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Spectral measures for subfactors and Lie groups

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Abstract

We compute the rooted laws of four small-index principal graphs. For a unitary fusion category we express the rooted law of the canonical Lagrangian inclusion in terms of formal codegrees. This explains why two Haagerup extensions have the same moments. We determine when the signed measures of the Deligne–de Man interpolation have positive support and nonnegative cumulative functions for all weights. For the adjoint tensor square, the second and fourth heat derivatives determine the simple Lie type. For the n-th tensor power, we prove an expansion in $\frac{1}{n}$, with two correction terms after the leading factor, of the Laplace transform of the Casimir law at time $\frac{s}{n}$. The second correction is an explicit function of the dimension and of the squared norm, divided by the dimension, of the symmetrized quartic trace tensor, affine in the latter.

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1 Introduction

Subfactors and their rooted laws A factor of type II_1 is an infinite-dimensional von Neumann algebra with trivial center which carries a faithful normal tracial state. If $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of such factors, Jones defined its index $[\mathcal{M} : \mathcal{N}]$ as the dimension of $L^2(\mathcal{M})$ as a module over $\mathcal{N}$, and proved in [29] that it belongs to $\{4\cos^2 \frac{\pi}{k} : k \geqslant 3\} \cup [4, \infty]$. Suppose that the index is finite. Iterating the basic construction gives the Jones tower $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$, and we put $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_{-1} = \mathcal{N}$. The relative commutants $\mathcal{N}' \cap \mathcal{M}_{n-1}$ and $\mathcal{M}' \cap \mathcal{M}_n$ are then finite-dimensional. Together with their inclusions, the Jones projections and the traces, they form the standard invariant, which Jones organized as a planar algebra [30]. We put $P_n = \mathcal{N}' \cap \mathcal{M}_{n-1}$. The simplest summary of this invariant is a bipartite graph Γ, the principal graph. Its even and odd vertices are the irreducible $\mathcal{N}$-$\mathcal{N}$ and $\mathcal{N}$-$\mathcal{M}$ bimodules which occur in the alternating relative tensor products $\mathrm{L}^2(\mathcal{M}) \otimes_{\mathcal{M}} \mathrm{L}^2(\mathcal{M}) \otimes_{\mathcal{N}} \mathrm{L}^2(\mathcal{M}) \otimes_{\mathcal{M}} \cdots$, and the number of edges between an even vertex X and an odd vertex Y is the multiplicity of Y in $X \otimes_{\mathcal{N}} \mathrm{L}^2(\mathcal{M})$.

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The root $*$ is the trivial bimodule. For an irreducible inclusion of finite index, the dimension $c_n = \dim P_n$ is the number of closed walks of length $2n$ on Γ starting at $*$ (Proposition 2.6).

The numbers c_n are therefore the moments of a probability measure on $[0, \infty)$, namely the spectral measure at the root of the square of the adjacency operator of Γ. We call it the rooted law of the inclusion. When Γ is finite, it is a finite sum of atoms, its largest atom is the index, and the mass of this atom is the inverse of the global dimension of the even part (Section 2.2). The rooted law is computable from the graph alone, and, when the index is larger than 4, it determines the annular multiplicities of Jones [28], summed over the rotation eigenvalues. They count the irreducible modules of the planar algebra under the action of the annular Temperley–Lieb category. On the other hand, the rooted law does not determine the standard invariant. For instance, let $\mathcal{R}$ be the hyperfinite factor of type II_1. For outer actions of $\mathbb{Z}/4\mathbb{Z}$ and of $(\mathbb{Z}/2\mathbb{Z})^2$, the inclusions $\mathcal{R} \subseteq \mathcal{R} \rtimes \mathbb{Z}/4\mathbb{Z}$ and $\mathcal{R} \subseteq \mathcal{R} \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ have the same principal graph, a star with one odd vertex and four even vertices, hence the same rooted law $\frac{3}{4}\delta_0 + \frac{1}{4}\delta_4$. In both cases the even vertices are the $\mathcal{R}$-$\mathcal{R}$ bimodules attached to the elements of the group, and their fusion rules are given by the group law. Since the standard invariant determines these fusion rules and the two groups are not isomorphic, the two standard invariants are not isomorphic. In this paper we compute rooted laws in several families and we determine, in each family, which part of the structure they detect and which part they lose. We also compare them with two kinds of measures from Lie theory, the signed root measures of Deligne and de Man [12] and the Casimir laws of tensor powers of the adjoint representation.

For subfactors of index at most 4, the rooted law has a circular lift, that is, it is the image by $u \mapsto (u + u^{-1})^2$ of a probability measure on the unit circle which is invariant under $u \mapsto u^{-1}$ and $u \mapsto -u$ [6]. The cyclotomic expansions of these lifts for the exceptional Dynkin diagrams are given in [5]. Evans and Pugh studied the corresponding joint spectral measures for nimreps of SU(3) and G_2 [18, 19, 20]. When the support extends above 4, the circular lift no longer exists, but a lift to the circle and the positive half-line still encodes the annular multiplicities (Proposition 2.4 and Theorem 2.7).

Fusion categories and canonical inclusions The even part of a subfactor of finite depth is a unitary fusion category $\mathcal{C}$. Such a category has finitely many simple objects, and every object is a finite direct sum of simple ones. The tensor product of two simple objects X and Y decomposes as $X \otimes Y \cong \bigoplus_Z N_{XY}^Z Z$, and the nonnegative integers N_{XY}^Z are the fusion rules. They define the based ring $K(\mathcal{C})$. The category contains more information than its fusion rules. For instance the associativity isomorphisms $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$, the associator, are given by matrices whose entries depend on choices of bases in the spaces of morphisms. The Drinfeld center $Z(\mathcal{C})$ is the braided category of pairs formed by an object X of $\mathcal{C}$ and a half-braiding, that is, a natural isomorphism $X \otimes - \cong - \otimes X$ compatible with the tensor product. A connected étale algebra of $Z(\mathcal{C})$ is a commutative separable algebra A with $\dim \mathrm{Hom}(\mathbf{1}, A) = 1$. Those of the largest possible dimension $\dim(\mathcal{C})$ are called Lagrangian algebras. They correspond to the indecomposable module categories over $\mathcal{C}$, and the category of modules over such an algebra is a fusion category Morita equivalent to $\mathcal{C}$ [9, Proposition 4.8 and Remark 4.9]. The Lagrangian algebra attached to $\mathcal{C}$ itself is $I(\mathbf{1})$, where I is the right adjoint of the forgetful functor $F\colon Z(\mathcal{C}) \to \mathcal{C}$. Every irreducible finite-index inclusion whose dual canonical endomorphism is $I(\mathbf{1})$ has the same rooted law $\mu_{\mathcal{C}}$. An example is the dual of the Longo–Rehren inclusion [43, 47], and another, when such an extension exists, is an inclusion of conformal nets given by a holomorphic extension whose algebra object is $I(\mathbf{1})$. Theorem 4.1 describes $\mu_{\mathcal{C}}$:

$$\mu_{\mathcal{C}} = \Big(1 - \sum_E \frac{1}{f_E^2}\Big)\delta_0 + \sum_E \frac{1}{f_E^2}\,\delta_{\dim(E) f_E},$$

where E runs through the irreducible representations of the complexified fusion ring $K(\mathcal{C})_{\mathbb{C}}$ and f_E is the formal codegree of Ostrik [52], that is, the scalar by which $\sum_X \operatorname{Tr}(X,E)X^*$ acts on E. For the one-dimensional representation given by the Frobenius–Perron dimensions, the formal codegree is the global dimension of $\mathcal{C}$. Equivalently, for any integer $n \geqslant 1$, we have

$$c_n = \sum_{X_1,\dots,X_n \in \operatorname{Irr}(\mathcal{C})} \bigl(\dim \operatorname{Hom}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)\bigr)^2 = \sum_E \frac{(\dim(E) f_E)^n}{f_E^2}.$$

This statement is a reformulation of results of Ostrik [53, Remark 2.11 and Theorem 2.13] in terms of subfactors. In particular, the rooted law, every dimension $\dim P_n$ and every annular multiplicity summed over the rotation eigenvalues depend only on the based ring $K(\mathcal{C})$, and not on the associator.

The Haagerup subfactor, of index $\frac{5+\sqrt{13}}{2}$, was constructed by Asaeda and Haagerup [4]. Grossman and Snyder proved that the Morita equivalence class of its even part contains exactly three fusion categories $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, that $\mathcal{H}_2$ and $\mathcal{H}_3$ have the same based ring and that they are not equivalent [22, Theorem 1.4 and Corollary 3.22]. In Corollary 4.2 we deduce that the canonical inclusions attached to $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, which are different from the Haagerup subfactor itself, have the rooted laws

$$\mu_1 = \frac{\delta_{D_+}}{D_+^2} + \frac{\delta_{D_-}}{D_-^2} + \frac{2}{9}\delta_3 + \frac{80}{117}\delta_0, \quad \mu_2 = \mu_3 = \frac{\delta_{D_+}}{D_+^2} + \frac{\delta_{D_-}}{D_-^2} + \frac{1}{9}\delta_6 + \frac{31}{39}\delta_0,$$

where $D_\pm = \frac{39\pm 9\sqrt{13}}{2}$, and that no braided autoequivalence of the common center relates the second and third Lagrangian algebras. In the unpublished manuscripts [1, 2, 3], a vertex operator algebra of central charge eight whose representation category is the center of the Haagerup categories is studied, together with three holomorphic extensions. Corollary 4.2 explains why two of these extensions have the same moments (Remark 4.3). For pointed categories Vec_G^ω, whose simple objects are the elements of a finite group G and whose associator is a 3-cocycle ω, the rooted law depends only on $|G|$ (Example 4.4). For a cubic family of cocycles on $\mathbb{F}_p^3$, the dimensions of the simple objects of the center still distinguish the trivial associator from the nontrivial ones (Corollary 4.7).

Small index In Section 3 we give closed formulas for the rooted laws of the four finite principal graphs of the Haagerup, Asaeda–Haagerup, Goodman–de la Harpe–Jones 3311 and Izumi 2221 subfactors (Proposition 3.7). We show in Proposition 3.6 that, for a principal graph without multiple edges which has more than one vertex at some depth, and for an index larger than 4, the first positive annular weight is the first depth at which the principal graph has more than one vertex, and that its multiplicity is the number of vertices at that depth minus one. For the exceptional Dynkin diagrams we recall the expansions of [6, 5], explain the coincidence between the ternary index of [5, Theorem 8.7] and the parameter of Deligne and de Man, and show that the circular law at the short tip of E_6 is $\frac{1+2\mathsf{h}_1}{3}\mathsf{d}_2''$, in the notation of Section 3. Deleting a vertex gives a second description of the weights of a rooted graph through the quotient of two characteristic polynomials.

What the associator contributes Since the rooted law of the canonical inclusion depends only on the based ring (Theorem 4.1), it does not see the associator, and we ask how much additional data recovers it. A change of bases in the one-dimensional spaces of morphisms of Vec_G^ω multiplies ω by a coboundary. The Laurent monomials in the entries of ω which are invariant under these gauge changes are exactly the evaluations on integral cycles of the bar

complex (Proposition 5.1). For $G = \mathbb{F}_p^r$ with p odd and a basis of G fixed, we exhibit in Theorem 5.6 a family of $r + \binom{r}{2} + \binom{r}{3}$ such products which recovers the cohomology class of ω, and we show that no smaller family suffices. A cubic coordinate of the class is read from six entries. For the cubic family on $\mathbb{F}_p^3$, with six marked entries and arbitrary normalized gauge changes, Theorem 5.10 shows that the sharp angular error threshold per entry is $\frac{\pi}{6p}$. Proposition 5.12 gives the corresponding three-entry test for a Tambara–Yamagami category, whose simple objects are the elements of a finite abelian group H and one more object m with $m \otimes m \cong \bigoplus_{a \in H} a$.

Relative root spectra and deletions Deligne observed that, for the simple Lie algebras of types A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8, the dimensions of the irreducible components of the low tensor powers of the adjoint representation are given by rational functions of one parameter [11], see also [12]. For a dominant weight λ of the simple Lie algebra of a connected compact Lie group K, the Weyl dimension formula is a product over the positive roots α of $\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$, where ρ is half the sum of the positive roots. The signed measure $L_K(\lambda)$ of [12, p. 578] is the difference of the counting measures of the two lists of numbers $12(\lambda + \rho, \alpha)$ and $12(\rho, \alpha)$, for the inner product induced by the Killing form. Its cumulative function $x \mapsto -L_K(\lambda)((-\infty, x])$, with the sign convention of Section 6, is nonnegative, and the measure is the spectral shift of a pair of positive diagonal matrices. For the seven-weight interpolation along the exceptional line, we prove in Theorem 6.2 that the cumulative functions are nonnegative with positive atom support for all weight labels precisely for $0 < \mu \leqslant \frac{2}{5}$, where $\mu = \frac{6}{h^\vee}$ is the Deligne–de Man parameter and $h^\vee$ is the dual Coxeter number. On the Cartan ray the range is $0 < \mu \leqslant 3$ by Theorem 6.3. Deleting r even vertices of a bipartite graph changes the square of the adjacency on the odd part by a positive operator of rank at most r. For a finitely supported signed measure on $[0, \infty)$ with integer coefficients, mass zero and nonnegative cumulative function, Proposition 6.6 shows that the maximum of the cumulative function is the minimum number of deleted vertices in a weighted bipartite realization, and Theorem 6.7 determines when the Cartan shifts are deletions of a single vertex or of a graph with integer adjacency.

Heat measures and Casimir laws Let K be a connected compact Lie group with simple Lie algebra of dimension d, and let χ be the character of its adjoint representation V. The positive Laplacian $\mathcal{L}$ of K, for the metric given by the negative of the Killing form, defines the heat semigroup $P_t = \mathrm{e}^{-t\mathcal{L}}$, and P_t is the convolution by the law at time t of the Brownian motion on K with generator $-\mathcal{L}$, started at the identity. The image of this heat measure by $\frac{\chi}{d}$ is a probability measure on $[-1, 1]$. As $t \to \infty$ it tends to the image of the Haar measure by $\frac{\chi}{d}$, which is the spectral measure at the trivial representation of the adjacency operator of the representation graph of V, after division of the spectral variable by d (Section 2.5). The n-th moment of the image of the heat measure at time t is the Laplace transform at t of the Casimir law σ_n of $V^{\otimes n}$, that is, of $\sigma_n = d^{-n} \sum_\lambda m_{n,\lambda} \dim(V_\lambda)\, \delta_{c_\lambda}$, where $m_{n,\lambda}$ is the multiplicity of V_λ in $V^{\otimes n}$ and c_λ is the eigenvalue on V_λ of the Casimir operator for the same metric, which is one on V (Proposition 7.1). In tensor degree two, the second and fourth derivatives at $t = 0$ identify the simple type, and the derivatives through order three do not suffice (Theorem 7.6). The fourth derivative is governed by the norm of the harmonic part of the quartic trace polynomial $Y \mapsto \mathrm{Tr}(\mathrm{ad}(Y)^4)$, and this harmonic part vanishes exactly on the Deligne series. In Theorem 7.9 we prove that, for every $s_1 > 0$ and uniformly for $0 < s \leqslant s_1$,

$$\int \mathrm{e}^{-\frac{s}{n}c}\, \mathrm{d}\sigma_n(c) = \left(1 + \frac{2s}{d}\right)^{-\frac{d}{2}} \left(1 + \frac{A_d(s)}{n} + \frac{B_{d,q_4}(s)}{n^2} + O(n^{-3})\right),$$

where A_d depends only on d, and B_{d,q_4} is an explicit function of d and of the squared norm, divided by d, of the symmetrized quartic trace tensor, which we denote by q_4. It is affine in q_4. The leading factor is the Laplace transform of $\frac{1}{d}$ times a chi-squared law with d degrees of freedom. The proof writes the moments as Gaussian integrals on the Lie algebra by means of the character formula of Kirillov (Lemma 7.8), and then controls the remainder of a Taylor expansion uniformly in n. At a finite level the classical Casimir eigenvalues, up to a rescaling of time, do not define a Markov semigroup on the spectrum of the $\mathrm{SU}(2)_k$ fusion algebra for $k \geqslant 2$, and, for $k \geqslant 1$, the nonzero Markov generators which are diagonal in the discretized characters and supported on nearest neighbours are the positive multiples of the generator of the fundamental fusion walk (Subsection 7.8).

Structure of the paper Section 2 fixes the conventions, and Section 3 treats rooted graphs of small index. Section 4 gives the canonical rooted law and compares it with examples from the Haagerup categories and from pointed centers. Section 5 recovers associator data from products of phases. Section 6 treats relative root spectra. Section 7 develops the Casimir laws, the expansion of the heat moments and the finite-level analogue of the heat semigroup. The questions about rooted laws, associators and heat measures appear at the ends of the corresponding sections. The script `cross_checks.py` computes the rooted laws of the graphs of small index and checks the laws of the Haagerup extensions, sample cases of Theorem 6.2, the exceptional square law and fourth Casimir traces. The programs are available at `https://github.com/cedricarhancet/`. The scripts `checks.py` and `check_interval.py`, in the folder `review`, were written by Codex (OpenAI) when it reviewed a previous version of this paper.

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2 Preliminaries

2.1 Rooted spectral measures and circular lifts

Let V be a countable vertex set, let $o \in V$, and let A be a bounded selfadjoint adjacency operator acting on ℓ^2_V. Multiple edges and loops are allowed, with their multiplicities included in the matrix coefficients. We use the probability measure

$$\mu_{A,o}(B) \overset{\text{def}}{=} \langle E_A(B)\delta_o, \delta_o\rangle, \quad B \subseteq \mathbb{R} \text{ Borel}, \tag{2.1}$$

where E_A is the spectral resolution of A.

Proposition 2.1 *For any $k \in \mathbb{N}$, we have $\int x^k \,\mathrm{d}\mu_{A,o}(x) = \langle A^k\delta_o, \delta_o\rangle$. For an adjacency matrix this is the number of length-k closed paths at o, counted with multiplicity. If V is finite, then*

$$\mu_{A,o} = \sum_{\lambda \in \mathrm{Spec}(A)} \|P_\lambda \delta_o\|^2 \,\delta_\lambda, \tag{2.2}$$

where P_λ is the orthogonal spectral projection. Moreover, for any $z \in \mathbb{C}$ with $|z| < \|A\|^{-1}$ when $A \neq 0$, we have

$$\sum_{k=0}^{\infty} \langle A^k\delta_o, \delta_o\rangle z^k = \langle (\mathrm{Id} - zA)^{-1}\delta_o, \delta_o\rangle = \int \frac{1}{1-zx} \,\mathrm{d}\mu_{A,o}(x). \tag{2.3}$$

The root measure of A^2 is the pushforward of $\mu_{A,o}$ by $x \mapsto x^2$.

Proof . The first assertion follows from the spectral theorem applied to the polynomial x^k. The matrix coefficient of A^k is obtained by summing products of edge multiplicities over intermediate vertices. Formula (2.2) follows from the orthogonal spectral decomposition. The Neumann series for $(\mathrm{Id} - zA)^{-1}$ converges in operator norm in the stated disk. Its scalar matrix coefficient can therefore be summed term by term. The same bound gives uniform absolute convergence of the corresponding geometric series on $\operatorname{supp} \mu_{A,o}$. Finally, the functional calculus gives $f(A^2) = (f \circ (x \mapsto x^2))(A)$ for bounded Borel functions f. This proves the assertion about the pushforward. ∎

Suppose that the graph is bipartite with even part $V_0 \ni o$ and odd part V_1, and write

$$A = \begin{bmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{bmatrix}, \quad K \stackrel{\mathrm{def}}{=} \Lambda\Lambda^* \colon \ell^2_{V_0} \to \ell^2_{V_0}. \tag{2.4}$$

The closed odd paths vanish. The root measure $\mu^{\mathrm{sq}}_{\Lambda,o} \stackrel{\mathrm{def}}{=} \mu_{K,o}$ of K is the image of $\mu_{A,o}$ by $x \mapsto x^2$, and its n-th moment is the number c_n of closed walks of length $2n$ at o. We call it the rooted law, or the squared law, of the rooted bipartite graph. This is the even-moment convention used for principal graphs in [6, 5].

Proposition 2.2 *Let $p_n(v)$ be the number of walks of length n from o to v, counted with edge multiplicities. Then for any integer $n \geqslant 0$ we have*

$$c_n = \langle K^n \delta_o, \delta_o \rangle = \sum_v p_n(v)^2. \tag{2.5}$$

Proof . Cutting a closed walk of length $2n$ after n steps and reversing its second half gives an ordered pair of walks of length n from o to the same vertex v. Conversely, such a pair determines the closed walk, including the choice of the edges. Equivalently, $\langle A^{2n}\delta_o, \delta_o \rangle = \|A^n \delta_o\|^2$ since A is selfadjoint. ∎

The vector $p_n = A^n \delta_o$ is computed by the recursion $p_{n+1}(v) = \sum_w A_{vw} p_n(w)$, and only the vertices at distance at most n from o enter. This gives a short integer algorithm for the moments.

Proposition 2.3 *A probability measure μ on $\mathbb{R}$ is the pushforward of an inversion-invariant probability measure on $\mathbb{T}$ by $u \mapsto u + u^{-1}$ if and only if $\operatorname{supp} \mu \subseteq [-2, 2]$. Under this support condition the inversion-invariant lift is unique.*

Proof . The image of $\mathbb{T}$ is $[-2, 2]$. Conversely, put $\theta(x) \stackrel{\mathrm{def}}{=} \arccos \frac{x}{2}$ and define the lift by

$$\int f \, \mathrm{d}\nu = \frac{1}{2} \int \bigl(f(\mathrm{e}^{\mathrm{i}\theta(x)}) + f(\mathrm{e}^{-\mathrm{i}\theta(x)}) \bigr) \, \mathrm{d}\mu(x), \quad f \in C(\mathbb{T}).$$

This is a probability measure with the asserted symmetry and pushforward. For uniqueness, an inversion-invariant measure gives the same integral to f and to the function $u \mapsto \frac{1}{2}(f(u) + f(u^{-1}))$. The latter function is constant on the fibers of $u \mapsto u + u^{-1}$ and therefore is determined by the pushforward. At the endpoints the two points in the displayed formula coincide, which causes no additional choice. ∎

For a rooted bipartite graph of norm at most 2, the circular measure is the unique probability ε_Γ on $\mathbb{T}$ which is invariant under $u \mapsto \overline{u}$ and $u \mapsto -u$ and whose image by $\Phi(u) \stackrel{\mathrm{def}}{=} (u + u^{-1})^2$ is the squared law [6, Definition 1.2], [5, Definition 3.2]. It is obtained by symmetrizing the lift of Proposition 2.3 under $u \mapsto -u$. Above index 4 a lift remains available if one allows the positive half-line, as in [5, Definition 3.2].

Proposition 2.4 *Put* $\Psi(z) \overset{\text{def}}{=} 2 + z + z^{-1}$ *for* $z \in \mathbb{C} \setminus \{0\}$. *Let* μ *be a probability measure on* $[0,\infty)$ *with bounded support. There exists a unique probability measure* ν *on* $\Psi^{-1}([0,\infty)) = \mathbb{T} \cup (0,\infty)$ *which is invariant under* $z \mapsto z^{-1}$ *and satisfies* $\Psi_*\nu = \mu$. *The part of* ν *above* $[0,4]$ *lies on* $\mathbb{T}$, *and the part above* $(4,\infty)$ *lies on* $(0,\infty)$.

Proof . For $x \geqslant 0$ the fiber $\Psi^{-1}(x)$ is the set of roots of $z^2 - (x-2)z + 1 = 0$. The two roots have product one. For $0 < x < 4$ they are complex conjugate points of $\mathbb{T}$, for $x > 4$ they are distinct positive reals, and for $x = 0$ or $x = 4$ the roots coincide at -1 or 1. Integrating against μ the uniform probability measure on each fiber gives an invariant lift. Conversely, if ν is an invariant lift, then $\int f \,\mathrm{d}\nu = \int \frac{1}{2}(f(z) + f(z^{-1}))\,\mathrm{d}\nu(z)$, and the integrand is a function of $\Psi(z)$. Hence $\int f \,\mathrm{d}\nu$ is determined by μ. ∎

On $\mathbb{T}$ we have $\Psi(u^2) = \Phi(u)$, so the measure ν is the image of the circular measure by $u \mapsto u^2$ when $\operatorname{supp}\mu \subseteq [0,4]$. The Laurent moments $\int z^k \,\mathrm{d}\nu(z)$ of ν, which are its Fourier coefficients when ν is carried by $\mathbb{T}$, are related to the moments by the following transform, which appears in [6, Proposition 1.2] and [5, Proposition 3.4].

Proposition 2.5 *Let* μ *be a probability measure on* $[0,\infty)$ *with bounded support and moments* c_n, *let* $f(t) = \sum_{n\geqslant 0} c_n t^n$ *and let* ν *be the lift of Proposition 2.4. For* $k \geqslant 1$, *put* $b_k \overset{\text{def}}{=} \int z^k \,\mathrm{d}\nu(z)$. *Then*

$$1 + 2\sum_{k\geqslant 1} b_k q^k = \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right) \tag{2.6}$$

as formal power series. In particular $2b_1 = c_1 - 2$ *and* $2b_2 = c_2 - 4c_1 + 2$. *If the moments* c_n *are integers, then the numbers* $2b_k$ *are integers.*

Proof . For $x = \Psi(z)$ and q small, we have $(1+q)^2 - qx = (1-qz)(1-qz^{-1})$. Hence

$$\frac{1-q}{1+q} \cdot \frac{1}{1 - \frac{q}{(1+q)^2}x} = \frac{1-q^2}{(1-qz)(1-qz^{-1})} = 1 + \sum_{k\geqslant 1} q^k (z^k + z^{-k}).$$

Integrating against ν and using its invariance under $z \mapsto z^{-1}$ gives (2.6) for $|q|$ small. Both sides are analytic near 0, so the identity holds as formal power series. Finally, $z^k + z^{-k}$ is a polynomial with integer coefficients in $z + z^{-1} = x - 2$, by the recurrence $P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$ with $P_0 = 2$ and $P_1 = t$. Hence $2b_k$ is an integer combination of the c_n. ∎

2.2 Subfactors, principal graphs and the annular decomposition

We refer to [32], [17] and [31, Section 2] for the background on subfactors. Let $\mathcal{N} \subseteq \mathcal{M}$ be an irreducible inclusion of II_1 factors with finite index $[\mathcal{M} : \mathcal{N}] = \delta^2$, and let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1 \subseteq \cdots$ be its Jones tower. The principal graph Γ is the bipartite graph whose even vertices are the irreducible $\mathcal{N}$-$\mathcal{N}$ bimodules and whose odd vertices are the irreducible $\mathcal{N}$-$\mathcal{M}$ bimodules appearing in the tensor powers of ${}_{\mathcal{N}}\mathrm{L}^2(\mathcal{M})_{\mathcal{M}}$ and of its conjugate, the number of edges being the multiplicity. It is rooted at the trivial bimodule $*$. The standard invariant is encoded in the subfactor planar algebra $P = (P_n)_{n\geqslant 0}$ of Jones, with $P_n = \mathcal{N}' \cap \mathcal{M}_{n-1}$, where $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_{-1} = \mathcal{N}$ [30, Theorem 4.2.1].

Proposition 2.6 *With the above notation, we have* $\dim P_n = c_n$ *for any integer* $n \geqslant 0$, *where* c_n *is the number of closed walks of length* $2n$ *at* $*$ *on* Γ. *If* $\theta = \bar{\iota}\iota$ *denotes the dual canonical endomorphism, viewed as an object of the* $\mathcal{N}$-$\mathcal{N}$ *tensor category generated by* ${}_{\mathcal{N}}L^2(\mathcal{M})_{\mathcal{N}}$, *then* $c_n = \dim \mathrm{Hom}(\mathbf{1}, \theta^{\otimes n})$. *The same numbers are obtained on the dual principal graph rooted at the trivial* $\mathcal{M}$-$\mathcal{M}$ *bimodule.*

Proof . Two steps on the principal graph starting from an even vertex correspond to tensoring by ${}_{\mathcal{N}}L^2(\mathcal{M})_{\mathcal{N}}$, so that K in (2.4) is the fusion matrix of θ restricted to the even vertices, and $c_n = \langle K^n\delta_*, \delta_*\rangle = \dim \mathrm{Hom}(\mathbf{1}, \theta^{\otimes n})$. The equality $\dim P_n = c_n$ is the description of the higher relative commutants by bimodules, see [17, Chapter 9] and [31, Section 2.2 and Fact 2.9]. The last assertion holds since $\dim P_{n,+} = \dim P_{n,-}$ for a subfactor planar algebra. ∎

Recall that $[\mathcal{M} : \mathcal{N}] \geqslant \|\Lambda\|^2$, with equality when Γ is finite, and that the positive bimodule dimension function d satisfies $Kd = [\mathcal{M} : \mathcal{N}]d$ pointwise on the even vertices [31, Facts 2.6 and 2.8]. Consequently, if Γ is finite, the largest atom of the squared law is the index and, by (2.2), its mass is $\left(\sum_{v\in V_0} d_v^2\right)^{-1}$, with $d_* = 1$. The sum in this formula is the global dimension of the even part.

For $\delta > 2$ the annular Temperley–Lieb category acts on P. Let a_k be the number of irreducible annular submodules of P of lowest weight $k \geqslant 1$ in a decomposition of P, counting all rotation eigenvalues. By [28, Proposition 5.3 and Theorem 5.7], see also [31, Theorem 2.12], an irreducible annular module of lowest weight $k \geqslant 1$ has dimension $\binom{2n}{n-k}$ in the n-box space for $n \geqslant k$, and the lowest weight zero part of a subfactor planar algebra is the Temperley–Lieb subalgebra. This gives the following identity.

Theorem 2.7 (Jones) *Let P be a subfactor planar algebra with $\delta > 2$. For any integer $n \geqslant 0$ we have*

$$c_n = C_n + \sum_{k=1}^{n} \binom{2n}{n-k} a_k, \quad C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.7}$$

Equivalently, with the notation of Proposition 2.5, we have $a_1 = 2b_1 + 1$ and $a_k = 2b_k$ for $k \geqslant 2$.

Proof . By [28, Definition 6.2] and the discussion preceding it, the planar algebra P is, as a module over the annular Temperley–Lieb category, the direct sum of its Temperley–Lieb part and of a_k irreducible modules of lowest weight k for $k \geqslant 1$. The dimension of the latter in P_n is $\binom{2n}{n-k}$ by [28, Proposition 5.3 and Theorem 5.7], and the Temperley–Lieb part has dimension C_n. This proves (2.7). For the second assertion, [28, Theorem 6.3] states that $\sum_{k\geqslant 0} a_k q^k = \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right) + q$ with $a_0 = 1$. Comparing with (2.6) gives the assertion. ∎

The coefficient of a_n in (2.7) is one, so the system can be solved recursively. For example $c_1 = 1 + a_1$, $c_2 = 2 + 4a_1 + a_2$ and $c_3 = 5 + 15a_1 + 6a_2 + a_3$. In particular $b_k = \frac{a_k}{2} \geqslant 0$ for $k \geqslant 2$ when $\delta > 2$; see also [5, Theorem 3.5]. Thus the annular multiplicities are the Laurent moments of the lift ν of Proposition 2.4, up to the factor 2 and to the correction at $k = 1$.

2.3 Fusion categories, centers and formal codegrees

We refer to [14], [9] and [53] for the following notions. Let $\mathcal{C}$ be a unitary fusion category with set of simple objects $\mathrm{Irr}(\mathcal{C})$, unit $\mathbf{1}$ and fusion coefficients $N_{XY}^Z = \dim \mathrm{Hom}(Z, X \otimes Y)$. Its based ring $K(\mathcal{C})$ is the Grothendieck ring with basis $\mathrm{Irr}(\mathcal{C})$ and involution $X \mapsto X^*$. The linear form τ on $K(\mathcal{C})_{\mathbb{C}} \overset{\text{def}}{=} K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ defined by $\tau(X) = \delta_{X,\mathbf{1}}$ satisfies $\tau(XY^*) = \delta_{X,Y}$ for $X, Y \in \mathrm{Irr}(\mathcal{C})$. Extending the involution antilinearly, we obtain $\tau(xx^*) = \sum_X |x_X|^2$ for $x = \sum_X x_X X$. Thus $K(\mathcal{C})_{\mathbb{C}}$ is a finite-dimensional $*$-algebra with a faithful positive trace, hence a C^*-algebra, hence a semisimple algebra. For an irreducible representation E of $K(\mathcal{C})_{\mathbb{C}}$, the element $\alpha_E = \sum_X \mathrm{Tr}(X, E)X^*$ is central, vanishes on the other irreducible representations and acts on E by a nonzero scalar f_E, called the formal codegree of E [52, Lemma 2.3], [53, Section 2.3]. Thus $e_E = f_E^{-1}\alpha_E$ is the minimal central idempotent of E, and $\tau(e_E) =$

$f_E^{-1}\operatorname{Tr}(\mathbf{1},E)=f_E^{-1}\dim(E)$. Since every trace on $K(\mathcal{C})_{\mathbb{C}}\cong\bigoplus_E\operatorname{End}(E)$ is a linear combination of the traces $\operatorname{Tr}(\cdot,E)$, we obtain, see also [53, Proposition 2.10],

$$\tau=\sum_E\frac{1}{f_E}\operatorname{Tr}(\cdot,E),\quad\text{in particular}\quad\sum_E\frac{\dim(E)}{f_E}=1, \tag{2.8}$$

and $f_E\geqslant\dim(E)$ [53, Remark 2.12]. For the dimension character E_0, given by the Frobenius–Perron dimensions, we have $f_{E_0}=\dim(\mathcal{C})=\sum_X\dim(X)^2$.

Let $Z(\mathcal{C})$ be the Drinfeld center, $F\colon Z(\mathcal{C})\to\mathcal{C}$ the forgetful functor and I its right adjoint. Then $I(\mathbf{1})$ is a connected étale algebra with $\dim I(\mathbf{1})=\dim(\mathcal{C})$, called the canonical Lagrangian algebra, and

$$F(I(Y))\cong\bigoplus_{X\in\operatorname{Irr}(\mathcal{C})}X\otimes Y\otimes X^*,\quad Y\in\mathcal{C}, \tag{2.9}$$

see [14, Proposition 5.4] and [53, Proposition 2.4]. We put

$$R_{\mathcal{C}}\stackrel{\text{def}}{=}[F(I(\mathbf{1}))]=\sum_{X\in\operatorname{Irr}(\mathcal{C})}XX^*\in K(\mathcal{C}). \tag{2.10}$$

The element $R_{\mathcal{C}}$ acts on the irreducible representation E by the scalar $\dim(E)f_E$ [52, Lemma 2.6], [53, Remark 2.11]. By [53, Theorem 2.13], the simple summands of $I(\mathbf{1})$ are in bijection $E\mapsto A_E$ with the irreducible representations of $K(\mathcal{C})_{\mathbb{C}}$, with multiplicity $[I(\mathbf{1}):A_E]=\dim(E)$ and $\dim(A_E)=\frac{\dim(\mathcal{C})}{f_E}$. The Lagrangian algebras of $Z(\mathcal{C})$ are in bijection with the indecomposable $\mathcal{C}$-module categories $\mathcal{M}$, and the category of modules over the Lagrangian algebra attached to $\mathcal{M}$ is equivalent to the dual category $\mathcal{C}_{\mathcal{M}}^*$ [9, Proposition 4.8 and Remark 4.9]. In particular, if $\mathcal{D}$ is Morita equivalent to $\mathcal{C}$, then the canonical Lagrangian algebra $I_{\mathcal{D}}(\mathbf{1})$ of $Z(\mathcal{D})\simeq Z(\mathcal{C})$ is a Lagrangian algebra of $Z(\mathcal{C})$ whose category of modules is $\mathcal{D}$.

Suppose that a unitary braided fusion category is realized by finite-index endomorphisms of a type III factor $\mathcal{N}$. A connected étale algebra in it which is a Q-system, that is, whose multiplication is a multiple of a coisometry, gives an irreducible finite-index inclusion $\mathcal{N}\subseteq\mathcal{M}$ whose dual canonical endomorphism is the underlying object of the algebra, see [43], [47] and [7]. For the canonical Lagrangian algebra of $Z(\mathcal{C})$ this is the dual of the Longo–Rehren inclusion [43, 47].

2.4 Compact quantum groups and characters

We use [48] for compact quantum groups. Let G be a compact quantum group with Haar state h. For a finite dimensional unitary representation $U\in B(H)\otimes C(G)$, the character $\chi(U)=(\operatorname{Tr}\otimes\operatorname{Id})(U)$ satisfies $\chi(U)^*=\chi(\overline{U})$ and $\chi(U)\chi(V)=\chi(U\otimes V)$ [48, Section 2.7]. By the orthogonality relations [48, Theorem 1.4.3], the operator $(\operatorname{Id}\otimes h)(U)$ is the orthogonal projection onto the space $\operatorname{Fix}(U)$ of invariant vectors, so that

$$h(\chi(U))=\dim\operatorname{Fix}(U). \tag{2.11}$$

Consequently, if u is a selfconjugate representation, the law of the selfadjoint element $\chi(u)$ for h has moments $\dim\operatorname{Fix}(u^{\otimes n})$, and these moments depend only on the fusion rules of G.

2.5 Representation graphs and the McKay series

Let K be a compact group and let X be a finite-dimensional selfdual unitary representation. The irreducible characters are an orthonormal basis of the space of square-integrable class functions. On this space multiplication by χ_X is the operator of tensor multiplication by X in the character basis. Its norm is at most $\dim X$ and its root vector is the constant character 1. Consequently, the root spectral measure of the representation graph of X at the trivial representation is the image of normalized Haar measure by χ_X. For nimreps, the vector state must instead be specified in the module on which the fusion matrices act, as in [19, 20].

More explicitly, a nimrep assigns to each simple object a a matrix N_a with nonnegative integer entries. If N_{ab}^{c} are the fusion coefficients, then

$$N_{\mathbf{1}} = \mathrm{Id}, \qquad N_{\bar{a}} = N_a^{t}, \qquad N_a N_b = \sum_c N_{ab}^{c} N_c.$$

An entry of N_a counts directed edges of colour a. For example $(N_a N_{\bar{a}})_{oo}$ counts the two-step paths of these colours at the chosen vertex o.

For a finite subgroup Γ of SU(2), let Δ be the McKay adjacency matrix of the defining representation. The restricted symmetric powers satisfy the SU(2) tensor recurrence. Thus the matrices of their multiplicities have generating series $(\mathrm{Id} - z\Delta + z^2\mathrm{Id})^{-1}$. If $G(w) = \langle (\mathrm{Id} - w\Delta)^{-1}\delta_o, \delta_o \rangle$, with o the trivial representation, the root entry is

$$\frac{1}{1+z^2} G\left(\frac{z}{1+z^2}\right). \tag{2.12}$$

This is the affine case of the calculation of Evans and Pugh [18, Theorem 1 p. 389], where the Hilbert, Molien and Kostant series are compared. In the finite Dynkin case the preprojective Hilbert matrix includes the numerator $\mathrm{Id} + Pz^h$, where P is the Nakayama permutation and h the Coxeter number [18, Section 5.3 pp. 387–388].

An actual map to a Lie group is available through a principal SU(2) homomorphism [34]. For example, for G_2 the adjoint representation restricted to this subgroup has character $\chi_{10} + \chi_2$, where the subscripts denote highest weights of SU(2). Indeed, the positive root heights are $1, 1, 2, 3, 4, 5$. The principal Cartan element has eigenvalues twice these heights on the positive root spaces, the negatives on the negative root spaces and zero with multiplicity two. Thus the second moment of the restricted character for the Haar measure of the principal SU(2) is two, whereas the second moment of the adjoint character of G_2 is one by irreducibility. Restricting further to a finite subgroup Γ of the principal SU(2) can only increase the dimensions of the invariant subspaces. Hence the image of the uniform measure on Γ by the adjoint character of G_2 is never the adjoint character law of G_2.

2.6 Simple Lie algebras and the exceptional series

Let K be a connected compact group with simple Lie algebra $\mathfrak{k}$ of dimension d, dual Coxeter number $h^\vee$ and Coxeter number h. We equip $\mathfrak{k}$ with the negative of the Killing form. For the induced pairing $(\cdot,\cdot)$ on weights, a long root θ satisfies $(\theta,\theta) = \frac{1}{h^\vee}$, the quadratic Casimir acts on the irreducible module V_λ by $c_\lambda = (\lambda, \lambda + 2\rho)$, and $c_\theta = 1$ for the adjoint representation. The Freudenthal–de Vries strange formula reads $(\rho,\rho) = \frac{d}{24}$. Vogel's parameters (α,β,γ), normalized by $\alpha = -2$ and $\alpha + \beta + \gamma = h^\vee$, give uniform formulas for the decomposition of the symmetric square of the adjoint representation and for the eigenvalues of the split Casimir operator. We use the conventions and tables of Isaev and Krivonos [25, Section 3 and Table 3]. The exceptional line is $\beta = \frac{h^\vee}{3} + 2$, $\gamma = \frac{2h^\vee}{3}$, and the Deligne–de Man parameter is $\mu = \frac{6}{h^\vee}$ [12, p. 577]. The Deligne series consists of A_1, A_2, G_2, D_4, F_4, E_6, E_7 and E_8 [11].

2.7 Character moments and the trivalent intertwiner for G_2

In the representation-graph model of Section 2.5, take the compact group G_2. Its two fundamental characters have dimensions 7 and 14. Weyl integration on the rank-two torus gives their joint Haar law, and in [20, Sections 3.3 and 4.2] a one-coordinate marginal leads to elliptic integrals. Their mixed moments are tensor product multiplicities. We now show that these moments do not change under the Drinfeld–Jimbo deformation. The projections onto the spaces of invariant vectors of the deformation are the object of Problem 2.8.

Let $0 < q < 1$, let $G_{2,q}$ be the Drinfeld–Jimbo deformation of the compact group G_2, with Haar state h_q, and let u, v be its fundamental unitary representations of dimensions 7 and 14. The fusion rules of $G_{2,q}$ are those of G_2 [49]. Both fundamental representations are selfdual, and $u \otimes v$ is equivalent to $v \otimes u$ by the fusion rules. Taking ordinary characters shows that $X_q = \chi(u)$ and $Y_q = \chi(v)$ are selfadjoint and commute. Applying (2.11) to their tensor products gives, for all integers $m, n \geqslant 0$,

$$h_q(X_q^m Y_q^n) = \dim \mathrm{Fix}(u^{\otimes m} \otimes v^{\otimes n}) = \int_{G_2} \chi_7(g)^m \chi_{14}(g)^n \, \mathrm{d}g.$$

Thus the joint law of (X_q, Y_q) does not depend on q, and it is the joint law computed by Evans and Pugh [20, Section 4.1]. On the other hand, the operator $P_q = (\mathrm{Id} \otimes h_q)(u^{\otimes 3})$ is the orthogonal projection onto the line $\mathrm{Fix}(u^{\otimes 3})$, which is one-dimensional since, by the fusion rules, $u \otimes u$ decomposes like $7 \otimes 7 = 1 \oplus 7 \oplus 14 \oplus 27$ for G_2. For G_2, the symmetric square of 7 is $1 \oplus 27$ and its exterior square is $7 \oplus 14$. The weights of 7 are 0 and the six short roots, each with multiplicity one. The script `g2_tensor_square.py` applies the Weyl character formula to these weights to obtain both decompositions. At $q = 1$ this line is spanned by the invariant three-form of the octonions, and in the G_2 spider it is the trivalent vertex [35].

Problem 2.8 *Compute the matrix coefficients of P_q in a weight basis of $u^{\otimes 3}$ as functions of q. For a finite G_2 nimrep realized by a module category at a root of unity, as in [20], determine the action of the trivalent vertex on the spaces of paths of length three. Which contractions of these matrices are determined by the rooted loop counts of the nimrep graphs?*

3 Rooted laws of graphs of small index

3.1 Grid measures and the position of the root

We use the following measures on $\mathbb{T}$, which are those of [6] and [5, Definitions 4.3 and 8.1] in a different notation. For any integer $n \geqslant 1$, let d_n be the uniform probability measure on the $2n$-th roots of unity, and let $\mathsf{d}'_n \stackrel{\mathrm{def}}{=} 2\mathsf{d}_{2n} - \mathsf{d}_n$ be the uniform probability measure on the $4n$-th roots of unity u with $u^{2n} = -1$. For $\ell \geqslant 1$, put $w_\ell \stackrel{\mathrm{def}}{=} \mathrm{e}^{\frac{2\pi \mathrm{i}}{12\ell}}$ and let d''_ℓ be the uniform probability measure on the points w_ℓ^j with $j \equiv \pm 1 \mod 6$. Let $\mathsf{d}'''_1 \stackrel{\mathrm{def}}{=} \frac{1}{2}(3\mathsf{d}_3 - \mathsf{d}_1)$, which is the uniform probability measure on the four sixth roots of unity different from ± 1. For any integer $j \geqslant 1$ we consider the density

$$\mathsf{h}_j(u) \stackrel{\mathrm{def}}{=} 1 - \Re(u^{2j}), \quad u \in \mathbb{T}. \tag{3.1}$$

The densities $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3$ are denoted α, β, γ in [5, Definition 5.2]. A product such as $\mathsf{h}_1 \mathsf{d}_n$ denotes the measure with density h_1 with respect to d_n. We denote by ε_Γ the circular measure of a rooted bipartite graph Γ of norm at most 2, see Section 2.1.

Proposition 3.1 *Let $j \geqslant 1$ be an integer.*

1. *The root measure of the adjacency operator of the half-line with vertices $1, 2, \ldots$, rooted at j, is the image of the probability measure $\mathsf{h}_j(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}$ on $\mathbb{T}$ by $u \mapsto u + u^{-1}$.*

2. *Let $n > j$. The root measure of the adjacency operator of the path A_{n-1} with vertices $1, \ldots, n-1$, rooted at j, is the image of $\mathsf{h}_j\mathsf{d}_n$ by $u \mapsto u + u^{-1}$.*

Proof . 1. By the reflection principle, the walks of length $2m$ on $\mathbb{Z}$ from j to j which visit 0 are in bijection with the walks from $-j$ to j, by reflecting the initial segment up to the first visit to 0. Hence the number of walks of length $2m$ from j to j on the half-line is $\binom{2m}{m} - \binom{2m}{m-j}$, with $\binom{2m}{m-j} = 0$ for $m < j$. The first term is the constant Fourier coefficient of $(u+u^{-1})^{2m}$ and the second one is its coefficient of u^{2j}, which is also its coefficient of u^{-2j}. Thus

$$\binom{2m}{m} - \binom{2m}{m-j} = \int_{\mathbb{T}} (u+u^{-1})^{2m}\Big(1 - \frac{u^{2j}+u^{-2j}}{2}\Big)\frac{\mathrm{d}\theta}{2\pi}.$$

The odd moments vanish on both sides by symmetry. Since both measures have compact support, they coincide.

2. For $1 \leqslant k \leqslant n-1$, the vector with coordinates $\sqrt{\frac{2}{n}}\sin(\frac{rk\pi}{n})$, $1 \leqslant r \leqslant n-1$, is a normalized eigenvector of eigenvalue $2\cos(\frac{k\pi}{n})$. By (2.2) the mass at this eigenvalue is $\frac{2}{n}\sin^2(\frac{jk\pi}{n})$. The two points $\mathrm{e}^{\pm\mathrm{i}\frac{k\pi}{n}}$ have mass $\frac{1}{2n}$ for d_n and density $\mathsf{h}_j = 2\sin^2(\frac{jk\pi}{n})$ there, which gives the same mass. At the points ± 1 the density h_j vanishes. The total mass of $\mathsf{h}_j\mathsf{d}_n$ is one since $2j$ is not a multiple of $2n$. ∎

For $j = 1$ the first assertion gives the semicircle law. In general, the distance j from the root to the missing vertex 0 is the order of the harmonic u^{2j} in the density h_j.

Theorem 3.2 *The circular measures of the following rooted graphs are given by:*

Γ	$A_{n-1},\ n \geqslant 2$	$\widetilde{A}_{2n-1},\ n \geqslant 2$	$D_{n+1},\ n \geqslant 3$
ε_Γ	$\mathsf{h}_1\mathsf{d}_n$	d_n	$\mathsf{h}_1\mathsf{d}'_n$

Γ	$\widetilde{D}_{n+2},\ n \geqslant 2$
ε_Γ	$\frac{1}{2}(\mathsf{d}_n + \mathsf{d}'_1)$

Here A_{n-1} is rooted at an endpoint, the cycle $\widetilde{A}_{2n-1}$ with $2n$ vertices at any vertex, D_{n+1} at the end of its long arm and $\widetilde{D}_{n+2}$ at an extending vertex.

Proof . These formulas are [6, Theorem 2.1, Theorem 3.1, Theorem 3.2 and Theorem 4.1]. We give short direct proofs. The case of A_{n-1} is Proposition 3.1 with $j = 1$. On the cycle, the adjacency operator is $S + S^{-1}$, where S is the cyclic shift. Each Fourier vector has squared modulus $\frac{1}{2n}$ at the root, so that the root law of S is d_n.

For D_{n+1}, the involution exchanging the two short tips fixes the root. On the symmetric subspace the adjacency operator is the path on n vertices whose last edge has weight $\sqrt{2}$. For $0 \leqslant k < n$, put $\theta_k \overset{\text{def}}{=} \frac{(2k+1)\pi}{2n}$. The vector with coordinates

$$\sin(\theta_k),\ \sin(2\theta_k),\ \ldots,\ \sin((n-1)\theta_k),\ \frac{\sin(n\theta_k)}{\sqrt{2}}$$

is an eigenvector of eigenvalue $2\cos(\theta_k)$. Indeed the equation at the vertex $n-1$ is the sine addition formula, and the equation at the last vertex reduces to $\cos(n\theta_k)\sin(\theta_k) = 0$. Since $\sum_{r=0}^{n-1}\mathrm{e}^{2\mathrm{i}r\theta_k} = \frac{2}{1-\mathrm{e}^{2\mathrm{i}\theta_k}}$ has real part 1, we have $\sum_{r=1}^{n-1}\cos(2r\theta_k) = 0$, so that the squared norm

of this vector is $\frac{n-1}{2}+\frac{1}{2}=\frac{n}{2}$. The root mass is therefore $\frac{2}{n}\sin^2(\theta_k)$. The two conjugate points $\mathrm{e}^{\pm \mathrm{i}\theta_k}$ of the support of d'_n give the same mass for $\mathsf{h}_1\mathsf{d}'_n$.

The graph $\widetilde{D}_{n+2}$ is the McKay graph of the binary dihedral group G of order $4n$, rooted at the trivial representation. By Section 2.5, its root law is the image of the uniform measure on G by the trace of the defining representation. The cyclic subgroup of order $2n$ consists of the matrices $\mathrm{diag}(\zeta,\bar{\zeta})$ with $\zeta^{2n}=1$, which gives d_n. The other coset consists of matrices with eigenvalues $\mathrm{i},-\mathrm{i}$, which gives d'_1. ∎

3.2 The exceptional diagrams

We root E_6 at the end of one of its two arms of length two, and E_7, E_8 at the end of their longest arm. The arms are counted from the trivalent vertex. The expansions below are given in [5, Theorems 7.1 and 8.7], see also [6, Theorem 6.2] for E_6.

Theorem 3.3 *The circular measures of the exceptional Dynkin diagrams are*

$$\varepsilon_{E_6}=\mathsf{h}_1\mathsf{d}_{12}+\frac{1}{2}(\mathsf{d}_{12}-\mathsf{d}_6-\mathsf{d}_4+\mathsf{d}_3),\qquad \varepsilon_{E_7}=\mathsf{h}_2\mathsf{d}'_9+\frac{1}{2}(\mathsf{d}'_1-\mathsf{d}'_3),$$
$$\varepsilon_{E_8}=(\mathsf{h}_1+\mathsf{h}_3)\mathsf{d}'_{15}-\frac{1}{2}(\mathsf{d}'_5+\mathsf{d}'_3). \tag{3.2}$$

Equivalently, on the ternary grids,

$$\varepsilon_{E_6}=\frac{1}{6}(\mathsf{d}''_2+2\mathsf{h}_1\mathsf{d}''_2+3\mathsf{d}'''_1),\quad \varepsilon_{E_7}=\frac{1}{3}(2\mathsf{h}_2\mathsf{d}''_3+\mathsf{d}'_1),\quad \varepsilon_{E_8}=\frac{1}{3}(2(\mathsf{h}_1+\mathsf{h}_3)\mathsf{d}''_5-\mathsf{d}''_1). \tag{3.3}$$

For the affine diagrams $\widetilde{E}_{n+3}$, $n=3,4,5$, rooted at the extending vertex, we have $\varepsilon_{\widetilde{E}_{n+3}}=\frac{1}{2}(\mathsf{d}_n+\mathsf{d}_3+\mathsf{d}_2-\mathsf{d}_1)$.

Proof . The formulas (3.2) are [5, Theorem 7.1] and (3.3) is [5, Theorem 8.7], see also [6, Theorem 6.2] for E_6. The affine formula is contained in [5, Theorem 7.1], see also [6, Theorem 6.1], and can also be obtained from the McKay correspondence of Section 2.5 for the binary tetrahedral, octahedral and icosahedral groups. As an illustration of the conversion between (3.2) and (3.3), consider E_7. Among the odd residues modulo 36, two thirds are prime to 3, and dividing the remaining six residues by three gives the odd residues modulo 12. Hence $\mathsf{d}'_9=\frac{1}{3}(2\mathsf{d}''_3+\mathsf{d}'_3)$. On the support of d'_3, the density h_2 vanishes at $\pm\mathrm{i}$ and equals $\frac{3}{2}$ at the four other points, so that $\mathsf{h}_2\mathsf{d}'_3=\frac{1}{2}(3\mathsf{d}'_3-\mathsf{d}'_1)$. Substituting these two identities in (3.2) gives the formula for ε_{E_7} in (3.3). ∎

For E_8 the cancellations in (3.2) can be seen on the grid of d'_{15}. At a primitive twelfth root, $\mathsf{h}_1+\mathsf{h}_3=5/2$. The term $(\mathsf{h}_1+\mathsf{h}_3)\mathsf{d}'_{15}$ gives mass $1/12$ there, exactly cancelled by $\mathsf{d}'_3/2$. At a primitive twentieth root, $\mathsf{h}_1+\mathsf{h}_3=3/2$ and the mass $1/20$ is cancelled by $\mathsf{d}'_5/2$. At $\pm\mathrm{i}$ both corrections apply, giving $1/12+1/20=2/15$, the mass of the first term. The remaining support consists of the primitive sixtieth roots.

The script `checks.py`, written by Codex, compares the moments of (3.2), (3.3) and the affine formulas through order 12 with the walk counts of Proposition 2.2. It also checks Cartan dimensions, deletion determinants and quartic trace identities. For the affine diagrams, the mass at $u=1$ is $\frac{1}{24}$, $\frac{1}{48}$ and $\frac{1}{120}$, respectively. By Section 2.5, the mass of the image law at $x=4$ is the probability that an element of the binary tetrahedral, octahedral or icosahedral group has trace ±2. The only such elements are ±1, and the symmetric lift splits this mass $\frac{2}{|\Gamma|}$ equally between $u=1$ and $u=-1$.

Remark 3.4 The ternary index ℓ of [5, Theorem 8.7] is 2, 3, 5 for E_6, E_7, E_8. The Coxeter numbers are $h = 12, 18, 30$, and the Deligne–de Man parameters are $\mu = \frac{6}{h^\vee} = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$ [12, p. 577]. Since these diagrams are simply laced, $h = h^\vee$, and we obtain

$$\ell = \frac{h}{6} = \frac{1}{\mu}.$$

The eigenvalues of the adjacency matrix of a Dynkin diagram of type X are the numbers $2\cos(\frac{\pi m}{h})$, where m runs through the exponents of X, so that the circular measure is carried by the $2h$-th roots of unity, which is the grid $w_\ell^{\mathbb{Z}}$ of order 12ℓ. On the other hand, the parameter of Deligne and de Man is $6(\theta,\theta) = \frac{6}{h^\vee}$. The equality $\ell = \frac{1}{\mu}$ is therefore not an additional coincidence between the cyclotomic expansions and the exceptional series. The numbers in (3.3) which are not determined by h are the coefficients of the measures, and these depend on the root. Proposition 3.5 below makes this dependence explicit for E_6.

Let a and b be the two tips of the arms of length two of E_6, let a', b' be their neighbours, let t be the trivalent vertex and let s be the tip of the short arm. Let σ be the involution exchanging a with b and a' with b'.

Proposition 3.5 *Let μ_{long} and μ_{short} be the squared laws of E_6 at a and at s.*

1. *On the σ-symmetric subspace, with basis $\frac{e_a+e_b}{\sqrt{2}}$, $\frac{e_{a'}+e_{b'}}{\sqrt{2}}$, e_t, e_s, the adjacency operator is the weighted path with edge weights $1, \sqrt{2}, 1$. On the antisymmetric subspace, with basis $\frac{e_a-e_b}{\sqrt{2}}$, $\frac{e_{a'}-e_{b'}}{\sqrt{2}}$, it is the path with two vertices.*
2. *We have*

$$\mu_{\text{short}} = \frac{1+\frac{1}{\sqrt{3}}}{2}\delta_{2-\sqrt{3}} + \frac{1-\frac{1}{\sqrt{3}}}{2}\delta_{2+\sqrt{3}}, \quad \mu_{\text{long}} = \frac{1}{2}\mu_{\text{short}} + \frac{1}{2}\delta_1. \tag{3.4}$$

3. *The circular measures are*

$$\varepsilon_{E_6,s} = \frac{1+2\mathsf{h}_1}{3}\mathsf{d}_2'', \quad \varepsilon_{E_6,a} = \frac{1}{2}\varepsilon_{E_6,s} + \frac{1}{2}\mathsf{d}_1'''. \tag{3.5}$$

Proof . 1. The two neighbours a', b' of t have equal components in $e_{a'} + e_{b'}$ and opposite components in $e_{a'} - e_{b'}$. This gives the coefficient $\sqrt{2}$ between $\frac{e_{a'}+e_{b'}}{\sqrt{2}}$ and e_t, and no coupling between the antisymmetric vectors and e_t.

2. The vector $e_a = \frac{1}{\sqrt{2}}\frac{e_a+e_b}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{e_a-e_b}{\sqrt{2}}$ has squared norm $\frac{1}{2}$ in each block. The square of the two-vertex path is the identity, which gives the term $\frac{1}{2}\delta_1$. The weighted path with weights $1, \sqrt{2}, 1$ is invariant under the reflection exchanging its endpoints, which exchanges $\frac{e_a+e_b}{\sqrt{2}}$ and e_s. Hence the symmetric part of μ_{long} is $\frac{1}{2}\mu_{\text{short}}$. The square of this weighted path, restricted to the span of its two even vertices e_s and $\frac{e_{a'}+e_{b'}}{\sqrt{2}}$, has matrix $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$. Its eigenvalues are $2 \pm \sqrt{3}$ and its first moment at e_s is 1. Writing $w_\pm$ for the masses, we have $w_+ + w_- = 1$ and $(2+\sqrt{3})w_+ + (2-\sqrt{3})w_- = 1$, which gives (3.4).

3. The support of d_2'' consists of the eight points $\mathrm{e}^{\frac{i\pi j}{12}}$ with $j \in \{1, 5, 7, 11, 13, 17, 19, 23\}$, each of mass $\frac{1}{8}$. For $j \in \{1, 11, 13, 23\}$ we have $\Re(u^2) = \frac{\sqrt{3}}{2}$, $\frac{1+2\mathsf{h}_1(u)}{3} = 1 - \frac{1}{\sqrt{3}}$ and $\Phi(u) = 4\cos^2(\frac{\pi}{12}) = 2 + \sqrt{3}$. For $j \in \{5, 7, 17, 19\}$ we have $\Re(u^2) = -\frac{\sqrt{3}}{2}$, $\frac{1+2\mathsf{h}_1(u)}{3} = 1 + \frac{1}{\sqrt{3}}$ and

$\Phi(u) = 2 - \sqrt{3}$. The measure $\frac{1+2\mathsf{h}_1}{3}\mathsf{d}_2''$ is invariant under $u \mapsto \overline{u}$ and $u \mapsto -u$, and its image by Φ is μ_{short} by (3.4). The measure d_1''' is the symmetric lift of δ_1, since $\Phi(u) = 1$ exactly when $u + u^{-1} = \pm 1$. This proves (3.5). ∎

Comparing (3.5) with (3.3), the antisymmetric block contributes the term $\frac{1}{2}\mathsf{d}_1'''$ of ε_{E_6} and the symmetric block contributes $\frac{1}{6}(\mathsf{d}_2'' + 2\mathsf{h}_1\mathsf{d}_2'')$. Moving the root from a to s removes the first term and doubles the second one. The symmetric block is the folded diagram of type F_4 with its natural weights.

There is a second description of the weights. If P_G denotes the characteristic polynomial of the adjacency operator A of a finite graph G, then $\langle (w\mathrm{Id} - A)^{-1}\delta_o, \delta_o\rangle = \frac{P_{G\setminus o}(w)}{P_G(w)}$ by Cramer's rule, where $G \setminus o$ is the graph obtained by deleting o. At a simple eigenvalue x of the adjacency operator, the root mass is the residue $\frac{P_{G\setminus o}(x)}{P_G'(x)}$, including the value zero when the numerator vanishes. At the root chosen in Theorem 3.3, deletion gives the successive diagrams D_5, E_6, E_7 for E_6, E_7, E_8, with

$$P_{D_5}(w) = w^5 - 4w^3 + 2w, \qquad P_{E_6}(w) = w^6 - 5w^4 + 5w^2 - 1,$$
$$P_{E_7}(w) = w^7 - 6w^5 + 9w^3 - 3w, \qquad P_{E_8}(w) = w^8 - 7w^6 + 14w^4 - 8w^2 + 1.$$

Suppose that G is bipartite, with V_0 and V_1 the sets of even and odd vertices, and write its adjacency as $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$, the even vertices being listed first. The Schur complement of the even block gives $P_G(w) = w^{|V_0|-|V_1|}\det(w^2\mathrm{Id} - B^*B)$. For a set S of r even vertices, the same formula for $G \setminus S$, whose adjacency has the block B_0 obtained by deleting the rows indexed by S, gives

$$w^r \frac{P_{G\setminus S}(w)}{P_G(w)} = \frac{\det(z\mathrm{Id} - B_0^*B_0)}{\det(z\mathrm{Id} - B^*B)}, \qquad z = w^2.$$

This is the quotient of Proposition 6.5 below.

3.3 Above index four

For a finite principal graph of index greater than 4, the squared law has an atom outside $[0, 4]$, and the lift of Proposition 2.4 has a pair of atoms on the positive half-line. The first annular weight can be read from the first branch of the graph.

Proposition 3.6 *Let Γ be a connected locally finite bipartite graph without multiple edges, rooted at o. Let $n \geqslant 1$ be an integer such that, for each $0 \leqslant r \leqslant n-1$, there is exactly one vertex v_r at distance r from o, and let $m \geqslant 1$ be the number of vertices at distance n.*

1. *We have $c_k = C_k$ for $0 \leqslant k \leqslant n-1$ and $c_n = C_n + m - 1$.*
2. *If Γ is the principal graph of a subfactor planar algebra with $\delta > 2$, then $a_k = 0$ for $1 \leqslant k \leqslant n-1$ and $a_n = m - 1$.*

Proof . 1. A walk of length k starting at o stays at distance at most k. By Proposition 2.2, the number c_k only depends on the ball of radius k around o. For $k \leqslant n-1$ this ball is the path $v_0, \dots, v_k$, and c_k is the number of Dyck paths of length $2k$, which is C_k. For $k = n$, a walk of length n from o to a vertex at distance $r < n$ never reaches distance n, so that $p_n(v_r)$ is the same number as on the half-line. A vertex v at distance n is a neighbour of v_{n-1}, and $p_n(v) = p_{n-1}(v_{n-1}) = 1$ since Γ has no multiple edges. On the half-line rooted at its endpoint there is exactly one vertex at distance n, with $p_n = 1$. Hence (2.5) gives $c_n = C_n - 1 + m$.

2. By Theorem 2.7 and the first part, we obtain recursively $a_1 = \cdots = a_{n-1} = 0$. Then (2.7) at the integer n reduces to $c_n = C_n + a_n$. ∎

For the principal graph of a subfactor, the number n in Proposition 3.6 with $m \geqslant 2$ is the supertransitivity plus one. For the following four finite principal graphs of index in $(4, 5)$, which are among those drawn in [31, Theorem 4.1], Proposition 3.6 gives the first positive annular weight n and its multiplicity $m - 1$:

	Haagerup	Asaeda–Haagerup	3311	2221
index	$\frac{5+\sqrt{13}}{2}$	$\frac{5+\sqrt{17}}{2}$	$3 + \sqrt{3}$	$\frac{5+\sqrt{21}}{2}$
(n, a_n)	$(4, 1)$	$(6, 1)$	$(4, 2)$	$(3, 2)$

Figure 1 shows the four pairs. Reading from left to right gives the successive depths from the root. The Haagerup principal graph has a path of length 3 followed by two arms of length 3. Its dual principal graph has a path of length 3 followed by two vertices at depth 4, one of which has two neighbours at depth 5. The Asaeda–Haagerup dual principal graph is the lower graph of [4, Figure 2]. The 3311 graph is the tree with four arms of lengths $3, 3, 1, 1$ from its unique vertex of degree four, rooted at the end of an arm of length 3, and the 2221 graph is the tree with four arms of lengths $2, 2, 2, 1$ from its unique vertex of degree four, rooted at the end of an arm of length 2. In each case the squared norm of the graph is the index.

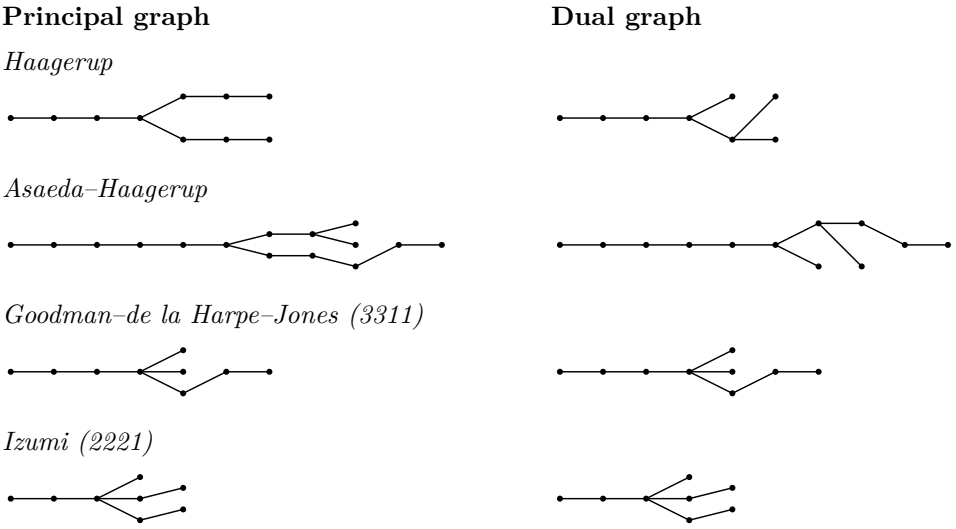


Figure 1: Four principal graph pairs from Jones, Morrison and Snyder [31, Theorem 4.1]. The root is the leftmost vertex in each diagram. In the 3311 example the two graphs coincide although the planar algebra is not self-dual.

Proposition 3.7 *The rooted squared laws of the Haagerup, Asaeda–Haagerup,* 3311 *and* 2221

principal graphs are

$$\mu_{\mathrm{H}} = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_2 + \sum_{\pm} \frac{1}{6}\left(1 \mp \frac{3}{\sqrt{13}}\right)\delta_{\frac{5\pm\sqrt{13}}{2}},$$

$$\mu_{\mathrm{AH}} = \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1 + \frac{1}{8}\delta_2 + \frac{1}{8}\delta_3 + \sum_{\pm} \frac{1}{8}\left(1 \mp \frac{4}{\sqrt{17}}\right)\delta_{\frac{5\pm\sqrt{17}}{2}},$$

$$\mu_{3311} = \frac{5}{12}\delta_0 + \frac{1}{4}\delta_2 + \sum_{\pm} \frac{2 \mp \sqrt{3}}{12}\delta_{3\pm\sqrt{3}},$$

$$\mu_{2221} = \frac{2}{3}\delta_1 + \sum_{\pm} \frac{1}{6}\left(1 \mp \frac{3}{\sqrt{21}}\right)\delta_{\frac{5\pm\sqrt{21}}{2}}.$$

Proof . By Proposition 2.6 we may use either graph of a principal graph pair. We use the dual principal graph for the Haagerup and Asaeda–Haagerup subfactors and the principal graph in the two other cases. Ordering the even vertices by their distance to the root, the matrix $K = \Lambda\Lambda^*$ of (2.4) is respectively

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The second matrix is written in the numbering $1, \dots, 6$ of the even vertices of [4, Figure 2]. The root entry of $(t\mathrm{Id} - K)^{-1}$ is the first diagonal cofactor divided by the determinant. A direct computation gives, in the four cases,

$$\frac{1}{3t} + \frac{1}{3(t-2)} + \frac{t-4}{3(t^2-5t+3)}, \quad \frac{1}{4t} + \frac{1}{4(t-1)} + \frac{1}{8(t-2)} + \frac{1}{8(t-3)} + \frac{2t-9}{8(t^2-5t+2)},$$
$$\frac{5}{12t} + \frac{1}{4(t-2)} + \frac{2t-9}{6(t^2-6t+6)}, \quad \frac{2}{3(t-1)} + \frac{t-4}{3(t^2-5t+1)}.$$

The masses of the statement are the residues of these rational functions at their poles. For instance, at the roots $x_\pm = \frac{5\pm\sqrt{13}}{2}$ of $t^2 - 5t + 3$ the residue of the last term of the first expression is $\frac{x_\pm - 4}{\pm 3\sqrt{13}} = \frac{1}{6}(1 \mp \frac{3}{\sqrt{13}})$. The other cases are similar. ∎

The largest atom has mass $\frac{1}{\dim \mathcal{C}}$, where $\mathcal{C}$ is the even part, in accordance with Section 2.2. For the Haagerup subfactor, we recover $\dim \mathcal{C} = \frac{39+9\sqrt{13}}{2}$, since $\frac{1}{6}(1 - \frac{3}{\sqrt{13}}) \cdot \frac{39+9\sqrt{13}}{2} = 1$. Only the largest atom lies outside $[0, 4]$. By Proposition 2.4, it gives two atoms of the lift ν on $(0, \infty)$ at the roots of $z + z^{-1} = x_+ - 2$, and all the other atoms of ν lie on $\mathbb{T}$.

The moments of the Haagerup and Asaeda–Haagerup laws and the annular multiplicities given by (2.7) start as follows:

n	1	2	3	4	5	6	7	8	9	10
$c_n(\mathrm{H})$	1	2	5	15	52	199	807	3374	14321	61227
$a_n(\mathrm{H})$	0	0	0	1	0	1	0	4	3	6
$c_n(\mathrm{AH})$	1	2	5	14	42	133	443	1552	5716	22067
$a_n(\mathrm{AH})$	0	0	0	0	0	1	0	2	2	6

For instance, for the Asaeda–Haagerup graph at $n = 8$, the vector p_8 of Proposition 2.2 on the even vertices of [4, Figure 2] is $(14, 28, 21, 7, 9, 1)$ and its squared norm is 1552. Subtracting the Temperley–Lieb dimension 1430 and the contribution $\binom{16}{2} a_6 = 120$ of the weight six vector gives $a_8 = 2$. The moments of μ_{AH} can also be written in closed form. For $n \geqslant 1$ we have

$$8c_n = 2 + 2^n + 3^n + r_n, \quad r_0 = 2,\ r_1 = 1,\ r_{n+2} = 5r_{n+1} - 2r_n,$$

since $8(w_+ x_+^n + w_- x_-^n)$ satisfies the recurrence of the characteristic polynomial $t^2 - 5t + 2$. The scripts `rooted_graphs.py`, `asaeda_haagerup_walks.py` and `cross_checks.py` check these values in exact arithmetic. The second uses both Asaeda–Haagerup graphs, and the third both Haagerup graphs. The script `rooted_graphs.py` computes the four root resolvents and the moments of the Haagerup and Asaeda–Haagerup graphs. For the Asaeda–Haagerup pair, `asaeda_haagerup_walks.py` counts loops separately on the two graphs and extracts the annular multiplicities at each degree.

3.4 Root weights and the exceptional series

Remark 3.4 shows that the grid of the cyclotomic expansions is fixed by the Coxeter number, and Proposition 3.5 shows that the coefficients depend on the root. The masses at the root are residues of quotients of characteristic polynomials of a diagram and of the diagram with one deleted vertex. On the Lie side, the exceptional series is related to $\mathfrak{e}_8$ through the dual pairs (A_1, E_7), (A_2, E_6), (G_2, F_4) and (D_4, D_4) in E_8 [13, p. 2 of the authors' version]. For the pair (A_2, E_6), the classical decomposition

$$\mathfrak{e}_8 = (\mathfrak{e}_6 \oplus \mathfrak{sl}_3) \oplus (27 \otimes 3) \oplus (\overline{27} \otimes \overline{3})$$

is a $\mathbb{Z}/3\mathbb{Z}$-grading of $\mathfrak{e}_8$. The automorphism of order three which acts by 1, w and w^2 on the three summands, where $w = \mathrm{e}^{2\pi \mathrm{i}/3}$, has the normalized spectral law $(86\delta_1 + 81\delta_w + 81\delta_{w^2})/248$, since the summands have dimensions 86, 81 and 81.

Question 3.8 *Find a uniform expression, as a function of the Deligne–de Man parameter $\mu \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$, of the masses of the squared laws of E_6, E_7, E_8 at the roots of Theorem 3.3. Can the coefficients of the ternary expansions (3.3) be obtained from a state on a representation graph attached to one of the above dual pairs, or to the $\mathbb{Z}_3$-grading of $\mathfrak{e}_8$?*

4 Canonical inclusions, Haagerup extensions and pointed centers

4.1 Rooted laws and formal codegrees

Let $\mathcal{C}$ be a unitary fusion category, let $L = I(\mathbf{1}) \in Z(\mathcal{C})$ be its canonical Lagrangian algebra and let $R_{\mathcal{C}} = \sum_X XX^*$ be the element (2.10). The fusion matrix N_L of L, acting on $\ell^2_{\mathrm{Irr}(Z(\mathcal{C}))}$, is selfadjoint since L is selfdual, the Frobenius algebra structure of L giving a nondegenerate pairing $L \otimes L \to \mathbf{1}$. We denote by $\mu_{\mathcal{C}}$ its spectral measure at the vector $\delta_{\mathbf{1}}$, and by $c_n = \dim \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, L^{\otimes n})$ its moments. By Proposition 2.6, $\mu_{\mathcal{C}}$ is the rooted law of any irreducible finite-index inclusion of type II_1 factors whose dual canonical endomorphism is L. For an inclusion of type III factors given by a Q-system as in Section 2.3, for instance the dual of the Longo–Rehren inclusion of $\mathcal{C}$ [43, 47], we define the rooted law by the same moments $c_n = \dim \mathrm{Hom}(\mathbf{1}, \theta^{\otimes n})$, where θ is the dual canonical endomorphism.

Theorem 4.1 *Let $\mathcal{C}$ be a unitary fusion category.*

1. *For any integer $n \geqslant 1$, we have*

$$c_n = \tau\big(R_{\mathcal{C}}^{n-1}\big) = \sum_{X_1,\dots,X_n \in \mathrm{Irr}(\mathcal{C})} \big(N_{X_1\cdots X_n}^{\mathbf{1}}\big)^2, \tag{4.1}$$

   where $N_{X_1\cdots X_n}^{\mathbf{1}} = \dim \mathrm{Hom}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)$.

2. *We have*

$$\mu_{\mathcal{C}} = \Big(1 - \sum_E \frac{1}{f_E^2}\Big)\delta_0 + \sum_E \frac{1}{f_E^2}\,\delta_{\dim(E) f_E}, \tag{4.2}$$

   where E runs through the irreducible representations of $K(\mathcal{C})_{\mathbb{C}}$ and f_E is the formal codegree of E.

3. *The measure $\mu_{\mathcal{C}}$ depends only on the based ring $K(\mathcal{C})$.*

Proof . 1. By rigidity, we have $\mathrm{Hom}(\mathbf{1}, Y \otimes L) \cong \mathrm{Hom}(Y^*, L)$ for any object Y of $Z(\mathcal{C})$. Using the adjunction between F and I, the monoidality of F and the equality $F(L) = R_{\mathcal{C}}$ in $K(\mathcal{C})$, we obtain for $Y = L^{\otimes(n-1)}$

$$\mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, L^{\otimes n}) \cong \mathrm{Hom}_{Z(\mathcal{C})}\big((L^{\otimes(n-1)})^*, I(\mathbf{1})\big) \cong \mathrm{Hom}_{\mathcal{C}}\big(F(L^{\otimes(n-1)})^*, \mathbf{1}\big).$$

The dimension of the last space is the coefficient of $\mathbf{1}$ in $[F(L)^{\otimes(n-1)}]^* = (R_{\mathcal{C}}^*)^{n-1} = R_{\mathcal{C}}^{n-1}$, that is, $\tau(R_{\mathcal{C}}^{n-1})$. This proves the first equality.

For the second equality, observe first that $R_{\mathcal{C}}$ is central in $K(\mathcal{C})$. Indeed $R_{\mathcal{C}}$ is the class of $F(I(\mathbf{1}))$, and the half-braiding of $I(\mathbf{1})$ gives isomorphisms $F(I(\mathbf{1})) \otimes Y \cong Y \otimes F(I(\mathbf{1}))$ for any $Y \in \mathcal{C}$. Let $x \in K(\mathcal{C})$ have nonnegative integer coefficients x_Y. Since $\tau(xX) = x_{X^*}$, we have

$$\sum_{X \in \mathrm{Irr}(\mathcal{C})} \tau(xX)^2 = \sum_Y x_Y^2 = \tau(xx^*). \tag{4.3}$$

Fix $n \geqslant 2$ and $X_1, \dots, X_{n-1}$. Applying (4.3) to $x = X_1 \cdots X_{n-1}$ gives $\sum_{X_n} (N_{X_1\cdots X_n}^{\mathbf{1}})^2 = \tau(X_1 \cdots X_{n-1} X_{n-1}^* \cdots X_1^*)$. Summing over X_{n-1} replaces $X_{n-1}X_{n-1}^*$ by the central element $R_{\mathcal{C}}$, so that the sum over X_{n-1} and X_n equals $\tau(R_{\mathcal{C}} X_1 \cdots X_{n-2} X_{n-2}^* \cdots X_1^*)$. Iterating this argument we obtain $\tau(R_{\mathcal{C}}^{n-1})$. For $n = 1$ both sides of (4.1) are equal to 1.

2. Recall from Section 2.3 that $K(\mathcal{C})_{\mathbb{C}} \cong \bigoplus_E \mathrm{End}(E)$, that $R_{\mathcal{C}}$ acts on E by $\dim(E) f_E$ and that $\tau = \sum_E f_E^{-1} \mathrm{Tr}(\cdot, E)$. Hence, for any integer $n \geqslant 1$, we have

$$c_n \overset{(4.1)}{=} \tau(R_{\mathcal{C}}^{n-1}) \overset{(2.8)}{=} \sum_E \frac{\dim(E)}{f_E} \big(\dim(E) f_E\big)^{n-1} = \sum_E \frac{1}{f_E^2} \big(\dim(E) f_E\big)^n.$$

Let $\rho = \sum_E f_E^{-2} \delta_{\dim(E) f_E}$, which is a positive measure on $(0, \infty)$. The compactly supported signed measures $x\mu_{\mathcal{C}}(x)$ and $x\rho(x)$ have the same moments of all orders, hence they are equal. Thus $\mu_{\mathcal{C}}$ and ρ coincide on $\mathbb{R} \setminus \{0\}$, and the mass of $\mu_{\mathcal{C}}$ at 0 is $1 - \rho(\mathbb{R})$. This proves (4.2).

3. Both expressions in (4.1) only involve the fusion coefficients of $\mathcal{C}$. ∎

In terms of the center, the nonzero atoms of $\mu_{\mathcal{C}}$ are the nonzero eigenvalues $[I(\mathbf{1}) : A_E] \frac{\dim(\mathcal{C})}{\dim(A_E)}$ of the fusion matrix of $I(\mathbf{1})$, see the proof of [53, Theorem 2.13]. If S is the normalized modular matrix of $Z(\mathcal{C})$ and m is the multiplicity vector of L, the Verlinde formula gives $\mu_{\mathcal{C}} = \sum_j S_{\mathbf{1}j}^2 \delta_{(Sm)_j / S_{\mathbf{1}j}}$, a sum over the simple objects j of $Z(\mathcal{C})$. Theorem 4.1 shows that the result can be computed in the much smaller ring $K(\mathcal{C})$. The numbers $\dim(E) f_E$ are the eigenvalues of the matrix of multiplication by $R_{\mathcal{C}}$ on $K(\mathcal{C})$, with multiplicities $\dim(E)^2$. This spectrum determines $\mu_{\mathcal{C}}$, but not always the individual pairs $(\dim(E), f_E)$.

4.2 The three Haagerup fusion categories

By [22, Theorem 1.4], the Morita equivalence class of the Haagerup fusion categories consists of exactly three fusion categories. Following [22, Section 2], $\mathcal{H}_1$ has four simple objects, which we denote by $\mathbf{1}, x, y, z$ instead of $\mathbf{1}, \nu, \eta, \mu$, with the fusion rules of [22, Table 2], and $\mathcal{H}_2$ has six simple objects $\mathbf{1}, \alpha, \alpha^2, \xi, \alpha\xi, \alpha^2\xi$ with the fusion rules

$$\alpha^3 = \mathbf{1}, \quad \alpha\xi = \xi\alpha^2, \quad \xi^2 = \mathbf{1} + \xi + \alpha\xi + \alpha^2\xi$$

of [22, Table 1]. The third category $\mathcal{H}_3$ has the same based ring as $\mathcal{H}_2$ [22, Theorem 1.4], and $\mathcal{H}_3$ is not equivalent to $\mathcal{H}_1$ or $\mathcal{H}_2$ [22, Corollary 3.22]. Moreover, the Brauer–Picard groupoid has exactly one Morita equivalence between any two of these categories [22, Theorem 1.5], so that the braided autoequivalence group of the common center $Z(\mathcal{H}) \stackrel{\text{def}}{=} Z(\mathcal{H}_i)$ is trivial by [15, Theorem 1.1]. We put

$$D_\pm \stackrel{\text{def}}{=} \frac{39 \pm 9\sqrt{13}}{2}, \quad D_+ + D_- = 39, \quad D_+ D_- = 117.$$

The number D_+ is the global dimension of each $\mathcal{H}_i$.

Corollary 4.2 *Let $L_i = I_{\mathcal{H}_i}(\mathbf{1})$ be the three Lagrangian algebras of $Z(\mathcal{H})$ and let $\mu_i = \mu_{\mathcal{H}_i}$.*

1. *The irreducible representations of $K(\mathcal{H}_1)_{\mathbb{C}}$ are one-dimensional, with formal codegrees $D_+, D_-, 3, 3$. The irreducible representations of $K(\mathcal{H}_2)_{\mathbb{C}} = K(\mathcal{H}_3)_{\mathbb{C}}$ have dimensions $1, 1, 2$, with formal codegrees $D_+, D_-, 3$.*

2. *We have*

$$\mu_1 = \nu_{\mathcal{H}} + \frac{2}{9}\delta_3 + \frac{80}{117}\delta_0, \quad \mu_2 = \mu_3 = \nu_{\mathcal{H}} + \frac{1}{9}\delta_6 + \frac{31}{39}\delta_0, \quad \nu_{\mathcal{H}} \stackrel{\text{def}}{=} \frac{\delta_{D_+}}{D_+^2} + \frac{\delta_{D_-}}{D_-^2}. \tag{4.4}$$

3. *For any integer $n \geqslant 2$, we have $c_n^{(1)} = D_+^{n-2} + D_-^{n-2} + 2 \cdot 3^{n-2}$ and $c_n^{(2)} = c_n^{(3)} = D_+^{n-2} + D_-^{n-2} + 4 \cdot 6^{n-2}$. Consequently*

$$c_n^{(2)} - c_n^{(1)} = \frac{6^n - 2 \cdot 3^n}{9}, \quad n \geqslant 1. \tag{4.5}$$

4. *No braided autoequivalence of $Z(\mathcal{H})$ sends L_2 to L_3, although $\mu_2 = \mu_3$.*

Proof . 1. Using the tables of [22], we obtain $R_{\mathcal{H}_1} = 4 \cdot \mathbf{1} + 4x + 3y + 2z$ and $R_{\mathcal{H}_2} = 6 \cdot \mathbf{1} + 3(\xi + \alpha\xi + \alpha^2\xi)$. The matrices of multiplication by these elements, in the bases $(\mathbf{1}, x, y, z)$ and $(\mathbf{1}, \alpha, \alpha^2, \xi, \alpha\xi, \alpha^2\xi)$, are

$$\begin{bmatrix} 4 & 4 & 3 & 2 \\ 4 & 20 & 13 & 9 \\ 3 & 13 & 13 & 7 \\ 2 & 9 & 7 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 6 & 0 & 0 & 3 & 3 & 3 \\ 0 & 6 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 3 & 3 & 3 \\ 3 & 3 & 3 & 15 & 9 & 9 \\ 3 & 3 & 3 & 9 & 15 & 9 \\ 3 & 3 & 3 & 9 & 9 & 15 \end{bmatrix},$$

with characteristic polynomials $(\lambda-3)^2(\lambda^2-39\lambda+117)$ and $(\lambda-6)^4(\lambda^2-39\lambda+117)$. The roots of $\lambda^2 - 39\lambda + 117$ are $D_\pm$. The ring $K(\mathcal{H}_1)$ is commutative [22, Section 2], so its irreducible

representations are one-dimensional and the formal codegrees are the eigenvalues of $R_{\mathcal{H}_1}$. The ring $K(\mathcal{H}_2)$ is not commutative since $\alpha\xi = \xi\alpha^2 \neq \xi\alpha$. A semisimple algebra of dimension 6 which is not commutative is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. The eigenvalue $\dim(E)f_E$ of $R_{\mathcal{H}_2}$ appears with multiplicity $\dim(E)^2$ in the left regular representation, so that the eigenvalue 6, of multiplicity 4, corresponds to the two-dimensional representation with $f_E = 3$. The formal codegrees of the two one-dimensional representations are D_+ and D_-. As a check, $\frac{1}{D_+} + \frac{1}{D_-} + \frac{2}{3} = \frac{39}{117} + \frac{2}{3} = 1$ and $\frac{1}{D_+} + \frac{1}{D_-} + \frac{1}{3} + \frac{1}{3} = 1$, in accordance with (2.8).

2. This follows from (4.2). The total mass of $\nu_{\mathcal{H}}$ is $\frac{(D_+ + D_-)^2 - 2D_+D_-}{(D_+D_-)^2} = \frac{1287}{13689} = \frac{11}{117}$, which gives the masses at 0.

3. The formulas for c_n follow from (4.4). Passing from μ_1 to μ_2 moves the two masses $\frac{1}{9}$ at 3 to 0 and 6, which gives (4.5).

4. By [15, Theorem 1.1] and [22, Theorem 1.5], the identity is the only braided autoequivalence of $Z(\mathcal{H})$ up to isomorphism. By [9, Proposition 4.8 and Remark 4.9], the categories of modules over L_2 and L_3 are equivalent to $\mathcal{H}_2$ and $\mathcal{H}_3$, which are not equivalent by [22, Corollary 3.22]. Hence L_2 and L_3 are not isomorphic algebras, and no braided autoequivalence of $Z(\mathcal{H})$ relates them. ■

The first moments are

	c_0	c_1	c_2	c_3	c_4	c_5	a_2	a_3
$\mathcal{H}_1$	1	1	4	45	1305	45684	2	28
$\mathcal{H}_2,\ \mathcal{H}_3$	1	1	6	63	1431	46494	4	34

The annular multiplicities are given by (2.7), with $a_1 = 0$. Thus c_n distinguishes the first inclusion from the other two for every $n \geqslant 2$, while the second and third inclusions cannot be distinguished by any moment, nor by the numbers a_k. The script `cross_checks.py` checks the characteristic polynomials of the two matrices in the proof of Corollary 4.2, the values c_n of the table and, for $2 \leqslant n \leqslant 8$, the formulas for $c_n^{(1)}$ and $c_n^{(2)}$ in part 3 of this corollary. Using the fusion rules of $K(\mathcal{H}_2)$, it also checks the second expression of (4.1) for $\mathcal{H}_2$ through $n = 5$.

Remark 4.3 This remark relies on the manuscripts [1, 2, 3], which are not published. Let S be the vertex operator algebra of central charge eight constructed in [1], contained in the lattice vertex operator algebra $V_{E_6 \oplus A_2}$. By [3, Theorem 6.1 and Corollary 6.2] we have $\mathrm{Rep}(S) \simeq Z(\mathcal{H})$, and [2, Theorem 3.9] describes three holomorphic extensions of S, all isomorphic to V_{E_8} as vertex operator algebras, whose algebra objects are

$$L_1 = \mathbf{1} + b + a + c_0, \quad L_2 = \mathbf{1} + b + 2a, \quad L_3 = \mathbf{1} + b + 2c_0,$$

where $\mathbf{1}, b, a, c_0$ are the simple objects of $Z(\mathcal{H})$ with trivial twist, labelled as in [16]. These are the three Lagrangian algebras of $Z(\mathcal{H})$, and $L_i = I_{\mathcal{H}_i}(\mathbf{1})$ for the labelling of [2, Section 2.2]. This is consistent with [53, Theorem 2.13]: the multiplicity 2 in L_2 and L_3 is the dimension of the two-dimensional representation of the common based ring of $\mathcal{H}_2$ and $\mathcal{H}_3$. Consequently, if these extensions are realized by inclusions of conformal nets whose dual canonical endomorphisms are the L_i, then these inclusions have the rooted laws (4.4) and the common index D_+. We do not address here the passage from vertex operator algebras to conformal nets. Corollary 4.2 shows that the equality $\mu_2 = \mu_3$ comes from the equality of the based rings of $\mathcal{H}_2$ and $\mathcal{H}_3$, and not from a symmetry of $Z(\mathcal{H})$.

4.3 The pointed case and the sector dimensions of the center

The last two Haagerup extensions give the same rooted law inside a common center. In a pointed category we can vary the associator on a fixed group and compute how the dimensions

of the simple objects in the center change.

Example 4.4 Let G be a finite group of order q, let $\omega \in Z^3(G,\mathbb{T})$ and let $\mathcal{C} = \mathrm{Vec}_G^\omega$. Then $K(\mathcal{C})_\mathbb{C}$ is the group algebra $\mathbb{C}[G]$, independently of ω, and $R_\mathcal{C} = \sum_g gg^{-1} = q \cdot 1$. Theorem 4.1 gives

$$\mu_{\mathrm{Vec}_G^\omega} = \left(1 - \frac{1}{q}\right)\delta_0 + \frac{1}{q}\delta_q, \quad c_n = q^{n-1} \quad (n \geqslant 1). \tag{4.6}$$

The same measure is the root measure of left multiplication by $X = \bigoplus_{g\in G} g$ on ℓ^2_G, whose matrix J_G has all entries equal to one. Indeed $J_G = qP$, where P projects onto the constant vectors. Thus the rooted law of the dual Longo–Rehren inclusion, the fusion graph of X and the dimensions of all simple objects are independent of ω, whereas the associator of Vec_G^ω, with the group labels fixed, is classified by $[\omega] \in H^3(G,\mathbb{T})$. The largest atom q of (4.6) is both the dimension of X and the global dimension of Vec_G^ω.

To compute the simple dimensions in the center, suppose now that G is abelian. For a group element g (the flux) and a normalized unitary 3-cocycle ω, put

$$\alpha_g(x,y) \stackrel{\mathrm{def}}{=} \frac{\omega(g,x,y)\omega(x,y,g)}{\omega(x,g,y)}. \tag{4.7}$$

This is a normalized projective factor set. The standard twisted-double convention with factor set (4.7) identifies the simple objects of the center with pairs (g,π), where π is an irreducible α_g-projective representation. Their dimensions are $\dim\pi$, since all conjugacy classes in G have size one and the forgetful functor to Vec_G^ω, which preserves dimensions, sends (g,π) to $\dim\pi$ copies of g. This is the abelian specialization of the transgression and centralizer description in [57, Sections 3.1–3.3 and Theorem 19, p. 23 of the arXiv version]. Reversing all half-braiding conventions can replace these factor sets by opposite or inverse ones. The dimensions calculated below are unchanged.

Write $A_\omega(g,x,y)$ for the product of the values of ω on the three even permutations of (g,x,y) divided by the product on the three odd permutations. The commutator of the factor set is

$$B_g(x,y) \stackrel{\mathrm{def}}{=} \frac{\alpha_g(x,y)}{\alpha_g(y,x)} = A_\omega(g,x,y). \tag{4.8}$$

Indeed, substituting (4.7) gives the three even permutations of (g,x,y) in the numerator and the three odd permutations in the denominator. Thus the alternating product of the six entries of ω at the permutations of (g,x,y) is the commutator of the projective factor set α_g.

For completeness, we give the algebra calculation used for the sector dimensions. Let α be a unitary 2-cocycle on a finite abelian group G. The twisted group algebra has basis $(u_x)_{x\in G}$ and product $u_x u_y = \alpha(x,y)u_{x+y}$. Its usual involution is chosen so that all u_x are unitary. The regular representation gives a finite-dimensional C^*-algebra.

Lemma 4.5 *Let $R_\alpha = \{x \in G\colon B(x,y) = 1 \text{ for all } y \in G\}$, where $B(x,y) = \frac{\alpha(x,y)}{\alpha(y,x)}$. Then*

$$\mathbb{C}^\alpha[G] \cong \bigoplus_{j=1}^{|R_\alpha|} M_d(\mathbb{C}), \quad d^2 = \frac{|G|}{|R_\alpha|}. \tag{4.9}$$

In particular there are $|R_\alpha|$ irreducible projective representations, all of dimension d.

Proof . Conjugating u_x by u_y multiplies it by $B(y,x)$. Linear independence of the basis shows that the center is the span of u_x for $x \in R_\alpha$. Products and inverses of central basis unitaries show that R_α is a subgroup. The center is a commutative C^*-algebra of dimension $|R_\alpha|$, so it has that many minimal central projections.

Fix one such projection e. For a set T of coset representatives of G/R_α, the elements eu_t, with $t \in T$, span the ideal $e\mathbb{C}^\alpha[G]$. Indeed, multiplying by an element of the central algebra acts as a scalar on this ideal. Each eu_t is nonzero since u_t is invertible. These elements are linearly independent, because their supports in the original group basis lie in disjoint cosets. Hence the ideal has dimension $\frac{|G|}{|R_\alpha|}$. A minimal central ideal in a finite-dimensional C^*-algebra is a full matrix algebra. Its dimension is the square of its matrix size, which proves (4.9). ∎

For an explicit family, let p be an odd prime, set $G = \mathbb{F}_p^3$, and put $\zeta \stackrel{\text{def}}{=} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}}$. In the standard coordinates on G, take

$$\omega_t(a,b,c) \stackrel{\text{def}}{=} \zeta^{t a_1 b_2 c_3}, \quad t \in \mathbb{F}_p. \tag{4.10}$$

The exponent is trilinear, so the cocycle identity follows by expanding sums in each variable. The identity element in any slot gives value one.

Proposition 4.6 *For the cocycle $\omega_t(a,b,c) = \zeta^{t a_1 b_2 c_3}$ on $G = \mathbb{F}_p^3$ with p odd, the simple objects of the center have the following dimensions. If $t = 0$, the center has p^6 simple objects, all of dimension one. If $t \neq 0$, it has p^3 simple objects of dimension one and $p(p^3-1)$ simple objects of dimension p. In both cases its global dimension is p^6.*

Proof . For (4.10), substitution in (4.7) gives

$$\alpha_g(x,y) = \zeta^{t(g_1 x_2 y_3 + x_1 y_2 g_3 - x_1 g_2 y_3)}, \quad B_g(x,y) = \zeta^{t \det(g,x,y)}.$$

The displayed exponent for α_g is bilinear in x, y, so it also verifies directly that α_g is a 2-cocycle. If $t = 0$, every flux has p^3 one-dimensional projective representations, which gives p^6 simple objects in all.

Assume $t \neq 0$. For $g = 0$, the factor set is trivial and again gives p^3 one-dimensional representations. If $g \neq 0$, the alternating form $(x,y) \mapsto t \det(g,x,y)$ has radical $\mathbb{F}_p g$. To see this, a multiple of g belongs to the radical. If x is not a multiple of g, extend g, x to a basis by choosing y, so $\det(g,x,y) \neq 0$. Thus the radical has cardinality p. Lemma 4.5 gives p matrix blocks of size p in every nonzero flux. There are $p^3 - 1$ such fluxes. Finally,

$$\operatorname{Dim} Z(\mathrm{Vec}_G^{\omega_t}) = p^3 + p(p^3-1)p^2 = p^6.$$

This proves the assertion. ∎

Let $\mathcal{D}_t = Z(\mathrm{Vec}_G^{\omega_t})$. Define a probability on its simple labels and a dimension law by

$$q_t(s) \stackrel{\text{def}}{=} \frac{d_s^2}{p^6}, \quad \nu_t \stackrel{\text{def}}{=} \sum_{s \in \mathrm{Irr}(\mathcal{D}_t)} q_t(s) \delta_{d_s}, \quad S_t \stackrel{\text{def}}{=} -\sum_s q_t(s) \log q_t(s). \tag{4.11}$$

Corollary 4.7 *We have*

$$\nu_0 = \delta_1, \quad \nu_t = p^{-3}\delta_1 + (1 - p^{-3})\delta_p \quad (t \neq 0), \tag{4.12}$$

and

$$S_0 = 6 \log p, \quad S_t = (4 + 2p^{-3}) \log p \quad (t \neq 0). \tag{4.13}$$

The rooted laws, the fusion graphs of $\bigoplus_g g$ and the dimensions of the simple objects of Example 4.4 agree, and all cyclic restrictions of these associators are trivial. Nevertheless the centers for $t = 0$ and $t \neq 0$ are not equivalent as braided categories.

Proof . Proposition 4.6 gives total probability p^{-3} on the dimension-one sectors and $1 - p^{-3}$ on the dimension-p sectors. This proves (4.12). Each individual weight is p^{-6} or p^{-4} respectively, which gives (4.13). To check the cyclic assertion directly, choose a generator v of a line and put $h \stackrel{\text{def}}{=} tv_1v_2v_3$. On this line we have $\omega_t(iv, jv, kv) = \zeta^{hijk}$. Since 2 is invertible in $\mathbb{F}_p$, define $b(i, j) = \zeta^{hij^2/2}$. Its coboundary is

$$\frac{b(j,k)b(i,j+k)}{b(i+j,k)b(i,j)} = \zeta^{hijk}.$$

Thus the restriction represents the trivial class on every line. A tensor equivalence preserves simple dimensions and their multiplicities, so the two center spectra exclude a braided equivalence. ∎

Example 4.8 For $p = 3$ the pointed category has global dimension 27 in every case. The fusion matrix J_G has measure $\frac{26}{27}\delta_0 + \frac{1}{27}\delta_{27}$. Its center has global dimension 729.

	$d_s = 1$	$d_s = 3$	dimension law ν_t
$t = 0$	729	0	δ_1
$t \neq 0$	27	78	$\frac{1}{27}\delta_1 + \frac{26}{27}\delta_3$

Remark 4.9 In this particular family all nonzero t become equivalent after relabelling. Pullback by $(a_1, a_2, a_3) \mapsto (sa_1, a_2, a_3)$ sends ω_t to ω_{st} for $s \in \mathbb{F}_p^\times$. Thus recovery of t is a marked problem, whereas the distinction between zero and nonzero survives after removing the marking.

Fiedler, Naaijkens and Osborne relate the total quantum dimension, under their hypotheses, to the index of an inclusion of cone algebras and to a secret-sharing task [21]. The family ω_t shows that equal global dimensions can coexist with different laws of the sector dimensions.

4.4 Annular rotation for the Haagerup extensions

Return to the two Haagerup inclusions of Corollary 4.2. They have the same moments c_n and the same annular multiplicities a_k. No braided autoequivalence of their common center carries L_2 to L_3. By Remark 4.3, which relies on unpublished manuscripts, keeping an endpoint distinguishes them: for $i = 1, 2, 3$ the multiplicities $\dim \mathrm{Hom}(a, L_i)$ are $1, 2, 0$, and $\dim \mathrm{Hom}(c_0, L_i)$ are $1, 0, 2$. These tests use the labels a, c_0 fixed by the forgetful functor to $\mathcal{H}_2$ in [2, Section 2.2]. An intrinsic test is suggested by rotation. For $\delta > 2$, write $a_{k,\omega}$ for the multiplicity of the rotation eigenvalue ω on the lowest-weight space of weight k. Then $a_k = \sum_{\omega^k=1} a_{k,\omega}$ by [28, Theorem 5.7 and Definition 6.2] and [31, Theorem 2.12].

Question 4.10 *Are the standard invariants of the two inclusions with dual canonical endomorphisms L_2 and L_3 isomorphic? Do their refined annular multiplicities $a_{k,\omega}$ coincide for all k and ω? If not, determine the smallest weight k at which they differ. More generally, for two unitary fusion categories $\mathcal{C}_1$, $\mathcal{C}_2$ with the same based ring, which planar algebra invariants of the dual Longo–Rehren inclusions are determined by the based ring?*

Remark 4.11 (Rotation in weights two and three) The algebra $P_2 = \mathrm{End}(L_i)$ is $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ for both $i = 2, 3$. At weight three both lowest-weight spaces have dimension 34. If R denotes the order-three rotation on either space, the traces of $1, R, R^2$ determine its three eigenvalue multiplicities by Fourier inversion. These traces are determined by the F-symbols of $\mathcal{H}_2$ and $\mathcal{H}_3$, for instance through the tube algebra, and we have not computed them.

5 Recovering associators from phases

For $G = \mathbb{F}_p^3$, the rooted law of Example 4.4 is constant throughout the family ω_t. The sector dimensions in Corollary 4.7 distinguish $t = 0$ from $t \neq 0$. To recover the actual parameter we must specify the group labels: on the ordered standard basis, the commutator in (4.8) takes the value $A_{\omega_t}(e_1, e_2, e_3) = \zeta^t$. Here a marking fixes the group labels, while the choice of basis in each one-dimensional fusion space is free. Which products of associator entries survive these changes of basis, and how many suffice to recover the class?

5.1 Integral cycles and gauge-invariant products

Write 1_G for the identity of a general finite group. Let $C_n(G)$ be the free abelian group on symbols $[g_1|\cdots|g_n]$ with every $g_j \neq 1_G$. A symbol with an identity entry is interpreted as zero. The normalized bar differential in degree three is

$$\partial_3[g|h|k] = [h|k] - [gh|k] + [g|hk] - [g|h]. \tag{5.1}$$

A normalized cochain $\omega\colon G^3 \to \mathbb{T}$ has value one when any entry is the identity. Its multiplicative differential is

$$(\delta\omega)(g,h,k,l) = \frac{\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k)}{\omega(gh,k,l)\omega(g,h,kl)}. \tag{5.2}$$

The cocycle equation is $\delta\omega = 1$. In terms of chains, $(\delta\omega)(g,h,k,l) = H_{\partial_4[g|h|k|l]}(\omega)$ with the notation (5.3) below, where $\partial_4[g|h|k|l] = [h|k|l] - [gh|k|l] + [g|hk|l] - [g|h|kl] + [g|h|k]$ is the normalized bar differential in degree four, extended linearly. A normalized 2-cochain β acts by $\omega \mapsto \omega\delta\beta$, with

$$(\delta\beta)(g,h,k) = \frac{\beta(h,k)\beta(g,hk)}{\beta(gh,k)\beta(g,h)}.$$

This fixes our gauge convention.

For an integral chain $z = \sum_t n_t[t] \in C_3(G)$, after combining equal symbols, put

$$H_z(\omega) \stackrel{\text{def}}{=} \prod_t \omega(t)^{n_t}, \quad L(z) \stackrel{\text{def}}{=} \sum_t |n_t|. \tag{5.3}$$

The number $L(z)$ is the degree of this Laurent monomial, counting an inverse as one occurrence. On $\mathbb{T}$ it is also the degree when negative powers are written using complex conjugates.

Proposition 5.1 *The monomial H_z is invariant under every normalized 2-cochain gauge change if and only if $\partial_3 z = 0$. This holds for arbitrary normalized 3-cochains. On cocycles it depends only on the class of z in $\mathrm{H}_3(G,\mathbb{Z})$ and the class of ω in $\mathrm{H}^3(G,\mathbb{T})$. Evaluations on all integral 3-cycles separate these cohomology classes.*

Proof . Evaluation of the coboundary gives

$$H_z(\omega\delta\beta) = H_z(\omega)\beta(\partial_3 z). \tag{5.4}$$

Here β is extended multiplicatively to $C_2(G)$. Its values on the normalized basis can be chosen independently in $\mathbb{T}$. Consequently $\beta(\partial_3 z) = 1$ for every β if and only if every coefficient of $\partial_3 z$ is zero. Also, $H_{\partial_4 w}(\omega) = (\delta\omega)(w) = 1$ for a cocycle, which proves the asserted homological dependence.

For separation, suppose that a cocycle v evaluates to one on every 3-cycle. Viewed as a homomorphism from $C_3(G)$ to $\mathbb{T}$, it factors through $\partial_3 C_3(G) \subseteq C_2(G)$. Every homomorphism from a subgroup of a finite-rank free abelian group to $\mathbb{T}$ extends to that free group. Indeed, a Smith basis reduces extension to taking integral roots in $\mathbb{T}$. Choose such an extension β on $C_2(G)$. Then $v = \delta\beta$, as required. Applying this to the quotient of two cocycles proves separation. ∎

The last argument is the divisible-coefficient form of the universal coefficient theorem. It gives a finite collection by choosing a finite generating set of the cycle lattice.

For $u, v \in \mathbb{T}$ define the angular metric $\operatorname{dist}_{\mathbb{T}}(u, v) \stackrel{\text{def}}{=} |\operatorname{Arg}(u\overline{v})| \in [0, \pi]$, where the absolute value is unambiguous at the negative real axis.

Proposition 5.2 *Let $z \in C_3(G)$, and let ω, u be normalized unitary cochains. Suppose that the entries of u and ω on the support of z have angular distance at most ε. Then*

$$\operatorname{dist}_{\mathbb{T}}(H_z(u), H_z(\omega)) \leqslant \min\{\pi, L(z)\varepsilon\}. \tag{5.5}$$

If instead $|u(t) - \omega(t)| \leqslant \eta$, then $|H_z(u) - H_z(\omega)| \leqslant L(z)\eta$.

Proof . The angular metric is invariant under multiplication. Its triangle inequality, applied to each repeated factor and its inverse, gives (5.5). The chordal estimate follows by telescoping the products. Every omitted factor has modulus one, and inversion preserves the chordal distance on $\mathbb{T}$. ∎

Remark 5.3 The product $A_\omega(g, x, y)$ in (4.8) is of the form $H_z(\omega)$. Its six exponents are the signs of the permutations of (g, x, y), and the corresponding chain is a cycle for $G = \mathbb{F}_p^r$ by Lemma 5.4 below. Thus this cycle evaluation is the commutator of a projective factor set of the center.

5.2 Reconstruction on an elementary abelian group

Let p be an odd prime, let $r \geqslant 1$, and put $G \stackrel{\text{def}}{=} \mathbb{F}_p^r$, written additively. We use representatives $0, \dots, p-1$ for field elements and write $[m]_p$ for the representative of an integer m. Put $\zeta \stackrel{\text{def}}{=} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}}$ and $C_p(b, c) \stackrel{\text{def}}{=} \frac{b+c-[b+c]_p}{p}$. For parameters $\alpha_i, \beta_{ij}, \gamma_{ijk} \in \mathbb{F}_p$, with $i < j$ and $i < j < k$, define

$$\omega_{\alpha,\beta,\gamma}(a, b, c) \stackrel{\text{def}}{=} \zeta^{\sum_i \alpha_i a_i C_p(b_i, c_i) + \sum_{i<j} \beta_{ij} a_j C_p(b_i, c_i) + \sum_{i<j<k} \gamma_{ijk} a_i b_j c_k}. \tag{5.6}$$

In the phase formulas for $\mathbb{F}_p^r$, exponents of ζ are read modulo p.

These are standard representatives, whose terms depend on one, two and three coordinates respectively, and are usually called of type I, II and III. Formula (5.6) also follows from the family in [55, equation (41)]. It need not coincide term by term with a representative using a different resolution in [24, equation (3.1), p. 12 of the arXiv version].

For $v \neq 0$ and $a, b, c \in G$, define

$$z_v \stackrel{\text{def}}{=} \sum_{j=1}^{p-1} [v|jv|v], \quad w(a, b, c) \stackrel{\text{def}}{=} \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)[a_{\sigma(1)}|a_{\sigma(2)}|a_{\sigma(3)}], \tag{5.7}$$

where $(a_1, a_2, a_3) = (a, b, c)$. Put $Q_\omega(v) \stackrel{\text{def}}{=} H_{z_v}(\omega)$. The evaluation $H_{w(a,b,c)}(\omega)$ is exactly the alternating product $A_\omega(a, b, c)$ of (4.8).

Lemma 5.4 *The chains in (5.7) are cycles. We have $L(z_v) = p - 1$. If a, b, c are linearly independent, then $L(w(a, b, c)) = 6$.*

Proof . For z_v, the first two terms in (5.1) telescope to $[v|v]$, since $pv = 0$ and an identity entry gives zero. The last two terms telescope to $-[v|v]$. For w, write $\partial_3[x|y|z] = [y|z] - [x + y|z] + [x|y + z] - [x|y]$ with $(x, y, z) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. The terms $[x + y|z]$ coming from σ and from σ composed with the transposition of the first two positions are equal and come with opposite signs, so they cancel. For the same reason the terms $[x|y + z]$ cancel in pairs, using the transposition of the last two positions. The remaining terms are $\sum_\sigma \operatorname{sgn}(\sigma)\big([a_{\sigma(2)}|a_{\sigma(3)}] - [a_{\sigma(1)}|a_{\sigma(2)}]\big)$. Replacing σ by $\sigma \circ c$, where c is the cyclic permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$, maps the second sum onto the first one, and $\operatorname{sgn}(\sigma \circ c) = \operatorname{sgn}(\sigma)$. Hence these terms cancel as well. The degree assertions follow by counting distinct nondegenerate symbols. ∎

Lemma 5.5 *The cochains in (5.6) are normalized 3-cocycles. Moreover, $H^3(\mathbb{F}_p^r, \mathbb{T}) \cong \mathbb{Z}_p^{d_r}$, where $d_r = r + \binom{r}{2} + \binom{r}{3}$.*

Proof . The carry function satisfies $C_p(b, c) + C_p([b + c]_p, d) = C_p(c, d) + C_p(b, [c + d]_p)$. Substituting this identity into (5.2) proves the assertion for a carry term. Replacing an integer sum $a_j + b_j$ by its representative introduces a multiple of p, which vanishes in the exponent. A trilinear term satisfies the additive cocycle identity by expansion over $\mathbb{F}_p$. Normalization follows directly.

We recall a count independent of the chosen representatives. The periodic resolution of a cyclic group gives $\mathrm{H}_{2m+1}(\mathbb{F}_p, \mathbb{Z}) = \mathbb{Z}_p$ for $m \geqslant 0$ and $\mathrm{H}_{2m}(\mathbb{F}_p, \mathbb{Z}) = 0$ for $m \geqslant 1$. The integral Künneth theorem, whose short exact sequence splits, noncanonically, as abelian groups, then gives

$$\mathrm{H}_1(\mathbb{F}_p^r, \mathbb{Z}) = \mathbb{Z}_p^r, \quad \mathrm{H}_2(\mathbb{F}_p^r, \mathbb{Z}) = \mathbb{Z}_p^{\binom{r}{2}}, \quad \mathrm{H}_3(\mathbb{F}_p^r, \mathbb{Z}) = \mathbb{Z}_p^{d_r}.$$

Indeed, adjoining one cyclic factor adds $r - 1$ to the exponent in degree two, from the tensor product of the first homologies. It adds $1 + (r - 1) + \binom{r-1}{2}$ to the exponent in degree three, from its own third homology, the torsion product of the first homologies, and the tensor product $\mathrm{H}_2(\mathbb{F}_p^{r-1}, \mathbb{Z}) \otimes \mathrm{H}_1(\mathbb{F}_p, \mathbb{Z})$, respectively. These are the recurrences for $\binom{r}{2}$ and d_r. Pairing with the divisible group $\mathbb{T}$ gives the stated count for cohomology. The same count is contained in the complete abelian formulas of [24]. ∎

Theorem 5.6 *Let $e_1, \dots, e_r$ be the fixed standard basis of $G = \mathbb{F}_p^r$. The following d_r monomial observables determine the marked cohomology class of any normalized unitary 3-cocycle:*

$$Q_\omega(e_i), \quad Q_\omega(e_i + e_j)\ (i < j), \quad A_\omega(e_i, e_j, e_k)\ (i < j < k). \tag{5.8}$$

For (5.6), their values are

$$Q_\omega(e_i) = \zeta^{\alpha_i}, \quad Q_\omega(e_i + e_j) = \zeta^{\alpha_i + \alpha_j + \beta_{ij}}, \quad A_\omega(e_i, e_j, e_k) = \zeta^{\gamma_{ijk}}. \tag{5.9}$$

No family with fewer than d_r integral cycle evaluations is complete. If each phase entry used in (5.8) has angular error at most ε, nearest-pth-root decoding recovers the class whenever

$$\varepsilon < \frac{\pi}{pL_r}, \quad L_r = \begin{cases} p - 1 & r \leqslant 2, \\ \max\{p - 1, 6\} & r \geqslant 3. \end{cases} \tag{5.10}$$

The unknown phase table may be expressed in any normalized gauge. The noisy table need not itself satisfy the cocycle equation.

Proof . For a nonzero coordinate v_i, the sequence $[jv_i]_p$, with $0 \leqslant j < p$, runs through all residues. The telescoping identity

$$\sum_{j=0}^{p-1} C_p([jv_i]_p, v_i) = v_i \tag{5.11}$$

also holds for $v_i = 0$. The cubic part has zero cyclic sum modulo p, since $\sum_{j=0}^{p-1}[jv_i]_p$ is either zero or $\frac{p(p-1)}{2}$, and p is odd. Consequently,

$$Q_{\omega_{\alpha,\beta,\gamma}}(v) \overset{(5.11)}{=} \zeta^{\sum_i \alpha_i v_i^2 + \sum_{i<j} \beta_{ij} v_i v_j}. \tag{5.12}$$

The value at $v = 0$ may be set equal to one.

The carry terms in an alternating evaluation cancel by interchanging the last two variables. On the ordered independent triple (e_i, e_j, e_k), only the corresponding ordered cubic term contributes to the alternating exponent. This proves (5.9). Thus the p^{d_r} parameter choices in (5.6) are pairwise noncohomologous, by Proposition 5.1. Lemma 5.5 shows that they exhaust the classes.

Every integral cycle evaluation is a character of $H^3(G, \mathbb{T})$, which has exponent p by Lemma 5.5. Hence it takes values among the pth roots of unity. A list of s such values has at most p^s possibilities, so completeness requires $s \geqslant d_r$.

Finally, distinct pth roots have angular separation at least $\frac{2\pi}{p}$. Proposition 5.2 and Lemma 5.4 show that every observation differs from its exact value by less than $\frac{\pi}{p}$ under (5.10). Decode the raw observations first. The recovered exponents give α_i, then β_{ij} by subtraction of the two recovered α parameters, and finally γ_{ijk}. Subtracting already decoded exponents introduces no additional noise factor. Gauge changes leave the exact observations unchanged by Proposition 5.1. ∎

Corollary 5.7 *All restrictions of a marked associator on $\mathbb{F}_p^r$ to cyclic subgroups determine its quadratic parameters α, β and discard its entire cubic parameter space. More precisely, the kernel of restriction to all cyclic subgroups has $p^{\binom{r}{3}}$ elements. For $r \geqslant 3$, the data of Example 4.4 together with all these cyclic restrictions are therefore incomplete.*

Proof . On a cyclic subgroup generated by v, the evaluation $Q_\omega(v)$ determines its third cohomology class, by Theorem 5.6 for $r = 1$. Formula (5.12) determines the restriction and contains no γ parameter. If all restrictions vanish, evaluations at e_i and $e_i + e_j$ show $\alpha_i = \beta_{ij} = 0$. The remaining parameters are free. Example 4.4 supplies the last assertion about the graph data. ∎

Remark 5.8 The number d_r is optimal only in the specified class of scalar monomial observations. It does not optimize their total degree or the union of their supports. It also concerns marked associators. Quotienting by $\mathrm{Aut}(G)$ can identify several recovered parameter arrays. For $p = 2$ the cyclic cubic sum used in the proof does not vanish, and we do not claim Corollary 5.7 in that case.

5.3 The sharp error bound for six entries

Theorem 5.6 uses one alternating product for each cubic coordinate. For the family (4.10), a single such product already recovers t in the chosen basis. We determine exactly how much angular error its six entries can tolerate.

Fix linearly independent $a, b, c \in \mathbb{F}_p^r$. Let $\mathcal{S}(a,b,c)$ be the six triples obtained by permuting these elements, and write $\operatorname{sgn}(\sigma)$ for the sign of $\sigma \in S_3$. The observed data lie in $\mathbb{T}^6$. The gauge group consists of all normalized 2-cochains on G, restricted to their action on these six triples. It is not restricted to pth-root-valued gauges.

Theorem 5.9 *An integer Laurent monomial in the six phases indexed by $\mathcal{S}(a,b,c)$ is gauge invariant if and only if it is an integral power of $A_\omega(a,b,c)$. Every nonconstant such monomial has degree at least six, and this degree is attained. The orbits of the restricted gauge action on $\mathbb{T}^6$ are precisely the fibers of*

$$A(u) \stackrel{\text{def}}{=} \prod_{\sigma \in S_3} u_\sigma^{\operatorname{sgn}(\sigma)}. \tag{5.13}$$

Proof . Write $z = \sum_\sigma n_\sigma [a_{\sigma(1)}|a_{\sigma(2)}|a_{\sigma(3)}]$. In its boundary the coordinate $[a+b|c]$ can arise only from the two orderings (a,b,c) and (b,a,c). Linear independence ensures that it cannot coincide with a coordinate having a single basis element in its first slot. Its coefficient is therefore $-n_{abc} - n_{bac}$. For the same reason, the coordinate $[a|b+c]$ arises only from the orderings (a,b,c) and (a,c,b), and its coefficient is $n_{abc} + n_{acb}$. The corresponding equations for every permutation show that the coefficients change sign under either adjacent transposition. Consequently,

$$\ker\bigl(\partial_3|_{\mathbb{Z}^{\mathcal{S}(a,b,c)}}\bigr) = \mathbb{Z}\bigl(\operatorname{sgn}(\sigma)\bigr)_{\sigma \in S_3}. \tag{5.14}$$

Conversely this vector is a cycle by Lemma 5.4. Proposition 5.1 proves the monomial assertion, and a nonzero multiple of this vector has degree at least six.

It remains to identify the full gauge image, including possible finite-index issues. A phase vector $v \in \mathbb{T}^6$ is in this image if it factors through the boundary map on the free group generated by $\mathcal{S}(a,b,c)$. By (5.14), this happens exactly when $A(v) = 1$. Indeed, in that case v defines a homomorphism on the image of the boundary map, and that homomorphism extends to $C_2(G)$ by the root-taking argument in Proposition 5.1. The resulting normalized 2-cochain realizes v. Thus the gauge image is $\ker A$, and its cosets are the asserted fibers. ∎

Define the distance between two restricted gauge orbits by

$$\operatorname{dist}_{\mathcal{S}}([u],[v]) \stackrel{\text{def}}{=} \inf_{\beta} \max_{\sigma \in S_3} \operatorname{dist}_{\mathbb{T}}\bigl(u_\sigma, v_\sigma(\delta\beta)_\sigma\bigr). \tag{5.15}$$

This definition compares only the six observed entries. It does not bound the difference of the full cocycle tables.

Theorem 5.10 *For any $u, v \in \mathbb{T}^6$, we have*

$$\operatorname{dist}_{\mathcal{S}}([u],[v]) = \frac{1}{6}\operatorname{dist}_{\mathbb{T}}(A(u), A(v)). \tag{5.16}$$

For the family $\omega_t(a,b,c) = \zeta^{t a_1 b_2 c_3}$ on $G = \mathbb{F}_p^3$, suppose that only the six entries at permutations of (e_1, e_2, e_3) are observed, in an arbitrary normalized gauge, with angular error at most ε on each entry. There exists a decoder recovering t for every such input whenever $\varepsilon < \frac{\pi}{6p}$. If $\varepsilon \geqslant \frac{\pi}{6p}$, no decoder can recover t on all such inputs.

Proof . Multiplicative invariance and the angular triangle inequality give $\operatorname{dist}_{\mathbb{T}}(A(u), A(v)) \leqslant 6 \max_\sigma \operatorname{dist}_{\mathbb{T}}(u_\sigma, v_\sigma(\delta\beta)_\sigma)$. This proves the lower bound in (5.16).

Choose $\theta \in [-\pi, \pi]$ with $\mathrm{e}^{\mathrm{i}\theta} = A(u)\overline{A(v)}$, and put $s_\sigma \stackrel{\text{def}}{=} \mathrm{e}^{\frac{\mathrm{i}\operatorname{sgn}(\sigma)\theta}{6}}$. Then $A(s) = \mathrm{e}^{\mathrm{i}\theta}$, so $u(vs)^{-1}$ belongs to $\ker A$. Theorem 5.9 supplies a normalized gauge such that $v\delta\beta = us^{-1}$ on the six entries. Every resulting angular discrepancy equals $\frac{|\theta|}{6}$. This proves (5.16).

For (4.10), formula (5.9) gives $A(\omega_t) = \zeta^t$. The minimum distance between the corresponding restricted gauge orbits is therefore $\frac{\pi}{3p}$. Proposition 5.2 shows that taking the alternating product of the observed phases and decoding its nearest pth root succeeds when $6\varepsilon < \frac{\pi}{p}$.

For the converse choose two adjacent roots. By the equality just proved, representatives of their restricted orbits can be chosen so that each of their six coordinates differs by signed angle $\frac{\pi}{3p}$. Choose the midpoint of each of these six shortest arcs. The resulting observation is at angular distance $\frac{\pi}{6p}$ from both exact, gauge-transformed phase tables. The full gauges exist by Theorem 5.9, so both tables are restrictions of genuine cocycles in the two specified classes. A decoder receiving this common observation cannot recover both parameters. This proves the impossibility assertion, including the threshold case. ∎

Remark 5.11 For $p = 3$ the threshold is $\frac{\pi}{18}$ radians per entry. Thus the marked six-entry test distinguishes the two nonzero parameters that give the same sector dimensions in Section 4.3. For $\mathbb{F}_3^3$, `cocycle_reconstruction.py` checks the cocycle identities, reconstructs all 3^7 marked classes from their cycle evaluations and computes the rank of the six-entry boundary matrix.

The gauge used in the proof of Theorem 5.10 changes the unobserved entries, so (5.16) gives no estimate on them. Moreover, the degree-six lower bound concerns monomials supported on these six permutations, whereas Problem 5.14 allows other triples.

5.4 Three entries in a Tambara–Yamagami category

In the pointed calculation a change of basis multiplies the six observed entries by phases whose alternating product equals one. A related calculation with three entries works for the noninvertible simple object of a Tambara–Yamagami category [54, Section 3 p. 703]. Let H be a finite abelian group, with simples labelled by H together with m, and $m \otimes m = \bigoplus_{a \in H} a$. In the usual normalized unit convention the m-to-m recoupling matrix for m, m, m has entries

$$F_{a,b} = \tau\overline{\chi_H(a,b)}, \quad a, b \in H,$$

where χ_H is a nondegenerate symmetric bicharacter and $\tau = \pm|H|^{-1/2}$. We use the convention in which the bicharacter is conjugated in this matrix. The entries have constant modulus. The simple dimensions and all fusion graph laws are fixed by H.

Proposition 5.12 *For any nonzero $a, b \in H$, the normalized gauge action, which fixes the unit constraints, that is, the bases of the fusion spaces $0 \otimes m \to m$ and $m \otimes 0 \to m$, acts on $(F_{a,0}, F_{0,b}, F_{a,b})$ by multiplication by (x, y, xy), with $x, y \in \mathbb{T}$ arbitrary. Its invariant Laurent monomials are generated by $\frac{F_{a,b}}{F_{a,0}F_{0,b}}$. For fixed τ this gives $\chi_H(a,b)$ from three entries. Suppose that $H = \mathbb{F}_p^r$ with $r \geqslant 2$, that a, b are linearly independent and that τ is known. Consider observations of the three entries above, for an unknown nondegenerate symmetric bicharacter and in an unknown normalized gauge, each with angular error at most ε. Then $\chi_H(a,b)$ can be recovered from every such observation if $\varepsilon < \frac{\pi}{3p}$, and not if $\varepsilon \geqslant \frac{\pi}{3p}$.*

Proof . A change of basis in a multiplicity-one fusion space is a unit complex scalar. In this recoupling block it gives row factors r_a and column factors c_b. The two factors for the unit channel are reciprocal since the two unit constraints are fixed, so $r_0 c_0 = 1$. More explicitly, the row factor for a contains the changes of basis for $m \otimes m \to a$ and $a \otimes m \to m$. The inverse column factor for b contains those for $m \otimes m \to b$ and $m \otimes b \to m$. For nonzero a, b, the changes for $a \otimes m \to m$ and $m \otimes b \to m$ are independent. Hence $r_a c_0 = x$ and $r_0 c_b = y$ can be chosen freely, and $r_a c_b = xy$. A monomial with integer exponents (l_1, l_2, l_3) is invariant exactly when

$l_1 + l_3 = l_2 + l_3 = 0$. This proves the assertion on monomials. Multiplying the generating ratio by τ gives $\overline{\chi_H(a,b)}$.

For the noise claim, measure the error of each phase by the length of its shortest arc in $\mathbb{T}$ and use the maximum of the three errors. The orbit distance between two triples is one third of the angular distance of their ratios. Indeed, the ratio changes by at most the sum of the three arc lengths. Conversely, an angular difference $\theta \in [-\pi, \pi]$ of the ratios can be produced by changing the three phases by $-\frac{\theta}{3}, -\frac{\theta}{3}, \frac{\theta}{3}$. The remaining difference has ratio one and lies in the gauge orbit. When a, b are independent, each p-th root of unity occurs as $\chi_H(a,b)$ among the nondegenerate symmetric bicharacters. Extend a, b to a basis. A zero off-diagonal coefficient of the corresponding bilinear form can be obtained from the identity matrix, and a nonzero coefficient ℓ from the block $\begin{bmatrix} 0 & \ell \\ \ell & 0 \end{bmatrix}$ and the identity on the other coordinates. The minimal orbit distance is $\frac{2\pi}{3p}$. Nearest-ratio decoding works strictly below half this distance, and a midpoint gives ambiguity at equality. This proves the radius. ∎

In the same unit convention, $F_{0,0} = \tau$ is gauge invariant. Thus for a marked elementary abelian group, the bicharacter and the sign of τ can both be read from the specified block. The radius of Proposition 5.12 concerns one independent pair and a known sign τ.

5.5 Annular phase tests and the cost of reconstruction

Proposition 5.12 uses selected entries of a recoupling block and the normalized unit channel. In the pointed case, (4.8) realizes the six-entry alternating product as the commutator of a projective factor set of the twisted double.

Question 5.13 (An annular phase test) *For the Tambara–Yamagami fusion rules on $\mathbb{F}_p^r$, can the three-entry invariant of Proposition 5.12 be obtained from a closed annular contraction with an effective bound on its degree and on its angular error? Which of these tests can be expressed intrinsically on the tube algebra, without a marking of the invertible objects?*

Theorem 5.6 gives the smallest possible number of integral cycle evaluations, $d_r = r + \binom{r}{2} + \binom{r}{3}$. Their cost also depends on the number $L(z)$ of phase factors in each observation, counted with multiplicity as in (5.3).

Problem 5.14 (Shortest lists of cycle evaluations) *For odd primes p and $r \geqslant 3$, let $\mathcal{B}$ range over lists of d_r integral normalized bar cycles on $\mathbb{F}_p^r$ whose evaluations separate $H^3(\mathbb{F}_p^r, \mathbb{T})$. Determine*

$$D(p,r) \stackrel{\text{def}}{=} \min_{\mathcal{B}} \max_{z \in \mathcal{B}} L(z), \quad C(p,r) \stackrel{\text{def}}{=} \min_{\mathcal{B}} \sum_{z \in \mathcal{B}} L(z). \tag{5.17}$$

In particular, what is the growth of $D(p,3)$ as $p \to \infty$? Can a complete list of minimal cardinality have uniformly bounded degree?

Theorem 5.6 gives $D(p,r) \leqslant \max\{p-1, 6\}$ and $C(p,r) \leqslant (r + \binom{r}{2})(p-1) + 6\binom{r}{3}$. The six-entry result proves optimality only for the restricted support. General cycles can use other group elements, and adding a boundary may shorten a representative. The resolution maps of [24, Section 2.4] give representatives without solving this optimization.

6 Relative root spectra on the exceptional line

6.1 A trace formula for the signed measure

Let K be connected and compact with simple Lie algebra. Fix a maximal torus of K and take the complexification of its Lie algebra as Cartan subalgebra. We use a system of positive roots Δ_+ and the Weyl vector ρ. In the root space let $(\cdot,\cdot)$ be the pairing induced by the inverse Killing form on the complex Cartan subalgebra. For a long highest root θ one has $(\theta,\theta)=\frac{1}{h^\vee}$. Let λ be a dominant integral weight of the complexified Lie algebra and put

$$L_K(\lambda) \stackrel{\text{def}}{=} \sum_{\alpha\in\Delta_+} \big(\delta_{12(\lambda+\rho,\alpha)} - \delta_{12(\rho,\alpha)}\big). \tag{6.1}$$

This is the signed measure of [12, equation (1) p. 578]. Its total mass is zero. If $\langle\cdot,\cdot\rangle_2$ denotes the pairing with long roots of squared length two, its atom locations can also be written as $\mu\langle\lambda+\rho,\alpha\rangle_2$ and $\mu\langle\rho,\alpha\rangle_2$, with $\mu=\frac{6}{h^\vee}$.

For any finitely supported signed measure L of total mass zero, define its right-continuous cumulative function with the sign $\xi_L(x) \stackrel{\text{def}}{=} -L((-\infty,x])$. It vanishes outside the interval spanned by the atoms.

Proposition 6.1 *Let L be a finitely supported signed measure of total mass zero. For any C^1 function f on an interval containing the atom locations,*

$$\int f\,\mathrm{d}L = \int f'(x)\xi_L(x)\,\mathrm{d}x. \tag{6.2}$$

For $L=L_K(\lambda)$ one has $\xi_L\geqslant 0$, and $\log\dim V_\lambda=\int\frac{\xi_L(x)}{x}\,\mathrm{d}x$, where V_λ is the irreducible module of highest weight λ of the complexified Lie algebra. By the theorem of the highest weight, it is a representation of K if and only if λ is a weight of the chosen maximal torus. If $\lambda\neq 0$, the number $Z_L\stackrel{\text{def}}{=}\int x\,\mathrm{d}L$ is positive, and $\frac{1}{Z_L}\xi_L(x)\,\mathrm{d}x$ is a probability measure. Its r-th moment is $\frac{1}{(r+1)Z_L}\int x^{r+1}\,\mathrm{d}L$ for $r\geqslant 0$.

Proof . Since L has mass zero, it is a finite real linear combination of measures $\delta_b-\delta_a$. With our sign, such a measure has cumulative function $\mathbf{1}_{[a,b)}$ if $a\leqslant b$ and $-\mathbf{1}_{[b,a)}$ if $b<a$. Integrating f' over this interval and using linearity proves (6.2). Now let A_λ be the diagonal operator on $\ell^2_{\Delta_+}$ with entries $12(\lambda+\rho,\alpha)$. The entries of A_0 are positive. Dominance gives $A_\lambda\geqslant A_0$, so that $L_K(\lambda)$ is a sum of measures $\delta_b-\delta_a$ with $0<a\leqslant b$, and $\xi_{L_K(\lambda)}\geqslant 0$. For $f(x)=\log x$, the left side is $\log\det A_\lambda-\log\det A_0$, which is $\log\dim V_\lambda$ by the Weyl dimension formula. The claim about Z_L follows from (6.2) with $f(x)=x$ and from $Z_L=24(\lambda,\rho)=12\sum_{\alpha\in\Delta_+}(\lambda,\alpha)$, which is positive for $\lambda\neq 0$: all the terms are nonnegative by dominance, and $(\lambda,\alpha)>0$ for some simple root α. The other moments follow by taking $f(x)=x^{r+1}$. For $\lambda=0$ the measure and its cumulative function vanish. ∎

The identity (6.2) is the finite-dimensional trace formula for a spectral shift. For a measure L with integer coefficients and mass zero, positivity of ξ_L is equivalent to the existence of diagonal matrices $A\leqslant B$ such that L is the spectral counting measure of B minus that of A. Indeed, list the negative atoms with their absolute multiplicities as $a_1\leqslant\cdots\leqslant a_N$ and the positive ones as $b_1\leqslant\cdots\leqslant b_N$. The cumulative inequality is equivalent to $a_i\leqslant b_i$ for all i. The diagonal matrices with these entries then give L.

The factors for the higher Casimirs in [12, equation (3) p. 578] can be checked directly. For $r\geqslant 1$ and $F_{2r}(X)=\mathrm{Tr}((\mathrm{ad}\,X)^{2r})$, X in the Cartan subalgebra, using the Killing pairing to identify a weight with a Cartan element, the difference $F_{2r}(\lambda+\rho)-F_{2r}(\rho)$ is $2\cdot 12^{-2r}\int x^{2r}\,\mathrm{d}L_K(\lambda)$.

The factor two accounts for the negative roots. The dimension and Casimir formulas in this subsection use the module V_λ of the complexified Lie algebra. For a representation of the full automorphism group, the dimension may include the length of restriction to the connected component, as in [12, Theorem 3 p. 579].

6.2 The affine atoms of Deligne and de Man

We recall the root construction of [12, equations (8)–(11) p. 580] in a form suited to ordering its atoms. Number the simple roots of E_8 so that the edges are

$$(1,3),(2,4),(3,4),(4,5),(5,6),(6,7),(7,8).$$

Write a positive root as $c = (c_1, \dots, c_8)$ in simple-root coordinates, put $b \stackrel{\text{def}}{=} c_4$, $H(c) \stackrel{\text{def}}{=} \sum_i c_i$ and $a \stackrel{\text{def}}{=} H(c) - 5b$. For $\boldsymbol{p} \in \mathbb{N}^7$, put

$$q(c) \stackrel{\text{def}}{=} (c_8, c_7, c_1, c_6, c_2, c_5, c_3), \qquad s(c) \stackrel{\text{def}}{=} q(c) \cdot \boldsymbol{p}. \tag{6.3}$$

Here the order is that of the fundamental weights used for E_8 in [12, p. 578]. In the notation $\{c, e, g, h, f, d, b, a\}$ of [12, p. 580] for the root $\sum_i c_i \alpha_i$, these letters are $c_1, \dots, c_8$ in this order, so that the label $(B; n; a_1, \dots, a_7)$ of the first Dirac mass in rule (A) there is $(c_4; H(c) - 5c_4; q(c))$. Under the reflection in the fourth simple root, a positive root is fixed, belongs to a pair of positive roots, or is the fourth simple root itself. In a pair take c to be the member with the larger fourth coordinate. The other member has fourth coordinate $b - 1$, height $H(c) - 1$ and the same $q(c)$.

The formal signed measure $L_\mu(\boldsymbol{p})$ is the sum of the following contributions, where $\mu > 0$.

$$\delta_{b+(a+s(c))\mu} - \delta_{b+a\mu} \qquad \text{for a fixed root,} \tag{6.4}$$

$$\sum_{j=1}^{s(c)} \left(\delta_{b+(a+j)\mu} - \delta_{b-1+(a+3+j)\mu} \right) \qquad \text{for a positive pair.} \tag{6.5}$$

The fourth simple root contributes zero. Let $\mathbf{G}$ be one of the groups A_1, $A_2.2$, G_2, $D_4.S_3$, F_4, $E_6.2$, E_7, E_8 of the exceptional series of [12, p. 577], with $\mu = \frac{6}{h^\vee}$ for the dual Coxeter number $h^\vee$ of its identity component, and suppose that $p_i = 0$ for $i > \min(\operatorname{rk} \mathbf{G}, 7)$. By [12, Theorem 1 p. 579], the evaluation of $L_\mu(\boldsymbol{p})$ at this value of μ is the measure (6.1) of a compact form of the identity component of $\mathbf{G}$ for the dominant weight $\sum_i p_i \Omega_i$, where the weights Ω_i are those listed in [12, pp. 577–578].

For clarity, let x denote the formal affine variable in [12] before evaluation at μ. For fixed b, the difference operator in that reference is convolution with $\delta_0 - \delta_{-x}$. For a fixed root the two terms of the difference are at slopes $a + s(c)$ and $a + s(c) - 1$. Subtracting their values for $s(c) = 0$, which is rule (C) of [12, p. 580], and applying the inverse difference gives (6.4). For a pair, the two terms before subtraction are at $(b, a + s(c))$ and $(b - 1, a + 3 + s(c))$ with opposite signs. The lower root has $H - 1 - 5(b - 1) = a + 4$, and the difference construction uses one less than this slope. For a slope A the identity

$$\left(\sum_{j=1}^{s} \delta_{(A+j)x} \right) * (\delta_0 - \delta_{-x}) = \delta_{(A+s)x} - \delta_{Ax}$$

now yields (6.5). This verifies the signs and the offsets in our form of the interpolation.

6.3 A sharp simultaneous positivity interval

Theorem 6.2 *Let $\mu > 0$. The measures $L_\mu(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathbb{N}^7$ have support in $(0,\infty)$ and satisfy $\xi_{L_\mu(\boldsymbol{p})} \geqslant 0$ if and only if $\mu \leqslant \frac{2}{5}$. On this interval their first moments are positive for $\boldsymbol{p} \neq 0$.*

Proof . Put $u \stackrel{\text{def}}{=} \frac{1}{\mu}$ and divide all atom locations by μ. A fixed root gives a positive interval starting at $bu+a$ of length $s(c)$. In a pair the positive atom minus the negative atom in (6.5) has distance $u-3$ in the rescaled variable. Thus for $u \geqslant 3$ all the contributions to the cumulative function are nonnegative.

For a fixed root, the sum of the coefficients at the neighbours of vertex 4 is $2b$. Hence $H(c) \geqslant 3b$. For the larger member of a positive pair, the neighbour sum is $2b-1$ and $H(c) \geqslant 3b-1$. For $u \geqslant \frac{5}{2}$ these inequalities give positive locations of the atoms in (6.4)–(6.5). For example, the pair locations in the rescaled variable are bounded below by $b(u-2)-1+j$ and $(b-1)(u-2)+j$, respectively. For a fixed root with $b=0$ the starting point is the positive height.

We treat $\frac{5}{2} \leqslant u \leqslant 3$ by an exact finite verification, which we describe in detail. Write z for the rescaled observation point. A fixed root whose starting point is at most z contributes one if

$$q(c) \cdot \boldsymbol{p} \geqslant \ell(c), \quad \ell(c) \stackrel{\text{def}}{=} \lfloor z - bu - a \rfloor + 1. \tag{6.6}$$

A pair can give a negative contribution. Set $y = z - bu - a$ for its larger member and $j = \lfloor y \rfloor$. It contributes minus one exactly when

$$j \geqslant 1, \quad y - j < 3 - u, \quad q(c) \cdot \boldsymbol{p} \geqslant j. \tag{6.7}$$

The intervals in a given pair have length at most $\frac{1}{2}$ in the rescaled variable. There is at most one such contribution from each pair.

For a fixed u, z, form a bipartite graph. Its first class consists of the pairs satisfying the first two tests of (6.7). Its second class consists of the fixed roots with starting point at most z. Join a pair c to a fixed root d if $q(d) \geqslant q(c)$ coordinatewise and $\ell(d) \leqslant j(c)$. A matching of the first class into the second proves nonnegativity for all $\boldsymbol{p}$. If no matching is found, the first class is nonempty. Let $M \geqslant 1$ be the maximum of the thresholds of the two classes. Replacing every coordinate p_i by $\min(p_i, M)$ preserves all the threshold tests. Indeed, a term with positive coefficient at a capped coordinate is already at least M. This gives a finite check in the absence of a matching.

Here is the reduction to finitely many u, z. The fourth coordinate of a positive E_8 root is at most six, and the height is at most 29. Put $z \stackrel{\text{def}}{=} k + r$, with k an integer and $0 \leqslant r < 1$. For fixed u, the tests and thresholds are constant in each half-open interval of r starting at a value in

$$\{bu - \lfloor bu \rfloor : 0 \leqslant b \leqslant 6\}. \tag{6.8}$$

This follows since the thresholds change only at $a + bu$ plus integers or at $a + 3 + (b-1)u$ plus integers. At $k \geqslant 30$ all fixed roots satisfy the starting point test and all pairs satisfy $j \geqslant 1$, since $bu + a + 1 = H(c) + b(u-5) + 1 \leqslant H(c) + 1 \leqslant 30$ for $u \leqslant 3$. In that range raising k by one raises every threshold by one and leaves the matching graph unchanged. We check $0 \leqslant k \leqslant 30$, requiring a matching at $k = 30$. We only need $z \geqslant 0$ by the positivity of the atom locations.

The integer parts and the order of the residues in (6.8) are constant on the open intervals of u cut out by values for which $(b-b')u$ is an integer with $0 \leqslant b' < b \leqslant 6$. On $[\frac{5}{2}, 3]$ the endpoints,

including the ends of this interval, are

$$\frac{5}{2},\ \frac{13}{5},\ \frac{8}{3},\ \frac{11}{4},\ \frac{14}{5},\ \frac{17}{6},\ 3. \tag{6.9}$$

The endpoint tests use the coincident residues as one value. For each open interval we use its midpoint. The thresholds and the first two tests in (6.7) depend only on the integer parts and the weak ordering just described when r is taken from (6.8). Indeed, for $z = k + r$ with $0 \leqslant r < 1$, write $\mathrm{fr}(x)$ for the fractional part of x and $[P]$ for the indicator of a condition P. For a fixed root, $\ell(c) = k - a - \lfloor bu \rfloor + 1 - [r < \mathrm{fr}(bu)]$, and the starting point test reads $\ell(c) \geqslant 1$. For the larger member of a pair, $j = k - a - \lfloor bu \rfloor - [r < \mathrm{fr}(bu)]$, and the second test in (6.7), which is $z < a + 3 + (b-1)u + j$, reads $k - a - 3 - \lfloor (b-1)u \rfloor - [r < \mathrm{fr}((b-1)u)] < j$. Thus this finite set covers all u, z.

The verification uses 63 fixed roots and 28 positive pairs. They are obtained by starting with the simple coordinate vectors and repeatedly applying the reflections

$$c_i \mapsto \sum_{j \sim i} c_j - c_i$$

in coordinate i, keeping the other coordinates fixed. Closing this set gives the 240 roots, of which 120 are positive. The following gives the full checking procedure after this root enumeration.

1. For each endpoint and midpoint of (6.9), for k from 0 to 30 and for each distinct residue r in (6.8), construct the two threshold lists at $z = k + r$.

2. Find a matching by augmenting paths with the edge test above. When a matching does not cover the pair class, check $k < 30$ and compare the counts in (6.6) and (6.7) for all $\boldsymbol{p} \in \{0, \dots, M\}^7$.

There are 2108 configurations. Matchings certify 2072. The other 36 have $M \leqslant 2$, and the capped checks all give nonnegativity. The script `certificate_uniform_interval.py` implements these steps in exact rational arithmetic and performs 41670 capped weight tests. It also recomputes the root counts, the affine atoms of the two necessity examples below, the vectors (f_i), (g_i), the measures U_1, U_2 and the endpoint lists of Theorem 6.3 for $1 \leqslant k \leqslant 8$, and the comparisons with the root formula at G_2 and E_8 in Section 6.4. This proves the sufficiency for $\mu \leqslant \frac{2}{5}$.

We prove necessity first for $\frac{2}{5} < \mu \leqslant \frac{1}{2}$. Take $\boldsymbol{p}_* = (0, 0, 0, 0, 1, 1, 1)$, which selects the three neighbours of vertex 4. Put $\eta \stackrel{\mathrm{def}}{=} 1 - 2\mu$, so that $0 \leqslant 2\eta < \mu$. Any negative atom from a pair has location at least μ. A fixed negative atom with location less than μ must have $H(c) = 3b$ and $b \geqslant 1$. This equality confines the root to the star of vertex 4. For the three neighbour coefficients v_i one then has $\sum_i v_i = 2b$ and $\sum_i v_i^2 - b^2 = 1$, by the squared length of the root. Cauchy–Schwarz gives $b \leqslant 1$. There are exactly three such roots, with $b = 1$ and two neighbour coefficients equal to one. Their atoms at η have total mass -3. For pairs, the positive atoms below μ must have $j = 1$ and $H(c) = 3b - 1$. Again these roots are supported on the star. Now $\sum_i v_i = 2b - 1$ and $\sum_i v_i^2 - b^2 + b = 1$, so $b \leqslant 2$. For $b = 1$ the three roots cancel the mass -3 at η. For $b = 2$ the unique root has all three $v_i = 1$, and contributes a positive atom at 2η. The positive endpoints from fixed roots with $s(c) > 0$ are at least μ. Thus below μ the measure is $\delta_{2\eta}$, and its cumulative function is negative on $[2\eta, \mu)$.

For $\mu > \frac{1}{2}$ take the seventh coordinate vector. In (6.4)–(6.5) the atoms with negative slopes, after affine cancellations, are

$$-\delta_{1-2\mu} - \delta_{1-\mu} + \delta_{2-4\mu} + \delta_{3-3\mu} + \delta_{3-2\mu} + \delta_{3-\mu} + \delta_{5-\mu}.$$

The remaining atoms have positive locations for $\mu > 0$. This list is obtained by the same root enumeration. The atom $2 - 4\mu$ is strictly smaller than every other location when $\mu > \frac{1}{2}$ and has coefficient one. This gives the failure of positivity.

Finally, the first moment is $\sum_i p_i(\mu f_i + (1 - 3\mu)g_i)$, where f_i and g_i are the sums of $q_i(c)$ over the fixed roots and over the larger members of pairs, respectively. The root enumeration gives

$$(f_i)_{i=1}^7 = (34, 66, 52, 96, 76, 124, 102), \quad (g_i)_{i=1}^7 = (12, 24, 20, 36, 30, 48, 40).$$

Each coefficient is positive at $\mu = 0$ and at $\mu = \frac{2}{5}$ since $g_i > 0$ and $2f_i > g_i$. By affinity it is positive throughout the interval. This finishes the proof. ∎

The script `check_interval.py`, written by Codex, independently constructs the E_8 roots in Euclidean coordinates and tests the cumulative function on both sides of $\mu = 2/5$.

6.4 Cartan powers and their intervals

Write $\boldsymbol{e}_i$ for the coordinate vectors in $\mathbb{N}^7$. The ray $k\boldsymbol{e}_1$ gives the Cartan powers of the adjoint. Its signed measures have the following particularly simple representation.

Theorem 6.3 *Let $k \geqslant 1$ be an integer. The negative endpoints of $L_\mu(k\boldsymbol{e}_1)$, after the formal cancellations and before any coincidence at particular values of μ, are*

$$j\mu, \quad 1 + j\mu, \quad 2 + (j-1)\mu, \quad 1 \leqslant j \leqslant k.$$

The positive endpoints are

$$4 + (j-1)\mu, \quad 5 + (j-2)\mu \quad (1 \leqslant j \leqslant k), \quad 6 + (j-1)\mu \quad (1 \leqslant j < k), \quad 6 + (2k-1)\mu.$$

Their signed difference has positive support and nonnegative cumulative function precisely for $0 < \mu \leqslant 3$. Its first moment is $k(12 - 2\mu)$.

Proof . Let $\tau_{j\mu}$ translate a measure by $j\mu$. In the fixed terms of (6.4) use the telescoping difference with endpoints $b + a\mu$ and $b + (a-1)\mu$. A root with $q_1(c) = i$ then contributes a sum of translates $\tau_{j\mu}$ for $1 \leqslant j \leqslant ik$. The pair terms are already in this form. Summing the terms of each type by the same enumeration as above gives

$$\begin{aligned} U_1 &= -\delta_0 - \delta_1 - \delta_{2-\mu} + \delta_{4-\mu} + \delta_{5-2\mu} + \delta_{6-2\mu}, \\ U_2 &= \delta_{6-\mu} - \delta_{6-2\mu}. \end{aligned}$$

The only other value of $q_1(c)$ is zero, with zero contribution. Thus $L_\mu(k\boldsymbol{e}_1) = \sum_{j=1}^{k} \tau_{j\mu} U_1 + \sum_{j=1}^{2k} \tau_{j\mu} U_2$. The second sum telescopes and cancels the first translate at $6 - \mu$ from the U_1 terms. This gives the endpoint list.

Pair $j\mu$ with $4 + (j-1)\mu$, pair $2 + (j-1)\mu$ with $5 + (j-2)\mu$, and pair $1 + j\mu$ with $6 + (j-1)\mu$ for $j < k$. For $j = k$ in the last pairing use $6 + (2k-1)\mu$. The first three distances before this replacement are $4 - \mu$, $3 - \mu$ and $5 - \mu$. The replacement adds $k\mu$. For $\mu \leqslant 3$ the intervals are nonnegative, their starting points are positive, and their total length is $k(12 - 2\mu)$. For $\mu > 3$ there is a positive atom at $5 - \mu < 2$ while all negative endpoints are at least 2. This proves the necessity. ∎

Example 6.4 For $k = 1$ the signed measure is

$$L_\mu(\boldsymbol{e}_1) = \delta_4 + \delta_{5-\mu} + \delta_{6+\mu} - \delta_\mu - \delta_{1+\mu} - \delta_2. \tag{6.10}$$

Its dimension product, that is, the product of the positive atom locations divided by the product of the negative ones, is $d(\mu) = \frac{2(5-\mu)(6+\mu)}{\mu(1+\mu)}$. At $\mu = 3$ the general formula reduces to $\delta_{6k+3} - \delta_3$, with dimension $2k+1$. At $\mu = \frac{3}{2}$, type G_2, the six positive roots in short-long simple coordinates are $(1,0), (0,1), (1,1), (2,1), (3,1), (3,2)$. With $\rho = (5,3)$ and $\theta = (3,2)$ in the same coordinates, and the Gram matrix of the simple roots with entries $\frac{1}{12}, -\frac{1}{8}, \frac{1}{4}$ for the pairing $(\cdot,\cdot)$ of Section 6.1, (6.1) gives the same signed measure as Theorem 6.3. The dimensions at $k = 1, 2, 3$ are $14, 77, 273$. At $\mu = \frac{1}{5}$, type E_8, they are $248, 27000, 1763125$. The script `certificate_uniform_interval.py` checks the equality of the signed measures and these dimensions for $k = 1, 2, 3$. The dimension products themselves agree with the Cartan dimension formulas of [38, Proposition 1.1 p. 2 of the arXiv version], whose parameter, denoted a there, is related to ours by $\mu = \frac{2}{a+2}$. The script `dynkin_cartan_casimir.py` checks the positivity, heights, first moments and integrality of the Cartan endpoint lists at selected weights and parameters, along with the Dynkin characteristic polynomials of Section 3 and split Casimir traces of Section 7.5.

There is a limit of the cumulative laws at the formal endpoint $\mu = 0$. For fixed k, (6.10) and the interval representation give

$$L_\mu(k\boldsymbol{e}_1) \to k(\delta_4 + \delta_5 + \delta_6 - \delta_0 - \delta_1 - \delta_2)$$

weakly. The corresponding cumulative functions converge in $\mathrm{L}^1(\mathbb{R})$ by the convergence of the finitely many interval endpoints. The logarithmic integral has a singularity at zero. Directly from the product of the endpoints, the dimension product is asymptotic to $\frac{60^k}{k!\mu^k}$. Thus the normalized cumulative law has a limit even though the relative determinant diverges.

6.5 Deleting vertices and taking a relative spectrum

We give a second source of positive cumulative functions. Let G be a finite bipartite graph with nonnegative edge weights. Its adjacency, with the even vertices listed first, has the form $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. Let S be a set of $r \geqslant 0$ even vertices. Write U for the coordinate inclusion of ℓ^2_S into the even space, $B_S = U^*B$, and B_0 for the matrix obtained by deleting the rows indexed by S. On the odd space put

$$Y \stackrel{\text{def}}{=} B^*B, \quad Y_0 \stackrel{\text{def}}{=} B_0^*B_0, \quad Y - Y_0 = B_S^*B_S. \tag{6.11}$$

Write $L_{G,S}$ for the spectral counting measure of Y minus that of Y_0, with multiplicities. On the even space use the matrix resolvent

$$G_S(z) \stackrel{\text{def}}{=} U^*(z\mathrm{Id} - BB^*)^{-1}U.$$

For a single vertex, this is the Stieltjes transform of the rooted measure of the squared adjacency.

Proposition 6.5 *For real $z > \|Y\|$, one has*

$$\det(zG_S(z)) = \frac{\det(z\mathrm{Id} - Y_0)}{\det(z\mathrm{Id} - Y)} = \exp\left(\int_{\mathbb{R}} \frac{\xi_{L_{G,S}}(x)}{z - x}\,\mathrm{d}x\right). \tag{6.12}$$

Moreover $0 \leqslant \xi_{L_{G,S}} \leqslant r$.

Proof . The resolvent identity gives $zG_S(z) = \mathrm{Id} + B_S(z\mathrm{Id} - Y)^{-1}B_S^*$. Applying the identity $\det(\mathrm{Id} + AB) = \det(\mathrm{Id} + BA)$ and then (6.11) gives the first equality. For the second, the logarithm of the quotient of determinants is $-\int \log(z-x)\,\mathrm{d}L_{G,S}(x)$. Apply (6.2) with the derivative of $\log(z-x)$. The inequalities for the cumulative function follow from the monotonicity of the eigenvalues under positive perturbations and from the interlacing inequalities for perturbations of rank at most r (both consequences of the min-max principle), since (6.11) is positive of rank at most r. ∎

The scalar identity is a form of the Markov–Krein correspondence for interlacing data [33, 45]. The factor z in $zG_S(z)$ makes the determinant tend to one at infinity, since $G_S(z) \sim z^{-1}\mathrm{Id}$. By Jacobi's identity for the complementary minors of an inverse matrix, for z outside the spectrum of BB^*, $\det(zG_S(z)) = z^r \frac{\det(z\mathrm{Id} - B_0B_0^*)}{\det(z\mathrm{Id} - BB^*)}$, and the factor z^r compensates for the passage from the even space to the odd space, since BB^* and B^*B have the same nonzero eigenvalues with multiplicities. For instance, the coefficient of z^{-1} in $\log\det(zG_S(z))$ is $\int x\,\mathrm{d}L_{G,S}(x) = \mathrm{Tr}(B_S^*B_S)$. More generally, the coefficient of z^{-j} is $\frac{1}{j}\int x^j\,\mathrm{d}L_{G,S}(x)$ for $j \geqslant 1$. This follows by expanding the integrand in (6.12) and using (6.2).

Proposition 6.6 *Let L be a finitely supported signed measure on $[0,\infty)$ with integer coefficients and mass zero. It has the form $L_{G,S}$ for a finite weighted bipartite graph if and only if $\xi_L \geqslant 0$. In that case the minimum number of deleted vertices is $\max_x \xi_L(x)$. If the support consists of nonnegative integers, there is a realization by a disjoint union of finite simple trees with unit edge weights.*

Proof . The necessity and lower bound follow from Proposition 6.5. First suppose $\max \xi_L = 1$. Merge any adjacent intervals of height one. The remaining intervals have endpoints $0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_q < b_q$, and $L = \sum_i (\delta_{b_i} - \delta_{a_i})$. The rational function $\prod_i \frac{z-b_i}{z-a_i}$ has strictly negative residues at the a_i. In fact the residue at a_i is $(a_i - b_i)\prod_{j\neq i}\frac{a_i-b_j}{a_i-a_j}$, with every ratio in the product positive. Write the residues as $-v_i^2$ with $v_i > 0$. Put $Y_0 \stackrel{\mathrm{def}}{=} \mathrm{diag}(a_i)$ and $Y \stackrel{\mathrm{def}}{=} Y_0 + vv^*$. The determinant lemma gives

$$\frac{\det(z\mathrm{Id} - Y)}{\det(z\mathrm{Id} - Y_0)} = 1 - \sum_i \frac{v_i^2}{z - a_i} = \prod_i \frac{z - b_i}{z - a_i}.$$

Thus the eigenvalues of Y are the b_i. A graph realization is obtained by taking the row v^* as the root and the diagonal rows with weights $\sqrt{a_i}$ as the remaining even vertices. Each such row joins only one odd vertex. A zero row can be dropped.

For general L, write $\xi_L = \sum_{j=1}^{\max \xi_L} \mathbf{1}_{\{\xi_L \geqslant j\}}$. Each summand is the indicator function of a finite union of half-open intervals. Its negative distributional derivative is a signed measure whose cumulative function is this indicator function, and it has a realization by the height-one construction. The disjoint union realizes L, with one deleted root for each summand. For $L = 0$ use no deleted vertices.

For the integer assertion there is a direct unit-weight realization of an interval $[a, b)$ with integer endpoints. Take $b - a$ odd vertices, each joined to a distinct pendant even vertices. Join all the odd vertices to an additional even root. Deleting the root gives $Y_0 = a\mathrm{Id}$. The added row changes one eigenvalue from a to b, so its counting difference is $\delta_b - \delta_a$. For general integer endpoints use one component per interval in the level-set decomposition. This gives the unit-weight realization. The minimum asserted in the proposition concerns the weighted problem. ∎

6.6 Cartan shifts as graph deletions

We apply Proposition 6.6 to the signed measures of Theorem 6.3. In this comparison we keep the atom locations of Deligne and de Man.

Theorem 6.7 *Let $k \geqslant 1$ be an integer and let $\mu > 0$. The following assertions hold.*

1. *The measure $L_\mu(k\boldsymbol{e}_1)$ is a weighted bipartite deletion measure, that is, of the form $L_{G,S}$ for a finite weighted bipartite graph, precisely for $0 < \mu \leqslant 3$. It is the deletion measure of a single vertex precisely for $\mu = 3$.*

2. *For $k = 1$, the minimum number of vertices in the weighted problem is 3 on $(0, 2)$, 2 on $[2, 3)$, and 1 at 3.*

3. *There is a bipartite deletion realization with integer adjacency coefficients precisely for $\mu \in \{1, 2, 3\}$. For each of these values there is a realization by simple trees with unit weights.*

Proof . For $0 < \mu \leqslant 3$, the first assertion about existence follows from Theorem 6.3 and Proposition 6.6. For $\mu > 3$, the smallest atom is $5 - \mu$, with a positive coefficient, by the proof of Theorem 6.3. Hence the cumulative function is negative just to its right, and Proposition 6.5 shows that the measure is not of the form $L_{G,S}$. For $\mu = 3$ the measure is $\delta_{6k+3} - \delta_3$, of height one. Suppose $0 < \mu < 3$. The three negative endpoints for $j = 1$ are μ, $1 + \mu$ and 2. The first positive endpoint is the smaller of 4 and $5 - \mu$. If $\mu \leqslant 2$, the negative endpoints μ and 2 occur before every positive endpoint. If $2 < \mu < \frac{5}{2}$, they still occur before the first positive endpoint since $\mu < 5 - \mu$. If $\frac{5}{2} \leqslant \mu < 3$, the three negative endpoints are at most $1 + \mu < 4$. There is just one positive endpoint below 4, namely $5 - \mu$. In each case the cumulative function reaches height at least two. This proves the single-vertex assertion.

For $k = 1$ and $0 < \mu < 2$, all three negative endpoints occur before the first positive one. For $2 \leqslant \mu < 3$, the endpoint $5 - \mu$ is at most $1 + \mu$, so the height is at most two. Together with the lower bound this proves assertion 2.

For integer adjacency, $\int x \, \mathrm{d}L_{G,S}(x)$ is an integer by (6.11). Since the first moment of $L_\mu(k\boldsymbol{e}_1)$ is $k(12-2\mu)$, this forces μ to be rational. Every atom of this measure is then rational. If it occurs with nonzero coefficient, it is an eigenvalue of an integer matrix, hence an algebraic integer. A rational algebraic integer is an integer. The negative endpoint μ survives the cancellations except when $\mu = \frac{5}{2}$, since only $5 - \mu$ can equal it in the range $0 < \mu \leqslant 3$ given by assertion 1. At $\mu = \frac{5}{2}$ the negative endpoint $1 + \mu = \frac{7}{2}$ survives instead. Thus integer adjacency forces $\mu \in \{1, 2, 3\}$. For these three parameters, the endpoint formula gives integer support and a nonnegative cumulative function. Proposition 6.6 constructs a simple unit-weight forest. ∎

Example 6.8 At $\mu = 3$ the single-vertex realization may be taken to have $6k$ odd neighbours of the root, each with three further even leaves. Its squared root law is

$$\frac{3}{6k+3}\delta_0 + \frac{6k}{6k+3}\delta_{6k+3}.$$

Indeed, the determinant quotient in (6.12) is $\frac{z-3}{z-(6k+3)}$. The squared root law is a probability measure, whereas the corresponding shift $\delta_{6k+3} - \delta_3$ is a signed measure of total mass zero; they are related through (6.12). For the adjoint representation at the E_6, E_7, E_8 parameters $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$, the height is three and the shift has a noninteger rational atom. By Theorem 6.7, these shifts are neither deletions of a single vertex nor deletions in a graph with integer adjacency at this

scale of the atoms. On the other hand, the squared rooted law of a finite bipartite graph at an even vertex o is always given by the deletion of o: by Proposition 6.5 with $S = \{o\}$, its Stieltjes transform G_S satisfies $zG_S(z) = \frac{\det(z\mathrm{Id}-Y_0)}{\det(z\mathrm{Id}-Y)}$. At the roots of Theorem 3.3, the deleted graphs are D_5, E_6, E_7 for E_6, E_7, E_8, and this quotient is $w\frac{P_{G\setminus o}(w)}{P_G(w)}$ with $z = w^2$, given explicitly in Section 3.2.

6.7 Restricted positivity and connected graph realizations

For $1 \leqslant r \leqslant 7$, let I_r be the set of $\mu > 0$ for which all $L_\mu(\boldsymbol{p})$ with $p_{r+1} = \cdots = p_7 = 0$ have positive support and nonnegative cumulative function. Theorems 6.2 and 6.3 give $I_7 = (0, \frac{2}{5}]$ and $I_1 = (0, 3]$.

Problem 6.9 (Restricted weight coordinates) *Determine I_r for $2 \leqslant r \leqslant 6$. Is I_r an interval? Can the signs for all allowed weights be explained by positive interval matchings compatible with the restriction of the exceptional weight coordinates in [12]?*

Remark 6.10 (Reduction to finitely many intervals) Every atom, after division by μ, has the form $a + bu$ with $a \in \mathbb{Z}$, $0 \leqslant b \leqslant 6$ and $u = \frac{1}{\mu}$. The order of the atoms, including the comparison with zero, is constant on each open interval between consecutive rational points u at which bu is an integer for some $1 \leqslant b \leqslant 6$, simultaneously for all $\boldsymbol{p}$. Since $I_r \subseteq I_1$ and $u \geqslant \frac{5}{2}$ is already covered by Theorem 6.2, only finitely many such intervals intervene.

Proposition 6.6 realizes by weighted graph deletions every measure L on $[0, \infty)$ with integer coefficients, mass zero and $\xi_L \geqslant 0$. The construction uses one component for each level. For integer endpoints the unit-weight construction may use several roots at the same level.

Question 6.11 (Connected graph deletions) *Let L be a finitely supported signed measure on $[0, \infty)$ with integer coefficients, mass zero and $\xi_L \geqslant 0$. When $\max \xi_L = 1$ the construction of Proposition 6.6 is already connected. If $\max \xi_L \geqslant 2$, determine when it is realized by a connected weighted bipartite graph with exactly $\max \xi_L$ deleted vertices. For integer support, determine the minimum under the additional condition of integer adjacency. Apply these questions to the Cartan shifts at $\mu = 1, 2$ and to the rational exceptional points after a common dilation of their atoms.*

The height-one residue formula gives a finite model for the weighted problem. For the integer problem the residues must be the spectral weights of a vector with integer entries for a Gram matrix with integer nonnegative factors.

7 Heat measures and Casimir laws of tensor powers

7.1 The heat moment identity

Let K be connected and compact with simple Lie algebra $\mathfrak{k}$. Equip $\mathfrak{k}$ with the negative Killing form, and let $X_1, \ldots, X_d$ be an orthonormal basis. We use the adjoint representation V on the complexification, of dimension d, and its character χ. Write $\mathcal{L}$ for the positive Laplacian, the closure of $-\sum_a X_a^2$ on smooth functions, where X_a also denotes the left-invariant vector field. The adjoint eigenvalue is one for this metric. Put $P_t \stackrel{\text{def}}{=} \mathrm{e}^{-t\mathcal{L}}$ and let $p_t(e, g)\,\mathrm{d}g$ be its heat probability measure at the identity, with $\mathrm{d}g$ normalized Haar measure. The character χ is real, and $\|\chi\|_\infty = d$.

Let σ_n be the spectral measure of the positive Casimir C_n on $V^{\otimes n}$ for the normalized trace $\tau_n = d^{-n}\,\mathrm{Tr}$. If $V^{\otimes n} = \bigoplus_\lambda m_{n,\lambda} V_\lambda$, and C_n acts on V_λ by c_λ, then

$$\sigma_n = d^{-n} \sum_\lambda m_{n,\lambda} \dim V_\lambda \delta_{c_\lambda}. \tag{7.1}$$

The convention for $n = 0$ is $\sigma_0 = \delta_0$. For $t > 0$, let ν_t be the image of the heat measure by $\frac{\chi}{d}$. Put $\nu_0 \stackrel{\mathrm{def}}{=} \delta_1$.

Proposition 7.1 *For any $n \geqslant 0$ and $t \geqslant 0$, we have*

$$m_n(t) \stackrel{\mathrm{def}}{=} \int x^n \,\mathrm{d}\nu_t(x) = d^{-n}(P_t(\chi^n))(e) = \int \mathrm{e}^{-tc} \,\mathrm{d}\sigma_n(c). \tag{7.2}$$

The limit of ν_t as $t \to \infty$ is the rooted adjacency law of the adjoint representation graph divided by d in the spectral variable.

Proof . The product χ^n is the character of $V^{\otimes n}$. Every χ_λ is an eigenfunction of $\mathcal{L}$ with eigenvalue c_λ and has value $\dim V_\lambda$ at e. This proves (7.2) by the finite decomposition. Since K is connected, the zero eigenspace on the characters is the trivial representation. It follows that each moment tends to $d^{-n} m_{n,0}$ as $t \to \infty$. These are the moments of the Haar image by $\frac{\chi}{d}$, which is the rescaled root law by Section 2.5. The measures have common support in $[-1, 1]$, so convergence of moments implies weak convergence. The same argument at $t = 0$ gives the stated convention. ∎

In what follows we aggregate the weights with equal Casimir eigenvalue in (7.1).

7.2 The exceptional tensor square

At $n = 2$, (7.2) connects the heat law to the tensor-square calculations of Deligne and to the split Casimir. In Vogel's normalization $\alpha = -2$, one writes $\beta = \frac{h^\vee}{3} + 2$ and $\gamma = \frac{2h^\vee}{3}$ on the exceptional line. The split Casimir is $\frac{1}{2}(C_2 - 2\mathrm{Id})$ [25, equations (5) and (45)]. We use the eigenvalue conventions of [25, equations (42)–(48)]. In particular, the two nontrivial symmetric channels on the exceptional line have total eigenvalues $2 + \frac{\mu}{3}$ and $\frac{5-\mu}{3}$. The channel with parameter γ has zero dimension away from the coincident-parameter specialization; at $\mu = 1$, type D_4, the weight $y_\beta(1) = 105$ already accounts for the whole eigenspace of eigenvalue $\frac{4}{3}$.

The following table gives the formal square law. Its entries are rational functions of μ, which we also evaluate at the Lie points; their positivity is proved below for $0 < \mu \leqslant \frac{2}{5}$. The weights in the right column are to be divided by $d(\mu)^2$.

channel	total Casimir	trace weight
unit	0	1
adjoint	1	$d(\mu)$
X_2	2	$\frac{d(\mu)(d(\mu)-3)}{2}$
$Y_2(\alpha)$	$2 + \frac{\mu}{3}$	$y_\alpha(\mu)$
$Y_2(\beta)$	$\frac{5-\mu}{3}$	$y_\beta(\mu)$

(7.3)

Here $d(\mu)$ is the product in Example 6.4, and

$$y_\alpha(\mu) = \frac{90(4+\mu)(5-\mu)}{\mu^2(1+\mu)(1+2\mu)}, \tag{7.4}$$

$$y_\beta(\mu) = \frac{90(3-\mu)(6+\mu)}{\mu(1+\mu)^2(1+2\mu)}. \tag{7.5}$$

The endpoint formula of Theorem 6.3 at $k = 2$ gives the product $\frac{4(4+\mu)(5-\mu)\cdot 5\cdot 6(6+3\mu)}{\mu\cdot 2\mu(1+\mu)(1+2\mu)\cdot 2(2+\mu)}$, which simplifies to (7.4). The products from (6.4)–(6.5) at $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ give the X_2 and $Y_2(\beta)$ weights. For the latter, the uncancelled product is

$$\frac{(3-3\mu)(3-\mu)4\cdot 5\cdot 6(6+\mu)}{\mu(1-\mu)(1+\mu)(1+2\mu)2(2+2\mu)}.$$

These products agree with the exceptional square formulas of [11, 8] and [25, equations (21) and (29)–(31)]. Direct substitution gives $1 + y_\alpha + y_\beta = \frac{d(d+1)}{2}$. All the weights are positive on the indicated interval since d is decreasing, with $d'(\mu) = -\frac{60(1+2\mu)}{\mu^2(1+\mu)^2}$, so that $d(\mu) \geqslant d(\frac{2}{5}) = \frac{736}{7}$ and the factors in (7.4)–(7.5) are positive there. Thus the table defines a probability measure with nonnegative support throughout this interval.

At E_8 the right column is $(1, 248, 30380, 27000, 3875)$, with eigenvalues $(0, 1, 2, \frac{31}{15}, \frac{8}{5})$. At $\mu = \frac{3}{2}$ the same rational formulas give the actual weights $(1, 14, 77, 77, 27)$ of G_2, with eigenvalues $(0, 1, 2, \frac{5}{2}, \frac{7}{6})$; the two weights 77 occur at different eigenvalues. This evaluation outside $(0, \frac{2}{5}]$ uses the tensor-square formulas, not the positivity statement of Theorem 6.2, which concerns all seven weight labels.

There is a simple test distinguishing the scalar conditions from a positive symmetric tensor trace. Suppose that the formal adjoint object has dimension d, that the permutations of the factors act on each tensor power by unitaries, with $\pi^* = \pi^{-1}$, and that a positive trace on each tensor power assigns $d^{\#\mathrm{cycles}(\pi)}$ to the permutation π. This is the ordinary symmetric diagram closure convention. The antisymmetrizer on the m-fold power is a selfadjoint idempotent with trace

$$\frac{1}{m!}\sum_{\pi\in S_m}\operatorname{sgn}(\pi)d^{\#\mathrm{cycles}(\pi)} = \frac{d(d-1)\cdots(d-m+1)}{m!}. \tag{7.6}$$

The polynomial identity follows, for instance, from the exterior-power dimension formula at every positive integer d. At $\mu = \frac{2}{5}$ the relative root tests hold for every $\boldsymbol{p}$ and (7.3) is positive. However $d = \frac{736}{7}$ lies strictly between 105 and 106, and (7.6) is negative at $m = 107$. This excludes a positive trace extending these permutation evaluations, since the trace of a selfadjoint idempotent is nonnegative.

The point $\mu = \frac{1}{4}$ provides another test. Here $d = 190$ and all five weights in (7.3) are integers. If a compact simple Lie algebra of dimension 190 gave this adjoint Casimir law, the classification would force type D_{10}. For D_{10}, $h^\vee = 18$, and the largest tensor-square eigenvalue is that of the Cartan square, $2 + \frac{2}{18} = \frac{19}{9}$. Indeed, 2θ dominates every other highest weight of a constituent, and the positive quadratic Casimir is increasing in the dominant order. The formal largest eigenvalue is $\frac{25}{12}$. They differ. Note that the intermediate algebra of dimension 190 in [39] is not semisimple.

7.3 Tensor degree identities and a scalar obstruction

In the adjoint representation put $A_a \overset{\mathrm{def}}{=} \mathrm{i}\,\mathrm{ad}(X_a)$. These operators are selfadjoint, and

$$\sum_a A_a^2 = \mathrm{Id}, \quad \tau_1(A_a) = 0, \quad \tau_1(A_aA_b) = \frac{\delta_{ab}}{d}. \tag{7.7}$$

In this display and below the Killing metric is used. For $i < j$ let $\Omega_{ij} = \sum_a A_a^{(i)}A_a^{(j)}$ on $V^{\otimes n}$, with the superscripts denoting tensor positions. Then

$$C_n = n\mathrm{Id} + 2\sum_{i<j}\Omega_{ij}. \tag{7.8}$$

Proposition 7.2 *For any integer $r \geqslant 1$, the quantity $\tau_n((C_n - n\mathrm{Id})^r)$ is a polynomial in n of degree at most r. Its expansion in $\binom{n}{q}$ has zero coefficients for $q = 0, 1$ and for $q > r$. In particular,*

$$\tau_n(C_n) = n, \tag{7.9}$$

$$\tau_n((C_n - n\mathrm{Id})^2) = \frac{4}{d}\binom{n}{2}, \tag{7.10}$$

$$\tau_n((C_n - n\mathrm{Id})^3) = -\frac{2}{d}\binom{n}{2} + \frac{48}{d^2}\binom{n}{3}. \tag{7.11}$$

Consequently the ordinary r-th moment of σ_n has degree at most r in n.

Proof . Expand the r-th power in (7.8) using ordered lists of r edges. The trace vanishes if a vertex occurs in exactly one edge. Indeed, the trace of a product of elementary tensors is the product of the traces over the tensor positions, and at that vertex the factor is $\tau_1(A_a) = 0$ by (7.7). A surviving list on q vertices then has $2q \leqslant 2r$. For any fixed q, its contribution depends on the list of edges after relabelling. Summing over the q chosen vertices gives the factor $\binom{n}{q}$. This gives the polynomial assertion and the mean. At order two the same edge must occur twice, with trace $\frac{1}{d}$ before multiplication by 2^2. This gives (7.10).

At order three there is either a triple edge or a triangle. Write $[X_a, X_b] = \sum_c f_{abc} X_c$. The numbers f_{abc} are real and alternating, and $\sum_{a,b,c} f_{abc}^2 = d$ by the Killing normalization. The matrices A_a are purely imaginary and skew-symmetric under transpose in a real orthonormal basis. Taking the transpose and using the trace identity gives $\tau_1(A_a A_b A_c) = \frac{1}{2}\tau_1([A_a, A_b]A_c) = \frac{\mathrm{i}}{2d} f_{abc}$. The repeated edge therefore has trace $\sum_{a,b,c} \tau_1(A_a A_b A_c)^2 = -\frac{1}{4d}$. A triangle gives $\frac{1}{d^2}$ by (7.7), for each of the six orders of its three edges. Multiplying by 2^3 proves (7.11). Expanding $C_n = (C_n - n\mathrm{Id}) + n\mathrm{Id}$ gives the last assertion. ∎

For each fixed n, the right side of (7.2) is completely monotone in t. For each fixed t, the left sides are moments of one probability on $[-1, 1]$. This gives the usual positive Hankel matrices and the localizing matrices for polynomials nonnegative on $[-1, 1]$. Proposition 7.2 imposes additional conditions on derivatives at $t = 0$ across different n.

Proposition 7.3 *There are probability laws $\widetilde{\sigma}_0, \ldots, \widetilde{\sigma}_3$ on $[0, \infty)$ with the following properties.*

1. *The first three laws, for $n = 0, 1, 2$, are the adjoint Casimir laws of $\mathrm{SO}(3)$. The law $\widetilde{\sigma}_3$ has the same mass at zero and the same moments of orders at most two as the actual law of C_3.*
2. *At each $t \geqslant 0$, the four Laplace transforms are the first four moments, from degree zero to degree three, of a probability on $[-1, 1]$.*
3. *The cubic identity (7.11) fails at $n = 3$.*

Proof . Use the spin-one representation of $\mathrm{SO}(3)$, so $d = 3$, with Casimir of spin j equal to $\frac{j(j+1)}{2}$. The decompositions at tensor degrees two and three give

$$\sigma_2 = \frac{\delta_0 + 3\delta_1 + 5\delta_3}{9}, \quad \sigma_3 = \frac{\delta_0 + 9\delta_1 + 10\delta_3 + 7\delta_6}{27}.$$

In fact the spin multiplicities at degree three, for spins $0, 1, 2, 3$, are $1, 3, 2, 1$. Put $\widetilde{\sigma}_i \stackrel{\mathrm{def}}{=} \sigma_i$ for $i \leqslant 2$, and set

$$\widetilde{\sigma}_3 \stackrel{\mathrm{def}}{=} \sigma_3 + \frac{1}{270}(-6\delta_1 + 15\delta_2 - 10\delta_3 + \delta_6).$$

All its coefficients are positive. The perturbation has mass zero, vanishing moments of orders one and two, and third moment $\frac{60}{270} = \frac{2}{9}$. This proves assertions 1 and 3.

To check assertion 2, let $z = \mathrm{e}^{-t}$. For the actual measures the moments are $1, z, m_2, m_3$ with $m_2 = \frac{1+3z+5z^3}{9}$ and $m_3 = \frac{1+9z+10z^3+7z^6}{27}$. The change in m_3 is

$$\Delta m_3 = -\frac{1}{270} z(1-z)^3(z^2+3z+6).$$

Put $H \stackrel{\text{def}}{=} \begin{bmatrix} 1 & z \\ z & m_2 \end{bmatrix}$ and $Q \stackrel{\text{def}}{=} \begin{bmatrix} z & m_2 \\ m_2 & m_3 \end{bmatrix}$, and let $\widetilde{Q}$ be Q after the perturbation. The actual matrices $H \pm Q$ are positive by (7.2), using $1 \pm x$. Thus $H - \widetilde{Q} \geqslant 0$ since $\Delta m_3 \leqslant 0$. For the other sign, the actual determinant is

$$\det(H+Q) = \frac{(1-z)^2}{81}(11+64z+27z^2+55z^3+38z^4+21z^5).$$

The perturbation of the determinant is $(1+z)\Delta m_3$, whose absolute value is at most $\frac{20}{270}(1-z)^2$. Since $\frac{20}{270} < \frac{11}{81}$, the determinant of $H+\widetilde{Q}$ is nonnegative. As its upper-left entry $1+z$ is positive, we obtain $H + \widetilde{Q} \geqslant 0$.

For $0 \leqslant z < 1$, one has $\det H = \frac{(1-z)^2(5z+1)}{9} > 0$. The selfadjoint matrix $B = H^{-1/2}\widetilde{Q}H^{-1/2}$ has spectrum in $[-1, 1]$. If e_0, e_1 are the two coordinate vectors and $v = H^{1/2}e_0$, then $\|v\| = 1$ and $Bv = H^{1/2}e_1$. The latter follows from $\widetilde{Q}e_0 = He_1$. Consequently its root moments of degrees zero through three are $1, z, m_2, m_3 + \Delta m_3$. The case $z = 1$ uses δ_1. This constructs the desired probabilities. ∎

The perturbation preserves the limits at $t = \infty$ as well as the lower jets at $t = 0$. It therefore separates these finite scalar conditions from a tensor trace identity, provided that the support of $\widetilde{\sigma}_3$ is not required to lie in the Casimir spectrum $\{0, 1, 3, 6\}$ of $V^{\otimes 3}$; indeed $\widetilde{\sigma}_3$ has an atom at 2. With this support condition assertion 1, on the mass at zero and the moments of orders at most two, would force $\widetilde{\sigma}_3 = \sigma_3$, since a measure on $\{1, 3, 6\}$ whose moments of orders 0, 1 and 2 vanish is zero. This is a truncated example through degree three.

7.4 The two highest weight data of the tensor square

Remark 7.4 The law σ_2 determines the pair $(d, h^\vee)$. Indeed, its mass at zero is $\frac{1}{d^2}$ since the adjoint representation has exactly one invariant pairing. Its largest atom is the Casimir eigenvalue of 2θ, since this weight occurs in $V \otimes V$ and dominates the highest weights of the other constituents, and for dominant weights $\lambda \geqslant \eta$ the difference $c_\lambda - c_\eta = (\lambda - \eta, \lambda + \eta + 2\rho)$ is positive if $\lambda \neq \eta$. As $c_\theta = 1$ and $(\theta, \theta) = \frac{1}{h^\vee}$, the largest atom is $2 + \frac{2}{h^\vee}$. At fixed $h^\vee$, the dimension is $(h^\vee)^2 - 1$ for type A, $\frac{(h^\vee+1)(h^\vee+2)}{2}$ for types B, D and $(h^\vee - 1)(2h^\vee - 1)$ for type C, and none of the exceptional pairs $(14, 4)$, $(52, 9)$, $(78, 12)$, $(133, 18)$, $(248, 30)$ is a classical pair. Two of the three classical formulas take a common value at the same $h^\vee$ only for $h^\vee = 2, 3, 4$, where they describe the isomorphisms $A_1 = C_1$, $B_2 = C_2$ and $A_3 = D_3$. Moreover $h^\vee$ is odd in type B_ℓ and even in type D_ℓ. Hence $(d, h^\vee)$ identifies the simple type. Since σ_2 has at most six atoms by the classical decompositions of the adjoint square [25, Section 3], the derivatives of m_2 of orders 0 to 11 at $t = 0$ already determine σ_2 among the laws with at most six atoms. Theorem 7.6 below shows that the derivatives of orders two and four suffice.

7.5 The fourth moment and the quartic adjoint tensor

Use the generators A_i of (7.7). Let $T_{ijkl} = \mathrm{Tr}(A_iA_jA_kA_l)$ and let S be its full symmetrization. These tensors are real in the adjoint model. Put

$$q_4 \overset{\text{def}}{=} \frac{\|S\|^2}{d}, \qquad P_4(Y) \overset{\text{def}}{=} \mathrm{Tr}\left(\left(\sum_i Y_iA_i\right)^4\right), \tag{7.12}$$

with the Euclidean norm on coefficient tensors. Thus S is the coefficient tensor of P_4. The trace in (7.12) is unnormalized.

Lemma 7.5 *Let $\Omega = \Omega_{12}$. Then*

$$\frac{\mathrm{Tr}(\Omega^4)}{d} = q_4 + \frac{1}{12}, \qquad m_2^{(4)}(0) = 16 + \frac{\frac{244}{3} + 16q_4}{d}. \tag{7.13}$$

Moreover $P_4(Y) = \frac{5}{2(d+2)}|Y|^4 + H_4(Y)$ with H_4 harmonic. If S_H is the coefficient tensor of H_4, then

$$\frac{\|S_H\|^2}{d} = q_4 - \frac{25}{12(d+2)}. \tag{7.14}$$

Proof . Cyclicity and reversal leave three orders in the symmetrization of T. Denote the corresponding tensors by T_1, T_2, T_3, with $T_1 = T$. They have the same norm and $S = (T_1 + T_2 + T_3)/3$. The squared norm of the difference of any two orders is $\frac{d}{4}$. Indeed, the difference between T_{ijkl} and T_{ijlk} is $\mathrm{Tr}(A_iA_j[A_k, A_l]) = -\frac{1}{2}\sum_a f_{ija}f_{kla}$, where we used the cubic trace in Proposition 7.2. The normalization gives $\sum_{i,j} f_{ija}f_{ijb} = \delta_{ab}$. Each of the three orders is obtained from each of the others by a rotation or a reversal of (i, j, k, l) followed by an exchange of two adjacent indices, so the same computation gives the other differences. Consequently $\sum_{\ell=1}^3 \|T_\ell - S\|^2 = \frac{1}{3}\sum_{\ell<m}\|T_\ell - T_m\|^2 = \frac{d}{4}$. Since S is the orthogonal symmetrization of each T_ℓ, we get $\|T\|^2 = \|S\|^2 + \frac{d}{12}$. On the other hand $\mathrm{Tr}(\Omega^4) = \sum_{i,j,k,l} T_{ijkl}^2$. This proves the first identity in (7.13). Expanding $C_2^4 = (2\mathrm{Id} + 2\Omega)^4$ and using (7.9)–(7.11) gives the second.

The adjoint Casimir acts on the span of the A_j by $c_\theta = 1$, so that $\sum_i [A_i, [A_i, A_j]] = A_j$. Expanding the double commutator and using $\sum_i A_i^2 = \mathrm{Id}$ gives $\sum_i A_iA_jA_i = \frac{1}{2}A_j$. Therefore the three pairings in the double contraction of S have traces d, $\frac{d}{2}$, d, and $\sum_{i,j} S_{iijj} = \frac{5d}{6}$. In the same notation, two of the three orders place the two indices i next to each other cyclically and one order separates them. Hence $\sum_i S_{iikl} = \frac{2}{3}\mathrm{Tr}(A_kA_l) + \frac{1}{3}\sum_i \mathrm{Tr}(A_iA_kA_iA_l) = \frac{2}{3}\delta_{kl} + \frac{1}{6}\delta_{kl} = \frac{5}{6}\delta_{kl}$. The coefficient tensor of $|Y|^4$ is $\Theta_{ijkl} = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/3$. Its partial contraction is $\frac{d+2}{3}\delta_{kl}$ and its squared norm is $\frac{d(d+2)}{3}$. Thus $S_H = S - \frac{5}{2(d+2)}\Theta$ has zero partial contraction and is orthogonal to Θ. The zero contraction is the harmonic condition, and taking squared norms gives (7.14). ∎

We next compute q_4 from the split Casimir formula. Let (α, β, γ) be the Vogel parameters with $\alpha = -2$ and $h = \alpha + \beta + \gamma = h^\vee$. In this subsection and in the proof of Theorem 7.6, the letter h denotes a dual Coxeter number, not a Coxeter number, and in the symplectic case $H = 2(\ell + 1)$ is twice the dual Coxeter number of $\mathfrak{sp}_{2\ell}$. Write $a = -\frac{\alpha}{2h}$, $b = -\frac{\beta}{2h}$ and $c = -\frac{\gamma}{2h}$. The spectrum on the nontrivial symmetric channels is contained in $\{a, b, c\}$, and the unit channel has eigenvalue -1. The antisymmetric part has eigenvalue $-\frac{1}{2}$ of multiplicity d, with zero on its complement [25, equations (14) and (19)–(21)]. These facts hold with absent channels and coincident values included in the polynomial identity [25, Section 3, equations (41)–(48)]. The characteristic identities in the adjoint representation are derived in [26].

Let s_j be the trace of Ω^j on the symmetric part with the unit removed. The moments through order three give

$$s_1 = 1 + \frac{d}{2}, \quad s_2 = \frac{3d}{4} - 1, \quad s_3 = 1 - \frac{d}{8}.$$

Put $p \stackrel{\text{def}}{=} ab + bc + ca$ and $q \stackrel{\text{def}}{=} abc$. Since $a + b + c = -\frac{1}{2}$, the cubic identity on this part gives $s_4 = -\frac{1}{2}s_3 - ps_2 + qs_1$. The trace of the cubic identity $(\Omega - a)(\Omega - b)(\Omega - c) = 0$ on this part, of dimension $\frac{d(d+1)}{2} - 1$, gives $s_3 + \frac{1}{2}s_2 + ps_1 - q\big(\frac{d(d+1)}{2} - 1\big) = 0$, that is, $(\frac{d}{2} + 1)(\frac{1}{2} + p) = \frac{(d+2)(d-1)}{2}q$. Hence $p = q(d-1) - \frac{1}{2}$. Adding the unit and antisymmetric contributions to s_4 and using $\frac{\mathrm{Tr}(\Omega^4)}{d} = q_4 + \frac{1}{12}$ from (7.13) gives

$$q_4 = \frac{5}{12} - \frac{3}{4}(d-3)q. \tag{7.15}$$

For this calculation repeated roots in the cubic cause no division. Formula (7.15) is an expression of the fourth split Casimir trace. The role of such traces in higher tensor powers is also discussed in [27].

The resulting values are as follows. In the orthogonal row $N \geqslant 5$ is the dimension of the defining module and $h = N - 2$. In the symplectic row the defining dimension is 2ℓ, with $\ell \geqslant 1$, and $H = 2(\ell + 1)$. In the row of $\mathfrak{sl}_N$, $N \geqslant 2$.

(7.16)

type	d	q_4
$\mathfrak{sl}_N$	$N^2 - 1$	$\dfrac{1 + 36/N^2}{24}$
$\mathfrak{so}_N$	$\dfrac{(h+1)(h+2)}{2}$	$\dfrac{1 - 9/h + 90/h^2 - 72/h^3}{24}$
$\mathfrak{sp}_{2\ell}$	$\dfrac{(H-2)(H-1)}{2}$	$\dfrac{1 + 9/H + 90/H^2 + 72/H^3}{24}$
G_2, F_4, E_6, E_7, E_8	$14, 52, 78, 133, 248$	$\dfrac{25}{12(d+2)}$

In the last row one can use $\gamma = \frac{2h}{3}$ and $\beta = \frac{h}{3} + 2$, giving $d + 2 = \frac{10h^2}{h+6}$ and $q = \frac{h+6}{18h^2}$. We use $\mathfrak{sl}_2$ for the rank-one type, $\mathfrak{sp}_4$ and $\mathfrak{so}_5$ for the same type, and $\mathfrak{sl}_4$ and $\mathfrak{so}_6$ for the same type. The two formulas agree in each overlap where the corresponding row is used.

By (7.14) the equality $H_4 = 0$ is exactly $q_4 = \frac{25}{12(d+2)}$. In type A the difference is $\frac{(N^2-4)(N^2-9)}{24N^2(N^2+1)}$. In the orthogonal row it is

$$\frac{(h-6)^2(h-1)(h+3)(h+4)}{24h^3(h^2+3h+6)}.$$

The symplectic formula is obtained from the orthogonal expression by $h \mapsto -H$. Thus $H_4 = 0$ precisely for the simple types in the Deligne list, including A_1, A_2, D_4. This agrees with the classical quartic trace identities [50, 51].

7.6 The optimal order of heat derivatives

Theorem 7.6 *The two numbers $m_2''(0)$ and $m_2^{(4)}(0)$ determine the compact simple Lie algebra. The derivatives through order three do not suffice. If two nonisomorphic simple types have the same dimension d, their fourth derivatives differ by at least $\frac{4}{25d^{3/2}}$. For B_ℓ and C_ℓ, the absolute difference multiplied by $d^{3/2}$ tends to $6\sqrt{2}$ as $\ell \to \infty$.*

Proof . We have $m_2''(0) = 4 + \frac{4}{d}$ by (7.10). Given d, the fourth derivative gives q_4 by (7.13). We prove separation and the stated estimates at fixed dimension.

Put $f(d) \stackrel{\text{def}}{=} \frac{1}{24} + \frac{3}{2(d+1)}$, which is the type A value when that type exists. For the dimensions of the orthogonal and symplectic rows in (7.16), respectively, direct factorization gives

$$f(d) - q_4(\mathfrak{so}_N) = \frac{3(h-4)(h-1)(h+2)(h+4)}{8h^3(h^2+3h+4)}, \tag{7.17}$$

$$q_4(\mathfrak{sp}_{2\ell}) - f(d) = \frac{3(H+4)(H+1)(H-2)(H-4)}{8H^3(H^2-3H+4)}. \tag{7.18}$$

In the first line $h \geqslant 5$ covers the orthogonal types after the small coincidences. The exceptional small case $\mathfrak{so}_5$ has dimension ten and is $\mathfrak{sp}_4$. In the second line $H \geqslant 6$ covers the symplectic types of rank at least two. The dimension is strictly increasing in each classical row as the rank grows. The two orthogonal series are distinguished by the parity of N, which itself is determined by the dimension. Thus the signs in (7.17)–(7.18) separate the classical collisions in dimension.

For $h \geqslant 5$, the first gap is at least $\frac{3}{100h}$, since $h-4 \geqslant \frac{h}{5}$, $h-1 \geqslant \frac{4h}{5}$ and $h^2+3h+4 \leqslant 2h^2$. This is larger than $\frac{1}{100\sqrt{d}}$. In the symplectic formula, $H-2 \geqslant \frac{2H}{3}$, $H-4 \geqslant \frac{H}{3}$ and $H^2-3H+4 \leqslant H^2$. Hence the gap is at least $\frac{1}{12H} > \frac{1}{100\sqrt{d}}$, using $H < 2\sqrt{d}$. By the dimension formulas of Section 7.4, the only exceptional dimension shared with a different simple type is 78, for E_6, B_6, C_6. Here the three values of q_4 are $\frac{5}{192}, \frac{145}{3993}, \frac{365}{4116}$. Their mutual gaps exceed $\frac{1}{100}$. This proves the uniform bound for q_4. Multiplying by $\frac{16}{d}$ gives the bound for $m_2^{(4)}(0)$.

For B_ℓ, C_ℓ at the same dimension, $h = 2\ell - 1$, $H = 2\ell + 2$ and $d = \ell(2\ell+1)$. The sum of the gaps in (7.17)–(7.18) is $\frac{3}{8}(\frac{1}{h} + \frac{1}{H}) + O(\ell^{-2})$. Multiplication by $\sqrt{d}$ gives $\frac{3\sqrt{2}}{8}$, and the assertion for the fourth derivatives follows. Finally, all derivatives through order three depend only on d by Proposition 7.2. The types B_ℓ, C_ℓ with $\ell \geqslant 3$ supply the lower bound on the order of the derivatives. ∎

For dimensions at most D, unequal dimensions are separated in $m_2''(0)$ by at least $\frac{4}{D^2}$. The same-dimension estimate in Theorem 7.6 therefore gives a separation of at least $\frac{4}{25D^2}$ for the pair of derivatives with the maximum norm.

Example 7.7 At the formal intermediate point $\mu = \frac{1}{4}$ the square law in (7.3) has dimension 190. The value inferred for q_4 from its fourth moment is $\frac{25}{2304}$. In fact the formal Vogel parameters on this line give $q_4 = \frac{25}{12(d+2)}$, or one can substitute the five weights directly in (7.13). The compact simple type of dimension 190 is D_{10}, with $q_4 = \frac{31}{972}$. This yields an obstruction using only the two derivatives of Theorem 7.6.

7.7 The second correction at time $\frac{s}{n}$

We first transfer the heat measure to the Lie algebra. Fix a maximal torus of K with Lie algebra $\mathfrak{t}$. We view the roots and the weights as real linear forms on $\mathfrak{t}$, a weight λ corresponding to the character $\exp H \mapsto \mathrm{e}^{\mathrm{i}\lambda(H)}$ of the torus, so that $\mathrm{ad}(H)$ has the eigenvalues 0 and $\pm\mathrm{i}\alpha(H)$ on $\mathfrak{k}_{\mathbb{C}}$, where α runs through a set Δ_+ of positive roots. Let ρ be half the sum of the positive roots. We identify $\mathfrak{k}$ with $\mathfrak{k}^*$ by the negative Killing form. This identifies $\mathfrak{t}^*$ with the forms vanishing on the orthogonal complement $[\mathfrak{t}, \mathfrak{k}]$ of $\mathfrak{t}$, and the norm of $\mathfrak{k}^*$ restricts on $\mathfrak{t}^*$ to the norm of the pairing $(\cdot, \cdot)$ of Section 2.6, for which $c_\lambda = (\lambda, \lambda + 2\rho)$. For $Y \in \mathfrak{k}$, let

$$J(Y) \stackrel{\text{def}}{=} \prod_{\alpha \in \Delta_+} \frac{\sin(\alpha(H_Y)/2)}{\alpha(H_Y)/2},$$

where $H_Y \in \mathfrak{t}$ is conjugate to Y under the adjoint action, and where a factor with $\alpha(H_Y) = 0$ is read as 1. The value does not depend on the choice of H_Y, since two such choices are conjugate under the Weyl group, which permutes the positive roots up to sign, and $\frac{\sin x}{x}$ is even. This is the analytic square root of the Jacobian $\det \frac{\sinh(\mathrm{ad}(Y)/2)}{\mathrm{ad}(Y)/2}$ of the exponential map in the left trivialization, with $J(0) = 1$ [40, Section 7.5, p. 33 of the arXiv version]. We have $|J(Y)| \leqslant 1$ since $|\sin x| \leqslant |x|$ for real x. For $t > 0$ we denote by Y_t a Gaussian random vector in $\mathfrak{k}$ with covariance $2t\mathrm{Id}_{\mathfrak{k}}$.

Lemma 7.8 *For any finite-dimensional representation of K with character ψ and any $t > 0$, we have*

$$\mathbb{E}\big[J(Y_t)\psi(\exp Y_t)\big] = \mathrm{e}^{-t\|\rho\|^2}(P_t\psi)(e). \tag{7.19}$$

Moreover $\|\rho\|^2 = \frac{d}{24}$.

Proof . By linearity we may assume that $\psi = \chi_\lambda$ is the character of an irreducible representation V_λ with highest weight λ. Kirillov's character formula [40, Theorem 57, p. 33 of the arXiv version], see also [44, Section 3], gives

$$J(Y)\chi_\lambda(\exp Y) = \int_{\mathcal{O}_{\lambda+\rho}} \mathrm{e}^{\mathrm{i}\beta(Y)}\,\mathrm{d}\nu_{\lambda+\rho}(\beta), \qquad Y \in \mathfrak{k},$$

where $\mathcal{O}_{\lambda+\rho} \subset \mathfrak{k}^*$ is the coadjoint orbit of $\lambda+\rho$ and $\nu_{\lambda+\rho}$ is a positive multiple of its Liouville measure. Here we have translated the conventions of [40], where the weights and the roots are the elements $\lambda_{\mathrm{L}} = \mathrm{i}\lambda$ and $\alpha_{\mathrm{L}} = \mathrm{i}\alpha$ of $\mathrm{i}\mathfrak{t}^*$, the index L referring to [40], so that $\rho_{\mathrm{L}} = \mathrm{i}\rho$ and the orbit $\mathcal{O}_{-\mathrm{i}(\lambda_{\mathrm{L}}+\rho_{\mathrm{L}})}$ of [40] is our $\mathcal{O}_{\lambda+\rho}$, and $\frac{\mathrm{e}^{\mathrm{i}a/2}-\mathrm{e}^{-\mathrm{i}a/2}}{\mathrm{i}a} = \frac{\sin(a/2)}{a/2}$ for real a. Taking $Y = 0$ shows that the total mass of $\nu_{\lambda+\rho}$ is $\dim V_\lambda$. The coadjoint action preserves the norm, so every $\beta \in \mathcal{O}_{\lambda+\rho}$ satisfies $\|\beta\| = \|\lambda+\rho\|$, and $\mathbb{E}[\mathrm{e}^{\mathrm{i}\beta(Y_t)}] = \mathrm{e}^{-t\|\beta\|^2}$. Since the integrand is bounded and the orbit is compact, the Fubini theorem gives

$$\mathbb{E}\big[J(Y_t)\chi_\lambda(\exp Y_t)\big] = \dim V_\lambda\,\mathrm{e}^{-t\|\lambda+\rho\|^2} = \mathrm{e}^{-t\|\rho\|^2}\dim V_\lambda\,\mathrm{e}^{-tc_\lambda},$$

where we used $c_\lambda = \|\lambda+\rho\|^2 - \|\rho\|^2$. The right-hand side is $\mathrm{e}^{-t\|\rho\|^2}(P_t\chi_\lambda)(e)$, as in the proof of Proposition 7.1. This proves (7.19).

For the trivial representation, (7.19) reads $\mathbb{E}[J(Y_t)] = \mathrm{e}^{-t\|\rho\|^2}$. The eigenvalues of $\mathrm{ad}(H_Y)$ give $\mathrm{Tr}(\mathrm{ad}(Y)^2) = -2\sum_{\alpha\in\Delta_+}\alpha(H_Y)^2$, and $\mathrm{Tr}(\mathrm{ad}(Y)^2) = -|Y|^2$ by the choice of the metric. Since $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + O(x^4)$ and $|\alpha(H_Y)| \leqslant \|\mathrm{ad}(Y)\|$, we obtain $J(Y) = 1 - \frac{1}{24}\sum_{\alpha\in\Delta_+}\alpha(H_Y)^2 + O(|Y|^4) = 1 - \frac{|Y|^2}{48} + O(|Y|^4)$ for $|Y| \leqslant 1$. For $|Y| \geqslant 1$ we have $|J(Y) - 1 + \frac{|Y|^2}{48}| \leqslant 2 + \frac{|Y|^2}{48} \leqslant 3|Y|^4$. Hence $|J(Y) - 1 + \frac{|Y|^2}{48}| \leqslant C|Y|^4$ on $\mathfrak{k}$, and $\mathbb{E}[J(Y_t)] = 1 - \frac{1}{48}\mathbb{E}|Y_t|^2 + O(t^2) = 1 - \frac{td}{24} + O(t^2)$ as $t \to 0$. Comparing the derivatives at $t = 0$ gives $\|\rho\|^2 = \frac{d}{24}$. This recovers the strange formula of Freudenthal and de Vries recalled in Section 2.6. ∎

For any $Y = \sum_i Y_i X_i \in \mathfrak{k}$, put $R \overset{\mathrm{def}}{=} |Y|^2$, $A(Y) \overset{\mathrm{def}}{=} \sum_i Y_i A_i$, $P_j \overset{\mathrm{def}}{=} \mathrm{Tr}(A(Y)^j)$ for $j = 4, 6$, and $\phi(Y) \overset{\mathrm{def}}{=} \frac{\chi(\exp Y)}{d}$. In the orthonormal basis (X_i), the matrix of $\mathrm{ad}(Y) = -\mathrm{i}A(Y)$ is real and skew-symmetric, so its odd powers have zero trace. We have $\mathrm{Tr}(\mathrm{ad}(Y)^2) = -R$ by (7.7), and $\mathrm{Tr}(\mathrm{ad}(Y)^4) = P_4$ and $\mathrm{Tr}(\mathrm{ad}(Y)^6) = -P_6$ since $\mathrm{ad}(Y) = -\mathrm{i}A(Y)$. The functions J and ϕ are even, equal to one at 0, and bounded by one in absolute value. We have $\phi(Y) = \frac{1}{d}\,\mathrm{Tr}\exp(\mathrm{ad}(Y))$ since $\exp(Y)$ acts on $\mathfrak{k}$ by $\exp(\mathrm{ad}(Y))$. Near 0, the eigenvalues 0 and $\pm\mathrm{i}\alpha(H_Y)$ of $\mathrm{ad}(Y)$ give $J(Y)^2 = \det\frac{\sinh(\mathrm{ad}(Y)/2)}{\mathrm{ad}(Y)/2}$ with $J(Y) > 0$, hence $\log J(Y) = \frac{1}{2}\,\mathrm{Tr}\log\frac{\sinh(\mathrm{ad}(Y)/2)}{\mathrm{ad}(Y)/2}$, where the

logarithm is given by its power series. Choose $\delta > 0$ such that J and ϕ are positive on the closed ball $\mathcal{B}_\delta$ of radius δ and such that $\log J$ and $\log \phi$ are given by even power series in the coordinates Y_i converging absolutely on a neighbourhood of $\mathcal{B}_\delta$. These expressions, together with $\log \frac{\sinh x}{x} = \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} + O(x^8)$, give a constant C_1 such that, for $Y \in \mathcal{B}_\delta$,

$$\left| \log J(Y) + \frac{R}{48} + \frac{P_4}{5760} + \frac{P_6}{362880} \right| \leqslant C_1 |Y|^8, \tag{7.20}$$

$$\left| \log \phi(Y) - g_2(Y) - g_4(Y) - g_6(Y) \right| \leqslant C_1 |Y|^8, \tag{7.21}$$

where $g_2 = -\frac{R}{2d}$, $g_4 = \frac{P_4}{24d} - \frac{R^2}{8d^2}$ and $g_6 = -\frac{P_6}{720d} + \frac{RP_4}{48d^2} - \frac{R^3}{24d^3}$ are the homogeneous parts of $\log(1 - \frac{R}{2d} + \frac{P_4}{24d} - \frac{P_6}{720d} + \cdots)$. We put

$$U \stackrel{\text{def}}{=} \frac{P_4}{24d} - \frac{R^2}{8d^2} - \frac{R}{48}, \qquad W \stackrel{\text{def}}{=} -\frac{P_6}{720d} + \frac{RP_4}{48d^2} - \frac{R^3}{24d^3} - \frac{P_4}{5760}. \tag{7.22}$$

For a Gaussian vector Z in $\mathfrak{k}$ with covariance $v\mathrm{Id}$ we write $\mathbb{E}_v$ for the expectation. We use the moments

$$\mathbb{E}_v R^k = d(d+2)\cdots(d+2k-2)v^k, \quad \mathbb{E}_v P_4 = \frac{5}{2}dv^2, \quad \mathbb{E}_v(RP_4) = \frac{5}{2}d(d+4)v^3, \quad \mathbb{E}_v P_6 = \frac{35}{4}dv^3. \tag{7.23}$$

The first formula is the moment formula of the chi-squared law. By Lemma 7.5, $P_4 = \frac{5}{2(d+2)}R^2 + H_4$ with H_4 harmonic. The average of a homogeneous harmonic polynomial of positive degree over every sphere centred at 0 is zero, so $\mathbb{E}_v(H_4 f(R)) = 0$ for every polynomial f. This gives the second and third formulas. For the last one, apply (7.19) to the trivial representation with $2t = uv$: we obtain $\mathbb{E}_v J(\sqrt{u}Z) = \mathrm{e}^{-\frac{udv}{48}}$ for $u > 0$. By (7.20), the Taylor polynomial of degree 6 of J at 0 is $1 - \frac{R}{48} - \frac{P_4}{5760} - \frac{P_6}{362880} + \frac{R^2}{2\cdot 48^2} + \frac{RP_4}{48\cdot 5760} - \frac{R^3}{6\cdot 48^3}$, and J differs from it by at most $C_2|Y|^8$ on $\mathfrak{k}$, since J is bounded. Hence $\mathbb{E}_v J(\sqrt{u}Z)$ has a Taylor expansion of order three in u, whose coefficient of u^3 gives

$$-\frac{\mathbb{E}_v P_6}{362880} + \frac{\mathbb{E}_v(RP_4)}{48\cdot 5760} - \frac{\mathbb{E}_v R^3}{6\cdot 48^3} = -\frac{d^3v^3}{6\cdot 48^3},$$

that is $\mathbb{E}_v P_6 = \frac{35}{4}dv^3$. Finally, we compute $\mathbb{E}_v H_4^2$ by Wick's formula, as a sum over the 105 pairings of the eight indices of $S_H \otimes S_H$. A pairing which joins two indices of the same copy of S_H gives zero, since S_H has zero partial contraction. The remaining 24 pairings join each index of the first copy to an index of the second one, and each of them gives $v^4 \|S_H\|^2$ since S_H is symmetric. Hence $\mathbb{E}_v H_4^2 = 24v^4 \|S_H\|^2$. With (7.14) this is $24dv^4(q_4 - q_0(d))$, where $q_0(d) = \frac{25}{12(d+2)}$.

For $s > 0$, put

$$a_s \stackrel{\text{def}}{=} 1 + \frac{2s}{d}, \quad v_s \stackrel{\text{def}}{=} \frac{2s}{a_s}, \quad A_d(s) \stackrel{\text{def}}{=} \frac{s^2(s-6)}{6da_s^2}.$$

Theorem 7.9 *Let $s_1 > 0$. For any integer $n \geqslant 1$ and any $0 < s \leqslant s_1$, we have*

$$\int \mathrm{e}^{-\frac{s}{n}c} \, \mathrm{d}\sigma_n(c) = a_s^{-d/2} \left(1 + \frac{A_d(s)}{n} + \frac{B_{d,q_4}(s)}{n^2} + O(n^{-3})\right), \tag{7.24}$$

where

$$B_{d,q_4}(s) = \frac{A_d(s)^2}{2} - \frac{d+16}{48d^2}v_s^3 + \left(\frac{(d+12)^2(d+3)}{576d^3(d+2)} + \frac{q_4 - q_0(d)}{48d}\right) v_s^4. \tag{7.25}$$

where the constant in $O(n^{-3})$ depends only on K and s_1. In particular, for two types of the same dimension, the difference of the coefficients $B_{d,q_4}(s)$ is $\frac{v_s^4}{768}$ times the difference of $m_2^{(4)}(0)$.

Proof . Let Y be a Gaussian vector in $\mathfrak{k}$ with covariance $2s\mathrm{Id}$, so that $\frac{Y}{\sqrt{n}}$ has the law of $Y_{s/n}$. By Proposition 7.1 and Lemma 7.8 applied to $W = V^{\otimes n}$, we have

$$\int \mathrm{e}^{-\frac{s}{n}c}\,\mathrm{d}\sigma_n(c) = d^{-n}(P_{s/n}(\chi^n))(e) = \mathrm{e}^{\frac{sd}{24n}}\,\mathbb{E}\Big[J\Big(\frac{Y}{\sqrt{n}}\Big)\phi\Big(\frac{Y}{\sqrt{n}}\Big)^n\Big]. \tag{7.26}$$

In what follows C denotes a constant which depends on K and s_1 but not on n, s or Y, and which may change from line to line. The left-hand side of (7.24) lies in $(0,1]$, and A_d and B_{d,q_4} are bounded on $[0,s_1]$. Hence it suffices to prove (7.24) for $n \geqslant n_1$, where n_1 is chosen below and depends only on K and s_1.

Expansion on the main region. Let n_0 be such that $n^{1/12} \leqslant \delta\sqrt{n}$ for $n \geqslant n_0$, and consider $n \geqslant n_0$ and $|Y| \leqslant n^{1/12}$. Then $\frac{Y}{\sqrt{n}} \in \mathcal{B}_\delta$, so that $J(\frac{Y}{\sqrt{n}})\phi(\frac{Y}{\sqrt{n}})^n = \exp(n\log\phi(\frac{Y}{\sqrt{n}}) + \log J(\frac{Y}{\sqrt{n}}))$. By homogeneity, (7.20) and (7.21) give

$$n\log\phi\Big(\frac{Y}{\sqrt{n}}\Big) + \log J\Big(\frac{Y}{\sqrt{n}}\Big) = -\frac{R}{2d} + T_n, \qquad T_n = \frac{U}{n} + \frac{W}{n^2} + E_n,$$

with $|E_n| \leqslant \frac{|P_6|}{362880n^3} + \frac{C_1|Y|^8}{n^3} + \frac{C_1|Y|^8}{n^4} \leqslant \frac{C(1+|Y|)^8}{n^3}$. Since $|U| \leqslant C(1+|Y|)^4$ and $|W| \leqslant C(1+|Y|)^6$, we have $|T_n| \leqslant C(1+|Y|)^8 n^{-1} \leqslant Cn^{-1/3}$ on this region, and in particular $|T_n| \leqslant 1$ if $n \geqslant n_1$ for a suitable $n_1 \geqslant n_0$ independent of s. For $|x| \leqslant 1$ we have $|\mathrm{e}^x - 1 - x - \frac{x^2}{2}| \leqslant |x|^3$. Writing $T_n^2 = \frac{U^2}{n^2} + (\frac{W}{n^2} + E_n)(\frac{2U}{n} + \frac{W}{n^2} + E_n)$, we obtain

$$\Big|\mathrm{e}^{T_n} - 1 - \frac{U}{n} - \frac{W + \frac{1}{2}U^2}{n^2}\Big| \leqslant \frac{C(1+|Y|)^{24}}{n^3} \qquad (|Y| \leqslant n^{1/12}). \tag{7.27}$$

The complement. Since $|J| \leqslant 1$ and $|\phi| \leqslant 1$, the integrand in (7.26) is bounded by one, and $|1 + \frac{U}{n} + \frac{W+U^2/2}{n^2}| \leqslant C(1+|Y|)^{12}$. By the Cauchy–Schwarz and Markov inequalities,

$$\mathbb{E}\big[(1+|Y|)^{12}1_{\{|Y|>n^{1/12}\}}\big] \leqslant \big(\mathbb{E}(1+|Y|)^{24}\big)^{\frac{1}{2}}\Big(\frac{\mathbb{E}|Y|^{72}}{n^6}\Big)^{\frac{1}{2}} \leqslant \frac{C}{n^3},$$

and the same bound holds for $\mathbb{P}(|Y| > n^{1/12})$. The moments of Y are bounded uniformly for $s \leqslant s_1$. We split the difference between the two sides of (7.28) below into its parts on $\{|Y| \leqslant n^{1/12}\}$ and on its complement. On the first set, $J(\frac{Y}{\sqrt{n}})\phi(\frac{Y}{\sqrt{n}})^n = \mathrm{e}^{-R/(2d)}\mathrm{e}^{T_n}$, and (7.27) together with $\mathrm{e}^{-R/(2d)} \leqslant 1$ bounds the contribution by $\frac{C}{n^3}\mathbb{E}(1+|Y|)^{24} \leqslant \frac{C}{n^3}$. On the complement, the two terms are bounded separately by the above estimates. We obtain

$$\mathbb{E}\Big[J\Big(\frac{Y}{\sqrt{n}}\Big)\phi\Big(\frac{Y}{\sqrt{n}}\Big)^n\Big] = \mathbb{E}\Big[\mathrm{e}^{-\frac{R}{2d}}\Big(1 + \frac{U}{n} + \frac{W+\frac{1}{2}U^2}{n^2}\Big)\Big] + O(n^{-3}), \tag{7.28}$$

uniformly for $0 < s \leqslant s_1$.

Gaussian change of variables. The density of Y is proportional to $\mathrm{e}^{-R/(4s)}$, and $\frac{1}{4s} + \frac{1}{2d} = \frac{1}{2v_s}$. Hence, for every polynomial f, $\mathbb{E}[\mathrm{e}^{-R/(2d)}f(Y)] = (\frac{v_s}{2s})^{d/2}\mathbb{E}_{v_s}f = a_s^{-d/2}\mathbb{E}_{v_s}f$. Since $a_s^{-d/2} \geqslant a_{s_1}^{-d/2} > 0$ and $\mathrm{e}^{\frac{sd}{24n}} = 1 + \frac{sd}{24n} + \frac{s^2d^2}{1152n^2} + O(n^{-3})$, (7.26) and (7.28) give (7.24) with

$$A_d(s) = \frac{sd}{24} + \mathbb{E}_{v_s}U, \qquad B_{d,q_4}(s) = \frac{A_d(s)^2}{2} + \mathbb{E}_{v_s}W + \frac{1}{2}\operatorname{Var}_{v_s}(U).$$

The coefficients. By (7.23), $\mathbb{E}_v U = -\frac{dv}{48} - \frac{(d+12)v^2}{48d}$. With $v = v_s$ we have $\frac{sd}{24} - \frac{dv_s}{48} = \frac{s^2}{12a_s}$ and $\frac{(d+12)v_s^2}{48d} = \frac{(d+12)s^2}{12da_s^2}$, whence $A_d(s) = \frac{s^2(da_s - d - 12)}{12da_s^2} = \frac{s^2(s-6)}{6da_s^2}$. Next, (7.23) gives

$$\mathbb{E}_v W = -\frac{dv^2}{2304} + \left(-\frac{7}{576} + \frac{5(d+4)}{96d} - \frac{(d+2)(d+4)}{24d^2} \right) v^3.$$

To compute the variance, write $U = -b_d R^2 - \frac{R}{48} + \frac{H_4}{24d}$ with $b_d = \frac{d+12}{48d^2(d+2)}$. The harmonic part is uncorrelated with the radial part, $\mathrm{Var}_v(R) = 2dv^2$, $\mathrm{Cov}_v(R^2, R) = 4d(d+2)v^3$ and $\mathrm{Var}_v(R^2) = 8d(d+2)(d+3)v^4$. With $\mathbb{E}_v H_4^2 = 24dv^4(q_4 - q_0(d))$ we get

$$\frac{1}{2}\,\mathrm{Var}_v(U) = \frac{dv^2}{2304} + \frac{d+12}{576d} v^3 + \left(\frac{(d+12)^2(d+3)}{576d^3(d+2)} + \frac{q_4 - q_0(d)}{48d} \right) v^4.$$

In the sum $\mathbb{E}_v W + \frac{1}{2}\,\mathrm{Var}_v(U)$ the terms in v^2 cancel, and the coefficient of v^3 is

$$\frac{-7d^2 + 30d(d+4) - 24(d+2)(d+4) + d(d+12)}{576d^2} = -\frac{d+16}{48d^2}.$$

This gives (7.25). Finally, at fixed d the coefficient $B_{d,q_4}(s)$ depends on q_4 through the term $\frac{q_4 v_s^4}{48d}$, whereas $m_2^{(4)}(0) = 16 + \frac{244/3 + 16q_4}{d}$ by (7.13). Since $48 \cdot 16 = 768$, this proves the last assertion. ∎

The script `heat_coefficients.py` recovers $A_d(s)$ and $B_{d,q_4}(s)$ symbolically from Gaussian moments.

The leading law is the chi-squared law with d degrees of freedom divided by d. This is in accordance with the tracial quantum central limit theorem [37], applied to the identity

$$\frac{C_n}{n} = \sum_a \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_a^{(i)} \right)^2.$$

The coefficient of s^3 in $a_s^{-d/2} B_{d,q_4}(s)$ is $-\frac{d+16}{6d^2}$. This agrees with the coefficient of n^{-2} in $-\frac{1}{6}\tau_n((C_n/n)^3)$ obtained from (7.11). At equal dimension the first two terms in (7.24) depend only on d, while the second correction separates the types by Theorem 7.6. At any fixed $s > 0$, the leading value $a_s^{-d/2}$ is strictly decreasing in d since $\log(1+x) > \frac{x}{1+x}$ for $x > 0$. Consequently the leading value and the second correction at that s also identify the type.

We have tested (7.24) for B_2, where $d = 10$ and $q_4 = \frac{2}{9} \neq q_0(10) = \frac{25}{144}$. With the multiplicities $m_{n,\lambda}$ computed in integer arithmetic by the Brauer–Klimyk rule for $n \leqslant 240$, we extrapolate the remainder $n^2\big(a_s^{d/2} \int \mathrm{e}^{-\frac{s}{n}c}\,\mathrm{d}\sigma_n(c) - 1 - \frac{A_d(s)}{n}\big)$ at $s = 2$ in powers of $\frac{1}{n}$. The Neville extrapolation of order 20 from the degrees $220 \leqslant n \leqslant 240$ differs by less than 10^{-46} from the value $B_{10,q_4}(2) = -\frac{460}{9261} = -0.0496706\ldots$ given by (7.25), whereas the value with q_4 replaced by $q_0(10)$ would be $-0.0564\ldots$. In `code/heat_B2`, `rootsys.py` supplies the integer multiplicities, `heat_values2.py` evaluates the heat moments at high precision and `fit_B.py` extrapolates them. The script `heat_B2_float.py` gives a separate floating-point least-squares check.

7.8 Positivity at finite affine level

The heat identity (7.2) uses the Casimir eigenvalues to define a positivity preserving convolution semigroup on a compact group. We examine whether the same eigenvalues, up to a rescaling

of time, still give a Markov semigroup when the characters are restricted to the finite fusion spectrum. For SU(2) with the negative of the Killing form, the Casimir eigenvalue of the irreducible representation of dimension $j+1$ is $c_j = \frac{j(j+2)}{8}$, since it equals one on the adjoint representation. Thus the multiplier $\mathrm{e}^{-tj(j+2)}$ below corresponds to time $8t$. For $\mathrm{SU}(2)_k$ this fails from level two on, for small times (Proposition 7.10). The fundamental fusion walk gives a positivity preserving semigroup with multipliers $\mathrm{e}^{-t\kappa_{k,j}}$, where $\kappa_{k,j} \to j(j+2) = 8c_j$ as k tends to infinity (Proposition 7.11).

For the $\mathrm{SU}(2)_k$ fusion algebra let $L = k+2$ and put

$$\theta_a \stackrel{\text{def}}{=} \frac{a\pi}{L}, \quad h_a \stackrel{\text{def}}{=} \sin\theta_a, \quad b_j(a) \stackrel{\text{def}}{=} \frac{\sin((j+1)\theta_a)}{h_a}, \quad 1 \leqslant a \leqslant L-1, \quad 0 \leqslant j \leqslant k.$$

These are the values of the basis of discretized characters, with $b_0 = 1$ [10, Section 2]. In rank one the formulas also follow from the fusion recurrence for the fundamental character and the boundary relation at level k. Indeed, for $k \geqslant 1$ multiplication by the fundamental basis element has the path matrix with $k+1$ vertices, and its eigenvectors are the discrete sines. The b_j form a basis of functions on the spectrum and are orthonormal for the measure with weights $w_a \stackrel{\text{def}}{=} \frac{2h_a^2}{L}$. For $k \geqslant 1$, its pushforward by b_1 is the rooted spectral measure of that adjacency.

Proposition 7.10 *The diagonal multiplier $T_t b_j = \mathrm{e}^{-tj(j+2)} b_j$ on the character spectrum is a Markov semigroup for $k = 0, 1$. For any $k \geqslant 2$, T_t does not preserve pointwise nonnegativity for sufficiently small $t > 0$.*

Proof . For $k = 0$ the operator is the identity. At $k = 1$ the values of b_1 are $1, -1$, and the two-state kernel has diagonal entries $\frac{1}{2}(1 + \mathrm{e}^{-3t})$ and off-diagonal entries $\frac{1}{2}(1 - \mathrm{e}^{-3t})$.

Suppose $k \geqslant 2$. The kernel is

$$T_t(a,b) = w_b \sum_{j=0}^{k} \mathrm{e}^{-tj(j+2)} b_j(a) b_j(b).$$

We examine the entry $a = 1$, $b = 3$, which vanishes at $t = 0$. Its derivative there is

$$-\frac{2h_3}{Lh_1} \sum_{r=1}^{L-1} (r^2 - 1) \sin\left(\frac{r\pi}{L}\right) \sin\left(\frac{3r\pi}{L}\right). \tag{7.29}$$

For an integer q not divisible by L, differentiation of the finite geometric series gives

$$\sum_{r=1}^{L-1} r^2 \cos\left(\frac{2\pi qr}{L}\right) = \frac{L}{2\sin^2(\pi q/L)} - \frac{L^2}{2}.$$

Here the differentiation is performed before the substitution of the root of unity. Taking the difference for $q = 1, 2$ and using sine orthogonality for the constant term shows that the sum in (7.29) is

$$\frac{L}{4}\left(\frac{1}{\sin^2(\pi/L)} - \frac{1}{\sin^2(2\pi/L)}\right) > 0.$$

The derivative is negative, so $T_t \mathbf{1}_{\{3\}}(1) < 0$ for sufficiently small positive t. This proves the failure. ∎

At level two the entry $T_t(1,3)$ is $\frac{1}{4}(1 - 2\mathrm{e}^{-3t} + \mathrm{e}^{-8t})$, which is negative exactly for $0 < t < t_0$, with $t_0 \approx 0.1305$. The obstruction comes from the off-diagonal generator sign. At every level T_t is selfadjoint on $\ell^2(w)$ with positive eigenvalues. What fails is the preservation of pointwise nonnegative functions.

Proposition 7.11 *Put*

$$\kappa_{k,j} \stackrel{\text{def}}{=} \frac{2L^2}{\pi^2}\left(\cos\left(\frac{\pi}{L}\right) - \cos\left(\frac{(j+1)\pi}{L}\right)\right). \tag{7.30}$$

The multipliers $\mathrm{e}^{-t\kappa_{k,j}}$ *in the* b_j *basis define a reversible Markov semigroup with weights* w_a*. The generator is nearest-neighbour on the ordered character spectrum. For* $k \geqslant 1$*, any nonzero nearest-neighbour Markov generator diagonal in this basis is a positive scalar multiple of it. For each fixed* j*,* $\kappa_{k,j} \to j(j+2)$ *as* $k \to \infty$*.*

Proof . Let A_{path} be the adjacency matrix on $1, \ldots, L-1$ and let H be the diagonal matrix with entries h_a. The sine vectors Hb_j have eigenvalues $2\cos(\frac{(j+1)\pi}{L})$ for A_{path}. The operator

$$Q = \frac{L^2}{\pi^2} H^{-1}\left(A_{\text{path}} - 2\cos\left(\frac{\pi}{L}\right)\mathrm{Id}\right) H$$

has eigenvalues $-\kappa_{k,j}$. It has nonnegative off-diagonal entries and $Q1 = 0$. Its matrix satisfies detailed balance for w_a since A_{path} is symmetric. Thus e^{tQ} has the claimed properties. This is the fundamental fusion walk with the corresponding continuous time scaling. Indeed, for $k \geqslant 1$, identify the spectral point a with the label $a-1$. Then $Q = \frac{2L^2}{\pi^2}\cos\left(\frac{\pi}{L}\right)(\widehat{P} - \mathrm{Id})$, where $\widehat{P}(a, a \pm 1) = \frac{h_{a\pm1}}{2\cos(\pi/L)h_a}$ is the transition kernel of [10, Section 2]. The eigenfunctions b_j are the discretized characters of that reference, since the matrix $\left(\sin\frac{(i+1)(m+1)\pi}{L}\right)_{i,m}$ is symmetric.

For uniqueness, let $\widetilde{Q}$ be a nearest-neighbour generator diagonal in the b_j basis. The matrix $H\widetilde{Q}H^{-1}$ is tridiagonal and diagonal in the eigenbasis of A_{path}. This adjacency has simple spectrum, so the matrix is a polynomial in it of degree at most $L-2$. A polynomial of degree $r \geqslant 2$ with nonzero leading coefficient has a nonzero entry at $(1, 1+r)$, since the unique path of length r contributes to the top degree and all lower powers vanish there. The polynomial must therefore be affine. The equation $\widetilde{Q}1 = 0$ fixes the constant term from the linear one, and positivity of the off-diagonal entries forces the linear coefficient to be positive for a nonzero generator. The convergence of the eigenvalues follows by Taylor expansion of the two cosines. ∎

Since the algebra of functions on the spectrum is commutative, the positive unital operators in Proposition 7.11 are completely positive. For noncommutative fusion algebras additional matrix positivity conditions appear, see [42, 23]. The script `finite_level.py` checks the negative derivative for $2 \leq k \leq 8$ and compares the spectrum of $-Q$ with $\kappa_{k,j}$ at $k = 1, 2, 5$.

Proposition 7.11 determines the local diagonal generators on the SU(2) character spectrum. At higher rank the affine alcove has several step directions. The fusion walks of [10, Definition 5.1 and Section 7] give positive examples with specified steps.

Problem 7.12 (Local fusion generators for G_2) *For* G_2 *at level* $k \geqslant 1$*, characterize the cone of Markov generators on the Verlinde spectrum which are diagonal in the discretized characters and whose off-diagonal support is contained in the support of the fusion matrix of the* 7*-dimensional representation. Does this cone consist of the multiples of the corresponding fusion generator? How do the cone and the approximation of the quadratic Casimir change when the steps of the adjoint fusion matrix are also allowed, at the levels where the adjoint representation is integrable?*

At fixed level the cone is defined by linear equalities and inequalities on the diagonal eigenvalues, and the rank two spectral computations of [20] give the character coordinates.

7.9 Trace tests on exceptional diagrams

In the Deligne–de Man parameter, the specialization $\mu = \frac{1}{4}$ has nonnegative cumulative functions for all weights and integral square weights. Section 7.2 excludes a compact simple adjoint model with that square law, and the example following Theorem 7.6 gives a second obstruction from two heat derivatives. The relevant diagrammatic framework is the trivalent category of [46, Section 1.1], with their parameter $\lambda = -\mu$. The functors of [56, Section 6] and the classical specialization of [46, Proposition 1.6] are defined at points which include the Lie points.

Question 7.13 (A positive trace at $\mu = 1/4$) *At $\mu = \frac{1}{4}$, find an explicit finite obstruction to a positive trace on the classical exceptional diagrams satisfying the bracket and symmetric closure relations. Can it be given by a negative value of a Gram determinant in tensor degree at most a specified N?*

At this integral point the antisymmetrizers (7.6) have nonnegative traces, positive for $m \leqslant 190$ and zero for $m > 190$. A positive trace satisfying the identities of Lemma 7.5 and giving the formal square law at $\mu = \frac{1}{4}$, so that $q_4 = \frac{25}{2304}$, would make the harmonic quartic component null, by (7.14). The identities of Lemma 7.5 alone do not force the harmonic quartic component to vanish: they hold for the adjoint representation of type D_{10}, whose dimension is also 190 and whose harmonic quartic component is nonzero.

The scalar data in Proposition 7.3 satisfy the moment conditions through degree three for all t, and they violate a trace identity on the same powers. At order four, the edge expansion of (7.8) leads to contractions supported on two, three or four tensor positions.

Problem 7.14 (Trace tests across tensor degrees) *For a fixed exceptional specialization of the diagram relations and a fixed tensor degree bound N, find a finite family of cross-degree trace identities and positivity tests which detects an inconsistency invisible to the separate laws σ_n and to the scalar heat moment tests for $n \leqslant N$.*

The traces of words in the operators Ω_{ij} and in the permutations of the tensor factors provide candidates.

7.10 Uniform estimates and separation of heat laws

At fixed dimension the term of order n^{-2} separates the simple types by Theorems 7.9 and 7.6, and its dependence on the harmonic quartic tensor is explicit. For B_ℓ and C_ℓ, the difference of the second coefficients is $\frac{v_s^4}{48d}$ times the difference of the values of q_4. By the proof of Theorem 7.6, it is of order $d^{-3/2}$ as $\ell \to \infty$, at fixed s.

Problem 7.15 (Uniformity in rank and tensor degree) *Give error bounds in (7.24) with an explicit dependence on the dimension, sufficient to compare B_ℓ and C_ℓ when ℓ and the tensor degree n grow together. What is the smallest growth of n for which the sign of the difference of the heat laws is governed by the second coefficient, uniformly for s in a compact subset of $(0, \infty)$?*

In the proof of Theorem 7.9, the constants depend on the type through the radius δ, the constant C_1 of the Taylor remainders (7.20)–(7.21) and the moments of a Gaussian vector in dimension d. The question asks for bounds on these quantities in terms of d, to be compared with the coefficient $\frac{v_s^4}{48d}$ of q_4 in (7.25).

There is an exact two-time version of (7.2). For $0 \leqslant m \leqslant n$, let C_m act on the first m factors of $V^{\otimes n}$. It commutes with the total representation and with C_n. Decomposing the first m factors, applying the heat factors there and then decomposing the total representation gives

$$\tau_n(\mathrm{e}^{-tC_m}\mathrm{e}^{-sC_n}) = d^{-n}\big(P_s((P_t(\chi^m))\chi^{n-m})\big)(e). \tag{7.31}$$

Indeed, $P_t(\chi^m)$ is the sum of the characters of the first block weighted by their heat factors, and multiplication by χ^{n-m} gives the total character expansion with these weights. On a terminal λ-isotypic block of positive multiplicity, the normalized block trace assigns to the intermediate label η the weight

$$\frac{m_{m,\eta}\, m_{\eta,\lambda}^{(n-m)}}{m_{n,\lambda}}.$$

Here $m_{\eta,\lambda}^{(n-m)}$ is the multiplicity of V_λ in $V_\eta \otimes V^{\otimes(n-m)}$.

Problem 7.16 (The intermediate Casimir at a fixed terminal label) *Suppose that $\frac{m}{n}$ ranges in a compact subset of $(0,1)$ and that $\frac{\lambda_n+\rho}{\sqrt{n}}$ ranges in a compact subset of the interior of the dominant chamber. Determine the expansion of the Laplace transform of $\frac{C_m}{n}$ under this terminal trace, with its first correction and a uniform error. How does the expansion change when the terminal label approaches a wall?*

Formula (7.31) gives the unconditioned joint moments and a starting point for the comparison with chamber paths. The local limit theorem of [36] describes the unconditioned shape distribution for random words in the defining representation of $\mathfrak{sl}_k$.

Remark 7.17 (Exact separation at one time) The optimal derivative order in Theorem 7.6 is four. There is also an exact observation at a single time. The atoms of σ_2 are rational by the Casimir formula, and its weights are rational by (7.1). For two different simple types, the difference of their heat functions at time t is a nonzero polynomial with rational coefficients evaluated at $\mathrm{e}^{-\frac{t}{M}}$, for a common positive integer M. Thus at any nonzero algebraic t the values differ, by the Hermite–Lindemann theorem [41].

Question 7.18 (Stable separation at positive time) *At a fixed positive algebraic time, for instance $t = 1$, estimate the smallest separation of the values of $m_2(t)$ for nonisomorphic types of dimension at most D. Can finitely many specified positive times yield a separation of polynomial order in D, with a reconstruction procedure for noisy values?*

For B_ℓ and C_ℓ of the same dimension d, Theorem 7.6 gives explicit rational exponents and coefficients, and the difference of the fourth derivatives is of order $d^{-3/2}$. Their explicit heat functions at positive time give a test case for the sharpness of lower bounds.

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