hexagon:2611.00589v1

Cluster monomials and finite type

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November 18, 2026

Abstract

We derive two characterizations of finite type for cluster algebras in the geometric setting of Fomin and Zelevinsky, with full-column-rank extended exchange matrix. In finite type, cluster monomials form a basis over the integral Laurent coefficient ring and its nonnegative counterpart describes the universally nonnegative elements. In infinite type, an explicit product of positive powers of two cluster variables lies outside the coefficient-linear span of all cluster monomials. The argument combines existing results and makes no priority claim.

1 Introduction and scope

Fomin and Zelevinsky [6, Theorem 4.27, Conjecture 4.28] considered a cluster algebra of geometric type with full-column-rank extended exchange matrix. They stated that in classical finite types the cluster monomials form a basis over the Laurent coefficient ring and that its positive cone is the corresponding nonnegative coefficient span. Their conjecture asserts separately that each property characterizes finite type, including the exceptional types.

The finite-type basis theorem of Ilten, Nájera Chávez, and Treffinger [8, Theorem 4.3.1] treats arbitrary geometric coefficients over an algebraically closed field of characteristic zero, with frozen variables initially non-invertible. We combine it with the proper Laurent monomial property of Cao and Li [1, Definition 6.5, Proposition 6.7 and Remark 6.8] to read off each coefficient in a cluster containing its monomial. This gives an integral and nonnegative descent without assuming that a field-basis theorem itself holds integrally. Earlier, Cerulli Irelli [2, Theorem 1.1 and Remark 5.1] proved atomicity in coefficient-free skew-symmetric finite type and observed its extension to arbitrary coefficient semifields when the cluster monomials are a basis. Felikson and Tumarkin [3, Theorem 10.2] proved atomicity in skew-symmetrizable finite type in their conventions. The latter's coefficient conventions are not silently substituted for the integral Laurent-coefficient assertion proved here.

For the converse we give a concrete universally nonnegative element outside the coefficient-linear span. The ingredients include the finite-type criterion of [4], enough g-pairs [1], and the rank-two fan description in [11]. Rupel, Stella, and Williams [10, Introduction, "Canonical bases"] had already stated that cluster monomials do not span outside finite type, in a paper focused on acyclic cluster algebras. The ordinary-basis characterization had also been stated in the literature: Mousavand [9, p. 2646] attributes the additive-basis equivalence to Gross, Hacking, Keel and Kontsevich [7]. That report does not identify a specific theorem there or separately address both the integral coefficient convention and the positive cone considered here. Our purpose is to make the present chain of implications explicit, not to claim a first proof or priority for the characterization or the witness.

2 Definitions and finite type

Let $n \geq 1$, $r \geq 0$, $m = n+r$, and let $\widetilde{B}$ be an integral $m \times n$ matrix of rank n, whose top square submatrix B is skew-symmetrizable. The initial extended cluster is $z = (x_1, \ldots, x_n, u_1, \ldots, u_r)$, whose entries are

algebraically independent, and the ambient field is $\mathcal{F} = \mathbb{Q}(z_1, \ldots, z_m)$. Mutations take place only in the first n directions. For a seed with extended cluster z and extended matrix $\widetilde{B} = (b_{ij})$, mutation in direction $k \le n$ replaces only x_k by

$$x_k' = x_k^{-1}\left(\prod_{i=1}^{m} z_i^{[b_{ik}]_+} + \prod_{i=1}^{m} z_i^{[-b_{ik}]_+}\right), \qquad [a]_+ = \max(a, 0),$$

and replaces $\widetilde{B}$ by $\widetilde{B}' = (b_{ij}')$, where

$$b_{ij}' = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+, & \text{otherwise.} \end{cases}$$

The u_i are frozen and invertible. The cluster algebra $\mathcal{A}$ is the $R = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$-subalgebra generated by all mutable cluster variables. A cluster monomial is a product of nonnegative powers of mutable variables belonging to one cluster; let $\mathcal{C}$ denote the set of distinct such elements, including 1. Define

$$R_+ = \mathbb{Z}_{\ge 0}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}], \quad \mathcal{A}_{\ge 0} = \mathcal{A} \cap \bigcap_t \mathbb{Z}_{\ge 0}[x_{1;t}^{\pm 1}, \ldots, x_{n;t}^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}].$$

The intersection runs over all seeds. Finite type means that there are finitely many mutable cluster variables; this is equivalent to the finite-seed definition used in the original formulation by [4, Theorem 1.8]. We use "positive" for membership of $\mathcal{A}_{\ge 0}$, including zero.

Theorem 1. *If $\mathcal{A}$ has finite type, then $\mathcal{C}$ is an R-basis of $\mathcal{A}$, and $\mathcal{A}_{\ge 0} = \sum_{M \in \mathcal{C}} R_+ M$.*

Proof. Let K be an algebraically closed field of characteristic zero. Form the corresponding cluster algebra over K without inverting frozen variables. By [8, Theorem 4.3.1], its mutable cluster monomials form a basis over $K[u_1, \ldots, u_r]$. Localization at the frozen variables shows that the same monomials form a $K[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$-basis after localization. This localization is the $K[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$-subalgebra generated by the mutable cluster variables. Their rational expressions are unchanged under extension from $\mathbb{Q}$ to K.

For $f \in \mathcal{A}$, write the finite expansion $f = \sum_M a_M(u) M$, where $a_M \in K[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$ and the sum indexes distinct elements only. Fix M and a cluster t containing it, and write $M = x_t^v$ with $v \in \mathbb{Z}_{\ge 0}^n$. The proper Laurent monomial property [1, Definition 6.5, proof of Proposition 6.7, Remark 6.8] applies to skew-symmetrizable cluster algebras with the coefficient group generated by the frozen variables. It states that the Laurent expansion of a cluster monomial not belonging to the chosen cluster is an R-linear combination of monomials with at least one negative *mutable* exponent. A different cluster monomial that does belong to the chosen cluster is a different ordinary monomial in its algebraically independent mutable variables. Thus the coefficient of x_t^v in the expansion of f, treating frozen Laurent monomials as coefficients, is exactly $a_M(u)$.

The Laurent phenomenon over $\mathbb{Z}$ implies that this coefficient lies in R, proving integral spanning; linear independence follows from the field-basis theorem. If f is universally nonnegative, the same coefficient lies in R_+. Conversely, all cluster variables are universally nonnegative by [7, Theorem 4.10]; hence each R_+-linear combination of cluster monomials is universally nonnegative. The R-linear independence already proved also makes the R_+-expression unique. This proves both assertions. □

3 An infinite-type obstruction

Assume $\mathcal{A}$ has infinite type. By [4, Theorem 1.8], after choosing a seed and relabeling, two mutable indices $I = \{1, 2\}$ have principal submatrix

$$B_I = \begin{pmatrix} 0 & c \\ -b & 0 \end{pmatrix}, \qquad b, c > 0, \quad bc \ge 4.$$

Use this seed as initial seed; hereafter z and $\widetilde{B}$ denote its extended cluster and exchange matrix. Mutation preserves the full column rank of the extended matrix [6, Section 4.2, before Definition 4.7]. The principal-coefficient pattern at this seed has initial extended matrix $\binom{B}{I_n}$ and independent frozen variables $y_1, \dots, y_n$. Give the initial variables degrees $\deg x_i = e_i$ and $\deg y_j = -B_{\bullet j}$ in $\mathbb{Z}^n$. A cluster variable is homogeneous in this grading; its degree is its g-vector, and the g-vector of a labeled cluster monomial is the sum of those of its factors. The corresponding F-polynomial is obtained by setting the initial mutable variables to 1 in its principal-coefficient Laurent expression. The G-matrix of a labeled seed is the matrix whose columns are the g-vectors of its mutable variables. All g-vectors below use this principal-coefficient pattern and a chosen mutation label; label-independence for a given original element is justified in the proof of Theorem 2. Let $\pi_I : \mathbb{Z}^n \to \mathbb{Z}^2$ be the coordinate projection, and let $\mathcal{F}(B_I)$ denote the g-fan of the rank-two principal-coefficient pattern whose initial extended matrix is $\binom{B_I}{I_2}$, consisting of the cones spanned by the g-vectors of its clusters and their faces; $|\mathcal{F}(B_I)|$ denotes its support, the union of these cones.

Lemma 1. *For any labeled mutable cluster monomial M, its projected g-vector belongs to the support $|\mathcal{F}(B_I)|$.*

Proof. Choose a seed t containing $M = x_t^v$, $v \in \mathbb{Z}^n_{\geq 0}$. The enough g-pairs property [1, Definition 4.5, Proposition 4.6, Theorem 4.8] gives a seed t' reachable from the initial seed using only I, and a matrix $Q \in \mathbb{Z}^{2\times n}_{\geq 0}$ with $G_t|_{I\times[1,n]} = G_{t'}|_{I\times I}Q$. The block-matrix identification in the proof of [1, Theorem 4.8] identifies $G_{t'}|_{I\times I}$ with the G-matrix of the rank-two seed s obtained by applying the same sequence of mutations in I to the rank-two principal-coefficient initial seed. Thus $\pi_I g(M) = G_s(B_I)Qv$, which belongs to a rank-two cluster cone. □

Lemma 2. *For the matrix B_I above, $(-2, b) \notin |\mathcal{F}(B_I)|$.*

Proof. This is precisely the rank-two exclusion in [11, Example 2.7]; its Definition 2.2 uses the standard principal-coefficient g-vector convention and the same matrix orientation. The assertion includes $bc = 4$, where the point lies on the nonzero common limiting ray, which is not in the union of the actual cones. □

Theorem 2 (Infinite type)**.** *If $\mathcal{A}$ has infinite type, a universally nonnegative element lies outside the R-span of $\mathcal{C}$. Thus neither the R-basis property nor the R_+-span property holds.*

Proof. Write $\widetilde{B} = \binom{B}{D}$. Full column rank permits a real vector $w \in \mathbb{R}^m$ such that $\widetilde{B}^T w = (1, \dots, 1)$. Assign to z^α the weight $w \cdot \alpha$. For a chosen mutation label of a cluster monomial M, write $g(M) \in \mathbb{Z}^n$ for its principal-coefficient g-vector. The following leading-term method is already anticipated in [5, Remark 7.11]. The separation formula [5, Corollary 6.3] takes the form

$$M = z^{(g(M),h_M)} F_M(z^{\widetilde{b}_1}, \dots, z^{\widetilde{b}_n}),$$

where $\widetilde{b}_j$ is column j, F_M is the product of the corresponding F-polynomials, and $h_M \in \mathbb{Z}^r$ records the reciprocal tropical denominator. Explicitly, $(h_M)_\ell = -\min\{(Dq)_\ell : q \in \operatorname{supp}(F_M)\}$, where $\operatorname{supp}(F_M) = \{q : [y^q]F_M \neq 0\}$; tropical addition is coordinatewise minimum. The constant coefficient of F_M is 1 and all its exponents are nonnegative [1, Theorem 2.7(ii)]. For an exponent $q \neq 0$ in F_M, its term has weight larger by exactly $\sum_j q_j > 0$, so M has unique lowest term $z^{(g(M),h_M)}$ with coefficient 1.

To justify this notation, first choose one mutation label for each distinct cluster monomial. The resulting g-vector is independent of the label: every label gives a unique lowest Laurent term of the same element, so its mutable exponent is the same. Distinct cluster monomials have distinct g-vectors. Indeed, if labeled monomials have the same principal g-vector, they coincide in the principal-coefficient algebra by [1, Theorem 3.2]. Setting initial mutable variables to 1 identifies their F-polynomials; the

separation formula then shows that their specializations to the given coefficients are equal. Consequently the lowest exponents of $u^a M$ for different pairs (a, M) are distinct.

Let x_1', x_2' be obtained by the two separate one-step mutations from the chosen seed and put $f = (x_1')^2 (x_2')^b$. From the exchange relations, the lowest term of x_k' has mutable exponent $-e_k + [-B_{\bullet k}]_+$, because the two numerator monomials differ in weight by 1. Their I-projections are $(-1, b)$ and $(0, -1)$, respectively. Thus the unique lowest term of f has projected mutable exponent $(-2, b)$. By Lemma 1 and Lemma 2, this exponent is distinct from that of every $u^a M$.

If f were a finite R-linear combination of cluster monomials, expand all coefficients as Laurent polynomials in u, move all terms to one side, and combine duplicate pairs. Among the participants choose the minimum of their lowest weights. At that weight the unique lowest Laurent monomials have distinct exponents, and cannot cancel. This contradicts the resulting zero Laurent polynomial. Finally $f \in \mathcal{A}_{\geq 0}$ by positivity of cluster variables [7, Theorem 4.10]; hence it is outside both specified spans. □

Theorem 3 (Characterizations)**.** *For the cluster algebras in the setting above, each of the following is equivalent to finite type: (i) $\mathcal{C}$ is an R-basis of $\mathcal{A}$; (ii) every element of $\mathcal{A}_{\geq 0}$ has a unique finite R_+-linear expansion in $\mathcal{C}$.*

Proof. The forward implications are Theorem 1, and the converses follow separately from Theorem 2. □

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