no T^2 term in F_2, precisely because $F_2(w) = 0$. The full coefficient of T is therefore $uv + \alpha u + \beta v + h_0$. Translations of u, v make it $uv + h$. The cubic terms outside T remain pure cubes. A constant shift of T removes the remaining quadratic uv term. This gives the displayed P. The direction of T has not changed, so $m \neq 0$; invertibility of the affine change preserves the nondegenerate Gaussian Hessian, while translations and the shifts of T produce the means and cross terms shown.

If $ad \neq 0$, the polynomial $Tuv + au^3 + dv^3$ is irreducible: an independent factor of a polynomial linear in T would divide both uv and $au^3 + dv^3$, whose gcd is one. Its gradient shows that its sole singular point is the stated ordinary node. If $a = 0, d \neq 0$, it is $v(Tu + dv^2)$, with a smooth conic meeting the line transversely at two points; the other one-zero case is symmetric. If both vanish, it is the coordinate triangle Tuv. □

This consolidates three leading-curve rows into one residual family. It is a reduction, not yet an exclusion of that family. In particular the full quadratic q, including its means, cannot be replaced by a convenient diagonal form during the remaining period argument.

11.3 Two controls on the scope of the criterion

Ordinary tameness of P alone is insufficient for the Gaussian conclusion. Let $P = \ell^T x$, with $\ell = (1, i)/\sqrt{2}$ in two variables. Its gradient is constant and nonzero, but $\Phi_1 = \exp(t^2 \ell^T \ell/2) = 1$. The phase $tP - q$ has its critical point at $x = t\ell$, escaping with action zero. Thus an isolated critical set for P, or the weaker gradient bound away from zero, would lose exactly the contribution that purity needs. Condition (27) with $J = 2$ prevents this.

The distinction between an original marker and a time-dependent amplitude also remains essential. For $P = (X_1^2 + X_2^2)/2$ one has $\Phi_1 = (1 - t)^{-1}$ and $(1 - t)\Phi_1 = 1$. This phase satisfies the gradient criterion. The unit in its rational differential module is represented by a time-dependent amplitude. No original fixed polynomial marker has a nonzero polynomial period, by Corollary 11.2. The abstract module alone does not record that distinction.

11.4 Ordinary nodes with a finite affine critical set

The three ordinary-node families admit one argument whenever the critical set of P is finite. In particular we need not solve for each escaping critical point, assume it is Morse, or separate its action from the actions of other escaping points. The Gaussian quadratic itself detects the unwanted contribution at infinity.

Retain the notation and the full pair (P, q) of Lemma 11.5. Write q_2 for the homogeneous quadratic part of q. The absence of a Gaussian frame is exactly the following condition at the leading nodes:

$$F_2(w) = 0 \quad \Longrightarrow \quad q_2(w) \neq 0. \tag{28}$$

Indeed the Gaussian bilinear form in these affine coordinates is $2q_2$; the value of F_2 at a leading singular direction is unchanged by translation. In the common normal form the T node already satisfies this condition because $m \neq 0$. The only extra conditions are $c \neq 0$ or $B \neq 0$ when $d = 0$, and $b \neq 0$ or $A \neq 0$ when $a = 0$.

Lemma 11.6 (The stationary action identity)**.** *Put $r = 1/t$, and let $x(r)$ be an algebraic critical branch of $S = P/r - q$. A prime means total differentiation along the branch. Then*

$$S' = -P(x(r))/r^2, \qquad q(x(r)) = -S - rS'. \tag{29}$$

If $x(r)$ is bounded, the polar part of S is α/r, where $\alpha = P(x(0))$. If it escapes and (28) holds, its polar part is neither zero nor α/r for any constant α.

Proof. Use a finite Puiseux cover so every branch is meromorphic. At a critical point the chain-rule term $\nabla_x S \cdot x'$ vanishes, giving both identities. For a monomial $cr^{-\kappa}$ of the polar part of S, its contribution to $-S - rS'$ is $(\kappa - 1)cr^{-\kappa}$. Thus bounded $q(x(r))$ permits only the pole α/r in S. Its coefficient is $P(x(0))$ when x is bounded.

For an escaping branch let $R = \|x\|$. The critical equation is $\nabla P = r\nabla q$, so $\|\nabla P\|/R \longrightarrow 0$. The leading projective direction is a singular point of F_3. The ordinary-node argument in the proof of the preceding coercive lemma works at this one direction: write $x = sw + y$, $y = o(s)$, in a fixed complementary plane. The injectivity there of $H = \operatorname{Hess} F_3(w)$ forces y bounded from

$$\nabla P/s = Hy + \nabla F_2(w) + O((1 + \|y\|)^2/|s|) = o(1).$$

Pairing with w then gives $F_2(w) = 0$. Condition (28) gives $q_2(w) \neq 0$, so $q(x(r))$ has a pole: its leading quadratic term cannot be cancelled by its lower-degree terms. Identity (29) now rules out both claimed polar parts. This argument includes all escape scales and all coincident actions. □

We specify the classical rank input before using it. This is a convergent algebraic application of formal stationary phase, not an extension to arbitrary nonconvergent coefficients or rings.

Lemma 11.7 (Rank accounting for a tame algebraic pencil)**.** *For the present no-frame pencil, shrink a punctured r disk to avoid its finitely many exceptional parameters. Its top twisted de Rham module has rank*

$$N = \sum_{\text{critical points of } S_r} \mu.$$

After a finite cover, the rank of its block with nonzero exponential polar part γ is the sum of these generic local Milnor numbers over critical branches with action polar part γ. Equal actions add their multiplicities. Consequently the rank of the zero-exponential block is the sum of the generic Milnor numbers of the branches with bounded action.

Proof. For fixed small nonzero r, the homogeneous quadratic term of S_r at a leading node w is $F_2(w)/r - q_2(w)$. It is nonzero at every node, by (28) and exclusion of finitely many parameters. The ordinary-node coercivity lemma gives $\|\nabla S_r\| \geq c_r\|x\|$ outside a compact set. The critical locus is therefore finite: a positive-dimensional affine algebraic component would be unbounded. Pham's tame Milnor decomposition [9, Part I, Sections 1–4] gives the displayed rank, retaining the Milnor number of every degenerate point.

Here are the checks needed to apply the direct-image rank formula. Take $U = \mathbb{C}_r^* \times \mathbb{C}_x^3$, $f = r$, $g = S_r$, and the regular algebraic module $\mathcal{O}_U$. Form the ordinary direct image by (f, g), followed by exponential integration in the g coordinate. All these are algebraic maps; g is regular on U. The ordinary direct image is regular holonomic. Over the punctured r disk its relevant cohomology is the middle, reduced Gauss–Manin module of the fibers of S_r. The other, constant fiber cohomology has zero exponential integral in the affine g line. Punctual parameter terms disappear on the punctured disk.

There are no extra finite-g singular curves from spatial infinity. The coercive estimate is locally uniform on a compact subset of the punctured r disk. For g in a compact set, the identity $P = r(g + q)$ on a fiber gives

$$\partial_r S_r = -(g + q)/r = O(1 + \|x\|^2).$$

Lifting r variation by the normalized gradient therefore uses a vector field of at most linear growth. Lifting g variation uses a field $O(1/\|x\|)$. These fields are complete outside a fixed ball on each such parameter neighborhood. They trivialize the outer fiber pairs, so all nonvertical finite-g discriminant curves come from affine critical values. Along such a curve the conormal multiplicity of the middle Gauss–Manin module is the sum of the local Milnor numbers: a transverse target disk computes precisely the corresponding isolated vanishing cycles. This also applies to persistent degenerate points and equal critical values.

First make a common finite Puiseux base change of all the critical-value curves. Apply the nonramified case of Roucairol's Theorem 0.1(2), proved in her Section 1.2, and Theorem 0.3 [10] to this identified ordinary direct image. They give exactly the stated nonzero polar parts and ranks. Count every graph after this base change, including distinct conjugates with the same polar part. For example the two graphs $g = r^{-1} \pm r^{1/2}$ must contribute twice after $r = \rho^2$; retaining just their one polar expression would lose rank. Their rank statement requires no separation of the constant terms of equal polar parts. The additional separation hypothesis in her Theorem 0.2 concerns its monodromy formula, which we do not use. Vertical components at $r = 0$ and the boundary $g = \infty$ do not add polar branches in Theorem 0.1. Subtracting the nonzero-block ranks from N proves the last assertion. In particular we have not inferred the regular rank from an Euler characteristic alone. □

Proposition 11.8. *Suppose the common ordinary-node pair satisfies* (28) *and P has finite affine critical set. Then P is not pure. More generally every original Gaussian marked period is either zero or is not a finite sum of polynomials in t times exponentials $e^{\beta t}$.*

Proof. Every bounded critical branch of S_r limits to a critical point of P. Disjoint small Milnor neighborhoods of these finitely many points contain, for the deformation $P - rq$, total critical multiplicity μ_p in the neighborhood of p. This is conservation of the local intersection multiplicity of the gradient, and does not require a Morse perturbation. By Lemma 11.6 its action has polar part $P(p)/r$. Every escaping branch has nonzero polar part. Hence Lemma 11.7 identifies the

entire zero-exponential rank as

$$\sum_{p:\, dP(p)=0,\, P(p)=0} \mu_p. \tag{30}$$

We construct that many independent local formal functionals and compute their monodromy over the unramified field $\mathbb{C}((r))$. At an isolated critical point at least one of u, v is nonzero. Indeed if both vanish, criticality requires $h = e = f = 0$, in which case the whole T axis is critical. In the chart $u \neq 0$ put

$$s = uv + h, \quad R(u) = G(u, -h/u), \quad T' = T + \frac{G(u, (s-h)/u) - G(u, -h/u)}{s},$$

where $G = P - T(uv + h)$. The quotient is analytic, and exactly $P = sT' + R(u)$. The coordinate Jacobian is $u \neq 0$. At a critical point p the one-variable critical multiplicity of R is μ_p. An analytic one-variable coordinate y, with $w = T'$, therefore gives the exact identity

$$P = \alpha + sw + y^k, \qquad k = \mu_p + 1, \quad \alpha = P(p).$$

The volume Jacobian and e^{-q} multiply every amplitude by an analytic unit $V(y, s, w)$; no Gaussian mean or cross term has been discarded.

For $j = 1, \ldots, k-1$ define a formal local functional on an analytic monomial $y^a s^b w^c$ as zero unless $b = c$ and $a = j - 1 + \ell k$ for an integer $\ell \geq 0$. Otherwise its value is

$$e^{\alpha/r} r^{1+j/k} (-1)^{b+\ell} b!\, (j/k)_\ell\, r^{b+\ell}. \tag{31}$$

Evaluate a polynomial form after multiplying its transformed amplitude by V. At each power of r only finitely many Taylor terms contribute. The recurrences

$$I_{a+k} = -\frac{a+1}{k} r I_a, \qquad J_{b+1,b+1} = -(b+1) r J_{b,b}$$

prove spatial integration by parts and parameter horizontality. Thus these are functionals on the full twisted quotient, not on an assumed cyclic image. They exist even if the corresponding asymptotic series diverge; their input coefficients here are the Taylor series of the stated analytic coordinates.

They are independent. Prescribe, at each p, the jets $1, y, \ldots, y^{k-2}$ on the line $s = w = 0$ and zero jets at the other points, using polynomial representatives of finite local jets and the Chinese remainder theorem. After dividing column j by $e^{\alpha/r} r^{1+j/k}$, the leading evaluation matrix is the jet map multiplied by the analytic unit V. It is triangular with nonzero diagonal. Its higher terms are in $r\mathbb{C}[[r]]$. For $\alpha = 0$ these independent characters fill exactly the rank in (30); there is no unaccounted regular piece. Their formal monodromy eigenvalues, or their inverses if one uses the dual module, are

$$\exp(2\pi i j/k), \qquad 1 \leq j < k.$$

None is one. This conclusion is over $\mathbb{C}((r))$. We do not replace a noninteger residue by its integer multiple after ramification. The unique zero-exponential projector commutes with descent, and the displayed unramified coefficient matrices identify its whole block. Every other block has a nonzero exponential polar part. The ambient module therefore has no trivial composition factor.

For completeness this excludes the formal Gaussian unit without choosing an actual analytic period. Formal Gaussian integration annihilates $\partial_i + tP_i - q_i$ coefficientwise. If its value on a fixed marker A were a nonzero constant, restriction to the rational differential submodule generated by

A would give a surjection to the trivial module: apply the functional to every rational relation between $A, \nabla_t A, \ldots$. A relation has zero constant coefficient because all positive derivatives of that constant vanish. Exact formal extension at infinity would preserve this composition factor, contradicting what was proved.

Finally a nonzero finite exponential-polynomial can be reduced to $ce^{\beta t}$ by a constant-coefficient differential operator. On an original marker this operator replaces A by the fixed polynomial marker $T(P)A$. Replace P by $P - \beta$ and divide the marker by c. The hypotheses and finite critical set are unchanged, so the same contradiction applies. Taking $A = 1$ proves the purity exclusion. □

11.5 An exhaustive critical-set split

The finite-critical hypothesis in Proposition 11.8 has a short exhaustive coefficient test. This prevents a generic argument from silently discarding the positive-dimensional cases.

If $h \neq 0$, the critical locus lies on $uv = -h$, and the equations above identify it with $R'(u) = 0$ in $\mathbb{C}^*$. The Laurent polynomial

$$R(u) = au^3 + bu^2 + eu + k - fh/u + ch^2/u^2 - dh^3/u^3$$

is nonconstant unless $a = b = c = d = e = f = 0$. A nonzero Laurent derivative has only finitely many roots in $\mathbb{C}^*$. In the constant case the critical set is the smooth conic $uv = -h$, $T = 0$.

If $h = 0$, outside the center the critical equations are

$$\begin{aligned} v = 0,\ u \neq 0: &\quad 3au^2 + 2bu + e = 0, \quad T = -f/u, \\ u = 0,\ v \neq 0: &\quad 3dv^2 + 2cv + f = 0, \quad T = -e/v. \end{aligned}$$

The center $u = v = 0$ is critical exactly when $e = f = 0$, and then the whole T axis is critical. Consequently the following table is exhaustive and disjoint:

Condition	Conclusion
$h \neq 0$, $(a, b, c, d, e, f) \neq 0$	Finite critical set; excluded above
$h \neq 0$, $a = b = c = d = e = f = 0$	Critical conic; Corollary 11.10
$h = 0$, $e = f = 0$	Center axes; Proposition 11.15
$h = 0$, $(e, f) \neq 0$, $a = b = e = 0$	Critical hyperbola; Corollary 11.10
$h = 0$, $(e, f) \neq 0$, $d = c = f = 0$	Symmetric hyperbola; same corollary
$h = 0$, $(e, f) \neq 0$, $(a, b, e) \neq 0$, $(d, c, f) \neq 0$	Finite critical set; excluded above

The two hyperbola rows cannot overlap because $(e, f) \neq 0$. All isolated degeneracies in the finite rows have already been included through their Milnor multiplicities. The conic and hyperbola rows are proved in the next subsection. The center-axis row is proved in Proposition 11.15, using a full formal frame or the uniform actual-cut criterion.

11.6 A critical conic in every polynomial degree

The conic and hyperbola rows of the preceding table are manifestations of the same calculation. It also works when one of the cubic terms is replaced by a polynomial of arbitrary degree. We prove that stronger statement to make clear what the mechanism actually uses.

Theorem 11.9. *Retain a full nondegenerate affine Gaussian quadratic*

$$q = \tfrac{1}{2}(Ax^2 + By^2 + mw^2 + 2Cxy + 2Dxw + 2Eyw) + \ell_x x + \ell_y y + \ell_w w + q_0.$$

Let $AB \neq 0$ and

$$P = wxy + F(w) + k, \qquad F(0) = 0, \quad F'(0) = h \neq 0.$$

Allow any polynomial degree $d_F = \deg F \geq 1$, requiring $m \neq 0$ when $d_F = 1$. Then P is not pure. Every original marked Gaussian period is either zero or is not a finite exponential-polynomial. The generic top twisted rank is $d_F + 3$ for $d_F \geq 2$ and five for $d_F = 1$. No Morse assumption on the Laurent phase obtained by restricting q to the conic, or on the isolated critical points of P, is needed.

The nonzero quadratic poles at both ends of the conic are essential to the rank-four construction below. The condition $AB \neq 0$ states this explicitly. Nondegeneracy of the full Gaussian alone would not imply it. This theorem concerns complex polynomials and valued-field formal frames; it does not assert an all-ring extension.

Proof. We first justify the full rank. Fix a generic $t \neq 0$ and set $S = tP - q$. Its gradient has a linear lower bound at complex spatial infinity. To see this, suppose along an unbounded sequence $\|\nabla S\| = o(R)$, where $R = \|(x, y, w)\|$. If w stays bounded, pass to a limit w_0 and normalize (x, y) by R. The first two gradient equations give

$$-AX + (tw_0 - C)Y = 0, \qquad (tw_0 - C)X - BY = 0.$$

At least one of X, Y is nonzero, and $AB \neq 0$ makes both nonzero. But the last gradient equation, divided by R^2, gives $tXY = 0$, a contradiction.

If w is unbounded, the first two equations have a matrix with inverse $O(1/|w|)$. Hence $x, y = O(1) + o(R)/|w|$. First $R \asymp |w|$, then $x, y = O(1)$. The last equation is therefore

$$tF'(w) - mw + O(1) = o(|w|).$$

This is impossible for $d_F \geq 3$, for $d_F = 2$ with $2t[w^2]F - m \neq 0$, and for $d_F = 1$ with $m \neq 0$. The same proof is uniform on compact subsets of these coefficient conditions. Thus all phase-critical points stay in one ball during any compact coefficient deformation inside that domain. They are isolated, and their total Milnor number is constant by conservation of local gradient-intersection multiplicity. Pham's tame Milnor decomposition [9, Part I, Sections 1–4] identifies that number with the full twisted rank.

For fixed t, the coefficient domain just used is the complement of finitely many complex algebraic hypersurfaces, and is path connected. We may compute the number at a generic diagonal Gaussian with zero means, retaining $ABm \neq 0$, and a generic F. The critical equations there are

$$twy = Ax, \qquad twx = By, \qquad t(xy + F'(w)) = mw.$$

There are $d_F - 1$ simple solutions with $x = y = 0$ if $d_F \geq 2$, and one if $d_F = 1$. The other solutions have $w = \pm\sqrt{AB}/t$. At each of these two values, x, y are proportional and their nonzero product is prescribed by the last equation; generically this gives two simple solutions. The counts are $d_F + 3$ and five, respectively. Uniform coercivity and the connected deformation justify this specialization; a convenient critical-point count by itself would not justify the rank for a degenerate member.

We now construct a full formal frame. Set $r = t^{-1/2}$ and write $F(w) = hw + bw^2 + \sum_{j \geq 3} f_j w^j$. Near the entire critical conic $w = 0$, $xy = -h$, use the exact coordinates

$$y = (rV - h)/x, \qquad w = rU.$$

They give

$$tP - q = k/r^2 + UV + bU^2 + \psi(x) + E_r(x, V, U), \qquad E_r \in r\mathbb{C}[x, x^{-1}, V, U][r], \tag{32}$$

$$dx\, dy\, dw = r^2 \frac{dx}{x}\, dV\, dU, \tag{33}$$

$$\psi(x) = -\frac{A}{2}x^2 - \frac{Bh^2}{2x^2} - \ell_x x + \frac{h\ell_y}{x} + Ch - q_0.$$

All Gaussian cross terms and means are in ψ or E_r. The normal quadratic $UV + bU^2$ has Hessian determinant -1 for every b. Its normalized formal Gaussian functional is well defined and takes one to one.

The logarithmic Laurent quotient for ψ has rank four, with basis $1, x, x^2, x^3$. Here is a direct proof, including independence. Its relation on x^j is

$$jx^j - Ax^{j+2} - \ell_x x^{j+1} - h\ell_y x^{j-1} + Bh^2 x^{j-2} = 0.$$

The top coefficient eliminates all exponents at least four without creating a negative exponent; the bottom eliminates all negative exponents without creating an exponent above three. If the image of a nonzero Laurent polynomial under $x\partial_x + x\psi_x$ were supported in $[0, 3]$, its highest exponent would be at most one and its lowest at least two, which is impossible.

Let $\lambda_0, \dots, \lambda_3$ be the dual basis on this quotient, tensored with the normal Gaussian functional. For a polynomial amplitude $a(x, y, w)$, define

$$L_i(a) = e^{k/r^2} r^2 \lambda_i\big(e^{E_r} a(x, (rV - h)/x, rU)\big)\,. \tag{34}$$

Every coefficient is a finite Laurent-polynomial calculation, so this is a well-defined formal series. Multiplication by e^{E_r} conjugates the fixed twisted differential to the perturbed one. Consequently these four functionals annihilate all spatial exact forms. The moving volume is retained in the factor r^2. For an explicit parameter check the coordinate velocity in the original variables is

$$Z = \left(0, \frac{xy + h}{rx}, \frac{w}{r}\right), \qquad \operatorname{div} Z = 2/r.$$

Cartan's identity $\mathcal{L}_Z = d\iota_Z + \iota_Z d$ shows that the pullback derivative differs from the original parameter connection only by a spatial exact form and this volume derivative. Thus (34) are horizontal for the original full module. In particular, for every fixed polynomial a,

$$L_i(a) \in e^{kt} r^2 \mathbb{C}[[r]]. \tag{35}$$

This bound holds for all original markers, not just for the unit.

Suppose first $d_F \geq 2$. The other critical points of P are $x = y = 0$, $F'(w_p) = 0$. Each w_p is nonzero because $h \neq 0$. If it has multiplicity μ_p, analytic coordinates give

$$P = k + F(w_p) + sz + y_0^{\mu_p + 1}.$$

Indeed absorb the nonvanishing factor w into x or y, then use the one-variable power coordinate for F at w_p. The explicit characters (31), including their integration-by-parts and time recurrences, apply to this exact local model. Their prefactors are

$$e^{(k + F(w_p))t} t^{-1 - j/(\mu_p + 1)}, \qquad 1 \leq j \leq \mu_p.$$

The Gaussian and coordinate Jacobian are analytic units in that construction. These characters retain multiple roots and equal critical values. Their total number is $\sum_p \mu_p = d_F - 1$.

The $d_F + 3$ characters just written are independent on polynomial amplitudes. Put $B_0(w) = \prod_p (w - w_p)^{\mu_p}$. For the four conic rows use $x^j B_0(w)$, $0 \le j \le 3$. Their leading conic matrix is $B_0(0) I_4$. Their leading finite jets vanish. For the remaining rows choose polynomials in w with value zero at $w = 0$ and the separate Hermite jets of orders $0, \dots, \mu_p - 1$ at the nonzero w_p. The Chinese remainder theorem supplies them. Their conic values have an additional positive power of r. Their normalized finite evaluation matrix is the invertible local jet map, multiplied by analytic units. After dividing each character column by its scalar prefactor, the leading matrix is therefore block diagonal and invertible. Since its size is the proved full rank, these are a full formal frame. No uncomputed image kernel or infinity block remains.

For $d_F = 1$ there is instead one further character. In the exact coordinates

$$x = X/t, \qquad y = Y/t, \qquad w = (h/m)t + W,$$

the phase is

$$\begin{aligned} tP - q &= \frac{h^2}{2m} t^2 + (k - h\ell_w/m)t + H(X, Y, W) + O(t^{-1}), \\ H &= \frac{h}{m} XY - \frac{h}{m}(DX + EY) - \frac{m}{2} W^2 - \ell_w W - q_0. \end{aligned}$$

The correction is polynomial in t^{-1}, X, Y, W, with positive t^{-1} order. The Hessian determinant of H is $h^2/m \neq 0$. Formal Gaussian evaluation gives a horizontal character with nonzero quadratic action and volume prefactor t^{-2}. To check independence set $w_* = (h/m)t$. Use $x^j(1 - w/w_*)$, $0 \le j \le 3$, and w/w_* as five polynomial amplitudes with rational parameter coefficients. The former have the original conic leading matrix and zero leading escape values; the latter has zero leading conic values and nonzero leading escape value. This gives the full rank-five frame.

In either degree case, after finite ramification the frame splits the module into rank-one factors. Every zero-exponential functional has positive order on every original polynomial marker, by (35) and the displayed finite prefactors. Lemma 10.2 rules out a nonzero constant original marked period. A constant-coefficient operator reduces any nonzero finite exponential-polynomial to $ce^{\beta t}$; it replaces the marker by another original polynomial in P. Shifting k by $-\beta$ and normalizing the marker gives the same contradiction. This proves the theorem. □

Corollary 11.10. *All the critical-conic and critical-hyperbola rows in the ordinary-node table are not pure. Proposition 11.15 treats its remaining center-axis row.*

Proof. For $h \neq 0$ and $a = b = c = d = e = f = 0$ the residual is $T(uv + h) + k$. Apply the $d_F = 1$ case with $w = T$, $x = u$, $y = v$. The no-frame conditions give $ABm \neq 0$.

For $h = 0$, $a = b = e = 0$, $f \neq 0$, permute the coordinates and write the residual as

$$v(Tu + dv^2 + cv + f) + k.$$

Take $w = v$, $x = T$, $y = u$, and $F(w) = dw^3 + cw^2 + fw$. The quadratic coefficients along T, u are nonzero: the original T square coefficient is nonzero, and the u node has $F_2 = 0$, so its Gaussian square coefficient is nonzero by the no-frame condition. If this F is linear, the remaining v node also has $F_2 = 0$, giving the extra w square coefficient required in degree one. The symmetric hyperbola row is identical after swapping u, v. These are all the indicated rows. □

11.7 A uniform cut criterion in any degree

Intersecting critical curves need not admit a simple formal normal form. The following criterion avoids choosing one. Its hypothesis concerns the full phase $tP - q$, uniformly in large complex t; bounded critical points alone would not suffice.

Theorem 11.11. *Let $P, q \in \mathbb{C}[x_1, \dots, x_n]$, with P nonconstant. Suppose there are fixed $c, R_0, t_0 > 0$ such that*

$$\|\nabla_x (tP - q)(x)\| \geq c\|x\| \quad (|t| \geq t_0,\ \|x\| \geq R_0). \tag{36}$$

Along every fixed ray $t = Re^{i\theta}$, the full rapid-decay homology has a flat cut basis $\Gamma_i(t)$, attached to the critical values $c_i(t)$ of $tP - q$, such that for every fixed polynomial a,

$$e^{-c_i(t)} \int_{\Gamma_i(t)} a(x) e^{tP-q}\, dx_1 \cdots dx_n \longrightarrow 0. \tag{37}$$

Each $c_i(t) = \alpha_i t + O(1)$, where α_i is a critical value of P. If all critical values of P lie in a closed half-plane through zero, a supporting t ray makes every flat period of every such fixed form tend to zero.

We give the completeness, tail and limiting-form arguments separately. None follows just from counting the phase-critical points.

Proof. For every sufficiently large t, (36) makes the phase-critical set finite and contained in one fixed ball. It also gives complex tameness at infinity. Pham's tame Milnor decomposition [9, Part I and Appendix] identifies the complete middle rapid-decay homology with the local vanishing homologies carried along cuts from its critical values. The rank is the sum of their Milnor numbers, including degenerate points. If this rank is zero the conclusion is immediate. Otherwise retain all these local generators; identical critical values are grouped, not discarded.

On the chosen ray, the critical points and values are algebraic functions of R, after a finite cover. Enlarge R_1 so branches and multiplicities stabilize. Choose $|\eta| = 1$ with $\Re\eta \geq 1/2$ so the cuts

$$S_t(x) = c_i(t) - \eta s, \qquad s \geq 0, \quad S_t = tP - q,$$

are disjoint for $R > R_1$, except at identical grouped endpoints. There are only finitely many forbidden choices: every nonzero difference of two algebraic critical values has one limiting direction. Choose this auxiliary cut direction after fixing θ; it imposes no genericity condition on θ itself. In particular a supporting t ray need not be a generic ray.

Along a cut, normalized complex gradient transport is

$$x_s = -\eta\, \overline{\nabla S_t} / \|\nabla S_t\|^2, \qquad S_t(x(s))_s = -\eta.$$

Outside the fixed ball, $|d\|x\|^2/ds| \leq 2/c$. All local vanishing generators start in that ball. With fixed sufficiently large A and $B > 2/c$, they therefore have representatives inside the carrier

$$\|x\|^2 \leq A + Bs. \tag{38}$$

At its outer boundary the transport points strictly inward relative to this inequality. No trajectory escapes at finite cut length, and the cuts avoid all other critical values.

We must control a single family of representatives. The gradient flow itself need not be semialgebraic. Instead compactify x by $x/\sqrt{1 + \|x\|^2}$ and s by $s/(1 + s)$. Take the closure of

each constrained cut carrier and mark its rapid boundary G, where $s = \infty$. Inequality (38) forbids spatial infinity at finite s. A compatible semialgebraic triangulation computes the relative homology of this compact pair by locally finite semialgebraic cut chains, with boundary allowed only in G. These representatives are admissible rapid-decay chains: on the carrier,

$$s \geq (\|x\|^2 - A)/B, \qquad |e^{-\eta s}| \leq e^{-s/2} \leq \exp\left(-\frac{\|x\|^2 - A}{2B}\right).$$

Semialgebraic approach to a boundary has at most power-law rates, so this decay dominates every boundary pole of a polynomial form. The chain-current boundary identity applies on finite phase bands; the tail estimate below justifies passage to the improper integral. Thus the compact-pair calculation supplies admissible cycle representatives, without identifying arbitrary singular chains with integration chains. Their exponential is exactly $e^{c_i - \eta s}$.

For the full unconstrained cut carrier, mark and remove any spatial boundary at finite s; it is not a rapid boundary. The constrained carrier, the full carrier, their rapid boundaries and this excluded boundary form a finite family of semialgebraic subsets in the same compactification. Apply the compatible finite-family form of Hardt triviality to projection onto R. It gives one trivialization on an unbounded interval, preserving these subsets and their inclusions. The precise form is stated in Drusvyatskiy–Lewis [3, Theorem 2.25]; it is Hardt's theorem. Semialgebraic triangulation and the chain-current boundary rule are as in Hardt–Lambrechts–Turchin–Volić [4, Sections 2.1 and 3, especially Theorem 3.5].

For precision, the pairs in this application are as follows. In the compactified (x, s) space write $\overline{K}_i(R)$ for the closure of the constrained cut carrier, $G_i(R) = \overline{K}_i(R) \cap \{s = \infty\}$, and write $\overline{K}_i^{\mathrm{full}}(R)$ for the unconstrained closure. Let $E_i(R)$ be its spatial boundary at finite s, and put $G_i^{\mathrm{full}} = \overline{K}_i^{\mathrm{full}} \cap \{s = \infty\}$. The relative pairs and their inclusion are

$$(\overline{K}_i, G_i) \longrightarrow (\overline{K}_i^{\mathrm{full}} \setminus E_i, G_i^{\mathrm{full}} \setminus E_i).$$

The constrained pair has no E_i, by (38). At $s = 0$ a local vanishing cycle closes by its collapsing thimble; that endpoint is included in K_i, rather than declared a rapid boundary. In degree n, projection to x takes these relative classes to the rapid-decay homology of e^{S_t}. The tame cut comparison above says that the images, summed over the cuts, contain the entire vanishing-homology decomposition. We use this image; we need not identify every relative group of an individual carrier with the full rapid-decay group.

The compatible Hardt trivialization preserves both pairs, their inclusion, and hence the induced maps on their relative homology. Gradient transport already showed that the constrained inclusion contains every local vanishing generator. At one parameter choose finitely many integral relative chains whose images are a basis over $\mathbb{C}$ of the full rapid-decay space. A nonprimitive integral image causes no problem over $\mathbb{C}$. Their number is the total tame Milnor rank, and their images are independent at this parameter by construction. Transport these chains by the compatible trivialization. They form a jointly semialgebraic family with fixed integer multiplicities and supports satisfying (38). Their images are a flat basis of the actual homology local system: on each compact parameter interval the transport preserves the relative pairs and is proper on every finite phase band. The induced comparison maps commute with this transport, so the image basis remains independent and spanning. This establishes flatness and completeness before any mass estimate is made; a uniform bound for arbitrary representatives would not establish either property.

Here is the needed uniform mass bound. In a fixed semialgebraic family of d-chains with bounded multiplicities, the mass inside a ball of radius $L \geq 1$ is at most CL^d, uniformly in the

parameters. Partition tangent planes by their largest coordinate d-minor. The Cauchy–Binet identity gives a lower bound $\binom{N}{d}^{-1/2}$ for some such minor in ambient dimension N. The area formula then bounds that piece by its coordinate projection volume times the number of preimages. Semialgebraic uniform finiteness bounds those finite fibers independently of the parameters; see, for example, [3, Theorem 2.32]. Sum the finitely many pieces and multiplicities. Apply the same argument to boundary chains. This proves uniformity, rather than merely pointwise finite volume.

Lift the cut chains to (x, s). Their bands $j \le s \le j+1$ lie in balls of radius $O(1+j)$ by (38). Their fixed semialgebraic descriptions and the mass estimate give, with constants independent of R,

$$\left| e^{-c_i(t)} \int_{s \ge L} a e^{S_t}\, dx_1 \cdots dx_n \right| \le C \sum_{j \ge \lfloor L \rfloor} (1+j)^M e^{-j/2}. \tag{39}$$

The polynomial amplitude changes only M. This tail tends to zero uniformly in R.

It remains to bound a fixed phase band. A bounded critical branch $x(r)$, $r = 1/t$, satisfies

$$q(x(r)) = -c_i(r) - r c_i'(r).$$

At a stationary point the chain rule gives $c_i'(r) = -P(x(r))/r^2$; eliminating P gives this identity. Since $q(x(r))$ is bounded, its Puiseux expansion forces the only possible polar term of c_i to be α_i/r. Its coefficient is $\alpha_i = P(x(0))$, and $\nabla P(x(0)) = 0$. Hence $c_i(t) = \alpha_i t + O(1)$ even at critical collisions.

Choose a generic fixed truncation level L. The truncated lifted chains have compact supports and uniformly bounded mass and boundary mass. The latter follows from the same semialgebraic estimate applied to their slices; after a finite stratification there is an arbitrarily large sequence of such levels. By the Federer–Fleming compactness theorem, every sequence $R \to \infty$ has a subsequence converging to an integral n-current. We use the version with both mass and boundary-mass bounds, Simon [5, Chapter 6, Section 3, Theorem 3.15]. Its rectifiability conclusion is essential.

On the lifted chain,

$$P(x) - \alpha_i = \frac{c_i(t) - \alpha_i t + q(x) - \eta s}{t}.$$

Every limiting current is therefore supported in $\{P = \alpha_i\} \times [0, L]$. Stratify this complex algebraic level set into smooth complex strata. Almost everywhere for the rectifiable limiting current, its tangent plane projects into the tangent of one of these strata. Each has complex dimension less than n. The holomorphic n-form $dx_1 \cdots dx_n$ thus vanishes on that tangent plane. Lower-dimensional strata carry no n-mass; the same argument covers singular strata of larger real dimension. Testing with the fixed smooth form $e^{-\eta s} a(x) dx_1 \cdots dx_n$, with a compact cutoff, gives zero on every subsequential limit. Consequently the normalized integral on $0 \le s \le L$ tends to zero. This does not assert that the real volume of the chains tends to zero.

First let $R \to \infty$, then let $L \to \infty$ through the allowed levels in (39). This proves (37). Finally choose θ with $\Re(e^{i\theta}\alpha) \le 0$ for all critical values α of P. The factors $e^{c_i(t)}$ are then bounded, so every basis period tends to zero. A flat cycle is a constant linear combination of this full basis, proving the last assertion. □

Corollary 11.12. *Let q be a full affine Gaussian quadratic and assume (36). If P has at most two critical values, every original marked Gaussian period is either zero or is not a finite exponential-polynomial. In particular P is neither pure nor safe.*

Proof. Any two complex numbers lie in a closed half-plane through zero. For two nonzero values normalize them to z_1, z_2 on the unit circle. If $z_1 + z_2 \neq 0$, use the direction $-\overline{z_1 + z_2}/|z_1 + z_2|$; if their sum is zero, use $i\overline{z_1}$. Zero or missing values add no condition.

A nonzero constant original marked period would, by the rational seed restriction and actual-period extension in Lemma 10.1, be represented by a flat period of that fixed polynomial form. Its generic exponential system has $f = t$, $g = tP - q$ on smooth affine space and regular input $\mathcal{O}$; f, g are algebraically independent because P is nonconstant. Remove the finite exceptional parameter set of the Hien–Roucairol comparison. Theorem 11.11 makes the resulting constant period tend to zero, a contradiction. No convergence of the formal moment series was assumed.

A constant-coefficient operator reduces a nonzero finite exponential-polynomial to $ce^{\beta t}$ while replacing its original marker by another fixed polynomial in P times that marker. Shift P by $-\beta$ and normalize. Its gradient is unchanged and its critical values are still at most two. The same contradiction applies. Since $\Phi_1(0) = 1$, it cannot be a polynomial, proving also the safety exclusion. □

11.8 Completing the ordinary-node center axes

The only remaining ordinary-node row is

$$P = Tuv + au^3 + dv^3 + bu^2 + cv^2 + k. \tag{40}$$

Keep the full q of Lemma 11.5, with $m \neq 0$. Under the no-frame assumption the other nodes satisfy

$$a = b = 0 \implies A \neq 0, \qquad d = c = 0 \implies B \neq 0.$$

We use a full formal frame when $bc \neq 0$ and the preceding uniform cut criterion when $bc = 0$. These branches are disjoint and include every zero face of the four coefficients.

Lemma 11.13. *For (40) with these Gaussian conditions, (36) holds.*

Proof. Otherwise take $\|x\| \to \infty$, $|t| \geq t_0$ and $\nabla(tP - q) = o(\|x\|)$. Division by $t\|x\|^2$ shows that a limiting projective direction is a leading node: the T axis, also the u axis if $a = 0$, and the v axis if $d = 0$.

Near any such node use height z and transverse coordinates $x, y = o(z)$. The phase polynomial has the form

$$P = zxy + \beta z^2 + H(x, y) + k,$$

where H has only terms of degrees two and three. There are no transverse linear terms. The transverse equations, divided by tz, give

$$|x| + |y| \leq C/|t| + o(|x| + |y|) + o(1/|t|).$$

The terms of $\nabla H/z$ are absorbed because $(x, y) = o(z)$; the Gaussian cross terms divided by tz are bounded by the same quantities. Thus $x, y = O(1/|t|)$. The height equation is

$$\partial_z(tP - q) = (2\beta t - M)z + O(1),$$

where M is the Gaussian square coefficient along this node. If $\beta \neq 0$, $|2\beta t - M|$ is uniformly bounded below for all sufficiently large $|t|$. If $\beta = 0$, the no-frame condition gives $M \neq 0$. This contradicts the assumed small gradient ratio. The proof also covers bounded-t subsequences within the chosen large-$|t|$ domain and therefore proves the required uniform estimate. □

Lemma 11.14. *The full generic twisted rank of* (40) *is*

$$N = 5 + \mathbf{1}_{a\neq 0} + \mathbf{1}_{d\neq 0}.$$

Proof. The three homogenized quadratic gradient equations have total intersection length eight in $\mathbb{P}^3$. Their points at infinity are exactly the leading ordinary nodes, in number $3 - \mathbf{1}_{a\neq 0} - \mathbf{1}_{d\neq 0}$. Each has length one. Indeed its local Jacobian has the two transverse columns of $\operatorname{Hess} F_3(w)$ and the column $\nabla(tF_2 - q_2)(w)$. The first two have rank two and are annihilated by pairing with w; the last has nonzero pairing $2(tF_2 - q_2)(w)$. This value is nonzero for large generic t by the stated conditions. Thus these infinity intersections are simple. The uniform gradient estimate gives a finite affine critical set; there are no other components. Bézout and Pham's tame comparison give the asserted full rank. □

Proposition 11.15. *Every phase* (40) *under the stated conditions is not pure. Every original marked Gaussian period is zero or is not a finite exponential-polynomial.*

Proof. First let $bc \neq 0$. Set $r = t^{-1/2}$, $u = rU$, $v = rV$, leaving T unchanged. Exactly,

$$\begin{aligned} tP - q &= k/r^2 + F_0(U,V,T) + E_r(U,V,T), \\ F_0 &= TUV + bU^2 + cV^2 - mT^2/2 - \ell_T T - q_0, \\ E_r &= r(aU^3 + dV^3 - DUT - EVT - \ell_u U - \ell_v V) \\ &\quad - r^2(AU^2 + BV^2 + 2CUV)/2, \\ du\,dv\,dT &= r^2 dU\,dV\,dT. \end{aligned}$$

The full polynomial twisted quotient for F_0 has rank five, including its degenerate parameter values. Here is a separate rank proof. Suppose $\nabla F_0 = o(\|(U,V,T)\|)$ on an unbounded sequence. If T stays bounded, the first two equations force both normalized U, V to be nonzero; the last, divided by the squared norm, gives their product zero, a contradiction. If T is unbounded, the first two equations have inverse $O(1/|T|)$, giving $U, V = o(1)$ and $\|(U,V,T)\| \asymp |T|$. The last equation then has leading term $-mT$, again a contradiction. This coercivity is uniform on compact sets with $bcm \neq 0$. The coefficient domain is connected, so conservation of critical intersection multiplicity and Pham's tame comparison make its rank constant. At generic ℓ_T, its critical equations give one point with $U = V = 0$ and four with $T^2 = 4bc$, all simple. Hence the rank is five, also when points merge at the exceptional values of ℓ_T.

Choose any polynomial basis $B_1, \dots, B_5$ of this fixed quotient and its dual $\lambda_1, \dots, \lambda_5$. Then

$$L_j(a_0) = e^{kt} r^2 \lambda_j\big(e^{E_r} a_0(rU, rV, T)\big)$$

defines five formal horizontal functionals on the original full module. Coefficients are finite polynomial evaluations at every order. Spatial exactness is the fixed-phase conjugacy; time horizontality retains the displayed r^2 volume and the coordinate velocity $(u/r, v/r, 0)$, whose divergence is $2/r$. Cartan's formula makes the remaining velocity term spatially exact. For every original fixed polynomial marker their values lie in $e^{kt} r^2 \mathbb{C}[[r]]$.

The other P-critical points are the off-axis points $p_u = (-2b/(3a), 0, 0)$ if $a \neq 0$ and $p_v = (0, -2c/(3d), 0)$ if $d \neq 0$. They are Morse, with Hessian determinants $8b^3/(9a^2)$ and $8c^3/(9d^2)$. Holomorphic Morse coordinates reduce their local phases to a nondegenerate quadratic. Expanding the analytic Gaussian amplitude and volume unit and applying the quadratic Wick recurrence gives all-order horizontal characters. These have prefactors $e^{P(p)t} r^3$, multiplied by power series in

r^2 on fixed markers. Their values may coincide with each other or with k; no separation hypothesis is used.

Here is an explicit completeness test. Let M bound the degree of the chosen B_j in U, V. Substitute $U = u/r, V = v/r$ and multiply each B_j by

$$C(u,v) = \prod_{p_u}(1 - u/u_p)^K \prod_{p_v}(1 - v/v_p)^K, \qquad K > M.$$

An empty product is one. At the central fixed model this is $1 + O(r)$, so the normalized five-by-five matrix is the identity at leading order. At each off-axis point its zero of order K overcomes the worst r^{-M} growth of the substituted basis; the normalized cross values tend to zero. For the off-axis rows use $(u/u_p)^K$ or $(v/v_p)^K$. They have zero central leading values, unit value at their own Morse point and zero leading value at the other. The resulting full matrix has invertible block diagonal leading term. Its size is exactly the rank of Lemma 11.14. These are therefore a full formal frame, including all fixed-model degeneracies and equal actions.

Every zero-exponential original-marker seed has positive order, either r^2 or r^3. Lemma 10.2 excludes a nonzero constant original marked period. Differential isolation and shifting k then exclude every nonzero finite exponential-polynomial as before.

It remains to treat $bc = 0$. The critical equations of P show that its values are among

$$k, \qquad k + 4b^3/(27a^2) \quad (ab \neq 0), \qquad k + 4c^3/(27d^2) \quad (dc \neq 0).$$

At a missing isolated point omit its displayed value. Any extra critical axes have value k. Since $bc = 0$, there are at most two values, including every intersecting-axis or degenerate case. Lemma 11.13 and Corollary 11.12 apply. This completes both disjoint branches and proves the proposition. □

Corollary 11.16. *Frame extraction is proved for all three ordinary-node leading types: irreducible nodal cubics, transverse line–conic unions, and three nonconcurrent lines. In fact none of these leading types permits a pure polynomial.*

Proof. If all leading nodes have $F_2 \neq 0$, use the coercive-node criterion. Otherwise an isotropic zero-F_2 node supplies the frame directly. In the absence of such a frame, the common normal form applies and satisfies the no-frame conditions at every other node. The exhaustive critical-set table is now entirely covered by Proposition 11.8, Corollary 11.10, and Proposition 11.15. Thus a pure polynomial must have a frame.

In that frame, the quadratic coefficient of W in the cubic leading form is the tangent quadratic at the chosen ordinary node. It has rank two, so its discriminant is nonzero. Proposition 10.10 excludes purity in that branch as well. This proves the stronger assertion. □

11.9 It is enough to complete the regular part

The cusp and tangent cases share a substantial simplification. We need not construct every escaping Gaussian character. It is enough to exclude escaping contributions to the regular part, compute that part's full rank, and construct precisely that many local functionals. This keeps the original marker while avoiding an unnecessary classification of the other exponential blocks.

Lemma 11.17 (A positive seed on the regular part)**.** *Let M and a seed a satisfy the formal Gaussian hypotheses of Lemma 10.1. After a finite cover at infinity, write M_0 for the zero-exponential summand. Evaluate its dual functionals on the canonical projection of a to M_0.*

Suppose M_0 has a full horizontal dual frame over $\mathbb{C}((\rho))$, and every member takes a into $\rho\mathbb{C}[[\rho]]$. Then the formal Gaussian functional cannot take a to one. No semisimplicity assumption on the nonzero-exponential summands is required.

The same conclusion holds if the regular frame exists over $\mathbb{C}((\rho))(\log\rho)$ and its seed values are finite polynomials in $\log\rho$ with coefficients of strictly positive ρ order.

Proof. The rational seed restriction supplies a quotient $U \to \mathbb{C}(t)$. The formal exponential decomposition is functorial and exact on submodules and quotients; different exponential types have no horizontal morphisms. These are the classical formal decomposition facts, recalled in Sabbah [11, Section 3, Corollary 3.3 and its proof]. Consequently the quotient can only use $U_0 \subset M_0$. Since M_0 is trivial over the displayed field, it is semisimple there, and this quotient extends to a horizontal functional on M_0. It is a constant linear combination of its full dual frame. Positive seed orders cannot sum to one.

Adjoining $\log\rho$, with derivative $1/\rho$, does not add constants or negative-power exponential solutions. The same argument applies over this differential field. A finite sum of positive powers times logarithmic polynomials still cannot be one. This also explains why a full regular frame, rather than semisimplicity of the whole ambient module, is the relevant hypothesis. For an elementary check, expand this field in $\mathbb{C}(\log\rho)((\rho))$. The operator $\rho\partial_\rho$ preserves its valuation. A nonzero exponential twist has a negative-valuation scalar term in that operator, which cannot be cancelled by a regular matrix. Its horizontal maps to the trivial module therefore vanish even over this field. Its constants are $\mathbb{C}$, since a rational function $a(z)$ satisfying $a' + na = 0$ is zero for integer $n \neq 0$ and constant for $n = 0$. □

Lemma 11.18 (No finite-target contribution from spatial infinity)**.** *Fix a small nonzero parameter ϵ_0. Suppose in a neighbourhood of ϵ_0 that $\|\nabla_x S\| \geq c\|x\|$ outside a compact set, where $S = P/\epsilon - q$ and q is quadratic. The outer fibre pairs of (ϵ, S) are locally trivial over bounded target neighbourhoods. Consequently no additional finite-target discriminant or conormal multiplicity arises from spatial infinity there.*

Proof. On $S = g$ with g bounded, $\partial_\epsilon S = -(g+q)/\epsilon = O(1 + \|x\|^2)$, locally uniformly in ϵ. A target-lifting vector field obtained by dividing the conjugate gradient by its squared norm has size $O(1/\|x\|)$. Correcting a parameter-lifting vector field by this field has size $O(1+\|x\|)$. After a cutoff inside a larger ball, these fields are complete for finite transport times: the radial growth inequality is at most linear. They lift the coordinate paths in the target and parameter neighbourhood and identify the exterior fibre pairs. All changes at a finite target are therefore supported inside a compact set. This is a fixed nonzero-parameter statement; it does not assert a uniform compact set as $\epsilon \to 0$. □

Proposition 11.19 (Bounded critical branches determine the regular rank)**.** *Let P have degree $D \geq 2$ and leading form F_D, and let q be quadratic. Suppose, locally uniformly for small nonzero ϵ, the phase $S = P/\epsilon - q$ has a complex linear gradient lower bound outside a compact set. The constants and compact set may depend on ϵ. Assume also*

$$q_2(w) \neq 0 \quad \textit{whenever } [w] \in \mathbb{P}^{n-1} \textit{ and } \nabla F_D(w) = 0. \tag{41}$$

Then the rank of the full zero-exponential summand equals the sum of the generic local Milnor numbers of bounded phase-critical branches whose limiting P value is zero. An escaping branch never has action polar part α/ϵ for any constant α.

Proof. At a stationary branch the chain rule gives $S' = -P/\epsilon^2$, hence $q = -S - \epsilon S'$. A bounded branch has bounded q, so its action has only the possible polar term $P(x(0))/\epsilon$. If a branch escapes, divide $\nabla P = \epsilon \nabla q$ by $\|x\|^{D-1}$. Its limiting direction is singular for F_D. Condition (41) makes the quadratic term of q along that direction nonzero. Thus $q(x(\epsilon))$ has a pole, whereas $-S - \epsilon S'$ would be bounded for either zero polar part or a single term α/ϵ. This excludes both possibilities, without an escape-scale or Morse assumption.

We spell out the global rank comparison. Fixed-parameter tameness gives full twisted rank equal to the total critical Milnor number by Pham's decomposition. For the regular algebraic input $\mathcal{O}$ on $\mathbb{C}_\epsilon^* \times \mathbb{C}^n$, form the ordinary direct image by (ϵ, S). Its relevant cohomology is the middle reduced Gauss–Manin module; the constant fiber cohomology has zero exponential integral in the affine S coordinate. Lemma 11.18 excludes any finite-target contribution from spatial infinity, using the locally uniform gradient hypothesis. Every nonvertical discriminant curve at finite target comes from an affine critical value, with conormal multiplicity equal to the sum of its local Milnor numbers.

First ramify all critical-value curves into graphs. Roucairol's nonramified rank formula, Theorem 0.1(2), Section 1.2, together with Theorem 0.3 [10], computes every nonzero exponential-block rank from those graph multiplicities. Count all conjugate graphs even when their polar parts agree. No distinct-action assumption or monodromy formula is used. More explicitly, on the ramified parameter disc let ℓ index the distinct critical-value graphs $c_\ell(\epsilon)$ and put

$$\mu_{\mathrm{tot}} = \sum_x \mu_x(S), \qquad m_\ell = \sum_{x:\, S(x)=c_\ell} \mu_x(S), \qquad Q_\ell = \mathrm{pp}_{\epsilon=0}\, c_\ell.$$

The first sum is over the generic affine critical scheme, and the second over the branches on that value graph. Tameness gives $\operatorname{rank} M = \mu_{\mathrm{tot}} = \sum_\ell m_\ell$. The conormal multiplicities entering Roucairol are exactly m_ℓ: the affine Milnor decomposition gives them locally and the preceding lemma excludes an extra term at spatial infinity. In the graph case the rank formula reads

$$\operatorname{rank} M_Q = \sum_{\ell: Q_\ell = Q} m_\ell \quad (Q \neq 0), \qquad \operatorname{rank} M_0 = \mu_{\mathrm{tot}} - \sum_{Q \neq 0} \operatorname{rank} M_Q = \sum_{\ell: Q_\ell = 0} m_\ell.$$

All conjugate graphs are included after the cover; equal polar parts are grouped only in the sum for M_Q. Exactly the bounded-action branches remain. By the first part they are precisely the bounded branches with limiting P value zero. This proves the assertion about the entire regular rank. □

We use the following explicit way to construct that regular part. Suppose a bounded group at P value α has coordinates, after a cover, in which

$$S = \alpha t + H_0(z) + E_\rho(z), \qquad \omega = \rho^\kappa J_\rho(z)\, dz, \quad E_\rho \in \rho\mathbb{C}[z][[\rho]], \quad \kappa > 0, \tag{42}$$

where J_ρ is a unit with polynomial coefficients at each order. If λ is a dual functional on the fixed twisted quotient of H_0, then

$$L(A) = e^{\alpha t} \rho^\kappa \lambda\big(e^{E_\rho} J_\rho A(x(\rho, z))\big)$$

is a formal functional on the original module. The product rule proves spatial exactness. Cartan's formula for the coordinate velocity, together with the displayed moving Jacobian, proves parameter horizontality. Every coefficient uses finitely many polynomial terms. Fixed original markers with nonnegative powers in these coordinates have positive seed order.

The rank check has two directions. First, every bounded critical branch in the group must have bounded scaled coordinates z. Its limiting z then belongs to the critical scheme of H_0, since $d_z E_\rho$ has positive order. Conversely, every isolated critical point of H_0 deforms with its full local intersection multiplicity in $H_0 + E_\rho$. In the applications below the coordinate map $x(\rho, z)$ is regular at $\rho = 0$ near these points, so all the deformed points give bounded original critical branches. No root of the fixed model is merely counted without being realized in the original family. Conservation of local intersection multiplicity therefore counts exactly the whole group, including collisions. One must also check that the fixed critical scheme is finite and that its total Milnor number equals the fixed twisted rank.

A polynomial basis of the fixed quotient then gives independent functionals at leading order. Distinct groups are separated by polynomial jets, with arbitrarily high vanishing orders to overcome the finite negative powers introduced by the scaled representatives. When their number equals the regular rank of Proposition 11.19, they exhaust M_0. These checks will be made explicitly below; the formal perturbation formula alone does not establish completeness.

11.10 The cusp and tangent families together

Both remaining irreducible or line–conic cases have a unique leading singular direction. If F_2 is nonzero there, a gradient argument excludes purity. If it is zero and the direction is isotropic, the frame test applies. We treat the nonisotropic alternative with its full Gaussian

$$q = \tfrac{1}{2}(Au^2 + Bv^2 + mT^2 + 2Cuv + 2DuT + 2EvT) + \ell_u u + \ell_v v + \ell_T T + q_0, \qquad m \neq 0.$$

Translations and linear changes give the two exact normal forms

$$P_C = T(u^2 + \gamma v + h) + v^3 + buv + eu + fv + k, \qquad \text{cusp}, \tag{43}$$

$$P_T = T(u^2 + gv + h) + (u + c)v^2 + eu + fv + k, \qquad \text{tangent line–conic}. \tag{44}$$

Here is the leading-form reduction. At the unique singular point, F_3 is T times a rank-one binary quadratic plus a binary cubic. Choose that quadratic as u^2. A linear shift of T removes the u^3, u^2v terms, leaving $Tu^2 + auv^2 + b_0v^3$. If $b_0 \neq 0$, a shift of v by a multiple of u removes uv^2, and rescaling gives the cusp. If $b_0 = 0$, tangency to a smooth conic requires $a \neq 0$; rescaling gives the tangent form. The direction of T is unchanged, so its Gaussian square coefficient remains nonzero. For the cusp, translate u to remove uT, translate T to remove u^2, and translate v to remove v^2. For the tangent case use the same first two translations, then a v translation to remove uv. These operations leave the displayed arbitrary coefficients and transform, rather than reset, the means and cross terms of q.

Lemma 11.20. *For both leading forms, F_2 nonzero at the singular direction implies $\|\nabla P\| \geq c\|x\|$ outside a complex compact set. In the nonisotropic forms above, each fixed nonzero phase $tP - q$ has the same type of lower bound, locally uniformly in t. Their full generic twisted ranks are six and five, respectively.*

Proof. Any failure of a linear gradient bound approaches the unique leading singular direction, so $u, v = o(T)$. For the cusp, the u equation divided by T makes u bounded. The T equation divided by T then has the nonzero limit $2F_2(0, 0, 1)$. For the tangent, the transverse equations give $uv = O(T)$ and $2Tu + v^2 = O(T)$. If u is unbounded these imply $u = O(|T|^{1/3})$; otherwise $u^2/T \to 0$ directly. The T equation again has the same nonzero limit. Apply the Gaussian gradient criterion. Replacing the quadratic value by $-m/2$ gives the fixed-phase bounds for the normal forms. Compactness of the coefficient estimates gives local parameter uniformity.

For clarity, the ranks do not come from an escape count. The three quadratic gradient equations have total projective length eight. Their only infinity point is the T direction. In the chart $T = 1$ with homogenizing coordinate z, the cusp u equation solves $u = O(z)$; the T equation then gives $z = 0$, and the last equation is $3tv^2 = 0$. Its length is two. For the tangent the T equation solves $z = O(u^2)$, the u equation gives $u = -v^2/2 + O(v^4)$, and the v equation has leading term $-tv^3$. Its length is three. Fixed-phase coercivity excludes affine components of positive dimension. Bézout and Pham's tame comparison therefore give ranks $8 - 2 = 6$ and $8 - 3 = 5$. □

Every escape in these two normal forms approaches the T direction, where $q_2 = m/2 \neq 0$. Proposition 11.19 therefore applies. We now list all possible bounded contributions; the other exponential blocks require no explicit construction.

Isolated critical points on a smooth constraint

Write $P = TA(u, v) + G(u, v)$. Where $A = 0$ is smooth, take $s = A$ and a coordinate y along that curve. The analytic shear

$$T' = T + \frac{G(y, s) - G(y, 0)}{s}$$

gives the exact identity $P = sT' + G(y, 0)$. At an isolated critical point this becomes $\alpha + sT' + y^{\mu+1}$. The already explicit characters (31) provide all μ dimensions with positive original-marker order.

For the cusp, $\gamma \neq 0$ gives a degree-six restriction to $v = -(u^2 + h)/\gamma$, so total isolated multiplicity is five. If $\gamma = 0, h \neq 0$, the two lines $u = \pm\sqrt{-h}$ each have a cubic restriction in v, giving multiplicity two each. For the tangent, $g \neq 0$ gives a degree-five restriction and multiplicity four. If $g = 0, h \neq 0$, each line $u = u_i$ gives

$$(u_i + c)v^2 + fv + eu_i + k.$$

It has one Morse point when $u_i + c \neq 0$, none when $u_i + c = 0, f \neq 0$, and a critical parabola when both vanish. The last case is included below. All repeated roots and equal critical values in the isolated cases retain their Milnor multiplicities; no Morse perturbation is assumed.

Four fixed models along the T lines

Here and below a scalar constant is harmless but retained in the phase value. Set $\epsilon = 1/t$. The critical equation in T is

$$u^2 = \epsilon(mT + Du + Ev + \ell_T) \tag{45}$$

whenever the constraint is the double line $u^2 = 0$. Thus every bounded phase-critical branch has $u = O(\epsilon^{1/2})$. This estimate will also verify that the following scalings lose no bounded contribution.

First consider a critical T line at $u = 0, v = v_0$ whose transverse quadratic part is $Tu^2 + \lambda(v - v_0)^2 + \beta u(v - v_0)$, with $\lambda \neq 0$. Scale $u = t^{-1/2}U$, $v = v_0 + t^{-1/2}V$, leaving T fixed. The volume is t^{-1} and the fixed phase is

$$H_0 = TU^2 + \lambda V^2 + \beta UV - mT^2/2 - \eta T - q_*.$$

Complete V, then T. This is a quartic in U, with leading coefficient $1/(2m)$, plus two nondegenerate Gaussians. Its twisted rank and critical Milnor number are both three, including all quartic collisions. Equation (45) and the v equation with nonzero linear coefficient 2λ give $v - v_0 = O(t^{-1/2})$. Every bounded branch in this group has bounded scaled variables. The model therefore supplies all three local dimensions, with original-marker values in $e^{\alpha t}t^{-1}\mathbb{C}[[t^{-1/2}]]$.

If instead $\lambda = 0, \beta \neq 0$, the same scaling gives

$$H_0 = TU^2 + \beta UV - mT^2/2 - \eta T - q_*.$$

The polynomial shear $V \mapsto V + TU/\beta$ removes TU^2. This is a nondegenerate three-variable Gaussian, of rank one. The u equation with coefficient β bounds $v - v_0$ by $O(u) + O(\epsilon)$, so bounded branches are again captured. It gives one dimension with the same positive volume order.

For the cusp's remaining line, $P = Tu^2 + v^3 + k$, use $u = t^{-1/2}U$, $v = t^{-1/3}V$. The fixed phase is

$$TU^2 + V^3 - mT^2/2 - \ell_T T - q_0.$$

Completing T gives a quartic, a cubic and one Gaussian, of total rank and Milnor number $3 \cdot 2 = 6$. The volume is $t^{-5/6}$. Bounded critical branches satisfy $u = O(t^{-1/2})$ and, from $3v^2 = \epsilon q_v$, $v = O(t^{-1/2})$, so their scaled coordinates are bounded.

For the tangent's remaining line, $P = Tu^2 + uv^2 + k$, use $u = t^{-1/2}U$, $v = t^{-1/4}V$. Completing T gives the fixed two-variable phase

$$U^4/(2m) + UV^2 - \ell_T U^2/m$$

and one Gaussian. Its rank and critical Milnor number are five. Here is a direct check: give U, V weights $1/4, 3/8$. The leading derivatives are $2U^3/m + V^2$ and $2UV$, with sole common zero the origin and quotient basis $1, U, U^2, V, V^2$. They form a regular sequence. The associated graded twisted de Rham complex is their Koszul complex, concentrated in top cohomology; all derivative and lower-phase terms have smaller weight. It follows that the same five classes give the twisted rank. The ordinary gradient quotient has the same weighted leading calculation, so its total Milnor number is also five for every ℓ_T. The volume is $t^{-3/4}$. Equation (45) bounds u, and $2Tu + v^2 = \epsilon q_u$ gives $v = O(t^{-1/4})$. This captures every bounded branch.

All remaining terms in these four scaled phases have positive order in a common finite cover. Their coefficients are polynomials at each order, and every fixed original marker has nonnegative order. Formula (42) constructs the stated complete local functionals. The Jacobian factors and coordinate-velocity terms are retained, exactly as in its proof.

The critical parabola

This occurs only for the tangent form, with $g = 0$, $h \neq 0$, $c^2 + h = 0$, $f = 0$, $c \neq 0$. Put

$$x = u + c, \qquad s = (x - 2c)T + v^2 + e, \qquad \alpha = k - ec.$$

Exactly $P = \alpha + xs$. Write $x = \rho X$, $s = \rho Y$, $\rho = t^{-1/2}$. The volume is

$$\frac{\rho^2}{\rho X - 2c}\, dX\, dv\, dY,$$

and the fixed phase is $XY + \psi(v)$, where

$$\psi(v) = -q\left(-c, v, \frac{v^2 + e}{2c}\right).$$

Its quartic coefficient is $-m/(8c^2) \neq 0$, so its fixed twisted rank and critical Milnor number are three. At every bounded phase-critical branch near this parabola, the exact normal equations give $x = \epsilon q_s = O(\epsilon)$ and $s = \epsilon q_x = O(\epsilon)$; the denominators stay away from zero. Thus X, Y are bounded. The scaled Jacobian is a unit times ρ^2, and its expansion and that of the phase have polynomial coefficients at each order. This gives all three dimensions, with strictly positive original-marker order. The other constraint line has its one isolated Morse point.

Exhaustive routing and completeness

The remaining double-line cases require only the following two tables. Their rows are disjoint. A missing common root means there is no bounded contribution at all, not an omitted escape.

Cusp with $\gamma = h = 0$	Bounded P-critical groups
$f \neq 0$	For each root $3v_i^2 + f = 0$ with $bv_i + e = 0$, one line with $\lambda = 3v_i$, $\beta = b$, rank three
$f = 0$, $e \neq 0$	None, for every b
$f = e = 0$, $b \neq 0$	One line with $\lambda = 0$, $\beta = b$, rank one
$f = e = b = 0$	Quartic–cubic model, rank six

The line values in the first row are $\alpha_i = v_i^3 + fv_i + k$; the other line values are k.

Tangent with $g = h = 0$	Bounded P-critical groups
$c \neq 0$	Put $v_0 = -f/(2c)$. None if $e + v_0^2 \neq 0$; otherwise one rank-three line with $\lambda = c$, $\beta = 2v_0$, value $k - f^2/(4c)$
$c = 0$, $f \neq 0$	None, for every e
$c = f = 0$, $e \neq 0$	Two lines $v_i^2 = -e$, each with $\lambda = 0$, $\beta = 2v_i$, rank one, value k
$c = f = e = 0$	Weighted quartic model, rank five, value k

To verify these tables directly, the double constraint forces $u = 0$. The cusp equations reduce to $3v^2 + f = 0$, $bv + e = 0$; the tangent equations reduce to $2cv + f = 0$, $v^2 + e = 0$. The remaining T coordinate is free. Together with the smooth constraint and separated-line alternatives already proved, this exhausts all coefficients of both normal forms.

For every fixed model the critical scheme is finite with the stated total Milnor number. The bounds above show that every bounded phase-critical branch has bounded scaled coordinates. Conservation of local critical multiplicity thus gives exactly the stated group counts. Proposition 11.19 shows that the groups at value zero account for the entire regular rank; all escaping branches are excluded from it.

We also need independence on the original polynomial module. Within a fixed group evaluate a polynomial basis of its fixed quotient: its normalized leading matrix is invertible by (42). Isolated groups use their finite power jets. Different line groups have distinct constant v_i; the parabola and the other isolated group have distinct constant u values. High-order Hermite cutoffs in these coordinates separate the groups. Choose their orders to exceed the largest pole order of the finitely many basis representatives, measured in the common cover. Then all cross evaluations have positive order. In a smooth isolated group the same assertion is ordinary finite jet interpolation. At the parabola chart, the leading denominator $-2c$ is a unit; multiplying by a high enough power of $u - c$ separates the other line without changing that unit. These constructions give an invertible block leading matrix of exactly the regular rank. There is no cyclic-seed assumption and no appeal to a count of unconstructed escape characters.

After a common cover this regular part is trivial, with all original-marker values of positive order: t^{-1} in the line and parabola models, $t^{-5/6}$ in the cusp's last model, $t^{-3/4}$ in the tangent's last model, and the positive powers of the isolated characters. Lemma 11.17 excludes a constant original marked period. Differential isolation and shifting k exclude every nonzero finite exponential-polynomial in the nonisotropic normal forms.

Proposition 11.21. *A pure cubic with cuspidal leading curve or tangent line–conic leading curve has an isotropic singular direction with $F_2 = 0$. It satisfies the mixed cutoff $m > 3 \deg Q$.*

Proof. The gate excludes nonzero F_2 at the singular direction. The exhaustive bounded-group construction excludes the nonisotropic alternative, including every collision and degree drop. In the remaining isotropic alternative, Lemma 7.1 supplies the Gaussian affine-linear coordinate. The already proved affine-linear theorem gives the asserted cutoff. □

11.11 Three concurrent distinct leading lines

Let the common vertex be the T direction, so the leading form is a squarefree binary cubic $F(u, v)$. Its binary gradient has only the zero at the origin. If F_2 at the vertex is nonzero, a sequence with $\nabla P = o(\|x\|)$ must approach that vertex; then $u, v = o(T)$ and the T equation has a nonzero linear leading term. The Gaussian gradient criterion excludes purity. An isotropic zero-F_2 vertex already gives a frame.

In the remaining alternative retain the full Gaussian with $m = q_{TT} \neq 0$ and write

$$P = TH(u, v) + G(u, v), \qquad \deg H \leq 1, \quad G_3 = F, \qquad q = mT^2/2 + TJ(u, v) + q_*(u, v),$$

where J is affine linear and q_* includes all lower terms. For each fixed t, the phase gradient is linearly coercive: its only possible limiting singular direction is the vertex, where its T derivative has leading term $-mT$. The estimate is locally uniform on compact parameter sets. Completing T exactly gives

$$tP - q = -\frac{m}{2}\left(T - \frac{tH - J}{m}\right)^2 + tG - q_* + \frac{(tH - J)^2}{2m}.$$

This change has volume Jacobian one. The binary phase has squarefree cubic leading form tF, whose two leading quadratic derivatives are a regular sequence of length four. Weighted leading reduction, or the same Koszul argument used above, gives full twisted rank four. The Gaussian factor has rank one. Condition (41) holds at the sole leading singular direction. Thus all escapes are excluded from every pure linear-action block by Proposition 11.19.

If H is a nonzero constant, P has no critical point. The full regular rank is zero and a constant original marked period is impossible. This remains true after a constant shift of P.

If $H = 0$, integrate the T Gaussian formally. A fixed polynomial marker becomes a fixed polynomial in u, v, using conditional mean $-J/m$ and covariance $1/m$. The effective Gaussian is

$$q_{\text{eff}} = q_* - J^2/(2m).$$

Its Hessian is nondegenerate by the Schur determinant identity $\det \operatorname{Hess} q = m \det \operatorname{Hess} q_{\text{eff}}$. The squarefree leading binary cubic gives $\|\nabla G\| \geq c\|(u, v)\|^2$ outside a compact set. The all-degree Gaussian gradient criterion therefore excludes every nonzero finite exponential-polynomial original period. The conditional normalization is independent of t.

Finally let H be nonconstant and use $u = H$ as an affine coordinate. Then $P = uT + G(u, v)$. If $R(v) = G(0, v)$ is nonconstant, its critical points give all the bounded groups. The exact smooth-constraint shear gives their local types $\alpha + sz + y^{\mu+1}$. This includes the empty critical set when R is linear. Their full power characters have positive original-marker orders, so the regular-rank and seed lemmas exclude a constant period without constructing the escaping blocks.

If $R = k$ is constant, write $G = k + uB(u, v)$, where B has degree at most two. The critical curve is

$$u = 0, \qquad T = -B(0, v).$$

The quadratic coefficient c of $B(0,v)$ is nonzero. Otherwise the squarefree cubic F would be divisible by u^2. Put $s = T + B(u,v)$, so exactly $P = k + us$ and the coordinate Jacobian is one. Scale $u = \rho U$, $s = \rho V$, $\rho = t^{-1/2}$. The fixed phase is

$$UV - q(0, v, -B(0,v)),$$

with quartic coefficient $-mc^2/2 \neq 0$ in v. Its rank and critical Milnor number are three. The exact critical equations $u = \epsilon q_s$, $s = \epsilon q_u$ show that every bounded branch has bounded scaled coordinates. This is precisely the complete quartic parabola construction above, now with Jacobian ρ^2. It supplies the entire regular block when $k = 0$, with positive original-marker orders; when $k \neq 0$ that block has nonzero exponential part. The polynomial representatives of v powers are original markers, so its leading matrix is invertible. The seed lemma applies.

These alternatives exhaust the affine polynomial H and the restriction R. They also exclude nonzero finite exponential-polynomial markers after differential isolation and shifting the constant term. We obtain the following conclusion.

Proposition 11.22. *Every pure cubic whose leading form is three distinct concurrent lines has an isotropic vertex with $F_2 = 0$. Hence it has a Gaussian affine-linear coordinate and satisfies $m > 3 \deg Q$.*

Proof. The gradient gate and the nonisotropic cases just proved leave only that vertex alternative. Apply Lemma 7.1 and the complete affine-linear theorem. □

11.12 Quadratic elimination and the repeated lines

The final two leading types have a repeated factor. They reduce to a polynomial in one variable after eliminating two Gaussian variables. The full leading-coefficient theorem makes an escape table unnecessary: only the actual degree, its leading coefficient and the determinant factor are needed.

Lemma 11.23 (The determinant factor in quadratic elimination)**.** *Over $\mathbb{C}(t)$, let a phase have the form*

$$-\tfrac{1}{2} y^T M(t) y + L(t,x)^T y + V(t,x),$$

where M is an invertible symmetric matrix and L, V are polynomial in x. Eliminating y identifies its top twisted module and constant volume seed with the one-variable phase and normalization

$$\varphi = V + \tfrac{1}{2} L^T M^{-1} L, \qquad h = (\det M)^{-1/2}.$$

The parameter connection is $\partial_t + \varphi_t + h'/h$. An algebraic square root is allowed. If the actual degree d of φ is at least two and $d\, v_\infty(h) > v_\infty([x^d]\varphi)$, its constant seed cannot have a rational unit quotient.

Proof. The change $y = z + M^{-1}L$ has Jacobian one. Its spatial twisted relations reduce the z factor to the one-dimensional Gaussian quotient, with normalized contractions $\ell(z_i z_j) = (M^{-1})_{ij}$. Parameter differentiation of its constant class inserts $-z^T M' z/2$, hence

$$-\tfrac{1}{2} \operatorname{tr}(M^{-1} M') = h'/h.$$

Terms from the moving center are spatial exact forms, by the product rule or Cartan's formula. This proves the identification of both connection and seed. After a finite cover at infinity, h is a

Laurent series and Theorem 8.4 applies. The cover multiplies both sides of the valuation inequality by the same positive integer.

The elimination is over the generic field. It need not define a conditional formal average at $t = 0$. The original Wick functional first gives its rational seed quotient by Lemma 10.1; this quotient is transported through the displayed module identification. No image-kernel calculation or convergent original period is needed. □

Lemma 11.24. *Let $u, Y_1, \dots, Y_s$ be independent standard Gaussian variables and*

$$P = Y^T A Y + B(u)^T Y + g(u), \qquad A = A^T,$$

with B, g arbitrary polynomials. Put

$$D(t) = \det(I - 2tA), \quad C(t) = (I - 2tA)^{-1}, \quad \varphi = tg - u^2/2 + (t^2/2)B^T C B.$$

If $\deg_u \varphi \geq 3$, then P is not pure.

Proof. Quadratic elimination gives $h = D^{-1/2}$. Let $d = \deg D$ and $r = \deg_u \varphi \geq 3$. Its leading coefficient is $N(t)/D(t)$, where N is a nonzero polynomial divisible by t: the degree-r coefficient vanishes at $t = 0$. Consequently

$$r v_\infty(h) - v_\infty(N/D) = (r/2 - 1)d + \deg N > 0.$$

Apply Lemma 11.23. This estimate is independent of the degrees of B, g and the dimension s. □

A nonisotropic repeated factor

Normalize the repeated factor to u with $\mathbb{E}[u^2] = 1$ and take an orthogonal Gaussian complement $Y \in \mathbb{C}^2$. Every such cubic has the form in Lemma 11.24, with

$$B(u) = pu^2 + qu + r, \qquad \deg g \leq 3.$$

For a double line $p \neq 0$. If its effective quartic coefficient is nonzero, the lemma excludes purity immediately. If that coefficient vanishes identically, then

$$0 = p^T \operatorname{adj}(I - 2tA)p = p^T p - 2t\big(\operatorname{tr}(A)p^T p - p^T A p\big).$$

Thus $p^T p = p^T A p = 0$. The nonzero vector $(0, p)$ is isotropic, lies in the leading singular plane $u = 0$, and satisfies $F_2(0, p) = 0$. It supplies the frame of Lemma 7.1. For a triple line, $p = 0$ and $[u^3]g \neq 0$. The effective phase has exact cubic coefficient $t[u^3]g$, so the same high-degree lemma excludes purity. This exhausts all matrices A, including nilpotent and defective ones, without a matrix case split.

An isotropic repeated factor

Use Gaussian circular coordinates with repeated factor Z. The singular plane is $Z = 0$, whose only isotropic direction is the W axis. Write $\lambda = [W^2]F_2$. If $\lambda = 0$, the frame test applies. Assume $\lambda \neq 0$.

For a double line write its other factor as $aW + bT + cZ$. The Gaussian null rotation

$$Z = Z', \quad T = T' + sZ', \quad W = W' - sT' - s^2 Z'/2$$

removes the T term if $a \neq 0$, or removes the Z term if $a = 0, b \neq 0$. Scaling P gives exactly these leading forms:

$$\mathrm{I} : Z^2W + a_0Z^3, \qquad \mathrm{II} : Z^2T, \qquad \mathrm{III} : Z^3.$$

The last is the triple line. The rotation preserves the Gaussian and keeps the W^2 coefficient nonzero. All lower terms remain arbitrary.

Write $P = \lambda W^2 + WH(Z,T) + G(Z,T)$. In type I, $H = Z^2 + bZ + cT + d$ and $G = a_0Z^3 + G_2$. In types II and III, $H = bZ + cT + d$ and $G = Z^2T + G_2$ or $Z^3 + G_2$ respectively. Here G_2 is an arbitrary polynomial of degree at most two. Eliminating W in $tP - ZW - T^2/2$ gives

$$\Psi = t\left(G - \frac{H^2}{4\lambda}\right) + \frac{ZH}{2\lambda} - \frac{T^2}{2} - \frac{Z^2}{4\lambda t}.$$

Its T^2 coefficient is $C_T = at - 1/2$, where $a = g_{02} - c^2/(4\lambda)$ and $g_{02} = [T^2]G_2$. Eliminate T as well. If $e = \deg_t C_T \in \{0, 1\}$, the total normalization has $v_\infty(h) = (1+e)/2$. The actual remaining leading coefficients are

type	$[Z^d]\varphi$	d
I	$-\dfrac{t}{4\lambda} - \dfrac{c^2t^2}{16\lambda^2 C_T} = -\dfrac{t(2g_{02}t-1)}{8\lambda C_T}$	4
II	$-\dfrac{t^2}{4C_T}$	4
III	t	3.

None is identically zero. Their valuations are at most $e-1$, $e-2$, and -1, respectively. Each is strictly less than $d(1+e)/2$. Lemma 11.23 excludes purity in all three types. In particular this includes the boundary where the quartic coefficient tends to a nonzero constant while lower coefficients grow.

Proposition 11.25. *Every pure ternary cubic with repeated leading factor has a Gaussian affine-linear coordinate. It therefore satisfies $m > 3\deg Q$.*

Proof. The nonisotropic and isotropic alternatives above exhaust the repeated factor. In every surviving branch the explicit frame test succeeds. Apply the complete affine-linear theorem. □

11.13 The geometric case split for frame extraction

For a cubic of degree exactly three, either the frame test (13) succeeds, or it does not. In the latter case, the projective curve $F_3 = 0$ belongs to exactly one of these types:

Leading curve	No-frame exclusion
Smooth cubic	Corollary 11.3
Irreducible nodal cubic	Corollary 11.16
Irreducible cuspidal cubic	Proposition 11.21
Line and smooth conic, transverse	Corollary 11.16
Line and smooth conic, tangent	Proposition 11.21
Three nonconcurrent lines	Corollary 11.16
Three concurrent distinct lines	Proposition 11.22
Double line and a distinct line	Proposition 11.25
Triple line	Proposition 11.25

The list is exhaustive by factorization of a plane cubic over $\mathbb{C}$. Every row has now been proved, including its lower terms and all critical degeneracies. In the repeated-line rows the singular set is a whole projective line; the proof checks its isotropic directions rather than choosing an arbitrary point on it.

Proof of Proposition 7.2. Suppose the frame test fails for a pure polynomial of degree three. Its leading curve is one of the nine types in the table. The smooth case is excluded by the gradient criterion, and the ordinary-node cases by Corollary 11.16. Propositions 11.21, 11.22 and 11.25 force a frame in every other case. All possibilities give a contradiction. This proves frame extraction. □

12 The ternary-cubic theorem and the boundary

12.1 The sharp mixed cutoff

Both Proposition 7.3 and the frame-extraction statement, Proposition 7.2, are now proved. For $\deg P \le 2$, Proposition 3.1 gives the result. For degree three, frame extraction and Lemma 7.1 put P in the family of Proposition 7.3. A complex orthogonal linear substitution preserves both total degree and Gaussian moments. The bound therefore transfers back to every original multiplier Q. Equation (12) proves optimality, completing the proof of Theorem 1.2.

12.2 The normal-form conclusion

Proof of Theorem 1.3. For degree at most two, Corollary 3.2 gives one-sided support. For degree three, Proposition 7.2 and Lemma 7.1 provide an affine-linear Gaussian coordinate. Proposition 7.3 then gives one-sided support or the exceptional orbit represented by P_*. Conversely, one-sided phases are pure by Lemma 2.1, and P_* is pure by (11). Gaussian linear changes and nonzero scalar multiples preserve purity. □

12.3 The dimension–degree deduction

Proof of Theorem 1.1. Wilson proves the assertion for $n \le 2$ in every degree. Proposition 3.1 proves it for $d \le 2$ in every dimension. Theorem 1.2 supplies $(n, d) = (3, 3)$.

Conversely, suppose none of these conditions holds. Then $n \ge 3$ and $d \ge 3$. If $n = 3$, the excluded pair $(3, 3)$ leaves $d \ge 4$, and P_3 from Section 5 is a counterexample. If $n \ge 4$, use P_4. Additional independent Gaussian variables can be ignored, and increasing the allowed degree does not remove an existing example. These possibilities exhaust the complement. □

The two minimal failures occur at degree four in three variables and degree three in four variables. Between them, the ternary-cubic bound is exact: a multiplier of degree q can expose a nonzero moment at power $3q$, but never beyond it.

Scope and further questions

The boundary theorem leaves open the classification of individual safe phases outside its universal range, and the general converse from safety to purity. Its proof combines classical comparison and decomposition results with the explicit compact-piece, neck and tail estimates above.

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