A Universal Square-Root-of-Six Bound for Kissing Numbers

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Abstract

The least positive number α for which the Euclidean kissing numbers satisfy $\tau_n \le \alpha^n$ in every positive dimension is $\sqrt{6}$. We show how this follows from a classical spherical-cap estimate and known low-dimensional bounds, then give two elementary proofs that require no tables of kissing numbers. A recurrence for cap areas gives strict inequality in every dimension except two. A second proof packs explicit bodies of revolution into the unit ball. The Lean 4/Mathlib certificate registered in Palomar uses this volume construction to prove the universal bound, the planar value $\tau_2 = 6$, and the optimality of the base.

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1 The universal base

A *kissing configuration* in $\mathbb{R}^n$ is a finite set X satisfying

$$\|x\| = 1 \quad (x \in X), \qquad \langle x, z\rangle \le \frac{1}{2} \quad (x, z \in X,\ x \ne z). \tag{1}$$

Multiplying its vectors by two gives the centers of nonoverlapping unit balls touching the unit ball at the origin. We write τ_n for the largest cardinality of a kissing configuration in $\mathbb{R}^n$. The bounds below also establish the existence of this maximum.

Theorem 1. *For every integer $n \ge 1$,*

$$\tau_n \le (\sqrt{6})^n.$$

Corollary 2. *The least real number $\alpha > 0$ such that $\tau_n \le \alpha^n$ for every $n \ge 1$ is $\sqrt{6}$. Equivalently,*

$$\sup_{n\ge 1} \tau_n^{1/n} = \sqrt{6}.$$

The planar regular hexagon forces every such α to be at least $\sqrt{6}$. The issue is whether that same base works with prefactor one in every dimension. The question was raised in 2017 [5], motivated by a result of Bakshi, Das, Liu, and Ren [2, Theorem 4.1]. For a finite-index irreducible inclusion $N \subset M$ of type II_1 factors, the number of minimal intermediate subfactors is at most τ_d, where $d = \dim_{\mathbb{C}}(N' \cap M_1)$ and M_1 is the basic construction. Theorem 1 thus gives the explicit bound $6^{d/2}$. Cohn [6] anticipated the value $\sqrt{6}$ and outlined an approach using the finite-dimensional bound of Kabatiansky and Levenshtein.

Throughout, put $s = \sqrt{3}$ and $a = s/2$. For a unit vector x, let

$$C_x = \{y \in \mathbb{R}^n : \langle x, y\rangle > a\|y\|\}.$$

This is the open cone of half-angle $\pi/6$ about x. Its intersection with the unit sphere is the corresponding open spherical cap.

Lemma 3. *For distinct members x, z of a kissing configuration, $C_x \cap C_z = \varnothing$.*

Proof. A vector in the intersection could be normalized to a unit vector u satisfying

$$s < \langle x+z, u\rangle \le \|x+z\| = \sqrt{2+2\langle x,z\rangle} \le \sqrt{3} = s,$$

a contradiction. □

In dimension one the only unit vectors are 1 and -1, so $\tau_1 = 2 < \sqrt{6}$.

Deduction from published bounds. For $n \ge 2$, write $\Omega_n(\theta)$ for the fraction of the area of S^{n-1} covered by a cap of angular radius θ. For $0 < \theta < \pi/2$, the estimate of Böröczky and Wintsche [3, Corollary 3.2(i)], also stated in [4, Lemma 3.4, (7)], gives

$$\Omega_n(\theta) \ge \frac{(\sin\theta)^{n-1}}{\sqrt{2\pi n}}.$$

By Lemma 3, the open caps of angular radius $\pi/6$ about the points of X are disjoint. Hence $|X| \le 2^{n-1}\sqrt{2\pi n}$. For $n \ge 6$ this is strictly smaller than $6^{n/2}$. Squaring reduces the comparison to $n\pi/2 < (3/2)^n$. It holds at $n = 6$ since $3\pi < (3/2)^6$, and $(3/2)^n/n$ increases for $n > 2$. In dimensions one and two the elementary bounds are 2 and 6 (in the plane the angles between consecutive directions are at least $\pi/3$). For dimensions three through five, the upper bounds 12, 24, and 45 in Bachoc and Vallentin [1, Table 5.1] suffice. Thus results already available before the 2017 question imply Theorem 1. The proofs below derive all the needed low-dimensional estimates directly.

2 First proof: a spherical-cap recurrence

For $n \ge 2$ and $0 < \theta < \pi/2$, polar coordinates about the cap center give

$$\Omega_n(\theta) = \frac{I_{n-2}(\theta)}{J_{n-2}}, \qquad I_m(\theta) = \int_0^\theta \sin^m t\, dt, \qquad J_m = \int_0^\pi \sin^m t\, dt. \tag{2}$$

Indeed, the polar area element has a factor $\sin^{n-2} t$, and the transverse area cancels. Put $c_n = \Omega_n(\pi/6)$. Disjointness gives

$$|X| c_n \le 1. \tag{3}$$

Lemma 4. *For $n \ge 2$ and $0 < \theta < \pi/2$,*

$$\frac{\Omega_{n+2}(\theta)}{\Omega_n(\theta)} > \frac{n}{n+1}\sin^2\theta.$$

In particular, $c_{n+2} > \dfrac{n}{4(n+1)} c_n \ge \dfrac{1}{6} c_n$.

Proof. Put $m = n-2$ and $h = \sin\theta$. Substituting $u = \sin t$ gives $I_m(\theta) = \int_0^h u^m w(u)\, du$, with $w(u) = (1-u^2)^{-1/2}$. Give $[0,h]$ the probability measure $d\mu(u) = (m+1)u^m\, du/h^{m+1}$ and denote its expectation by E. Both u^2 and $w(u)$ are strictly increasing. Thus

$$2(E[u^2 w] - E[u^2]E[w]) = \iint (u^2 - v^2)(w(u) - w(v))\, d\mu(u)\, d\mu(v) > 0,$$

since the integrand is positive off the diagonal. It follows that

$$\frac{I_{m+2}(\theta)}{I_m(\theta)} = \frac{E[u^2 w]}{E[w]} > E[u^2] = h^2 \frac{m+1}{m+3}.$$

Integration by parts gives $J_{m+2}/J_m = (m+1)/(m+2)$. Dividing the preceding inequality by this ratio gives the result. □

First proof of Theorem 1. Direct integration gives $c_2 = 1/6$ and $c_3 = (2-s)/4 > 1/15$, since $s < 26/15$ by squaring. From c_2 the strict recurrence gives $c_n > 6^{-n/2}$ in every even dimension $n \geq 4$. In dimension three, (3) gives $|X| < 15$. Integrality yields $|X| \leq 14 < 6\sqrt{6}$. To start the odd induction we use the stronger coefficient at $n = 3$:

$$c_5 > \tfrac{3}{16} c_3 > \tfrac{1}{80} > 6^{-5/2},$$

where the last inequality follows from $80^2 < 6^5$. Thus $c_n > 6^{-n/2}$ for every odd $n \geq 5$ as well. Equation (3) proves the result, with strict inequality for $n \neq 2$. In each dimension the set of realizable cardinalities is nonempty and bounded in the nonnegative integers, so it has a greatest element. □

3 Second proof: explicit Euclidean volumes

The second proof uses the bodies in the registered Lean certificate. For any unit direction e, radial integration identifies the fraction of the unit ball in C_e with the cap fraction c_n of Section 2. The common radial factor $\int_0^1 r^{n-1}\,dr$ cancels. To obtain a packing bound it suffices to inscribe a body of sufficiently large volume in that sector. We now do this using only Euclidean integration and Lemma 3.

Let B_n be the open unit ball and $V_n = \mathrm{vol}(B_n)$. Fix $e = (1, 0)$ in orthogonal coordinates $(t, v) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Rotating any measurable body $K \subset C_e \cap B_n$ toward the points of X gives disjoint bodies of equal volume by Lemma 3. Thus

$$|X|\,\mathrm{vol}(K) \leq V_n. \tag{4}$$

For a continuous nonnegative profile g on $[b, c]$, Fubini's theorem gives

$$\mathrm{vol}\{(t, v) : b < t < c,\ \|v\| < g(t)\} = V_{n-1} \int_b^c g(t)^{n-1}\,dt. \tag{5}$$

The coordinate conditions for containment are

$$(t, v) \in C_e \iff t > 0 \text{ and } 3\|v\|^2 < t^2, \qquad (t, v) \in B_n \iff t^2 + \|v\|^2 < 1. \tag{6}$$

3.1 A bicone and the dimensions $n \geq 5$

Consider the union $K_n = L_n \cup U_n$, where

$$L_n = \{(t, v) : 0 < t < a,\ \|v\| < t/s\},$$
$$U_n = \{(t, v) : a < t < 1,\ \|v\| < (2 + s)(1 - t)\}.$$

It is a bicone of total height one and common base radius $1/2$, with the base section omitted. More precisely, each point is a strict convex combination of a point of the disk

$$D = \{(a, v) : \|v\| < 1/2\}$$

with either 0 or e. The disk lies in $C_e \cap B_n$, whereas $0, e$ satisfy the weak inequalities defining their closures. The triangle inequality therefore gives $K_n \subset C_e \cap B_n$.

By (5), the two cones have volumes $aV_{n-1}/(n2^{n-1})$ and $(1 - a)V_{n-1}/(n2^{n-1})$. Hence

$$\mathrm{vol}(K_n) = \frac{V_{n-1}}{n2^{n-1}}, \qquad |X| \leq A_n := n2^{n-1}\frac{V_n}{V_{n-1}}. \tag{7}$$

The standard ball-volume formula gives

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}, \qquad V_{d+2} = \frac{2\pi}{d + 2} V_d, \tag{8}$$

and consequently

$$A_{n+2} = \frac{4(n+1)}{n} A_n \le 6A_n \qquad (n \ge 2). \tag{9}$$

The initial estimates

$$A_5 = \frac{256}{3} < 36\sqrt{6} = (\sqrt{6})^5, \qquad A_6 = 60\pi < 216 = (\sqrt{6})^6$$

follow by squaring the first comparison and using $\pi < 22/7$ for the second. Induction on each parity proves $|X| < (\sqrt{6})^n$ for all $n \ge 5$. In dimension two, the same bicone gives $|X| \le A_2 = 2\pi < 7$, so integrality gives $|X| \le 6$.

3.2 Dimension three

Retain L_3 and replace the upper cone by

$$W_3 = \{(t, v) : a < t < 1,\ \|v\| < \sqrt{1 - t^2}\}.$$

Together with L_3 this fills the entire sector apart from the joining slice $t = a$, which has volume zero. Indeed, below a the cone gives the smaller cross-section, and above a the ball does. In particular W_3 lies in the cone because $t > a$ implies $3(1 - t^2) < t^2$. Since $V_2 = \pi$, formula (5) gives

$$\mathrm{vol}(L_3 \cup W_3) = \pi \left(\frac{s}{24} + \int_a^1 (1 - t^2)\, dt \right) = \frac{(2 - s)\pi}{3} > \frac{4\pi}{45} = \frac{V_3}{15}.$$

Here $V_3 = 4\pi/3$, and the strict inequality uses $s < 26/15$. Thus (4) yields $|X| < 15$, hence $|X| \le 14 < 6\sqrt{6}$.

3.3 Dimension four

The bicone alone gives $A_4 = 12\pi > 37$. To bring the packing bound below 37 while keeping the volume integral polynomial, retain L_4 and raise the upper chord $q(t) = (2 + s)(1 - t)$ by a quadratic that vanishes at its endpoints. More precisely, set

$$p(t) = q(t) + 14(1 - t)(t - a) = (1 - t)(14t + 2 - 6s), \qquad W_4 = \{(t, v) : a < t < 1,\ \|v\| < p(t)\}.$$

The coefficient 14 can be seen from the tangent at a. There q meets the circular profile $\sqrt{1 - t^2}$ at height $1/2$. For the raised chord with coefficient k to stay below the circle on the right, its derivative must satisfy $-(2 + s) + k(1 - a) \le -s$, or $k \le 2/(1 - a) = 8 + 4s$. The largest integral candidate is 14. The following calculation checks containment on the entire interval.

For $a \le t \le 1$ we have $p(t) \ge 0$, since $14t + 2 - 6s \ge s + 2 > 0$. Writing $u = t - a$, a factorization gives

$$1 - t^2 - p(t)^2 = (1 - t)(t - a)(4s - 6 + (126s - 140)u + 196u^2) \ge 0. \tag{10}$$

Indeed, $u \ge 0$ and the constant and linear coefficients in the last factor are positive because $s > 3/2$. Thus W_4 lies in the ball, and $t > a$ puts it in the cone as in dimension three.

A lower bound for the volume avoids cancellation in the exact integral. Write $\delta = 1 - a$, $x = (t - a)/\delta$, and $b = 14\delta^2$. Then $p(t) = (1 - x)(1/2 + bx)$. The lower cone contributes $a/32$ to the profile integral in (5). For the upper piece use $(1/2 + bx)^3 \ge 1/8 + 3bx/4$ on $[0, 1]$. Integrating gives

$$\begin{aligned} T := \frac{\mathrm{vol}(L_4 \cup W_4)}{V_3} &\ge \frac{a}{32} + \delta \int_0^1 (1 - x)^3 \Big(\frac{1}{8} + \frac{3b}{4} x \Big)\, dx \\ &= \frac{a + \delta}{32} + \frac{3b\delta}{80} = \frac{1}{32} + \frac{21}{40}\delta^3. \end{aligned}$$

Since $s < 7/4$, we have $\delta > 1/8$, so $T > 1/32 + \frac{1}{2}(1/8)^3 = 33/1024$. Also $V_4/V_3 = 3\pi/8 < 33/28$ by $\pi < 22/7$. The packing inequality (4) therefore gives

$$|X| \le \frac{V_4/V_3}{T} < \frac{33/28}{33/1024} = \frac{256}{7} < 37.$$

Thus $|X| \le 36 = (\sqrt{6})^4$.

Completion of the second proof of Theorem 1. The volume estimates cover every $n \ge 2$, and the dimension-one observation covers $n = 1$. Thus every kissing configuration satisfies the claimed bound, independently of the first proof. As before, the finite nonempty set of realizable cardinalities has an attained maximum, which is τ_n. □

4 Sharpness and the equality case

Proof of Corollary 2. The vertices of a regular hexagon are

$$(1,0), \quad (\tfrac{1}{2}, \tfrac{s}{2}), \quad (-\tfrac{1}{2}, \tfrac{s}{2}), \quad (-1,0), \quad (-\tfrac{1}{2}, -\tfrac{s}{2}), \quad (\tfrac{1}{2}, -\tfrac{s}{2})$$

with every inner product between distinct vertices at most $1/2$. These six unit vectors form a kissing configuration. Theorem 1 gives the reverse bound, so $\tau_2 = 6$.

The base $\sqrt{6}$ works by the theorem. Conversely, any positive universal base α must satisfy $6 = \tau_2 \le \alpha^2$, hence $\alpha \ge \sqrt{6}$. Likewise, $\tau_n^{1/n} \le \sqrt{6}$ in every dimension, with equality at $n = 2$, proving the supremum statement. □

Corollary 5. *Dimension two is the unique maximizer of $\tau_n^{1/n}$ among positive integers n.*

Proof. The regular hexagon gives equality in dimension two, and the first proof of Theorem 1 gives $\tau_n < 6^{n/2}$ for every $n \ne 2$. □

5 Formal statements and provenance

The Palomar registration `PALOMAR-2026-09-05-000002`, version 1 of September 5, 2026, certifies eight statements in Lean 4/Mathlib. They bound every finite kissing configuration, establish the existence of a greatest realizable cardinality, and give the planar value and equivalent formulations of the optimal base. The proof uses the Euclidean bodies of Section 3, including the exceptional profiles in dimensions three and four. For the four-dimensional body the certificate evaluates the integral exactly, while the lower estimate in Section 3.3 suffices here. The cap recurrence and its strict consequence, Corollary 5, are additional informal results.

The registered sources contain the formal statements, proofs, and instructions for reproducing the checks. Comparator matched the challenge statements to the solution. The recorded verification includes checking by Lean's kernel and independent replay by NanoDa.

The author's provenance correction of September 5, 2026 attributes the original argument, manuscript, formalization, and registration preparation to OpenAI's GPT-6 Astra, working from the author's prompts. The archived metadata records the earlier credits to GPT-5.6 Pro and ChatGPT Work for the proof and to GPT-6 Pro for packaging. OpenAI's GPT-6.2 Astra assisted with the present manuscript revision. The Lean development and Palomar record are those of the registered version.

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