A FINITE-COLOR PARTITION RELATION FOR ω_1^2

WILL HOWE, GRANT HUFFMAN, JENSEN KOHLMEYER, AND LIAM KRUER

Abstract. We prove in ZFC that

$$\omega_1^2 \longrightarrow (\omega_1 \cdot \omega, \underbrace{3, \ldots, 3}_{k})^2_{k+1} \qquad (k < \omega),$$

answering Erdős problem 1171. Under a Baire category hypothesis, a branch through a countable tree of reservoirs enables an induction on the positive colors. We then show that Cohen forcing supplies the category hypothesis and that a homogeneous set of order type $\omega_1 \cdot \omega$ in a ground-model coloring reflects to the ground model.

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1. Introduction

The binary case of the partition relation below is due to Erdős and Hajnal [4], and the case of two positive colors is due to Baumgartner and Hajnal [2]. Under MA($\aleph_1$), Baumgartner proved $\omega_1 \cdot \omega \to (\omega_1 \cdot \omega, 3)^2$ [1]. Applying that relation to each positive color in turn gives the finite-color conclusion under the same hypothesis. The question for arbitrary finite palettes in ZFC appears as problem 7.84 in the 1999 booklet *Some of Paul's Favorite Problems* [7] and as problem 1171 in Bloom's catalogue [3]. See Komjáth [6, pp. 442–443] for a discussion of the small palettes.

Theorem 1.1. *For every $k < \omega$, every coloring $c : [\omega_1^2]^2 \to \{0, 1, \ldots, k\}$ has a color-0 homogeneous set of inherited order type $\omega_1 \cdot \omega$, or a monochromatic triangle in one of the colors $1, \ldots, k$.*

The proof first reduces to colorings whose positive edges satisfy a condition on the row and column coordinates. A category argument then produces a color-0 set in these colorings. The final step transfers this conclusion from a Cohen extension to the ground model.

2. Rows and the reduction to nested edges

Put $\Omega = \omega_1^2$ and $B = \omega_1 \cdot \omega$. Write (a, α) for the point $\omega_1 a + \alpha$, where $a, \alpha < \omega_1$. The row of this point is a, and its column is α. Thus the ordinal order on Ω is lexicographic, with the row compared first. A *thick slice* is an uncountable subset of one row. A *B-stock* is a union of thick slices in a strictly increasing sequence of rows of length ω. For distinct points x, y, write $c(x, y) = c(\{x, y\})$.

A set contains an inherited copy of B if and only if it has infinitely many thick rows. In that case its first ω thick rows give a B-stock, since every thick slice has order type ω_1. For the converse, suppose it has exactly $m < \omega$ thick rows. All rows before or between those thick rows contribute countable order types, which are absorbed by the following thick slice. The remaining tail is an ω_1-sum of countable order types, of order type at most ω_1. Thus the set has order type at most $\omega_1(m + 1) < B$. In particular, a color-0 copy of B exists precisely when there is a color-0 B-stock.

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Call a set *full* if it has uncountably many thick rows and *small* otherwise. Small sets form a σ-ideal. Indeed, in a countable union of small sets, any row outside the countable union of their thick-row sets still has a countable slice. A tail of a full set above any fixed row is full.

Lemma 2.1 (Thinning)**.** *Let $\mathcal{U}$ be a family of thick slices in distinct rows. Specify forbidden unordered edges between different slices. If each point has at most countably many forbidden neighbors in every other slice, then each slice has a thick subslice and the union of the subslices contains no forbidden edge.*

Proof. There are at most $\aleph_1$ slices. Schedule $\aleph_1$ choices from each slice in a recursion of length ω_1. At a given stage there are only countably many previous choices. In the current slice, avoid the points already chosen and the forbidden neighbors of all points chosen in other slices. This excludes only countably many points. Each slice receives $\aleph_1$ distinct choices, and any forbidden edge would have been excluded when its second endpoint was chosen. □

Lemma 2.2. *Let G be a triangle-free graph on a B-stock. Suppose each vertex has at most countably many neighbors in any fixed earlier row of the stock. Then G has an independent B-stock.*

Proof. Suppose there is no independent B-stock. The neighborhood of a vertex is independent, so it is thick in only finitely many rows. For each row of the stock, pass to a thick slice on which the finite set of later rows with thick neighborhoods is constant. This is possible because the countable row set has only countably many finite subsets.

Choose an increasing infinite subsequence of the rows by avoiding, at each step, the finitely many later rows marked by the previously chosen rows. Between any two chosen slices the degrees in both directions are now countable. Within each slice we can also find a thick independent set. If a vertex has an uncountable neighborhood in the slice, that neighborhood is independent. Otherwise an ω_1-recursion gives an independent set, since at each stage only countably many vertices are excluded by the earlier choices. After making each slice independent, apply Lemma 2.1 to the cross edges. This gives an independent B-stock, a contradiction. □

A coloring is *nested* if every positive edge from (a, α) to (b, β) with $a \leq b$ satisfies

$$a < b < \beta < \alpha.$$

In a nested coloring each row is color-0 homogeneous, and each point has at most countably many positive neighbors in any fixed later row.

Lemma 2.3. *If there is a counterexample to Theorem 1.1 for a given k, there is a nested counterexample for that same k.*

Proof. Given a counterexample c, recolor every nonnested edge with color 0. The new coloring cannot acquire a positive monochromatic triangle. Suppose it has a color-0 B-stock. In each slice, discard the points whose column is at most their row, leaving a thick slice. For retained points $(a, \alpha), (b, \beta)$ with $a < b$, an originally positive edge must satisfy $\alpha \leq \beta$. Otherwise $a < b < \beta < \alpha$, so the edge would have remained positive. For a fixed later point, its original positive neighbors in any fixed earlier row thus lie in a countable set of columns.

Apply Lemma 2.2 successively to the graphs of colors $1, \ldots, k$ of the original coloring, retaining a B-stock at each step. The graphs are triangle-free and the backward degree bound persists under restriction. The final stock is color-0 homogeneous for c, contradicting the choice of c. □

3. Full sets and backward incidence

Fix a nested coloring c with positive palette $K = \{1, \ldots, k\}$. Suppose it has neither a positive monochromatic triangle nor a color-0 B-stock. The case $k = 0$ is immediate, so we may take $k \geq 1$.

Lemma 3.1. *There is a full set $X \subseteq \Omega$ such that, for every thick slice $U \subseteq X$ and every full $Z \subseteq X$ entirely above the row of U, there is a positive edge joining U to Z. For such an X and any thick slice $U \subseteq X$, the set of later $y \in X$ for which*

$$\{x \in U : c(x, y) \in K\}$$

is countable is small.

Proof. If no full set has the first property, start with Ω. Choose a thick slice and a full later subset with no positive cross edge between them, then repeat inside that later set. The chosen slices lie in strictly increasing rows and have only color-0 edges between them. Each slice also has only color-0 edges within its row, by nesting. This gives a color-0 B-stock.

Fix X with the first property. If the exceptional set in the second assertion is full, choose one thick slice in each of its thick rows. Use Lemma 2.1 on these slices and U, with the positive edges incident to U as the forbidden edges. The backward degrees into U are countable by the definition of the exceptional set, and the forward degrees into each later slice are countable by nesting. The thinned slice of U and the full union of the other thinned slices contradict the first property. □

Since small sets form a σ-ideal, for any countable family of thick slices of X one can exclude one small set and require every remaining point above the slices to have uncountably many positive neighbors in each of them.

4. The category argument

The category hypothesis is that every family of at most $\aleph_1$ dense open subsets of ω^ω has nonempty intersection.

Proposition 4.1. *Assume the category hypothesis. Then the conclusion of Theorem 1.1 holds.*

Proof. Suppose there is a counterexample. Apply Lemma 2.3 and choose X as in Lemma 3.1. We introduce the data used in the induction.

For a B-stock $A \subseteq X$, write $R(A)$ for its row set and A_a for its slice in row a. If $y \in X$ is above all its rows and $\ell \in K$, put

$$M_\ell(A, y) = \{a \in R(A) : |\{x \in A_a : c(x, y) = \ell\}| = \aleph_1\}.$$

We say that the ℓ-fiber at y is *thick* on row a if $a \in M_\ell(A, y)$, and *thin* otherwise. With A we use a proper ideal J on $R(A)$ containing all finite sets, a set $L \subseteq K$, and a cutoff $\gamma < \omega_1$ bounding the rows of A. We require

$$M_\ell(A, y) \in J \qquad \text{for all } \ell \in L \text{ and } y \in X \text{ with } \mathrm{row}(y) > \gamma. \tag{4.1}$$

A row set outside J is J-positive. On a substock whose row set S is J-positive, the restriction $J \restriction S = \{E \subseteq S : E \in J\}$ is proper. Restriction to such a substock preserves (4.1), since thinning cannot enlarge any mark set on the retained rows.

Data satisfying (4.1) cannot have $L = K$. To see this, take a point $y \in X$ above γ and outside the countable union of the exceptions from Lemma 3.1 for the slices of A. Every slice contains uncountably many positive neighbors of y, so in every row some positive color has a thick fiber. Thus $R(A) = \bigcup_{\ell \in K} M_\ell(A, y) \in J$, contradicting properness.

We prove by induction on $r \le k$ that every full $Y \subseteq X$ contains a B-stock with data satisfying (4.1) and $|L| \ge r$. For $r = 0$, choose any B-stock in Y, take J to be the ideal of finite sets, and let $L = \varnothing$. We give the induction step in three parts.

A countable tree of reservoirs. Assume the assertion for r and fix a full $Y \subseteq X$. The root reservoir is Y. At each node s choose, in its full reservoir Y_s, a stock A_s and data J_s, L_s, γ_s given by the induction

hypothesis. For each $a \in R(A_s)$ set

$$C_{s,a} = \{y \in Y_s : \mathrm{row}(y) > \gamma_s,\ a \notin M_\ell(A_s, y) \text{ for all } \ell \in L_s\}. \tag{4.2}$$

If $C_{s,a}$ is full, include a child indexed by a with that reservoir. The row index distinguishes children even when their reservoirs coincide. Each level is countable and there are only countably many levels. Along a branch, every stock lies above the preceding stock's cutoff.

Choose $\gamma_* < \omega_1$ bounding all the γ_s. From the full tail of Y above γ_*, exclude all nonfull classes $C_{s,a}$ and all the exceptional sets in Lemma 3.1 for the slices of all A_s. These are countably many small sets. Fix a point h that remains.

At a node s with $h \in Y_s$, the rows on which every color in L_s is thin at h form a J_s-positive set. Its complement is a finite union of sets in J_s. On each of these rows, some color in $K \setminus L_s$ is thick at h, because h has uncountably many positive neighbors in each slice and K is finite. Since J_s is closed under finite unions, there is some $i_s \in K \setminus L_s$ for which the set D_s of rows having both properties is J_s-positive. For every $a \in D_s$ we have $h \in C_{s,a}$, and $C_{s,a}$ is full by the choice of h. Starting at the root, keep just these children. The resulting tree branches countably infinitely at every node, since the finite sets belong to J_s. Enumerating the children at each node identifies its branch space, with the cylinder topology, with ω^ω.

A branch meeting all finite requirements. A node s has type (L_s, i_s). For any finite set $F \subseteq X$ of points above γ_*, at each node s one can choose a row in D_s on which all fibers at points of F in colors in L_s are thin. Indeed, the union of the mark sets $M_\ell(A_s, y)$ for $y \in F$, $\ell \in L_s$, belongs to J_s by (4.1), whereas D_s is J_s-positive.

For each such F, each type τ, and each $n < \omega$, impose the following requirement on a branch. At a node of type τ of depth at least n, it chooses a row with all these fibers thin, or it has a prefix with no extension node of type τ at depth at least n. Here an extension may be the prefix itself. Both alternatives are witnessed by finite prefixes, so the requirement is open. It is dense because from any given prefix we can extend to a node of the specified type and depth and make the required choice, if such a node exists. If none exists, the prefix already witnesses the second alternative.

There are at most $\aleph_1$ requirements because X has cardinality $\aleph_1$ and there are finitely many types. The category hypothesis gives a branch meeting all of them. For any type occurring infinitely often along this branch, the second alternative is impossible. Hence for every finite F as above, the branch makes the simultaneous thin choice at arbitrarily late occurrences of that type.

Increasing the set of marked colors. Choose a type (L, i) that occurs infinitely often on this branch. At each occurrence take, in its chosen row, the thick i-fiber at h. These slices form a B-stock in Y. Its vertices are all i-neighbors of h, so it contains no i-edge.

For a retained slice chosen at node s in row a, every vertex of a later retained slice lies in $C_{s,a}$. Thus (4.2) gives countable backward degrees into the earlier slice for every color in L. Nesting gives countable forward degrees into each later slice. Apply Lemma 2.1 to remove all cross edges in colors from L. Denote the resulting B-stock by A. Its positive internal colors lie in $K \setminus (L \cup \{i\})$.

For every finite $F \subseteq X$ above γ_*, there are infinitely many rows of A outside all $M_\ell(A, y)$ with $y \in F$ and $\ell \in L$. These are rows at which the branch made the simultaneous thin choice, a property preserved by taking subsets of the slices. Let J be the ideal on $R(A)$ generated by the finite sets and all $M_\ell(A, y)$ for $\ell \in L$ and $\mathrm{row}(y) > \gamma_*$. Every finite union of generators has infinite complement, so J is proper.

Among the substocks of A on J-positive row sets choose one, A', with the fewest positive internal colors. Write P for its positive palette and $J' = J \restriction R(A')$. Any substock of A' on a J'-positive row set uses every color of P, by minimality. Also $P \neq \varnothing$, since otherwise A' is color-0 homogeneous. For $p \in P$ and $y \in X$ above γ_*, suppose $M_p(A', y) \notin J'$. The thick p-fibers at y in those rows form a substock on a J'-positive row set. It contains no p-edge, by triangle exclusion. This contradicts minimality, so $M_p(A', y) \in J'$.

The mark condition therefore holds on A' for $L \cup P$ with cutoff γ_*. Since P is nonempty and disjoint from L, we have $|L \cup P| \geq r + 1$. This completes the induction. At $r = k$ we obtain data with $L = K$, which was shown to be impossible. □

5. Cohen forcing and reflection

For an index set I, let $\mathbb{P}_I$ consist of finite partial functions $I \times \omega \to \omega$, ordered by extension, with a larger function a stronger condition. By the finite Δ-system lemma, every uncountable family of conditions contains an uncountable subfamily whose domains form a Δ-system and whose conditions agree on its root. If one domain occurs uncountably often, restrict to that domain and then to a common assignment. Otherwise choose uncountably many distinct domains, form a Δ-system, and restrict to a common assignment on its finite root. In either case, the subfamily is pairwise compatible. In particular, $\mathbb{P}_I$ is ccc and preserves ω_1 and ω_2.

Lemma 5.1 (Reflection)**.** *If $\mathbb{P}_I$ adds a color-0 B-stock for a ground-model coloring of $[\Omega]^2$, then that coloring already has a color-0 B-stock in the ground model.*

Proof. Work in the ground model. Let p force that $\dot{H}$ is a color-0 B-stock. Set $r_{-1} = p$. For each $n < \omega$, choose q_n extending r_{n-1} that decides the n-th row of $\dot{H}$, say as a_n. There are uncountably many distinct ground vertices x in row a_n for which some $p_{n,x}$ extending q_n forces $x \in \dot{H}$. If there were only countably many possible members, q_n would force the slice into that countable ground set, contrary to thickness.

Apply the Δ-system argument to the domains of these witness conditions, retaining an uncountable family with a common root assignment r_n. Each domain contains $\mathrm{dom}(q_n)$, so r_n extends q_n. The petals in this family are pairwise disjoint. The recursion yields increasing roots r_n, and the decided rows a_n are strictly increasing because each q_n extends the earlier deciding conditions.

Let $D = \bigcup_{n<\omega} \mathrm{dom}(r_n)$, a countable set. In each row family discard the at most countably many witnesses whose petals meet D. In a recursion of length ω_1, select uncountably many of the remaining witnesses in each row while keeping the petals pairwise disjoint across all the selections. At each step only countably many coordinates have been used, each of which excludes at most one witness in the current row family. Also exclude the countably many witnesses already selected.

The resulting conditions are pairwise compatible. Their roots agree on overlaps, and all petals are disjoint from D and from each other. A common extension of any two witnesses forces both of their vertices into $\dot{H}$. The ground color of their pair must therefore be 0. The selected vertices give a ground-model color-0 B-stock. □

This argument uses compatible witness conditions, as in Golshani's preservation analysis for a full-square target [5, Section 2]. Here the increasing root sequence handles the countably many thick rows required for the order type B.

Lemma 5.2. *For $I = \omega_2$, the forcing $\mathbb{P}_I$ forces the category hypothesis of Section 4.*

Proof. In the extension, let a family of at most $\aleph_1$ dense open sets be given. Index it by a sequence of length at most ω_1. Code each open set U by the set of finite strings $s \in \omega^{<\omega}$ whose basic cylinder is contained in U. The indexed codes have at most $\aleph_1$ membership bits in total. Working with a name for the sequence of codes, take a maximal antichain deciding each membership bit. These antichains are countable by the ccc, and the conditions on them use only finitely many coordinates each. Thus there is $S \subseteq I$ of size at most $\aleph_1$ in the ground model such that all the codes belong to the intermediate extension by $\mathbb{P}_S$. If an initial condition is used to force the specified indexing, include its coordinates in S.

Choose $i \in I \setminus S$. The product factorization of $\mathbb{P}_I$ makes the function at coordinate i Cohen-generic over the intermediate extension. The density test for a code is that every finite string has an extension

in the code. This test is absolute between the intermediate and full extensions. The Cohen function at i extends a string in each code. Thus in the full extension it belongs to every open set in the family. □

Proof of Theorem 1.1. Suppose a ground-model coloring is a counterexample. Force with $\mathbb{P}_{\omega_2}$. The ordinal square is unchanged, and a positive monochromatic triangle would already be present in the ground model, since it consists of three ground vertices with fixed edge colors. In the extension, Lemma 5.2 and Proposition 4.1 give a color-0 homogeneous set of order type B, hence a color-0 B-stock. By Lemma 5.1, such a stock exists in the ground model. This is the desired contradiction. □

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Email address: howew@purdue.edu

Email address: huffma97@purdue.edu

Email address: jenwin@purdue.edu

Email address: kruerl@purdue.edu