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CONVEX-BODY VOLUME COUNTEREXAMPLES TO HYPERPLANE EQUIPARTITION

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ABSTRACT. For every $d \geq 4$ and every ellipsoid in $\mathbb{R}^d$, we construct real-analytic convex bodies of positive curvature, converging smoothly to that ellipsoid and having the same volume, whose volume admits no equipartition by d affine hyperplanes. For $d \geq 6$, the bodies can be chosen centrally symmetric. The four-dimensional construction uses Soberón's cubic–quartic tensor nonvanishing theorem. We prove a boundary-perturbation criterion that realizes this obstruction for the uniform volume of a convex body; higher-dimensional examples follow from a finite-dimensional transversality argument. We also construct counterexamples to orthogonal equipartition near every ellipsoid with distinct semiaxes in dimensions $d \geq 3$, with central symmetry when $d \geq 4$. The counterexample property is open in the Hausdorff topology, giving polytopal examples as well.

1. INTRODUCTION

An equipartition of a convex body $K \subset \mathbb{R}^d$ by d affine hyperplanes requires each of the 2^d sign regions to have Lebesgue volume $2^{-d}\operatorname{vol}_d(K)$. The hyperplanes have arbitrary offsets and directions. A convex body is a compact convex set with nonempty interior.

Grünbaum's question originates in [3, Section 4(v)], where it is phrased for continuous mass distributions. Its convex-body volume formulation is stated explicitly in [2, Section 1]. The general mass problem has a positive answer in dimensions at most three by the elementary planar argument and Hadwiger's theorem [4]. Avis [1] constructed high-dimensional point-set counterexamples, and Soberón [8] recently constructed a smooth positive density counterexample in dimension four. The surveys [2, 6] discuss the mass-partition framework; the measurable-set volume formulation in dimension four is recorded in [9, Problem 21.2.6].

Here the measure is required to be the uniform volume of one convex body. A negative result for unrestricted densities does not by itself supply such a body. We realize the four-dimensional tensor obstruction of [8] by perturbing the boundary of a ball, while retaining convexity and positive curvature.

If $c \in \operatorname{int} K$, write

$$\rho_{K,c}(v) = \max\{r \geq 0 : c + rv \in K\}, \qquad v \in \mathbb{S}^{d-1}.$$

Smooth convergence below means convergence of these radial functions in every fixed $C^k(\mathbb{S}^{d-1})$ norm.

Theorem 1.1. *Let $d \geq 4$, and let $E \subset \mathbb{R}^d$ be an ellipsoid with center c. There are $\varepsilon_0 > 0$ and convex bodies K_ε, $|\varepsilon| < \varepsilon_0$, such that $K_0 = E$ and:*

1. *∂K_ε is real analytic and all its principal curvatures are positive;*
2. *$\operatorname{vol}_d(K_\varepsilon) = \operatorname{vol}_d(E)$, and $\rho_{K_\varepsilon,c} \to \rho_{E,c}$ in $C^\infty(\mathbb{S}^{d-1})$;*
3. *for every $0 < |\varepsilon| < \varepsilon_0$, K_ε has no equipartition by d affine hyperplanes.*

If $d \geq 6$, the bodies may be chosen centrally symmetric about c.

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Together with the low-dimensional positive results, this gives the dimension threshold for uniform convex-body volume.

Corollary 1.2. *Every convex body in $\mathbb{R}^d$ admits an equipartition by d affine hyperplanes if and only if $d \le 3$.*

We also consider the restriction that the hyperplane normals be mutually orthogonal. Maldonado and Roldán-Pensado [5] constructed measure counterexamples in dimensions at least three. The following result realizes uniform-volume counterexamples near a prescribed nonspherical reference shape.

Theorem 1.3. *Let $d \ge 3$, and let $E \subset \mathbb{R}^d$ be an ellipsoid with pairwise distinct semiaxis lengths and center c. There are real-analytic, positively curved convex bodies K_ε, smoothly converging to E, of the same volume as E, which admit no equipartition by d mutually perpendicular affine hyperplanes whenever $0 < |\varepsilon|$ is sufficiently small. For $d \ge 4$, they may be chosen centrally symmetric about c.*

Corollary 1.4. *In every dimension $d \ge 4$ there are convex polytopes whose volume has no equipartition by d affine hyperplanes. They can be chosen centrally symmetric for $d \ge 6$. For orthogonal equipartitions, polytopal counterexamples exist for $d \ge 3$, with central symmetry for $d \ge 4$. In each assertion the polytopes can be chosen arbitrarily close in Hausdorff distance to the reference ellipsoid in the corresponding theorem, and with its volume.*

Contribution and proof outline. The use of sign correlations, reduction to frames near $\mathrm{O}(4)$, and the cubic–quartic obstruction in dimension four come from Soberón's work. We use his tensor theorem as an explicit external input, restated in Theorem 4.1. The contribution here is its boundary realization for uniform convex-body volume, together with the proximity, symmetry, higher-dimensional and orthogonal statements above.

The main technical point is Proposition 3.4. Any sequence of equipartitions of small radial perturbations of a ball must converge to an orthogonal frame through the origin. For sign correlations involving at least three hyperplanes, changes of angles and offsets contribute only $o(|\varepsilon|)$. The remaining first-order term is the spherical boundary perturbation. A nonvanishing boundary obstruction therefore rules out every affine equipartition, including configurations whose hyperplanes move with ε.

Section 2 establishes the geometry and configuration estimates. Section 3 proves the transfer criterion with the offset and boundary limits written separately. Sections 4 and 5 prove Theorem 1.1. The orthogonal and polytopal assertions are proved in Sections 6 and 7.

2. Boundary perturbations and equipartition coefficients

Throughout, B^d is the unit ball, $\kappa_d = \mathrm{vol}_d(B^d)$, and $d\sigma$ is unnormalized surface measure on $\mathbb{S}^{d-1}$. A frame $q = (q_1, \dots, q_d) \in \mathrm{O}(d)$ is identified with the orthogonal matrix having these columns. Constants may depend on d and on a fixed perturbation function, but not on its parameter.

2.1. **A volume-exact radial perturbation.** Let $\phi \in C(\mathbb{S}^{d-1})$. For sufficiently small real ε, define the radial body

$$\rho_\varepsilon(v) = (1 + d\varepsilon\phi(v))^{1/d}, \qquad K_\varepsilon = \{rv : v \in \mathbb{S}^{d-1},\ 0 \le r \le \rho_\varepsilon(v)\}. \tag{1}$$

We take $|d\varepsilon|\|\phi\|_\infty < 1/2$. Polar integration gives the exact identities

$$\mathrm{vol}_d(K_\varepsilon) = \kappa_d + \varepsilon \int_{\mathbb{S}}^{d-1} \phi \, d\sigma, \tag{2}$$

$$\mathrm{vol}_d(K_\varepsilon \triangle B^d) = |\varepsilon| \int_{\mathbb{S}}^{d-1} |\phi| \, d\sigma. \tag{3}$$

Lemma 2.1. *If ϕ is C^∞, then for sufficiently small $|\varepsilon|$, K_ε is a convex body with positive principal curvatures and $\rho_\varepsilon \to 1$ in C^∞. If ϕ is real analytic, its boundary is real analytic. If ϕ is even, $K_\varepsilon = -K_\varepsilon$. If $\int \phi \, d\sigma = 0$, its volume is exactly κ_d.*

Proof. Convergence in each fixed smooth norm follows from (1). To justify convexity directly, define the squared gauge

$$g_\varepsilon(x) = \frac{|x|^2}{\rho_\varepsilon(x/|x|)^2} \quad (x \neq 0), \qquad g_\varepsilon(0) = 0.$$

Its Hessian on the unit sphere converges uniformly to 2Id. Positive homogeneity makes the Hessian uniformly positive definite on $\mathbb{R}^d \setminus \{0\}$. Also g_ε is C^1 at zero, with gradient zero. Its restriction to any line is therefore convex, including a line through zero, so its sublevel set $K_\varepsilon = \{g_\varepsilon \leq 1\}$ is convex. On $g_\varepsilon = 1$, Euler's identity gives $x \cdot \nabla g_\varepsilon = 2$, and the second fundamental form is the positive Hessian restricted to the tangent space, divided by $|\nabla g_\varepsilon|$. This proves positive curvature. Analyticity follows from the analytic radial parametrization; symmetry follows from evenness. The volume assertion is (2). □

2.2. **Sign correlations and all affine configurations.** For unit normals $u = (u_1, \dots, u_d)$, offsets $t = (t_1, \dots, t_d)$, and $I \subset [d] = \{1, \dots, d\}$, put

$$W_I(K; u, t) = \int_K \prod_{i \in I} \operatorname{sgn}(u_i \cdot x - t_i) \, dx. \tag{4}$$

The empty product is one. The value of $\operatorname{sgn}(0)$ is irrelevant.

Lemma 2.2. *The hyperplanes $u_i \cdot x = t_i$ equipartition K if and only if $W_I(K; u, t) = 0$ for every nonempty $I \subset [d]$.*

Proof. For $s = (s_1, \dots, s_d) \in \{-1, 1\}^d$, the indicator of its sign region equals, outside the hyperplanes,

$$2^{-d} \prod_{i=1}^{d} (1 + s_i \operatorname{sgn}(u_i \cdot x - t_i)).$$

Its integral is $2^{-d} \sum_{I \subset [d]} (\prod_{i \in I} s_i) W_I(K; u, t)$. The characters $\prod_{i \in I} s_i$ form a basis on the sign cube; their orthogonality proves both implications. □

Lemma 2.3. *Suppose $\varepsilon_m \to 0$ through nonzero values and (u^m, t^m) equipartitions the radial bodies (1). Then*

$$\max_i |t_i^m| = O(|\varepsilon_m|), \qquad \max_{i \neq j} |u_i^m \cdot u_j^m| = O(|\varepsilon_m|). \tag{5}$$

After passage to a subsequence, $u_i^m \to q_i$, where $q \in \mathrm{O}(d)$.

Proof. Write $D = B^d$. Since every hyperplane bisects K_{ε_m}, (3) and Lemma 2.2 imply

$$|W_{\{i\}}(D; u^m, t^m)| \leq \operatorname{vol}_d(K_{\varepsilon_m} \triangle D) \leq C|\varepsilon_m|.$$

For a unit normal and $|s| < 1$,

$$W_{\{i\}}(D; u, s) = -2\kappa_{d-1} \int_0^s (1 - r^2)^{(d-1)/2} \, dr.$$

This is strictly decreasing, vanishes only at zero, and has derivative bounded away from zero near zero. Consequently each $t_i^m = O(|\varepsilon_m|)$. The same conclusion follows first by forcing $|t_i^m| < 1/2$ and then using the displayed derivative bound, so the estimate is uniform in the normal.

Changing one offset to zero changes a sign product only in the corresponding slab. A slab of width $|s|$ in D has volume at most $\kappa_{d-1}|s|$. Hence

$$|W_{\{i,j\}}(D; u^m, t^m) - W_{\{i,j\}}(D; u^m, 0)| \leq 2\kappa_{d-1}(|t_i^m| + |t_j^m|).$$

The two-sign correlation in a rotationally invariant ball is

$$\kappa_d^{-1} W_{\{i,j\}}(D; u, 0) = 1 - \frac{2}{\pi}\arccos(u_i \cdot u_j) = \frac{2}{\pi}\arcsin(u_i \cdot u_j). \tag{6}$$

Indeed, in the plane spanned by the two normals, the angular proportion on which the signs disagree is $\arccos(u_i \cdot u_j)/\pi$; rotational invariance gives the same proportion of ball volume. The coefficient over K_{ε_m} is zero, and its difference from the coefficient over D is bounded by (3). Since $|\arcsin r| \geq |r|$ for $|r| \leq 1$, this proves the second estimate in (5). Compactness of $(\mathbb{S}^{d-1})^d$ gives a convergent subsequence, whose limiting normals are pairwise orthogonal. □

3. The first-order boundary obstruction

For $|I| \geq 3$, define

$$T_I(\phi, q) = \int_{\mathbb{S}}^{d-1} \phi(v) \prod_{i \in I} \operatorname{sgn}(q_i \cdot v)\, d\sigma(v), \qquad q \in \mathrm{O}(d). \tag{7}$$

The map $q \mapsto T_I(\phi, q)$ is continuous by dominated convergence: the exceptional great spheres have surface measure zero.

3.1. **Centered correlations have no linear angular term.**

Lemma 3.1. *Let $u_1, \ldots, u_d$ be unit vectors whose Gram matrix is $G = \mathrm{Id} + E$, with $\|E\|$ sufficiently small. For $|I| \geq 3$,*

$$|W_I(B^d; u, 0)| \leq C_d \|E\|^2.$$

Proof. A uniformly distributed point of the ball and a standard Gaussian have the same uniformly distributed direction. Signs of homogeneous linear forms depend only on this direction. Thus, if Y_G is a centered Gaussian vector with covariance G,

$$\kappa_d^{-1} W_I(B^d; u, 0) = \mathbb{E} \prod_{i \in I} \operatorname{sgn}((Y_G)_i).$$

The covariance is positive definite near Id, and its density is

$$p_G(y) = (2\pi)^{-d/2} (\det G)^{-1/2} \exp(-\tfrac{1}{2} y^T G^{-1} y).$$

At $G = \mathrm{Id}$, its directional derivative in a symmetric direction E is

$$Dp_{\mathrm{Id}}[E](y) = \tfrac{1}{2}(y^T E y - \operatorname{tr} E) p_{\mathrm{Id}}(y). \tag{8}$$

The integral of the sign product against p_{Id} is zero. Its integral against every term in (8) is also zero: a monomial of degree at most two leaves at least one of the three or more specified coordinates available for an odd reflection.

For completeness, on a fixed neighborhood $\|G - \mathrm{Id}\| < 1/2$, second derivatives of p_G in the matrix entries are bounded in absolute value by $C_d(1 + |y|^4) e^{-c_d |y|^2}$. This is integrable uniformly in G. Taylor's formula in L^1 therefore bounds the remaining integral by $C_d \|E\|^2$. □

3.2. **Small offsets do not restore the higher correlations.**

Lemma 3.2. *Under the assumptions of Lemma 2.3, for every $|I| \geq 3$,*

$$W_I(B^d; u^m, t^m) = o(|\varepsilon_m|). \tag{9}$$

Proof. By Lemmas 2.3 and 3.1,

$$W_I(B^d; u^m, 0) = O(|\varepsilon_m|^2).$$

Change the offsets with indices in I, one at a time. At the step changing index i, the change of the coefficient is exactly

$$-2\int_0^{t_i^m} F_{i,m}(s)\,ds, \qquad F_{i,m}(s)=\int_{B^d\cap\{u_i^m\cdot x=s\}}\prod_{j\in I\setminus\{i\}}\operatorname{sgn}(u_j^m\cdot x-\widehat{t}_j^m)\,d\mathcal{H}^{d-1}(x). \tag{10}$$

Here each $\widehat{t}_j^m$ is either zero or t_j^m. The signed integral also covers $t_i^m<0$.

We claim that $F_{i,m}(s)=o(1)$ uniformly for s between zero and t_i^m. To see this, choose any sequence of such $s=s_m$. Choose orthogonal maps $R_m\to\mathrm{Id}$ with $R_mq_i=u_i^m$. The parametrization

$$x=s_mu_i^m+\sqrt{1-s_m^2}\,R_my, \qquad y\in B^d\cap q_i^{\perp},$$

has surface Jacobian $(1-s_m^2)^{(d-1)/2}$. For almost every y, its sign product converges to $\prod_{j\in I\setminus\{i\}}\operatorname{sgn}(q_j\cdot y)$. The excluded sets $q_j\cdot y=0$ have zero $(d-1)$-dimensional measure in this disk. The limiting integral is zero: reflection in any one of the directions q_j, $j\in I\setminus\{i\}$, reverses the product and preserves the disk. Dominated convergence proves convergence to zero for every choice of s_m, which also proves the claimed uniformity. Since $t_i^m=O(|\varepsilon_m|)$, (10) is $o(|\varepsilon_m|)$. There are only finitely many changes of offsets, proving (9). □

3.3. **The shell term and the transfer criterion.**

Lemma 3.3. *Suppose $u^m\to q\in\mathrm{O}(d)$, $t^m\to 0$, and $\varepsilon_m\to 0$ through nonzero values. For every $I\subset[d]$,*

$$\frac{W_I(K_{\varepsilon_m};u^m,t^m)-W_I(B^d;u^m,t^m)}{\varepsilon_m}\longrightarrow T_I(\phi,q),$$

where (7) is used also for smaller index sets.

Proof. The difference is the signed shell integral

$$\int_{\mathbb{S}}\int_1^{\rho_{\varepsilon_m}(v)} r^{d-1}\prod_{i\in I}\operatorname{sgn}(r\,u_i^m\cdot v-t_i^m)\,dr\,d\sigma(v).$$

For v outside the finitely many great spheres $q_i\cdot v=0$, all signs on the shrinking radial interval eventually agree with the limiting signs. Its signed radial volume is exactly

$$\int_1^{\rho_{\varepsilon_m}(v)} r^{d-1}\,dr=\varepsilon_m\phi(v).$$

The inner integral divided by ε_m is bounded in absolute value by $|\phi(v)|$, including negative parameters and inward perturbations. Dominated convergence gives the result. □

Proposition 3.4 (Boundary transfer)**.** *Let $\phi\in C(\mathbb{S}^{d-1})$. If a sequence $\varepsilon_m\to 0$, $\varepsilon_m\neq 0$, admits affine equipartitions of K_{ε_m}, then some $q\in\mathrm{O}(d)$ satisfies*

$$T_I(\phi,q)=0 \qquad \text{for every } |I|\geq 3. \tag{11}$$

Consequently, if the functions in (11) have no common zero on $\mathrm{O}(d)$, then K_ε has no affine equipartition whenever $0<|\varepsilon|$ is sufficiently small.

Proof. Take the limiting orthogonal frame supplied by Lemma 2.3. The coefficient over K_{ε_m} is zero. Divide it by ε_m, use Lemma 3.2 for the ball contribution and Lemma 3.3 for the difference. This proves (11). If the last conclusion failed, one could choose such a sequence with $\varepsilon_m\to 0$, contradicting the absence of a common zero. □

Remark 3.5. The argument excludes moving affine configurations, not merely centered orthogonal ones. Lemma 3.1 controls the change of directions and Lemma 3.2 controls the change of offsets. These estimates are what permit the spherical obstruction to be used for the full affine problem.

4. The four-dimensional construction

The sole nonvanishing input needed in dimension four is the following theorem. We state its attribution separately from the boundary-transfer argument.

Theorem 4.1 (Soberón [8, Theorem 2])**.** *There exist symmetric multilinear forms $A : (\mathbb{R}^4)^3 \to \mathbb{R}$ and $B : (\mathbb{R}^4)^4 \to \mathbb{R}$ such that, for every $q = (q_1, q_2, q_3, q_4) \in \mathrm{O}(4)$, the five numbers*

$$A(q_1, q_2, q_3), \quad A(q_1, q_2, q_4), \quad A(q_1, q_3, q_4), \quad A(q_2, q_3, q_4), \quad B(q_1, q_2, q_3, q_4)$$

do not all vanish.

The cited version is v2, whose Section 3 gives a non-computational proof of this theorem. We do not reprove that external theorem here.

Remark 4.2 (An explicit choice of the perturbation). The tensors in [8, Theorem 3] make the construction concrete. In coordinates $x = (x_1, x_2, x_3, x_4)$, set

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}, \qquad N = M + e_1 e_1^T,$$

and define the symmetric tensors by their diagonal polynomials

$$A_\tau(x, x, x) = x_1 x^T M x + 6\tau x_2 x_3 x_4, \qquad B(x, x, x, x) = (x^T N x)^2.$$

Soberón's theorem states that A_τ, B have the required nonvanishing property for every sufficiently small fixed $\tau > 0$. For such a fixed τ, one may therefore use

$$\phi(v) = v_1 v^T M v + 6\tau v_2 v_3 v_4 + (v^T N v)^2 - 5, \qquad v \in \mathbb{S}^3. \tag{12}$$

The cubic terms have zero mean. The standard fourth spherical moments give

$$\frac{1}{\sigma(\mathbb{S}^3)} \int_{\mathbb{S}^3} (v^T N v)^2 \, d\sigma(v) = \frac{(\operatorname{tr} N)^2 + 2 \operatorname{tr}(N^2)}{4 \cdot 6} = \frac{64 + 56}{24} = 5.$$

Thus (12) has zero mean, and $\rho_\varepsilon(v) = (1 + 4\varepsilon\phi(v))^{1/4}$ is an explicit defining formula for the family in dimension four. The parameter τ is fixed first; ε is then chosen sufficiently small. Both smallness assertions are existence statements, with no numerical lower bound on their allowed ranges.

Proposition 4.3. *There is a real-analytic family of positively curved convex bodies in $\mathbb{R}^4$, of volume κ_4, converging smoothly to B^4, with no equipartition by four affine hyperplanes for all sufficiently small nonzero parameters.*

Proof. Take A, B from Theorem 4.1, and set

$$\phi_0(v) = A(v, v, v) + B(v, v, v, v), \qquad \phi(v) = \phi_0(v) - \frac{1}{\sigma(\mathbb{S}^3)} \int_{\mathbb{S}^3} \phi_0 \, d\sigma.$$

The subtracted constant has zero correlation with every nonempty orthogonal sign product, so $T_I(\phi, q) = T_I(\phi_0, q)$.

To compute these correlations, write $v = \sum_{i=1}^4 z_i q_i$. Multiplication by a triple sign product leaves, in a cubic polynomial, only the monomial with exponent one in the three specified coordinates. All other monomials integrate to zero by coordinate reflection. The quartic contribution to a triple correlation is zero by the antipodal map. Thus, for distinct i, j, k,

$$T_{\{i,j,k\}}(\phi, q) = c_3 A(q_i, q_j, q_k), \qquad c_3 = 6 \int_{\mathbb{S}^3} |z_1 z_2 z_3| \, d\sigma(z) > 0. \tag{13}$$

For the fourfold correlation, only the quartic monomial with all four exponents one survives:

$$T_{[4]}(\phi,q)=c_4B(q_1,q_2,q_3,q_4),\qquad c_4=24\int_{\mathbb{S}^3}|z_1z_2z_3z_4|\,d\sigma(z)>0. \tag{14}$$

The constants 6 and 24 are the numbers of permutations of the corresponding distinct arguments in the symmetric multilinear expansions.

Equations (13)–(14) and Theorem 4.1 show that the higher correlations have no common zero. Proposition 3.4 rules out all affine equipartitions of (1) for small nonzero parameters. Lemma 2.1 gives analyticity, positive curvature, smooth convergence and the exact volume. □

5. Higher dimensions and prescribed ellipsoids

5.1. **A finite-dimensional genericity argument.** Let $\mathcal{P}_d^0$ be the finite-dimensional space of restrictions to $\mathbb{S}^{d-1}$ of real polynomials of degree at most d, with zero spherical mean. No choice of polynomial representation is needed; this is a space of functions on the sphere.

Proposition 5.1. *If $d\ge 5$, there is an open subset of full Lebesgue measure in $\mathcal{P}_d^0$ whose elements have no common zero in (11). If $d\ge 6$, the same assertion holds in the even subspace of $\mathcal{P}_d^0$, using only even index sets $|I|\ge 4$. Lebesgue measure here means Lebesgue measure after any linear identification of the relevant function space with a Euclidean space.*

Proof. Put $\mathcal{I}=\{I\subset[d]:|I|\ge 3\}$ and consider

$$\mathcal{T}:\mathcal{P}_d^0\times \mathrm{O}(d)\longrightarrow\mathbb{R}^{\mathcal{I}},\qquad \mathcal{T}(\phi,q)=(T_I(\phi,q))_{I\in\mathcal{I}}.$$

Changing variables $v=qz$ gives

$$T_I(\phi,q)=\int_{\mathbb{S}}^{d-1}\phi(qz)\prod_{i\in I}\operatorname{sgn} z_i\,d\sigma(z).$$

Since ϕ is a polynomial restriction, this is smooth jointly in its coefficients and in q, despite the sign functions.

For a fixed q and $J\in\mathcal{I}$, use the test polynomial

$$p_{J,q}(v)=\prod_{j\in J}(q_j\cdot v).$$

Its mean is zero and its degree is at most d. Coordinate reflection gives

$$T_I(p_{J,q},q)=\begin{cases}0, & I\neq J,\\ \displaystyle\int_{\mathbb{S}}^{d-1}\prod_{j\in J}|z_j|\,d\sigma(z)>0, & I=J.\end{cases} \tag{15}$$

Therefore the derivative of $\mathcal{T}$ in the polynomial variable is surjective at every (ϕ,q). Its zero set Z is a smooth submanifold of codimension

$$r=|\mathcal{I}|=2^d-1-d-\binom{d}{2}.$$

For $d\ge 5$,

$$r>\dim \mathrm{O}(d)=\binom{d}{2},\quad\text{equivalently}\quad 2^d>d^2+1. \tag{16}$$

The inequality holds at $d=5$, and its strictness persists on increasing d. The projection $Z\to\mathcal{P}_d^0$ has lower-dimensional domain than target. Its image has Lebesgue measure zero by Sard's theorem, applied on a countable atlas. That image is also closed: from a convergent sequence of polynomials

in the image, extract a convergent sequence of frames using compactness of $\mathrm{O}(d)$. Its complement is therefore open and has full measure.

For even polynomials, let $\mathcal{I}_{\mathrm{ev}} = \{I \subset [d] : |I| \geq 4,\ |I| \text{ even}\}$. The polynomials $p_{J,q}$ are now even, and the same diagonal test proves surjectivity. The target dimension is

$$r_{\mathrm{ev}} = 2^{d-1} - 1 - \binom{d}{2} > \binom{d}{2} \qquad (d \geq 6).$$

At $d = 6$ this is $16 > 15$; the equivalent inequality $2^{d-1} > d(d-1) + 1$ persists for larger d. The identical projection argument proves the assertion. □

5.2. **Affine transfer and proof of the main theorem.**

Lemma 5.2. *An invertible affine map preserves the existence of an equipartition by d affine hyperplanes. It preserves real-analytic boundaries and positive principal curvatures of convex bodies. For a fixed such map, it also preserves smooth convergence of radial functions about the corresponding interior points.*

Proof. Let $x = c + Ay$. The inverse image of a hyperplane is a hyperplane, and all sign-region volumes are multiplied by the same number $|\det A|$. For a smooth boundary parametrization, the second fundamental form after the change of variables is the original positive form divided by a positive normalizing factor. This proves preservation of positive curvature; analyticity follows from the analytic change of variables. If K is radial about zero, then

$$\rho_{c+AK,c}(v) = \frac{\rho_{K,0}(A^{-1}v/|A^{-1}v|)}{|A^{-1}v|}.$$

The sphere map and its positive denominator are fixed and smooth, so the formula proves the convergence assertion. □

Proof of Theorem 1.1. For $d = 4$, Proposition 4.3 provides the required mean-zero polynomial perturbation of the unit ball. For $d \geq 5$, choose a polynomial from Proposition 5.1. If $d \geq 6$, choose it in the even subspace. Proposition 3.4 excludes equipartitions for all sufficiently small nonzero parameters, and Lemma 2.1 supplies the remaining properties.

Write the prescribed ellipsoid as $E = c + AB^d$, with A invertible, and apply $x = c + Ay$ to the constructed family. Lemma 5.2 gives the assertions about equipartitions, curvature and smooth convergence. Equation (2) and the zero mean give the exact volume of E. An even perturbation gives symmetry about c. □

Proof of Corollary 1.2. The one-dimensional assertion follows by bisecting an interval. In the plane, bisect the body by one line and apply the ham-sandwich theorem to its two parts to find a second line bisecting both. Hadwiger's theorem [4] gives the assertion in dimension three. Theorem 1.1 supplies counterexamples in each dimension $d \geq 4$. □

6. Orthogonal equipartitions near ellipsoids

An affine map need not preserve orthogonality. The distinct axes of the reference ellipsoid allow us to control precisely which orthogonal configurations can survive in the limit.

Proof of Theorem 1.3. Translations and orthogonal transformations preserve the problem. We may therefore assume that the center is zero and

$$E = AB^d, \qquad A = \mathrm{diag}(a_1, \ldots, a_d), \quad a_i > 0, \quad a_i \neq a_j\ (i \neq j).$$

For the first assertion take $\phi(v) = v_1 v_2 v_3$. This polynomial has zero mean. Construct K_ε by (1), and consider AK_ε.

Suppose that along a sequence of nonzero parameters tending to zero, AK_{ε_m} admits an orthogonal equipartition. Its unit normals form an orthogonal matrix $U^m = (u_1^m, \dots, u_d^m)$. Pulling back its hyperplanes under A gives an equipartition of K_{ε_m}, with unit normals

$$v_i^m = \frac{Au_i^m}{|Au_i^m|}.$$

The offsets are divided by the same positive normalization factors. Take a subsequence $U^m \to U \in \mathrm{O}(d)$. Proposition 3.4 says that the limiting v_i are orthonormal and that their higher sign correlations with ϕ all vanish.

For $i \neq j$, their orthogonality implies

$$(u_i)^T A^2 u_j = 0.$$

Thus $U^T A^2 U$ is diagonal. Since the eigenvalues a_i^2 are distinct, the columns of U are a signed permutation of the coordinate vectors. Positive diagonal normalization leaves these vectors unchanged, so v is that same signed coordinate frame.

Choose the three indices I whose limiting normals correspond to the first three coordinate axes. Then

$$T_I(\phi, v) = \pm \int_{\mathbb{S}}^{d-1} |z_1 z_2 z_3| \, d\sigma(z) \neq 0,$$

contradicting Proposition 3.4. This excludes all orthogonal equipartitions for sufficiently small nonzero parameters.

For $d \geq 4$, take instead $\phi(v) = v_1 v_2 v_3 v_4$. It is even and has zero mean. The same argument uses the four indices corresponding to those axes and the nonzero integral of $|z_1 z_2 z_3 z_4|$. The resulting body is centrally symmetric. Lemma 2.1, scaling by A, and the radial formula in Lemma 5.2 prove all geometric and volume assertions. Finally undo the initial translation and rotation. □

7. Hausdorff stability and polytopal examples

Lemma 7.1. *Within the space of convex bodies with the Hausdorff metric, the bodies admitting an equipartition by d affine hyperplanes form a closed set. The same is true when the hyperplanes are required to be mutually perpendicular.*

Proof. Suppose $K_m \to K$ in Hausdorff distance and K has nonempty interior. All bodies lie in one ball of radius R. For an equipartition (u^m, t^m), each hyperplane bisects a body of positive volume and therefore intersects that ball. Consequently $|t_i^m| \leq R$. The normals and offsets have a convergent subsequence, say $(u^m, t^m) \to (u, t)$.

Indicators of Hausdorff-convergent convex bodies with an interior limit converge almost everywhere; their boundaries have Lebesgue measure zero. Outside the finitely many limiting hyperplanes, the sign products converge as well. Dominated convergence in a common bounding ball gives

$$W_I(K_m; u^m, t^m) \longrightarrow W_I(K; u, t) \qquad (I \subset [d]).$$

Every nonempty coefficient on the left is zero. The limiting configuration therefore equipartitions K, by Lemma 2.2. If the normals were orthonormal, their limits remain orthonormal. No independence assumption on a general limiting configuration is needed. □

Proof of Corollary 1.4. By Lemma 7.1, the absence of an equipartition is open. Approximate a counterexample in Theorem 1.1 or Theorem 1.3 by convex polytopes in Hausdorff distance. Such approximations are standard for convex bodies [7]; in the centrally symmetric case, take convex hulls of finite antipodally paired boundary nets. All sufficiently close approximants retain the absence of the corresponding equipartition.

Their volumes converge to the desired volume. Dilating each approximant about the reference center by a factor tending to one makes the volumes exactly equal. Positive dilations preserve

both ordinary and orthogonal equipartitionability, and preserve central symmetry. First choosing the smooth counterexample close to the reference ellipsoid and then choosing the polytopal approximation proves the stated proximity. □

Remark 7.2. The smooth approximations in the main theorems are in every fixed C^k norm. The polytopal approximation is in Hausdorff distance. The central-symmetry assertion for unrestricted affine hyperplanes is proved here only for $d \geq 6$; the even-polynomial dimension count used above does not establish it in dimensions four or five.

Use of artificial intelligence

ChatGPT assisted with mathematical exploration, checking arguments, organizing the exposition, and preparing the LaTeX source. The mathematical claims, the attribution of prior work, and the final manuscript are the responsibility of the author. The four-dimensional proof invokes Soberón's cited tensor nonvanishing theorem; the present article introduces no numerical certificate or computer-assisted existence step.

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