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Plane Algebraic Curve Families from Normalized Zeta Series: Prime-Base Extensions and Composite Breakdown

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Abstract

In a companion study of Euler's alternating series $\eta(s) = (1 - 2^{1-s})\zeta(s)$, distributing the prefactor across individual summands $t_n(s) = (-1)^{n-1}(1 - 2^{1-s})^{-1}n^{-s}$ exposed two distinct families of plane curves along vertical lines $s = a + ib$ in the critical strip. On the critical line $\mathrm{Re}(s) = \frac{1}{2}$, the pure powers $t_{2^k}(\frac{1}{2} + ib)$ trace an infinite hierarchy of closed, irreducible plane algebraic curves of degree $2k$ in $\mathbb{Z}[x, y]$ whose enclosed signed areas sum to zero, whereas weighting non-power terms by $(1 - 2^{-s})$ yields concentric circles centered at $(0, 0)$ uniquely at $a = \frac{1}{2}$ and open, precessing rosettes for $a \neq \frac{1}{2}$. This sequel investigates whether the emergence of these curve families is specific to base 2 or extends to general integer normalizations $\eta_q(s) = (1 - q^{1-s})\zeta(s)$ for $q \geq 2$. We show that every prime base p generates an analogous two-family structure: the pure-power terms $t_{p,p^k}(\frac{1}{2} + ib)$ form irreducible degree-$2k$ algebraic curves over $\mathbb{Z}$ governed by a two-step polynomial recurrence, a Blaschke-involution elliptic perimeter integral, and a zero-sum signed-area law; their seed quartics $t_{p,p^2}(\frac{1}{2} + ib)$ evolve geometrically from figure-eights ($p = 2, 3$) to teardrops and asymptotically circular ovals ($p \geq 5$) while retaining degree-4 integer structure; their rational equations in $\mathbb{Q}[x, y]$ uniquely single out $a = \frac{1}{2}$ among rational abscissas $a \in (0, 1) \cap \mathbb{Q}$; and all paired non-power curves ($n \neq p^k$) transition from concentric circles of radius $\mathcal{R}_{p,n} = |c_p(n)|/\sqrt{pn}$ on $a = \frac{1}{2}$ to two-dimensional rosettes off it. When q is composite, however, this classification fractures. Proper prime powers $q = p^m$ ($m \geq 2$) produce rational algebraic curves on $2m - 1$ vertical lines $a = \frac{r}{2m}$ and create a hybrid third family of resonant sub-powers p^j ($m \nmid j$) that remain closed algebraic loops off $\mathrm{Re}(s) = \frac{1}{2}$, while multi-prime composites fail to gather their prime divisors into algebraic towers and break the multiplicative structure of Euler's product.

1 Introduction

For $\mathrm{Re}(s) > 1$, the Riemann zeta function is defined both additively and multiplicatively by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}. \tag{1}$$

To continue $\zeta(s)$ into the critical strip $0 < \mathrm{Re}(s) < 1$, Euler introduced the alternating Dirichlet series $\eta(s) = (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$, canceling the simple pole at $s = 1$ so that the series converges conditionally throughout $\mathrm{Re}(s) > 0$ [3, 7].

In our companion paper [4], absorbing $(1 - 2^{1-s})^{-1}$ directly into each summand gave the termwise representation $\zeta(s) = \sum_{n=1}^{\infty} t_n(s)$ with $t_n(s) = (-1)^{n-1}(1 - 2^{1-s})^{-1}n^{-s}$. Tracking the trajectories $b \mapsto t_n(a + ib)$ along vertical lines $s = a + ib$ separated the positive integers into two contrasting curve families:

- **Pure-power algebraic curves ($n = 2^k$):** Sharing the single frequency $\ln 2$ in numerator and denominator, $t_{2^k}(\frac{1}{2} + ib)$ closes into an irreducible degree-$2k$ curve in $\mathbb{Z}[x, y]$ with complete elliptic perimeter and signed areas summing to zero across $k \geq 1$.

- **Paired circles and rosettes ($n \neq 2^k$):** Pairing odd-index neighbors $P_m(s) = t_m(s) + t_{2m}(s) = m^{-s}S_2(s)$ and multiples-of-4 differences $D_{4m}(s) = t_{4m}(s) - t_{8m}(s) = -4^{-s}P_m(s)$ via $(1-2^{-s})$ yields origin-centered circles of constant radius on $a = \frac{1}{2}$ and two-dimensional rosettes for $a \neq \frac{1}{2}$.

Because Euler's alternating signs rely on even and odd parity, a natural question is whether generating integer algebraic curves from $\zeta(s)$ requires base 2, extends to every prime p, or works equally well for any integer $q \geq 2$. Here we show that the two-family curve classification extends intact to every prime base $p \geq 2$, examine how the non-circular seed quartics $t_{p,p^2}(\frac{1}{2} + ib)$ evolve geometrically across increasing primes, and determine how composite bases $q \geq 4$ fracture both the rational vertical-line pin and the circle-versus-rosette split.

2 Base-q Normalizations and the Prime Circle–Rosette Split

For any integer base $q \geq 2$, subtracting $q\sum_{m=1}^{\infty}(qm)^{-s}$ from $\sum_{n=1}^{\infty} n^{-s}$ on $\mathrm{Re}(s) > 1$ cancels the pole at $s = 1$ by assigning weight $+1$ to integers indivisible by q and weight $-(q-1)$ to multiples of q [1, 5].

Definition 2.1 (Base-q Series and Normalized Curve Terms). Fix an integer $q \geq 2$ and define the q-periodic weight

$$c_q(n) := \begin{cases} +1, & q \nmid n, \\ -(q-1), & q \mid n. \end{cases} \tag{2}$$

Set $\eta_q(s) := \sum_{n=1}^{\infty} c_q(n)n^{-s}$, and away from the prefactor poles $1 - q^{1-s} = 0$ on $\mathrm{Re}(s) = 1$, define the normalized curve terms

$$t_{q,n}(s) := \frac{c_q(n)}{\big(1-q^{1-s}\big)n^s} \qquad (n \geq 1). \tag{3}$$

Because every block of q consecutive weights sums to $(q-1)(+1) - (q-1) = 0$, Dirichlet's test guarantees conditional convergence across the right half-plane.

Proposition 2.1 (Termwise Continuation Across $\mathrm{Re}(s) > 0$)**.** *For every integer $q \geq 2$, $\eta_q(s)$ converges conditionally on $\mathrm{Re}(s) > 0$ to $(1-q^{1-s})\zeta(s)$, giving $\zeta(s) = \sum_{n=1}^{\infty} t_{q,n}(s)$ whenever $\mathrm{Re}(s) > 0$ and $1 - q^{1-s} \neq 0$.*

To compare non-power terms across vertical lines, introduce the base-q pairing factor $(1-q^{-s})$, which converts $(1-q^{1-s})^{-1}$ into the Möbius ratio [6]

$$S_q(s) := \frac{1-q^{-s}}{1-q^{1-s}} = t_{q,1}(s) + \frac{1}{q-1}\,t_{q,q}(s), \qquad q_{q,n}(s) := \big(1-q^{-s}\big)t_{q,n}(s) = c_q(n)n^{-s}S_q(s). \tag{4}$$

At $q = 2$, $t_{2,n}(s) = t_n(s)$ and $q_{2,n}(s) = (-1)^{n-1}n^{-s}S_2(s)$ unify the companion paper's odd-index sums $q_{2,m}(s) = P_m(s)$ (m odd, radius $\mathcal{R}_m = 1/\sqrt{2m}$) and multiples-of-4 differences $q_{2,4m}(s) = D_{4m}(s)$ [4]. On $\mathrm{Re}(s) > 1$, summing $q_{q,n}(s)$ across $n \geq 1$ also matches the index-sieved Dirichlet series $(1-q^{-s})\zeta(s) = \sum_{q\nmid n} n^{-s}$ by canceling multiples of q.

Proposition 2.2 (Critical-Line Circle Criterion for S_q)**.** *For any integer $q \geq 2$ and any vertical line $s = a + ib$ ($a \neq 1$, $b \in \mathbb{R}$), $S_q(a+ib)$ traces an exact Euclidean circle of period $\frac{2\pi}{\ln q}$ with real center and radius*

$$h_q(a) = \frac{1-q^{1-2a}}{1-q^{2-2a}}, \qquad R_q(a) = \frac{q^{-a}(q-1)}{|1-q^{2-2a}|}. \tag{5}$$

This circle is centered at $(0,0)$ if and only if $a = \frac{1}{2}$, where $|S_q(\frac{1}{2}+ib)| = \mathcal{R}_q := \frac{1}{\sqrt{q}}$ is constant.

When $q = p$ is prime, irrationality of $\frac{\ln n}{\ln p}$ for every $n \neq p^k$ produces a clean dichotomy across all non-power indices [8].

Theorem 2.3 (Circle-Versus-Rosette Family for Prime Bases)**.** *Let p be a prime and let $n \geq 2$ satisfy $n \neq p^k$ for all $k \geq 1$ (including $n = 2, 4, 8, \ldots$ when p is odd).*

1. ***Concentric Critical-Line Circles:*** *At $a = \frac{1}{2}$, every curve $b \mapsto q_{p,n}(\frac{1}{2} + ib)$ traces a circle centered at $(0, 0)$ of constant radius $\mathcal{R}_{p,n} := \frac{|c_p(n)|}{\sqrt{pn}}$.*

2. ***Off-Line Precessing Rosettes:*** *For every $a \in (0, \infty) \setminus \{\frac{1}{2}, 1\}$, the curve $b \mapsto q_{p,n}(a + ib)$ unravels into an open, non-periodic rosette densely filling the annulus with inner and outer radii $|c_p(n)|n^{-a}\big||h_p(a)| - R_p(a)\big|$ and $|c_p(n)|n^{-a}\big(|h_p(a)| + R_p(a)\big)$.*

3 Pure-Power Algebraic Curve Towers for Prime Bases

Unlike the non-power terms, the unpaired pure powers $t_{p,p^k}(s) = -(p-1)p^{-ks}(1-p^{1-s})^{-1}$ $(k \geq 1)$ oscillate exclusively at integer harmonics of $\ln p$, so $b \mapsto t_{p,p^k}(a + ib)$ closes into a periodic curve of period $\frac{2\pi}{\ln p}$ on every vertical line $a \neq 1$. Defining the epicyclic step circles $\mathcal{C}_{p,p^k}(s) := (p-1)p^{-ks}S_p(s)$ (recovering $\mathcal{C}_{2^k}(s) = 2^{-ks}S_2(s)$ at $p = 2$ [4]), each curve in the tower is generated from its predecessor by phase-locked circle addition.

Proposition 3.1 (Epicyclic Construction of the Pure-Power Family)**.** *For every prime p and every s with $1 - p^{1-s} \neq 0$, the pure-power terms satisfy*

$$t_{p,p}(s) = 1 - S_p(s), \qquad t_{p,p^{k+1}}(s) = t_{p,p^k}(s) + \mathcal{C}_{p,p^k}(s) \quad (k \geq 1). \tag{6}$$

Along $\mathrm{Re}(s) = \frac{1}{2}$, $t_{p,p}$ is a circle of radius $\mathcal{R}_p = \frac{1}{\sqrt{p}}$ centered at $(+1, 0)$, and each step $k \mapsto k+1$ attaches an origin-centered circle $\mathcal{C}_{p,p^k}$ of radius $\frac{p-1}{p^{(k+1)/2}}$ rotating clockwise at frequency $(k+1)\ln p$.

On the critical line $s = \frac{1}{2} + ib$, setting $\theta = b \ln p$ writes $t_{p,p^k}(\frac{1}{2} + ib) = -(p-1)p^{-k/2}e^{-ik\theta}(1 - \sqrt{p}\, e^{-i\theta})^{-1}$. Despite the appearance of $\sqrt{p}$ in this parameterization, eliminating θ clears every radical simultaneously. Scaling out the integer factor $p-1$ via $u_{p,k}(s) := \frac{1}{p-1}t_{p,p^k}(s) = -p^{-ks}(1 - p^{1-s})^{-1}$ with coordinates $(u, v) = (\mathrm{Re}\, u_{p,k}, \mathrm{Im}\, u_{p,k})$—related to $(x, y) = (\mathrm{Re}\, t_{p,p^k}, \mathrm{Im}\, t_{p,p^k})$ by $(x, y) = (p-1)(u, v)$, so $(u, v) = (x, y)$ directly at $p = 2$—places all prime bases under a single recurrence.

Theorem 3.2 (Integer Polynomial Hierarchy on $\mathrm{Re}(s) = \frac{1}{2}$)**.** *Let p be a prime, let $k \geq 1$, and write $(u, v) = (\mathrm{Re}\, u_{p,k}(\frac{1}{2} + ib), \mathrm{Im}\, u_{p,k}(\frac{1}{2} + ib))$. Putting $W := p^k(u^2 + v^2)$, the locus of $u_{p,k}(\frac{1}{2} + ib)$ traces an irreducible plane algebraic curve of degree $2k$ in $\mathbb{Z}[u, v]$ defined by*

$$Q_{p,k}(W) - 2p^k W^{k-1}u = 0, \tag{7}$$

where $\{Q_{p,k}(W)\}_{k \geq 1} \subset \mathbb{Z}[W]$ is determined by the initial polynomials

$$Q_{p,1}(W) = (p-1)W + 1, \qquad Q_{p,2}(W) = (p-1)W^2 + (p+2)W - 1, \tag{8}$$

and the two-step recurrence

$$Q_{p,k+1}(W) = \big((p+1)W - 1\big)Q_{p,k}(W) - pW^2 Q_{p,k-1}(W) \qquad (k \geq 2). \tag{9}$$

Every $Q_{p,k}(W)$ has degree k, leading coefficient $p-1$, and constant term $(-1)^{k-1}$; substituting $(u, v) = \frac{1}{p-1}(x, y)$ and clearing $(p-1)^{2k}$ gives the irreducible degree-$2k$ equation of $t_{p,p^k}(\frac{1}{2} + ib)$ in $\mathbb{Z}[x, y]$ (matching $Q_k(W) - 2^{k+1}W^{k-1}x = 0$ at $p = 2$ [4]).

Across every prime tower, the positively oriented base circle $t_{p,p}$ and negatively oriented higher-order loops t_{p,p^k} $(k \geq 2)$ satisfy an exact signed-area conservation law, while the unit-circle Blaschke involution $w = \frac{\sqrt{p} - z}{1 - \sqrt{p}z}$ reduces every perimeter to a complete elliptic integral [2]. Notably, just as base 2 produces the (t_8, t_{16}) isometric arc-length twin pair [4], base 3 yields a second and final prime isometric twin pair $(t_{3,9}, t_{3,27})$ before third-kind Π terms persist across all higher primes $p \geq 5$.

Theorem 3.3 (Signed-Area Conservation and Elliptic Perimeter Across Prime Towers)**.** *Let p be a prime and orient $t_{p,p^k}(\frac{1}{2}+ib)$ by increasing $b \in [0, \frac{2\pi}{\ln p}]$ $(k \geq 1)$.*

1. ***Zero-Sum Signed-Area Law:*** *The enclosed signed area of $t_{p,p^k}(\frac{1}{2}+ib)$ is*

$$\mathcal{A}_{\text{signed}}\big(t_{p,p^k}\big) = \frac{p-k(p-1)}{p^k}\,\pi, \qquad \sum_{k=1}^{\infty} \mathcal{A}_{\text{signed}}\big(t_{p,p^k}\big) = 0, \tag{10}$$

vanishing for an individual curve if and only if $k = \frac{p}{p-1} \in \mathbb{Z}$ (occurring uniquely for the Bernoulli lemniscate t_4 at $p = 2, k = 2$ [4]).

2. ***Blaschke-Involution Elliptic Perimeter:*** *Setting $\alpha_{p,k} := k-(k-1)p$, the fundamental-period arc length of $t_{p,p^k}(\frac{1}{2}+ib)$ is given by*

$$\mathcal{L}\big(t_{p,p^k}\big) = \frac{1}{p^{k/2}} \int_0^{2\pi} \left| \frac{\alpha_{p,k} - \sqrt{p}\, e^{i\phi}}{1 - \sqrt{p}\, e^{i\phi}} \right| d\phi. \tag{11}$$

4 Geometric Evolution of Seed Quartics Across Prime Bases

In base $p = 2$, the companion study [4] centered on the $k = 2$ term $t_4(\frac{1}{2}+ib)$, which serves as the first non-circular curve in the dyadic tower after the shifted circle $t_2 = 1 - S_2$ and anchors all higher inner-loop curves t_{2^k} $(k \geq 3)$ under circle addition. For a general prime p, the quadratic seed $t_{p,p^2}(\frac{1}{2}+ib) = t_{p,p}(\frac{1}{2}+ib) + \mathcal{C}_{p,p}(\frac{1}{2}+ib)$ occupies this exact structural position: it is the first folded degree-4 algebraic curve generated by $Q_{p,2}(W) = (p-1)W^2 + (p+2)W - 1$ and the second initial seed required to launch the two-step recurrence for $k \geq 3$.

Plotting the curve $t_{p,p^2}(\frac{1}{2}+ib) = \frac{-(p-1)e^{-2i\theta}}{p(1-\sqrt{p}\,e^{-i\theta})}$ $(\theta = b\ln p)$ across increasing primes shows how the denominator $1-\sqrt{p}\,e^{-i\theta}$ governs the curve's silhouette. Rationalizing the denominator shows that $\operatorname{Im} t_{p,p^2}(\frac{1}{2}+ib)$ vanishes away from the apsides $\theta \in \{0,\pi\}$ if and only if $\cos\theta = \frac{\sqrt{p}}{2}$, while expanding $(1-\sqrt{p}\,e^{-i\theta})^{-1}$ in powers of $p^{-1/2}e^{i\theta}$ gives the convergent Fourier representation

$$t_{p,p^2}\left(\frac{1}{2}+ib\right) = \frac{p-1}{p^{3/2}}\,e^{-i\theta} + \frac{p-1}{p^2} + \frac{p-1}{p^{5/2}}\,e^{i\theta} + O\big(p^{-2}\big) \qquad (p \geq 2). \tag{12}$$

As illustrated across the first nine primes ($p = 2$ through $p = 23$) in Figure 1, this produces a smooth three-stage geometric progression:

- **Self-intersecting figure-eights ($p = 2, 3$):** Only $p = 2$ and $p = 3$ satisfy $\frac{\sqrt{p}}{2} \leq 1$, creating a real-axis double point at $x = \frac{p-1}{p}$. At $p = 2$, t_4 forms the symmetric Bernoulli lemniscate crossing at $(\frac{1}{2}, 0)$ with $\mathcal{A}_{\text{signed}} = 0$; at $p = 3$, $t_{3,9}$ crosses at $(\frac{2}{3}, 0)$ with a smaller right-hand lobe and net signed area $-\frac{\pi}{9}$.

- **Unfolded teardrops and ovals ($p = 5, 7, 11$):** Once $p \geq 5$, $\frac{\sqrt{p}}{2} > 1$ admits no real angle θ, erasing the self-intersection node so that $t_{5,25}$ unfolds into a simple pointed teardrop that rounds progressively into a smooth convex oval across $p = 7$ and $p = 11$.

- **Asymptotic circularity ($p = 13, 17, 19, 23$):** As p increases, the leading Fourier mode $\frac{p-1}{p^{3/2}}e^{-i\theta}$ dominates, rounding each seed curve toward a clockwise circle of radius $\frac{1}{\sqrt{p}}$ shifted right by $\frac{1}{p}$ with real-axis apsides at $-\frac{\sqrt{p}-1}{p}$ and $+\frac{\sqrt{p}+1}{p}$.

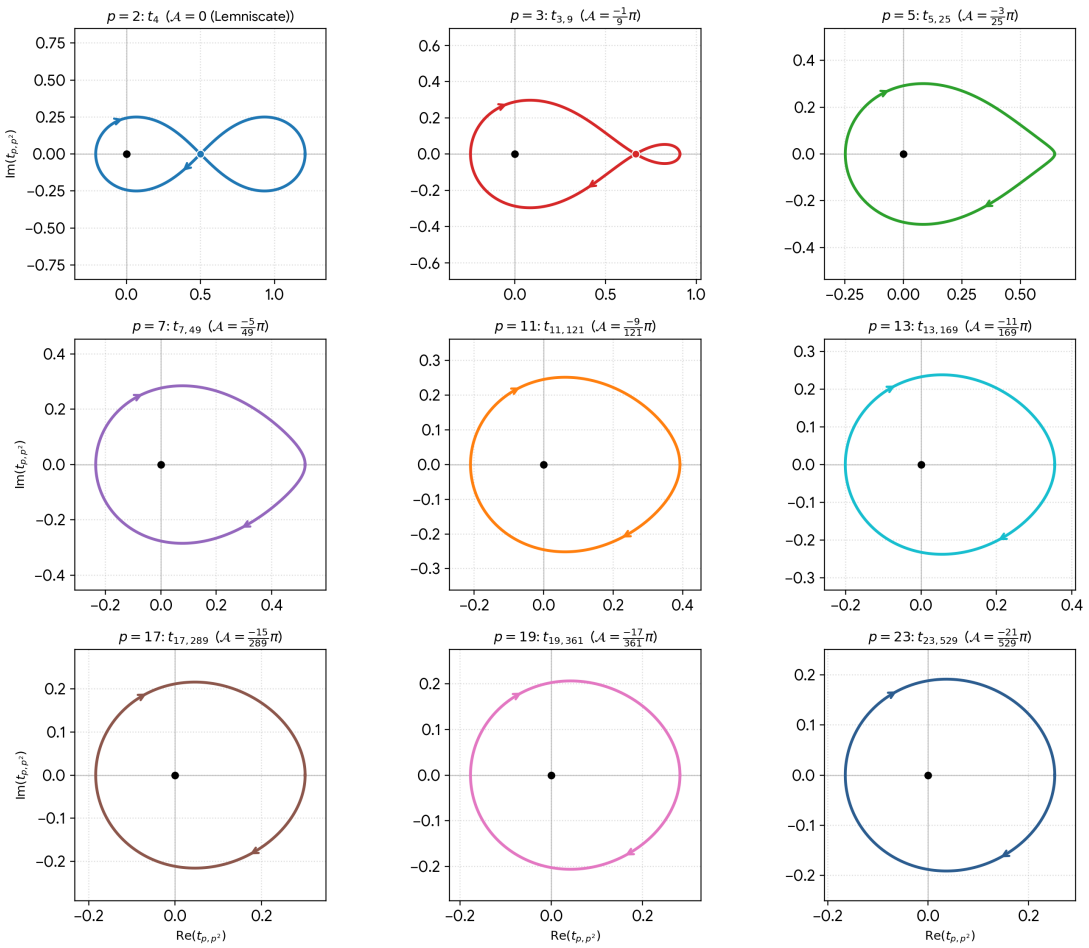


Figure 1: Geometric evolution of the degree-4 seed quartics $t_{p,p^2}(\frac{1}{2}+ib)$ across the first nine primes ($p = 2, 3, 5, 7, 11, 13, 17, 19, 23$), with the origin $(0, 0)$ marked in black and real-axis double points $(\frac{p-1}{p}, 0)$ marked for $p = 2, 3$. Although each panel is scaled to display its silhouette, the curves contract in true diameter as $O(p^{-1/2})$ and oscillate at $\mathbb{Q}$-linearly independent frequencies $\ln p$.

Figure 1 also clarifies why primality is an arithmetic and whole-series property rather than a silhouette property: even as $t_{p,p^2}(\frac{1}{2}+ib)$ becomes visually indistinguishable from a circle at larger primes, each curve remains an irreducible degree-4 locus in $\mathbb{Z}[x, y]$ built from cancelling the quadratic radical $\sqrt{p} \notin \mathbb{Q}$ on its own independent clock $\theta = b \ln p$, from which higher powers $k \geq 3$ fold $(k-1)$ nested loops around the origin.

5 Field-Theoretic Characterization of the Critical Line

When a moves off $\frac{1}{2}$ inside $(0, 1)$, the pure-power curves $u_{p,k}(a+ib)$ remain closed real algebraic curves in $\mathbb{R}[u, v]$, but their defining coefficients leave $\mathbb{Q}$ at every other rational abscissa.

Theorem 5.1 ($\mathbb{Q}$-Rationality and Dual Integrality of Prime Curve Towers)**.** *Let p be a prime, let $k \geq 1$, and fix $a \in (0, 1)$. Define the base parameter $w_p(a) := p^{2a}$ and write $V := w_p(a)^k(u^2+v^2)$ for $(u, v) = (\mathrm{Re}\, u_{p,k}(a+ib), \mathrm{Im}\, u_{p,k}(a+ib))$.*

1. ***Off-Line Cartesian Equations:*** *Along $s = a+ib$, the curve $u_{p,k}(a+ib)$ satisfies the*

degree-$2k$ equation

$$P_{p,k}\big(V;\, w_p(a)\big) - 2\big(pw_p(a)\big)^k V^{k-1} u = 0, \tag{13}$$

generated from $P_{p,1}(V;w) = (p^2-w)V+w$ *and* $P_{p,2}(V;w) = w(p^2-w)V^2+w(p^2+2w)V-w^2$ *by the recurrence*

$$P_{p,k+1}(V;w) = \big((w+p^2)V - w\big)P_{p,k}(V;w) - p^2 w V^2 P_{p,k-1}(V;w) \qquad (k \geq 2), \tag{14}$$

matching $P_k(V;w)$ *of [4] identically at* $p = 2$ *and reducing to* $p^k Q_{p,k}(W)$ *at* $w_p(\frac{1}{2}) = p$.

2. ***Uniqueness Among Rational Vertical Lines:*** *For* $a \in (0,1)\cap\mathbb{Q}$, *the curve* $t_{p,p^k}(a+ib)$ *is defined over* $\mathbb{Q}[x,y]$ *if and only if* $a = \frac{1}{2}$. *For any* $a = \frac{r}{d} \in (0,1) \setminus \{\frac{1}{2}\}$ *(*$\gcd(r,d) = 1$*),* $w_p(a) = p^{2r/d} \notin \mathbb{Q}$ *forces coefficients in the radical extension* $\mathbb{Q}(p^{2r/d})$.

3. ***Dual Integrality Under*** $s \leftrightarrow 1-s$***:*** *Across all real* $a \in (0,1)$, $a = \frac{1}{2}$ *is the unique abscissa at which both functional-equation partner parameters* $w_p(a) = p^{2a}$ *and* $w_p(1-a) = p^{2(1-a)}$ *belong to* $\mathbb{Z}$ *(both equaling* p*).*

6 How the Curve Families Break Down for Composite Bases

Formally, the circle $S_q(s)$ and polynomial sequence $Q_{q,k}(W)$ remain well-defined for any composite integer $q \geq 4$. However, the clean two-family architecture—one tower of rational algebraic curves pinned to $a = \frac{1}{2}$, plus a complementary family of non-power curves that unravel into rosettes off $a = \frac{1}{2}$ while preserving Euler's prime product—breaks down in two distinct ways depending on the prime factorization of q.

6.1 Proper Prime Powers ($q = p^m$, $m \geq 2$)

When $q = p^m$ is a higher power of a single prime (such as $q = 4, 8, 9, 16, 25$), fractional powers of q exist as integers p^j ($m \nmid j$) and alter both curve families simultaneously.

Theorem 6.1 (Extra Rational Lines and Leaked Algebraic Curves for $q = p^m$)**.** *Let* $q = p^m$ *for a prime* p *and an integer* $m \geq 2$.

1. ***Proliferation of Rational and Integer Curve Lines:*** *Because* $w_q(a) = q^{2a} = p^{2ma}$, *the pure-power curves* $t_{q,q^k}(a+ib)$ *belong to* $\mathbb{Q}[x,y]$ *(with integer parameter* $w_q(a) = p^r \in \mathbb{Z}$*) on all* $2m-1$ *rational vertical lines*

$$a \in \left\{ \frac{1}{2m},\ \frac{2}{2m},\ \ldots,\ \frac{2m-1}{2m} \right\} \subset (0,1). \tag{15}$$

Moreover, whenever m *is even, off-critical partner pairs* $a = \frac{r}{m}$ *and* $1-a = \frac{m-r}{m}$ *(*$1 \leq r \leq m-1$*) satisfy* $w_q(a), w_q(1-a) \in \mathbb{Z}$ *simultaneously (for instance,* $w_4(\frac{1}{4}) = 2$ *and* $w_4(\frac{3}{4}) = 8$ *at* $q = 4$*).*

2. ***Emergence of a Third Hybrid Family Off*** $\mathrm{Re}(s) = \frac{1}{2}$***:*** *Every sub-power* $n = p^j$ *with* $m \nmid j$ *(such as* $n = 2, 8, 32$ *when* $q = 4$*) satisfies* $n \neq q^k$ *and therefore belongs to the paired non-power family* $q_{q,n}(s)$. *Because* $\frac{\ln(p^j)}{\ln(p^m)} = \frac{j}{m} \in \mathbb{Q}$, $b \mapsto q_{q,p^j}(a+ib)$ *fails to unravel into an area-filling rosette for* $a \neq \frac{1}{2}$, *tracing instead a closed, periodic one-dimensional algebraic curve on every vertical line* $a \neq 1$.

6.2 Multi-Prime Composites ($q = 6, 10, 12, \dots$)

When q has at least two distinct prime factors (such as $q = 6 = 2 \cdot 3$), and is not itself a perfect power, $q^{2a} \in \mathbb{Q}$ once again holds uniquely at $a = \frac{1}{2}$ for rational $a \in (0, 1)$, and no rational frequency ratios leak into $n \neq q^k$. Instead, the curve decomposition loses alignment with the prime factorization of the integers.

For a prime base p, every prime power p^k belongs to the closed algebraic curve family $\{t_{p,p^k}\}$, while the index-sieved series $(1-p^{-s})\zeta(s) = \sum_{p\nmid n} n^{-s} = \prod_{\ell\neq p}(1-\ell^{-s})^{-1}$ ($\mathrm{Re}(s) > 1$) is supported on integers indivisible by p—a set closed under multiplication by Euclid's lemma [5]. Multi-prime bases forfeit both properties.

Theorem 6.2 (Prime-Tower Stranding and Multiplicative Breakdown for Multi-Prime q)**.** *Suppose $q \geq 6$ is divisible by at least two distinct primes $p_1, \dots, p_\omega$ ($\omega \geq 2$).*

1. ***Stranded Prime-Power Rosettes:*** *Only joint powers $n = q^k$ enter the closed algebraic curve tower $\{t_{q,q^k}\}$. For every constituent prime $p_i \mid q$ and every $k \geq 1$, the pure prime power $t_{q,p_i^k}(a + ib)$ has irrational frequency ratio $\frac{k \ln p_i}{\ln q} \notin \mathbb{Q}$ and unravels into an open two-dimensional rosette even on the critical line $a = \frac{1}{2}$.*

2. ***Loss of Multiplicative Closure and Euler Factorization:*** *Writing $q = AB$ with $\gcd(A, B) = 1$ and $A, B > 1$, the indicator $f_q(n) := \mathbf{1}_{q\nmid n}$ satisfies $f_q(A) = f_q(B) = 1$ but $f_q(AB) = 0$. Consequently, the paired series $(1 - q^{-s})\zeta(s) = \sum_{n=1}^{\infty} f_q(n)n^{-s}$ lacks multiplicative coefficients and does not factor as a sub-product of Euler's prime product.*

7 Classification of Curve Families Across Integer Bases

Table 1 summarizes how the algebraic and geometric curve families generated by $\eta_q(s) = (1 - q^{1-s})\zeta(s)$ depend on the arithmetic type of $q \geq 2$.

Table 1: Classification of algebraic and geometric curve families across integer bases $q \geq 2$.

Base Type	$\mathbb{Q}[x, y]$ **Lines**	$n \neq q^k$ $(a \neq \frac{1}{2})$	**Powers** p_i^k $(p_i \mid q)$	**Set** $q \nmid n$
Prime $(2, 3, 5)$	Unique $a = \frac{1}{2}$	All 2D rosettes	In $\mathbb{Z}[x, y]$ tower	Completely mult.
Prime power $(4, 8, 9)$	$2m - 1$ lines $\frac{r}{2m}$	Rosettes + 1D loops	Split across families	Coprime mult. only
Multi-prime $(6, 10, 12)$	$a = \frac{1}{2}$ (if non-power)	All 2D rosettes	Stranded 2D rosettes	Non-multiplicative

In Table 1, "$\mathbb{Q}[x, y]$ Lines" lists rational abscissas $a \in (0, 1) \cap \mathbb{Q}$ on which $t_{q,q^k}(a + ib)$ satisfies a rational polynomial equation, "$n \neq q^k$ $(a \neq \frac{1}{2})$" classifies the paired curves $q_{q,n}(a + ib)$ off the critical line, "Powers p_i^k $(p_i \mid q)$" tracks the fate of individual prime-factor powers $t_{q,p_i^k}(\frac{1}{2} + ib)$, and "Set $q \nmid n$" records the multiplicativity of $f_q(n) = \mathbf{1}_{q\nmid n}$.

This trichotomy clarifies what is special about prime normalizations of $\zeta(s)$: primes are the unique integer bases $q \geq 2$ for which $(1 - q^{1-s})^{-1}$ simultaneously gathers all powers of q's prime divisors into an irreducible $\mathbb{Z}[x, y]$ algebraic curve tower on $\mathrm{Re}(s) = \frac{1}{2}$, confines rational curve equations exclusively to $a = \frac{1}{2}$, forces every remaining paired term off $a = \frac{1}{2}$ to unravel into a two-dimensional rosette, and preserves multiplicative closure across the surviving integers.

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