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THE SHARP CONSTANT IN THE ERDŐS MAXIMUM-TERM PROBLEM

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Abstract. For a transcendental entire function f, let $\mu(r,f)$ be its maximum term and $M(r,f)$ its maximum modulus. We determine the supremum of $\liminf_{r\to\infty}\mu(r,f)/M(r,f)$ as the reciprocal of the minimum circle norm in the He–Tang theta family. The minimum is characterized by a unique solution of three real equations, and the supremum is attained. The proof combines coefficient transport at radii where two terms tie, a classification of the resulting gap sequences, and a uniform quadratic comparison for arithmetic gaps. The local comparisons use exact finite arithmetic.

1. The maximum-term problem

Given $f(z)=\sum_{n\ge0}a_nz^n$, set

$$\mu(r,f)=\max_{n\ge0}|a_n|r^n,\qquad M(r,f)=\max_{|z|=r}|f(z)|,$$
$$Q(f)=\limsup_{r\to\infty}M(r,f)/\mu(r,f).$$

Erdős Problem 513 asks for the supremum $B=\sup_f Q(f)^{-1}$ over transcendental entire functions [1]. Here the reciprocal of $+\infty$ is zero. We construct an extremizer in the two-parameter family of He and Tang [2], which extends the scaling construction of Clunie and Hayman [3]. The upper bound requires comparisons valid for arbitrary entire functions.

Theorem 1.1. *Put $T_n=n(n+1)/2$, $e(t)=(t^2-1+2it)/(1+t^2)$, and define $S_0=1$, $S_1(y)=4y-3$, and $S_{n+1}=(4y-2)S_n-S_{n-1}$. For*

$$R(q,t,y)=4y\left|\sum_{n\ge0}(qe(t))^{T_n}S_n(y)\right|^2$$

let (q_*,t_*,y_*) denote the unique solution of

$$\begin{aligned}R_y(q,t,y)&=0,\\ R(q,t,1)&=R(q,t,y),\\ R_q(q,t,1)R_t(q,t,y)&=R_t(q,t,1)R_q(q,t,y)\end{aligned}\tag{1.1}$$

in the box

$$\mathcal{B}=[0.28010,0.28040]\times[0.43450,0.43472]\times[0.633,0.643].\tag{1.2}$$

Then $B=1/A_*\approx0.58507882$, where $A_*=\sqrt{R(q_*,t_*,1)}$ and $1.70917142<A_*<1.70917143$. The function $f_{q_*,e(t_*)}$ defined below attains B.

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2. A bilateral scaling identity

Write $T_n = n(n+1)/2$ for $n \in \mathbb{Z}$. For $0 < q < 1$ and $|\epsilon| = 1$, define

$$f_{q,\epsilon}(z) = \sum_{n\geq 0} q^{T_n}\epsilon^{n(n-1)/2}z^n, \qquad k(z) = \sum_{n\in\mathbb{Z}} q^{T_n}\epsilon^{n(n-1)/2}z^n.$$

The first series is transcendental entire and the second converges uniformly on compact annuli. For $K = q^{-1}$ their coefficients give

$$k(Kz) = z\,k(\epsilon z), \qquad k(K^m z) = K^{m(m-1)/2}\epsilon^{m(m-1)/2}z^m k(\epsilon^m z).$$

At $r = K^m$, $m \geq 1$, the maximum term of $f_{q,\epsilon}$ is $K^{m(m-1)/2}$, attained at $n = m-1, m$. On this circle the discarded negative-degree part is bounded by $\sum_{j\geq 1} K^{-mj}$, so it tends uniformly to zero even before normalization. Thus M/μ tends to $\|k\|_{|z|=1}$ along the radii K^m. Between two consecutive such radii, $\log\mu$ is affine in $\log r$ and $\log M$ is convex by the three-circles theorem. The ratio is bounded above by the greater endpoint ratio. Consequently $Q(f_{q,\epsilon}) = \|k\|_{|z|=1}$.

Parameterize the phase by

$$e(t) = \frac{t^2 - 1 + 2it}{1+t^2}.$$

Writing $z = \epsilon e^{2ix}$ and multiplying by e^{ix} pairs the indices n and $-n-1$ in k. This gives $2\sum_{n\geq 0}(q\epsilon)^{T_n}\cos((2n+1)x)$. Define polynomials $S_0 = 1$, $S_1(y) = 4y - 3$, and $S_{n+1}(y) = (4y-2)S_n(y) - S_{n-1}(y)$. The cosine addition formula gives $\cos((2n+1)x) = \cos(x)S_n(\cos^2 x)$ for every x. Hence

$$Q(f_{q,e(t)})^2 = \max_{0\leq y\leq 1} R(q,t,y), \qquad R(q,t,y) = 4y\left|\sum_{n\geq 0}(qe(t))^{T_n}S_n(y)\right|^2. \tag{2.1}$$

3. An explicit rational norm bound

Take

$$q = \frac{200000000000000}{713637088446737}, \qquad t = \frac{4346017547}{10000000000}, \qquad U = \frac{17091714299}{10^{10}}.$$

We give a finite rational inequality for the polynomial $R_8(y)$ obtained from $R(q,t,y)$ by retaining $0 \leq n \leq 8$. In the remainder of this section write $q = a/b$, $t = u/v$, $D = b(u^2+v^2)$, and $W = a(u^2 - v^2 + 2iuv)$. Form the Gaussian-integer polynomial

$$L(y) = \sum_{n=0}^{8} W^{T_n}D^{36-T_n}S_n(y), \qquad \sum_{i=0}^{17} C_i y^i = 4yL(y)\overline{L(y)}.$$

Here conjugation acts on the coefficients. Then $R_8(y) = D^{-72}\sum_i C_i y^i$.

For integers d, j in the table below, use the interval $[j/2^d, (j+1)/2^d]$. Its Bernstein coefficient of index k for $\sum_i C_i y^i$ is

$$\sum_{m=0}^{k}\frac{\binom{k}{m}}{\binom{17}{m}}\sum_{i=m}^{17} C_i\binom{i}{m}j^{\,i-m}2^{-di}. \tag{3.1}$$

This follows by substituting $y = (j+s)/2^d$ and expanding each power s^m in the degree-17 Bernstein basis. In each row, all the coefficients (3.1) are bounded above by $D^{72}U^2$. The table lists the interval

indices for these integer comparisons.

d	j	d	j
1	0	7	80
2	3	8	162
3	4	9	327
4	11	10	652
5	21	11	1306
6	41	11	1307

For an integer form of the inequalities, let $L_0 = \operatorname{lcm}_{0\le m\le 17}\binom{17}{m}$. For each row and each $0 \le k \le 17$, multiply the left side of (3.1) by $L_0 2^{17d}$, replacing $\binom{k}{m}/\binom{17}{m}$ with $\binom{k}{m}(L_0/\binom{17}{m})$ and 2^{-di} with $2^{d(17-i)}$. Multiplying this integer by 10^{20} gives at most $L_0 2^{17d} D^{72}(17091714299)^2$. Equivalently the comparisons follow by starting with coefficients $L_0 \sum_{i\le k} C_i\binom{k}{i}/\binom{17}{i}$ and repeatedly using adjacent sums for midpoint subdivision, multiplying by 2^{17} in the common denominator at each subdivision. The intervals have disjoint interiors and cover $[0,1]$. The Bernstein basis is nonnegative and sums to one on each interval, giving $R_8(y) \le U^2$.

In the cosine representation the omitted part has sup norm at most $2q^{45}/(1-q^{10}) < 10^{-20}$, since $q < 3/10$ and $T_{n+1} - T_n = n+1$. Thus $Q(f_{q,e(t)}) \le U + 10^{-20} < 1.70917143$. This supplies the upper side of the claimed enclosure for the minimum norm in the family.

4. The theta minimax

The norm is invariant under conjugating the phase. The upper semicircle is covered by $(1+it)/(1-it)$ and $e(t)$, $0 \le t \le 1$. For $q \le 1/8$, evaluation at $x = 0$ and the triangle inequality give a norm of at least $2(1 - q/(1-q^2)) \ge 110/63$. For $q \ge 7/10$, Parseval gives squared norm at least $2 + 2q^2 \ge 2.98$. Thus the preceding rational upper bound permits restriction to the two compact rectangles $1/8 \le q \le 7/10$, $0 \le t \le 1$. The norm is continuous there by uniform convergence.

We detail the finite inequalities used in the minimization. In this paper decimal endpoints designate exact rationals. Interval operations enclose the full range of each operation. A directed fixed-point enclosure at scale S rounds lower endpoints down and upper endpoints up to multiples of $1/S$. For squaring, use lower endpoint zero if the input straddles zero, and otherwise the smaller endpoint square. We use rational arithmetic for the following two-parameter covers.

Let B be a parameter rectangle, with center (q_c, t_c), half-widths h_q, h_t, and $b = \max_B q$. Write $e_0(t)$ for the arc in use and $P_x(w) = 2\sum_{n=0}^{8} w^{T_n}\mathcal{T}_{2n+1}(x)$, with $\mathcal{T}_0 = 1, \mathcal{T}_1 = x, \mathcal{T}_{j+1} = 2x\mathcal{T}_j - \mathcal{T}_{j-1}$. For a rational $x \in [0,1]$, set $V = |P_x(qe_0(t))|^2$. Let a_j be the sum obtained by taking absolute coefficients in $P_x^{(j)}$ and evaluating at b, $j = 0, 1, 2$. Bounds for $|V_{qq}|, |V_{qt}|, |V_{tt}|$ are

$$2(a_1^2 + a_0a_2), \quad 2(2ba_1^2 + a_0(2ba_2 + 2a_1)), \quad 2(4b^2a_1^2 + a_0(4b^2a_2 + 4ba_1)).$$

They follow from $|e_0'| \le 2$, $|e_0''| \le 4$. Denote this triple by J. For the tuple $D = (V, V_q, V_t, J)$, evaluated at the center except for J, put

$$L(D) = V - |V_q|h_q - |V_t|h_t - J_{qq}h_q^2/2 - J_{qt}h_qh_t - J_{tt}h_t^2/2.$$

Taylor's theorem gives $V \ge L(D)$ on the rectangle. For two tests use also $L(\alpha D_0 + (1-\alpha)D_1)$ with $\alpha = -D_{1,j}/(D_{0,j} - D_{1,j}) \in (0,1)$, $j = q, t$. This bounds the corresponding weighted squared value and hence the larger of the two squared values. A test rejects the rectangle at threshold c if its lower bound exceeds $(c + 2b^{45}/(1-b^{10}))^2$.

Here are the rectangle tests and their counts. Bisect in q if $h_q \ge 3h_t/5$, and in t otherwise. In the first cover test 1 and the interior sampler attaining the largest center value among $\{j/80 : 41 \le j < 80\} \cup \{.794, .796, .798, .800, .802, .804\}$, using also the two convex combinations above. Keep

a rectangle lying in $B_1 = [.27, .29] \times [.429, .441]$ on the arc $e(t)$. For each of the two refinement covers start at B_1, test 1 and an interior sampler chosen at the center by maximizing the truncated value over $350/500, \dots, 450/500$. Bracket the best point with its two adjacent grid points (clipping at the grid ends), separate the bracket into thirds, retain the two thirds toward the better interior division point, and repeat forty times. Round the midpoint to the nearest multiple of 10^{-10}. Only the rational lower inequalities at the final sample points are used. Keep a rectangle in $B_2 = [.28015, .28036] \times [.43454, .43467]$ in the upper refinement.

initial region	c	rejected	kept	depth	lower margin on rejections
both arcs	1.70919	149	2	15	$3.58\,10^{-5}$
B_1	1.70917142	75	0	18	$4.72\,10^{-9}$
B_1	1.70917143	73	6	19	$6.06\,10^{-9}$

All row margins are for the squared test after the tail allowance. A parent is the union of its two children, so the counts and positive comparisons give $A_* > 1.70917142$ and place every minimizer in the interior of the parameter part of (1.2).

For the angular and derivative inequalities, keep $0 \le n \le 8$ and differentiate the recurrences for S_n and $(qe(t))^{T_n}$. For example the recursion for (S_n, S_n', S_n'') is multiplication by $4y - 2$, subtraction of the preceding triple, and addition of $(0, 4S_n, 8S_n')$. The absolute coefficient sum of S_n is bounded by 7^n. On the parameter rectangle, all needed partial derivatives of the omitted complex sum are bounded by

$$\sum_{n \ge 9} 16(T_n + 1)^2 n^2 7^n (3/10)^{T_n} < 10^{-8}.$$

Indeed the ratio of consecutive terms is at most $112(3/10)^{10} < 1/2$, and twice the first term is smaller than 10^{-8}. The bounding factor $16(T_n + 1)^2$ follows by twice differentiating with $q \ge 1/4$ and the arc derivative bounds. Add $[-10^{-8}, 10^{-8}]$ to each real and imaginary sum before differentiating $R = 4y|S|^2$ by the product rule on the enclosed sums. The following are lower bounds using interval operations at scale 2^{60}. Bisect the displayed y-interval until the signed quantity has positive lower endpoint. The q, t rectangle stays fixed.

y	quantity	leaves	lower bound
$[0, .62]$	R_y	8	.00032
$[.62, .66]$	$-R_{yy}$	1	3.93
$[.66, .86]$	$-R_y$	16	.0152
$[.86, .90]$	R_{yy}	1	4.03
$[.90, 1]$	R_y	5	.0389
$\{.633\}$	R_y	1	.0205
$\{.643\}$	$-R_y$	1	.0188

It follows that only the endpoint 1 and a unique interior maximum $Y(q, t)$ in $(.633, .643)$ can attain the global angular maximum. The implicit function theorem gives smoothness because $R_{yy} < 0$ there.

Write $F_0 = R(q, t, 1)$, $F_1 = R(q, t, Y(q, t))$, $D_0 = F_0 - F_1$. The same enclosure with $y \in [.633, .643]$ gives $(F_0)_t > 0 > (F_1)_t$, $(D_0)_q < 0 < (D_0)_t$. In this interval evaluation the first derivatives of F_1 are R_q, R_t, and its Hessian entries are $R_{ij} - R_{iy}R_{jy}/R_{yy}$, $i, j \in \{q, t\}$. Set $s = -(D_0)_q/(D_0)_t$ and $\lambda = -(F_1)_t/(D_0)_t$. Substitution gives $0 < \lambda < 1$ and

$$(1, s)\big(\lambda \nabla^2 F_0 + (1 - \lambda)\nabla^2 F_1\big)(1, s)^{\mathsf{T}} > 18.75. \tag{4.1}$$

For the interval evaluation of the F_1 term, subtract $(R_{qy} + sR_{ty})^2/R_{yy}$ from $R_{qq} + 2sR_{qt} + s^2 R_{tt}$. The lower endpoint for (4.1) exceeds 18.7552.

At an interior minimizer both branches must be active, since moving t slightly in the decreasing direction of a sole active branch would lower the maximum. Between any two points of $D_0 = 0$ the locus is a graph $t = t(q)$ entirely in the rectangle. In fact $t'(q) = s > 0$, and at every intermediate

q the two t-values bracket the unique zero by the signs of the derivatives of D_0. Differentiating the common value on this graph twice gives precisely (4.1), as the choice of λ cancels the t'' term. The first derivative vanishes precisely at the last equation of (1.1). Existence follows from the interior global minimizer. Strict convexity along the graph gives uniqueness, and the first equation specifies its unique interior angular maximum. This proves the restricted minimax and its enclosure.

5. Reduction for arbitrary entire functions

Assume $Q(f) < A_*$, and put $C_0 = 85459/50000$ and $C_1 = 171/100$. By the norm enclosure, there is an eventual strict norm-ratio cap less than C_0. The upper envelope of the affine functions $\log|a_n| + n\tau$ with $a_n \neq 0$, where $\tau = \log r$, is locally the maximum of finitely many such functions by absolute convergence. The maximizing indices tend to infinity because the support is unbounded. The transition radii therefore tend to infinity, by local finiteness of the upper envelope. At every sufficiently late transition exactly two terms tie, since three tied terms would give squared norm ratio at least three by Parseval. Denote the increasing sequence of exposed indices by n_k, the double log-radius transitions by τ_k, and define

$$d_k = n_k - n_{k-1}, \qquad \Delta_k = \tau_{k+1} - \tau_k.$$

The following comparisons, proved in Sections 6–12, give the unrestricted upper bound.

(1) Theorems 9.1 and 10.1 show that three successive double transitions at cap C_0 exclude both $d_{k-1} < d_k > d_{k+1}$ and $d_{k-1} = d_k > d_{k+1}$.
(2) By Theorem 6.1, for a decrease d, E with $E < d$, the cap $171/100$ bounds d times the log-length of the preceding plateau, uniformly in the relative size E/d.
(3) By Lemma 7.1, if the four residues $d_{k-1}, -d_{k-1}, d_{k+1}, -d_{k+1} \pmod{d_k}$ are distinct and nonzero, the two-sided test below has $N_k > (43/25)\|T_k\|_1$. When $d_{k-1} < d_k < d_{k+1}$, $d_{k-1} + d_{k+1} = 2d_k$, and $d_k/\gcd(d_k, d_{k-1}) > 10000$, the same test has $N_k > (12/7)\|T_k\|_1$.
(4) By Theorem 12.1, on arbitrarily late runs of equal exposed gaps of unbounded length, one has $Q(f) \geq A_*$.

We prove that these comparisons yield the unrestricted upper bound. The first two comparisons are used also in reversed finite windows. Indeed, a long Taylor polynomial retains every stipulated maximum term at finitely many selected radii and approaches the norms there uniformly. The strict cap allows this truncation. For a Taylor polynomial p of degree m, $z^m p(1/z)$ reverses the exponents, and its M/μ at r^{-1} equals that of p at r. The local comparisons then apply in the reverse order.

A late decrease must be preceded by a decrease by the first comparison. Positive integer gaps cannot decrease indefinitely. The gaps are consequently eventually nondecreasing. Once a late strict increase occurs, the reflected equal-pair exclusion makes the next comparison strict as well. Thus the gaps are eventually constant or eventually strictly increasing.

In the increasing case, comparison 2 applied in reverse to $d_{k-1} < d_k$ bounds $d_k\Delta_k$, since the preceding plateau in the reversed order is the one following gap d_k in the original order. Since $\sum_k \Delta_k = \infty$, this entails $\sum_k 1/d_k = \infty$. For any fixed length and any $\varepsilon > 0$ there are arbitrarily late blocks of that length in which every successive ratio is at most $1 + \varepsilon$. Otherwise every sufficiently late block of that length contains a ratio above $1 + \varepsilon$, giving geometric growth and a convergent reciprocal sum.

Take 224 interior knots in such a block with $\varepsilon = 10^{-4}$. At one such knot write $P = d_{k-1} < D = d_k < R = d_{k+1}$. The four residues are distinct and nonzero unless $P + R = 2D$, by the ratio bounds. In that exception $D/\gcd(D, P) \geq D/(D - P) > 10000$. Thus comparison 3 gives $N_k > (12/7)\|T_k\|_1$ at each interior knot.

For clarity, let $H_k(\theta)$ denote the circle function at τ_k divided by the lower tied complex term. Write $H_k = \sum_j h_k(j)e^{ij\theta}$, so $h_k(0) = 1$, $h_k(d_k) = A_k$ for a unit complex A_k. Set $z_k = A_k e^{id_k\theta}$,

$x_k = e^{-d_{k-1}\Delta_{k-1}}$, and $y_k = e^{-d_{k+1}\Delta_k}$. The tests and their known contributions are

$$T_k = \tfrac{1}{3}z_k^{-1}(1+z_k)^3 + \tfrac{1}{3}x_k(z_{k-1}^{-1} - z_{k-1}) + \tfrac{1}{3}y_k z_k(z_{k+1} - z_{k+1}^{-1}),$$
$$N_k = 2 + \tfrac{1}{3}(x_k^2 + y_k^2).$$

The coefficient transport is $h_{k+1}(j) = \overline{A_k} e^{j\Delta_k} h_k(j + d_k)$. It cancels $z_k^2/3$ at knot k against $-x_{k+1}z_k/3$ at knot $k+1$ in the real pairings. It also cancels $z_{k+1}^{-1}/3$ at knot $k+1$ against $-y_k z_k z_{k+1}^{-1}/3$ at knot k. The four unmatched boundary terms contribute in absolute value at most $4/3$. Hölder's inequality and $N_k \geq 2$ now require

$$2 \cdot 224\big(1 - C_0/(12/7)\big) \leq \tfrac{4}{3}.$$

The left side is $800576/600000 > 4/3$, a contradiction. The remaining case of an eventually constant gap gives $Q(f) \geq A_*$ by comparison 4, which contradicts the starting assumption. Together with the equality for $f_{q_*,e(t_*)}$, this proves Theorem 1.1.

6. A decrease preceded by a long plateau

Theorem 6.1. *Let $t < r < s$ be positive radii. Suppose that index N is maximal at t, r, that $N + d$ is also maximal at r, s, and that $N + d + E$ is maximal at s, with $0 < E < d$. If the normalized norms at r, s are at most C_0, then $(t/r)^d > 1/2500$. More generally, with any fixed cap strictly below C_1, the same quantity is bounded away from zero uniformly in d, E.*

Proof. Project at r, s to the two residue classes N and $N + E$ modulo d. At a fixed value of the high variable $z = e^{id\theta}$, orthogonality on the d root-of-unity shifts bounds the Euclidean norm of the component vector by the original cap. Use the Taylor coefficients divided by the positive maximum terms in each profile, with independent unit phases for the two components and a rotation of z fixing the first three displayed ties. Take real parts of the resulting coefficients by averaging the vector with its conjugate at $\bar{z}$. These operations commute with positive coefficient transport and preserve the anchors.

Delete the terms of high degree at most -1 in the first component and at most -2 in the second. Each deleted coefficient has normalized size at r at most $(t/r)^{N-j}$, where j is its original index. The same bound holds at s because the maximum term moves to $N + d$ and $j < N$. The distances $N - j$ in each class start at least at d and increase by d. The deletion cost at each knot is at most $2q/(1-q)$, $q = (t/r)^d$. If $q \leq 1/2500$, the cost is at most $2/2499 < C_1 - C_0$. With cap strictly below C_1, the cost fits the slack for all sufficiently small q. It suffices to exclude the modified vector profiles at cap C_1.

Write $x = (r/s)^d$, $\beta = (r/s)^E$, and multiply each vector by z to make it analytic. Its initial coefficients at the two radii have the form

	D	A	B	F
r	$(0, g)$	$(1, a)$	$(1, \beta)$	(c, e)
s	$(0, x^2 g/\beta)$	$(x, xa/\beta)$	$(1, 1)$	$(c/x, e/(x\beta))$.

All scalar coefficients have magnitude at most one, since every retained normalized Taylor coefficient has that bound. The entry g is retained from index $N + E - d$, whose distance before N can be small relative to d. Also $0 < x < \beta < 1$.

We first restrict x, β, c. Before multiplication by z, the first scalar component at r is $G(z) = 1 + z + cz^2 + uz^3 + \cdots$ with norm at most $C = C_1$. Let $A_0 = (C^2 - 1)^{-1}$, $L = C - 1/C$, $s_0 = 1 - 1/L^2$, $r_0 = Ls_0$, and $\xi = (c + A_0)/r_0$. Applying the Schwarz lemma to $(G - 1)/(C(1 - G/C^2))$ and dividing by z gives a unit-disk map with constant coefficient $1/L$. Apply its disk automorphism taking $1/L$ to zero and divide again by z. The resulting map has constant coefficient ξ and linear coefficient $(u + 2A_0c + A_0^2 + s_0\xi^2)/r_0$, by expansion. Schwarz–Pick yields

$$|\xi| \leq 1, \qquad |u - c_2(c)| \leq r_0(1 - \xi^2), \qquad c_2(c) = -2A_0c - A_0^2 - s_0\xi^2.$$

In particular $-2A_0 + \ell \le c \le -\ell$, with $\ell = 343889/1214100 = A_0 - r_0$.

At s the coefficient of degree j of the first scalar component, before multiplication by z, is the corresponding coefficient G_j times x^{1-j}. Parseval, with the two ties at s removed, implies for $p = x^2$ that

$$R_p = C^2 - 2 - p - c^2/p - u^2/p^2 \ge \sum_{j\ge 4} |G_j|^2/p^{j-1} \ge 0.$$

Take $z_0 = (3+4i)/5$ and multiply by $(24-7i)/25$ before taking the real part of $G(z_0)$. The real weights of its first four coefficients are $24/25, 4/5, 0, -4/5$. The tail contributes at most $\sqrt{R_p p^3/(1-p)}$, by Cauchy–Schwarz, so $1/20 - 4u/5 \le \sqrt{R_p p^3/(1-p)}$. If $p \le P = (41/100)^2$, the function R_p is increasing up to P because $|c| \ge \ell > P$. The factor $p^3/(1-p)$ also increases. These give the necessary inequalities with p replaced by P.

Here is the resulting two-variable arithmetic check. Start with $(c,u) \in [-1,0] \times [-1,1]$. For a rectangle use J for the interval of ξ^2, $[A,B]$ for the interval of $c_2(c)$, $b = r_0(1 - \inf J)$, $U = C^2 - 2 - P - (\inf c^2)/P - (\inf u^2)/P^2$, and $l = 1/20 - (4/5)\sup u$. Reject if $\inf J > 1$, if $[A-b, B+b]$ misses the u-interval, if $U < 0$, or if $l > 0$ and $P^3U/(1-P) < l^2$. The 41 rectangles in the supplementary table P (Table 1) cover the initial rectangle and each passes one of these rational tests. Thus $x > .41$. Since $x^2 + c^2/x^2 \le C^2 - 2$ and $p + \ell^2/p$ increases for $p > \ell$, its value at $p = .91^2$ implies $x < .91$. At r the two ties and the next exposed term give $\beta^2 \le C^2 - 2 < .97^2$.

For either analytic vector, compress multiplication by that vector from the input frequencies $0, 1, 2$ to the output frequencies $0, 1, 2, 3$. The norm is at most C by integration of the pointwise bound and orthogonal projection. Its columns are $(D, A, B, F), (0, D, A, B), (0, 0, D, A)$. Their squared norms and pairwise inner products give the Gram matrix G_{v}. Thus $C^2 I - G_{\mathrm{v}}$ has nonnegative principal minors. The following interval recurrence verifies that both sets of minors cannot be nonnegative at any point in the six-parameter box.

Use scale 2^{24}, ten slabs $x \in [(41+5j)/100, (46+5j)/100]$, $0 \le j < 10$, $\beta \in [.41, .97]$, $c \in [-2A_0 + \ell, -\ell]$, and $a, e, g \in [-1, 1]$. Discard a box if $\sup\beta < \inf x$. In each remaining box intersect the upper x bound with $\sup\beta$ and the lower β bound with $\inf x$ for evaluation. For a row $D = (0, g), A = (a_0, a_1), B = (b_0, b_1), F = (f_0, f_1)$, form the diagonal intervals

$$d_2 = C^2 - g^2 - |A|^2, \qquad d_1 = d_2 - |B|^2, \qquad d_0 = d_1 - |F|^2$$

and the positive-sign Gram entries

$$o_{12} = ga_1 + A \cdot B, \quad o_{01} = o_{12} + B \cdot F, \quad o_{02} = gb_1 + A \cdot F.$$

A negative upper endpoint of a diagonal rejects the box. In the remaining test intersect the diagonals with $[0, \infty)$ and reject on a negative upper endpoint of any $d_i d_j - o_{ij}^2$ or of

$$d_0 d_1 d_2 - 2o_{01}o_{02}o_{12} - d_0 o_{12}^2 - d_1 o_{02}^2 - d_2 o_{01}^2.$$

Use the first row before the second. For the second, intersect each transformed coefficient with $[-1, 1]$ and reject if any intersection is empty. Apply interval operations in the displayed order, using $o_{12} = ga_1 + a_0b_0 + a_1b_1$, $o_{01} = o_{12} + b_0f_0 + b_1f_1$, and $o_{02} = gb_1 + a_0f_0 + a_1f_1$. Bisect an unrejected original box on its widest coordinate, using the first such coordinate in the order x, β, a, c, e, g. Use the

lower midpoint on the fixed-point grid.

j	$\sup \beta < \inf x$	first matrix	empty	second matrix	depth
0	1247	110516	67	58309	36
1	3157	125088	64	59044	36
2	3537	87613	0	38368	35
3	1707	37405	0	16684	35
4	808	13589	0	6284	32
5	416	4953	0	1968	29
6	12	1687	0	540	26
7	11	535	0	107	23
8	4	105	0	6	15
9	6	38	0	0	13

Each branch of the subdivision ends in a rejection by the displayed inequalities. The two matrices are therefore incompatible. This excludes the modified profiles and proves the theorem. □

7. Two-sided tests for increasing gaps

Lemma 7.1. *Use T_k, N_k from Section 5 at a double knot of norm at most $\sqrt{3}$, and write P, D, R for its left, central and right gaps. If $P, -P, R, -R \bmod D$ are distinct and nonzero, then $N_k > (43/25)\|T_k\|_1$. If $P + R = 2D$ and $D/\gcd(D, P) \geq 2048$, then $N_k > (12/7)\|T_k\|_1$.*

Proof. Write $x = x_k,\ y = y_k,\ S = x^2 + y^2 \leq 1$ by Parseval. Put $P_0(z) = z^{-1}(1 + z)^3/3$. Averaging over the shifts by $2\pi/D$ fixes the central variable. In the separated case the other four terms are orthogonal and have zero mean on this fiber. Taking the square root of the mean square bounds their contribution together with the central polynomial by

$$I(S) = \int_0^{2\pi} \sqrt{|P_0(e^{i\theta})|^2 + 2S/9}\, \frac{\mathrm{d}\theta}{2\pi}.$$

Use $S \in [j/256, (j + 1)/256]$, the lower endpoint in $2 + S/3$ and the upper endpoint in $I(S)$. For $z = (1+it)/(1-it)$ and $-\bar{z}, 0 \leq t \leq 1$, the squared moduli of P_0 are $64/(9(1+t^2)^3)$ and $64t^6/(9(1+t^2)^3)$. Therefore $I(S)$ is $1/\pi$ times the integral over $[0, 1]$ of the sum of the two roots, each with $2S/9$ added under the root, times $2/(1 + t^2)$. Each root is 1-Lipschitz in $|P_0|$, is bounded by $19/6$, and the angular derivative of P_0 has modulus at most 2. With $2/(1 + t^2) \leq 2$ and the sharper bound $|(2/(1 + t^2))'| \leq 3$, the integrand is $124/3$-Lipschitz. At the 4096 midpoints $(j + 1/2)/4096$, take upward roots at scale 2^{32}, multiply by the rational weights and round upward. Add $(124/3)/(4 \cdot 4096)$ to the mean and divide by a lower bound for π from Machin's identity. Substitution in $2 + j/768 - (43/25)I((j + 1)/256)$ gives a lower margin exceeding 0.0051476 in all 256 tests.

For $P + R = 2D$, write $z = z_k,\ w = z_{k-1}$. The third phase is $z_{k+1} = \gamma z^2 w^{-1}/y$ for $|\gamma| = y > 0$, and the test equals

$$T(z, w) = P_0(z) + \tfrac{1}{3}x(w^{-1} - w) + \tfrac{1}{3}(\gamma z^3 w^{-1} - \bar{\gamma} z^{-1} w).$$

Let $D_0(x, \gamma)$ be the two-circle mean of $|T|$, convex in $(x, \Re\gamma, \Im\gamma)$. By conjugation consider $\Im\gamma \geq 0$. Take all cubes of side $1/5$ in $[0, 1] \times [-1, 1] \times [0, 1]$ whose points of shortest distance to the origin have squared distance at most one. There are 192 such cubes with 333 distinct vertices. At each vertex use the rectangular equiangular mesh 128×64, including $z = w = 1$. Bound the mean above by the mean of the sampled moduli plus

$$\frac{10}{3}\left(\frac{(8 + 10|\gamma|)/3}{128^2} + \frac{2(x + |\gamma|)/3}{64^2}\right).$$

Indeed, interpolating linearly between endpoints of an angular segment of length h costs at most $h^2 \sup|T''|/12$ in the mean by the linear interpolation remainder. The two derivative bounds are the sums of coefficient magnitudes times squared frequency. Use $\pi^2 < 10$.

For the vertex arithmetic, enclose the real and imaginary root coordinates to within 2^{-30} by Machin's identity and the degree-30 cosine and degree-31 sine polynomials on $[0, \pi/2]$, with quadrant rotations. For each monomial use the root table for the grid of order 128, since the order 64 divides it. Add to each sample coordinate the interval error bounded by 2^{-30} times the sum of the real and imaginary absolute coefficient parts. The upper modulus is the upward square root of the sum of the larger squared endpoints in the two coordinates. For $|\gamma|$ use an upward root as well. These substitutions in $2 + (x^2 + |\gamma|^2)/3 - (12/7)D_0(x, \gamma)$ give a least lower bound greater than 0.0133108. Multilinear interpolation of the vertex inequalities loses at most $3(1/5)^2/12 = 1/100$ from the squared parameter term. Thus the margin exceeds $3/1000$ throughout the ball.

At fixed z, the actual w's are a rotated regular polygon of order $D/\gcd(D, P)$. The angular Lipschitz constant of $|T|$ in w is at most $2(x + |\gamma|)/3 < 1$. Averaging this bound on the equal intervals centered at the polygon points gives error at most $\pi/(2D/\gcd(D, P)) < 2/(D/\gcd(D, P))$ between the polygon mean and the continuous mean. Multiplying the error at order 2048 by $12/7$ gives $3/1792 < 3/1000$. The strict inequality survives, proving the second claim. □

8. Fourier comparisons at three knots

Use brackets $\langle\cdot\rangle$ for normalized angular average. Let $n_0 < n_1 < n_2 < n_3$ be the consecutive tied indices at increasing radii r_0, r_1, r_2, and set $E = n_2 - n_1$ and $L = E\log(r_1/r_0)$, $R = E\log(r_2/r_1)$. Normalize by the maximum term in modulus and by the monomial of index n_1. Relative frequency λ in units of E then transports from the first to the middle knot by $e^{\lambda L}$ and from the middle to the last by $e^{(\lambda-1)R}$. These identities follow because n_1 is shared at the first and middle knots and n_2 at the middle and last. An angular rotation and unit component factors can fix the phases of independent anchors.

Represent each selected residue by a polynomial in $z = e^{iE\theta}$, times the appropriate residue monomial. At fixed z, averaging on the E root-of-unity shifts extracts a test with scalar values $\sum_i p_i(z)\chi_i$, where χ_i are the corresponding distinct characters. An angular mean of a test with fixed z is bounded by any magnitude-dependent norm $\Psi(p)$ that majorizes this cyclic scalar mean. In particular Jensen gives the Euclidean norm by orthogonality. Multiplying each component by a factor of modulus at most one remains valid for a coordinatewise increasing Ψ. Integrating the pairing between the entire function and the finite test bounds its real part by the actual cap times the mean of Ψ. This uses the Fourier coefficients of the full function, with unselected modes having zero test weight.

For a model denominator h, set $a = e^{-L/h}$, $b = e^{-R/h}$. In an integer radial model use $m = h\lambda$; the same formulas hold when m is a real number. Transport relative to the pivot knot has factors

$$(g_0, g_1, g_2) = \begin{cases} (1, a^{-m}, a^{-m}b^{h-m}), & m \le 0, \\ (a^m, 1, b^{h-m}), & 0 < m \le h, \\ (a^m b^{m-h}, b^{m-h}, 1), & m > h. \end{cases} \tag{8.1}$$

The pivot is 0, 1, 2, respectively. A table of test weights lists three raw entries in every selected coefficient column. For an unknown column replace its pivot entry by minus the sum of the other two entries times their factors. Its total pairing then vanishes. On a known unit anchor, leave the three entries unchanged and sum their transported real parts into a numerator N, multiplying all weights in that column by the anchor phase in the test. This convention gives the correct real pairing for complex weights since the weights in the pairing are conjugated. If the normalized norms are at most C, the resulting three

polynomial vectors P_k satisfy

$$N - C\sum_{k=0}^{2}\langle\Psi(P_k)\rangle \le 0. \tag{8.2}$$

8.1. **Cyclic mean estimates and norm properties.** Let $S = \sum_i |p_i|^2$. In formulas below each norm is extended by zero at $S = 0$. For an edge set $\mathcal{G}$ of unordered pairs, use

$$\Psi_{\eta,\mathcal{G}} = \sqrt{S} - \eta \sum_{\{i,j\}\in\mathcal{G}} |p_i p_j|^2/S^{3/2}.$$

For an arithmetic progression of three residues with middle magnitude x and endpoint magnitudes y, z, use

$$\Psi_{\mathrm{AP}} = \sqrt{S} - \big(y^2z^2/4 + x^2(y-z)^2/16\big)/S^{3/2}.$$

A two-component norm with an extra correction is

$$\Psi^{(2)}_{\eta,\zeta} = \sqrt{S} - \eta|p_0p_1|^2/S^{3/2} - \zeta|p_0p_1|^4/S^{7/2}.$$

For the cyclic bounds, normalize $S = 1$ and let Z be the squared scalar test minus one. Then $\langle Z\rangle = 0$. Write $v_2 = \langle Z^2\rangle$, $m_3 = \langle Z^3\rangle$, and $H(t) = 1/[2(1+\sqrt{1+t})^2]$ on $t \ge -1$. The tangent identity and Jensen with the probability weight Z^2/v_2 give

$$1 - \langle\sqrt{1+Z}\rangle = \langle Z^2H(Z)\rangle \ge v_2H(m_3/v_2), \qquad 1 - \langle\sqrt{1+Z}\rangle \ge v_2/8 - m_3/16. \tag{8.3}$$

Here H is decreasing and convex by differentiation. The second inequality uses its tangent at zero. If $v_2 = 0$ the correction is zero.

With two residues of relative order o, write X, Y for their squared magnitudes. The fluctuation is $2\sqrt{XY}\cos\theta$ on an o-gon. Order two uses the Euclidean bound. For $o \ge 3$ its second moment is $2XY$ and its third at most $2(XY)^{3/2}$, yielding $\eta = 1/5$ from $m_3/v_2 \le 1/2$ and (8.3). If $o \ge 4$ the third moment vanishes, giving $\eta = 1/4$. If $o \ge 6$, all odd moments through order five vanish and $\langle Z^4\rangle = 6X^2Y^2$. The degree-five Taylor polynomial of $\sqrt{1+Z}$ at zero is an upper bound on $[-1,1]$ because the sixth derivative is negative on $(-1,1]$, with the left endpoint by continuity. Its even coefficients give $(\eta,\zeta) = (1/4, 15/64)$.

For three residues $0, u, v$, let A, B, C be the normalized squared magnitudes in that order. In any moment of degree j, the character products have frequency $mu + nv$, where $\max(|m|,|n|,|m+n|) \le j$. If no nonzero relation of this size occurs through degree three, the only surviving products are opposite edges and oriented triangles. Thus $v_2 = 2(AB + AC + BC)$, $m_3 = 12ABC$. The ratio $m_3/v_2 \le 2/3$ follows from $1/A + 1/B + 1/C \ge 9$, with zero cases by continuity. This yields the all-pair loss $21(AB + AC + BC)/125$ from (8.3).

For progression $0, j, 2j$ of step order at least seven, the positive harmonics of Z are $U = p_1\bar{p}_0 + p_2\bar{p}_1$, $W = p_2\bar{p}_0$, so $v_2 = 2(|U|^2 + |W|^2)$, $m_3 = 6\Re(U^2\bar{W})$. The norm Ψ_{AP} follows from $|W| \le 1/2$, $|U| \ge x|y-z|$, and the second inequality of (8.3). For $0, j, 3j$ of order at least ten, distinguish $U = p_1\bar{p}_0$, $V = p_2\bar{p}_1$ and the third harmonic $p_2\bar{p}_0$. The only extra cubic products arise from $1 + 1 - 2 = 0$, and $m_3 \le 12ABC + 6AB\sqrt{BC}$. Putting $u_0 = \sqrt{B/C}$ and $v_0 = C/A$, the ratio is bounded by

$$\frac{3u_0^2(u_0+2)v_0}{(1+(1+u_0^2)v_0)(1+u_0^2+u_0^2v_0)} \le \frac{3u_0^2(u_0+2)}{(1+u_0+u_0^2)^2} \le \frac{17}{16}.$$

The first inequality is the arithmetic–geometric mean inequality on the constant and quadratic terms of the denominator divided by v_0. For the second, the cleared difference is $17s^4 + 54s^3 + 15s^2 - 30s + 9$ at $s = u_0 - 1$. For $s \in [-1, 0]$ it is bounded below by $(1+s)(9-39s) + 17s^4$. For $s \ge 0$, the cubic part is convex, has derivative -2 at $s = 1/3$ and positive derivative at $s = 1/2$. Its value at $1/3$ is $8/3$, so its

tangent is positive on $[0, 1/2]$ and the cubic is increasing thereafter. The resulting loss in (8.3) again exceeds $21(AB + AC + BC)/125$.

A third estimate has edge set consisting of the two pairs containing a distinguished coefficient of squared magnitude A, and weight $1/5$. Suppose the samples, after a unit rotation, have the form $p_0 + (p_1 + \delta p_2)\chi$ where δ ranges over the roots of order two or three, and conditional on δ the character χ ranges over a rotated polygon of order at least four. Put $\Delta(\delta) = 2\Re(p_1\bar{p}_2\bar{\delta})$. Conditional second and third moments of the first harmonic arise from its two opposite frequencies, and its odd moments through degree three vanish. For order two put $J = |\Delta(1)|^2$. Then $v_2 = 2A(1 - A) + J$, $m_3 = 6AJ$. With $t = J/(2A(1 - A))$ and $D = (1 + t)(1 + 2t)$, the bound $J \le (1 - A)^2$ gives $m_3/v_2 \le 6t/D$. The identity

$$(1 + 5t)D^2 - 12tD - 9t^2 = 1 - t - 2t^2 + 53t^3 + 64t^4 + 20t^5 \ge 0$$

holds for $t \ge 0$, by using $1 - t - 2t^2 \ge 0$ on $[0, 1/2]$ and $53t^3 \ge t + 2t^2$ thereafter. It implies $(2 + 3t/D)^2 \le 5(1 + t)$. Then $1 + \sqrt{1 + 6t/D} \le 2 + 3t/D$ and (8.3) imply loss at least $A(1 - A)/5$. The cases $A = 0, 1$ follow from Jensen without a correction. For order three, $\langle\Delta^2\rangle = 2BC$ and $\langle\Delta^3\rangle = 2\Re((p_1\bar{p}_2)^3) \le (B + C)BC$. Thus $v_2 = 2(A(B + C) + BC)$ and $m_3 \le 12ABC + (B + C)BC$. The tangent lower bound for the loss minus $A(B + C)/5$ is

$$A(1 - A)/20 + BC(3 - 11A)/16.$$

If $3 - 11A < 0$, replace BC by $(1 - A)^2/4$, giving $(1 - A)(55A^2 - 54A + 15)/320 \ge 0$. This proves the stated bound for order three as well.

For use in interpolation, each formula is a norm, invariant under independent phases and increasing in component magnitudes. For $\Psi_{\eta,\mathcal{G}}$, at $S = 1$ a unit tangent direction v satisfies $\sum u_i = 0$ for $u_i = \Re(\bar{p}_i v_i)$. With $P = \sum_{\mathcal{G}} |p_i p_j|^2$,

$$D^2P[v, v] = 2\sum_i |v_i|^2 \sum_{j:\{i,j\}\in\mathcal{G}} |p_j|^2 + 8\sum_{\mathcal{G}} u_i u_j \le 4.$$

In fact the sum of the positive u_i equals the sum of absolute negative u_i, at most $1/2$, so the positive pair products total at most $1/4$. The tangent Hessian of the norm is at least $1 + 3\eta P - 4\eta \ge 0$. Homogeneity gives zero radial Hessian. The partial derivative in a magnitude $p \ge 0$ is at least $(1 - 2\eta)p/\sqrt{S}$ since the adjacent magnitudes have total squared sum at most S. For Ψ_{AP} use $P = y^2z^2/4 + x^2(y - z)^2/16$. The largest Hessian eigenvalue of u^2v^2 is at most $3(u^2 + v^2)$, by its two-by-two matrix. Apply this to (y, z) and to $(x, (y - z)/\sqrt{2})$ to obtain $D^2P \le 15/16$ in unit directions on $S = 1$. For coordinate derivatives of the correction use $(y - z)^2 \le S$ in the x-derivative and $(y - z)_+ \le y$ in the y-derivative, and interchange y, z for its last derivative. These yield nonnegative derivatives of the norm. Hence this formula is convex and increasing on the nonnegative orthant. The triangle inequality in complex coordinates follows from coordinatewise magnitude domination and convexity. Continuity gives the assertions at zero.

For the extra two-component norm on real magnitudes x, y, it equals

$$\left\langle |x\cos t + y\sin t|\,\frac{\pi}{16384}(7891 - 4740\cos 4t + 945\cos 8t)\right\rangle_t .$$

Indeed the three integrals of $|\cos t|$ times $1, \cos 4t, \cos 8t$ over a full period are $4, -4/15, -4/63$. In polar coordinates this gives $\sqrt{S}(7891 + 316\cos 4\theta - 15\cos 8\theta)/8192$, which is the claimed formula. The density is positive and invariant under coordinate reflections. This gives convexity and coordinatewise monotonicity on magnitudes, and again the triangle inequality in complex coordinates. For all norms used here $0 \le \Psi \le \sqrt{S}$. Consequently $|\Psi(p) - \Psi(q)| \le \|p - q\|_2$.

8.2. **Directed integration and interpolation.** If $P = (\sum_n c_{jn} z^n)_j$, use an equiangular grid of J nodes with J larger than each component's frequency span. Independently shift each component by a unit monomial of integer degree $-s_j$. The exact angular bound is

$$\langle \Psi(P) \rangle \le J^{-1} \sum_{v=0}^{J-1} \Psi(P(e^{2\pi i v/J})) + \frac{(2\pi/J)^2}{12} \left(\sum_{j,n} |c_{jn}|^2 (n - s_j)^4 \right)^{1/2}. \tag{8.4}$$

To see this, interpolate P linearly over each angular step of length $h_0 = 2\pi/J$. Convexity controls the norm of the interpolant. At relative position s, its difference from P is bounded by integrating $|P''|$ against the absolute Green kernel $h_0^2 \min(s, v)(1 - \max(s, v))$ for $0 \le v \le 1$. For each v the mean over all steps of the derivative magnitude is at most its root mean square, which is the displayed coefficient sum by discrete orthogonality. Integrating over s, v gives $h_0^2/12$.

The directed version uses coefficient real and imaginary parts rounded down at scale $S_0 = 2^{30}$, with the total absolute rounding error added. Grid roots are approximated with complex error $< 3/S_0$ and the root-error allowance is $3/S_0$ times the sum of the absolute rounded real and imaginary parts of the coefficients. The same root bound with rational turn argument applies to $e^{i\pi t}$ at a corner. One construction uses $\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$, alternating sums of 24 and 8 terms and directed truncations at 2^{64}, with errors bounded by $16 \cdot 25 + 4 \cdot 9 < 500$ units at that scale. After quadrant reduction, cosine and sine on $[0, 8/5]$ use the terms of degree at most 32 at that scale. The argument error is below 251 units. Term truncation and the Taylor remainder add at most 17 units per component. Truncating the result at scale 2^{30} therefore gives the stated root error. Evaluate the sample norms by their squared-component formulas. In Ψ_{AP}, use the lower bound $\max(0, y^2 + z^2 - 2\sqrt{y^2 z^2}_{\mathrm{up}})$ for $(y - z)^2$. For any rational $t \ge 0$ an upper root at scale S_1 is $(\lfloor \sqrt{\lfloor t S_1^2 \rfloor} \rfloor + 1)/S_1$. For exact integer sample energies combine the rational correction factor with the upper integer root and round upward in the sample units. Use $44/7$ for 2π and upper roots at scale 2^{50} in the curvature term, minimizing each component's sum over s_j in its mode range. These bounds also apply to the zero sample.

The three-knot tables use rectangular parameter boxes. For a fixed raw test row, let $D_{N,i}$ bound the absolute pure second derivative in coordinate i of N, and let $D_{k,i}$ bound the sum of squares of the magnitudes of the corresponding coefficient derivatives in P_k. Successive linear interpolation contributes at most

$$\sum_i \frac{w_i^2}{8} \left(D_{N,i} + C \sum_k \sqrt{D_{k,i}} \right) \tag{8.5}$$

to the loss of the minimum corner margin, where w_i are the widths. This follows from the interval interpolation kernel, convexity of the means, and the Euclidean coefficient error bound by Parseval. For a column, sum the absolute raw weights times the pure derivative bounds of (8.1). Sum over known columns for $D_{N,i}$, and square and sum at the replaced pivots for $D_{k,i}$. A known phase column with phase $e^{i\pi t}$ contributes at most $(22/7)^2$ times its raw weight magnitude in the phase derivative and is independent of the phase in N. All other columns are phase independent.

For the fractional monomial factors $a^{p(u)} b^{q(u)}$ used below, p, q are nonnegative affine functions of u. The a-second derivative is $p(p-1)a^{p-2}b^q$, and analogously for b. Bound the factor $|p(p-1)|$ by its largest value at the exponent endpoints or $p = 1/2$ if present. For the remaining power use the lower exponent, taking the lower base endpoint if that exponent is negative and the upper endpoint otherwise. For u the derivative bound is $a_{\mathrm{hi}}^{p_{\min}} b_{\mathrm{hi}}^{q_{\min}} (|p'|\Lambda_a + |q'|\Lambda_b)^2$, where $\Lambda_a \ge -\log a_{\mathrm{lo}}$. With $z = (1 - a_{\mathrm{lo}})/(1 + a_{\mathrm{lo}})$ take $2(\sum_{j=0}^{5} z^{2j+1}/(2j+1) + z^{13}/(13(1-z^2)))$; likewise for b. Endpoint powers with rational exponent j/k are bracketed by comparing k-th powers to the j-th power of the base. Equivalently enclose the exponent by dyadic rationals and bracket the powers by repeated directed square roots. At a corner the enclosure midpoint can be used for a factor, charging its radius times the

absolute raw weights in the numerator or the row error. In all cases the full corner and interpolation allowances must be subtracted before testing positivity.

For tables using vertex-dependent raw rows, interpolate the raw entries by the tensor weights before doing coefficient transport. For one coordinate of width w, let M bound the endpoint magnitudes and D the edge differences of the raw entry. With first and second derivative bounds B_1, B_2 of its factor, the pure interpolation error for the product is at most $w^2MB_2/8 + wDB_1/4$. This is the second derivative estimate using $|w'_{\mathrm{raw}}| \le D/w$. Use $(22/7)$, $(22/7)^2$ for the phase factor. Sum over coordinates and send these errors through the same linear numerator and Euclidean coefficient bounds as before. If F_v are directed lower bounds on the corner margins, the margin throughout the box is at least $\min_v F_v$ minus these errors. Fixed rows are the special case $D = 0$.

Finally a finite set of closed positive-width boxes with disjoint interiors lying in an ambient box and having total volume equal to that ambient volume covers every point of it. Otherwise a relatively open portion of its complement would have positive volume. We use this check for the listed tables, including shared faces and corners.

9. Exclusion of a strict local maximum of the gaps

Theorem 9.1. *If all three knot norms are strictly below C_0, then the four selected exposed indices cannot have consecutive gaps d, E, D with $d < E$ and $D < E$.*

9.1. **Two-point bounds for the radial parameters.** Write $u = d/E$, $v = D/E$, $y = e^{-L}$, $\beta = e^{-R}$, $T = C_1^2 - 2$, $T_0 = C_0^2 - 2$. Theorem 6.1 applied using the middle and right knots gives $y > 1/2500$. Applying it in reverse gives $\beta > 1/2500$, using finite reversal as in Section 5. At the middle knot, project to the common residue of the tied terms and gauge the two coefficients to 1. If the projection is $F(z)$, take the real part of $e^{-i\theta/2}F(e^{i\theta})/2 + e^{i\theta/2}F(e^{-i\theta})/2$. It is at most C_0. The ties contribute $2c$, $c = \cos(\theta/2)$, and the weights on the two tails at displacements $j \ge 1$ are $\cos((j + 1/2)\theta)$. For $0 \le Y < 1$, set

$$\begin{aligned} S(Y, c) &= \sum_{j\ge1} Y^{2j} \cos^2((j + 1/2)\theta) \\ &= \frac{Y^2}{2(1 - Y^2)} + \frac{Y^2(4h^3 - 3h - Y^2h)}{2(1 - 2Y^2(2h^2 - 1) + Y^4)}, \quad h = 2c^2 - 1. \end{aligned}$$

The identity follows by summing the doubled-angle geometric series. The left tail comes from distinct coefficients at the left knot outside its two ties and the middle knot's upper tie, whose normalized magnitude there is y. Transport weights are y^j; the right side has the analogous weights β^j. Parseval and Cauchy–Schwarz bound the adverse tails by $B(U, c; T_i) = \sqrt{(T_i - U^2)S(U, c)}$ at $U = y, \beta$, with either $T_i = T_0$ or $T_i = T$. If $2Y^2 \le T_i$, this bound is increasing in $U \in [0, Y]$ since each $(T_i - U^2)U^{2j}$ is increasing there.

We also need a refined right tail bound when $D/g \ge 6$ and $E/g \ge 7$ for $g = \gcd(D, E)$. Reflect the right knot and index from its lower tied term in the reflected order. Its two ties now have indices $0, D$, with phases made 1 by a rotation. The known coefficient at index $D + E$ has magnitude β. The coefficients transporting to the tail have reflected indices $D - jE$, with factors β^j. Let $G(\theta)$ be the normalized full profile at this reflected knot. Take $P(\theta) = 3(1 + e^{iD\theta})$, $W = \max(1, |P|/C_1)$, $V = P/W$. Expanding squared moduli pointwise gives

$$|G - P|^2 = |P - V|^2 + W|G - V|^2 + (W - 1)(C_1^2 - |G|^2).$$

The two ties give $\langle |G - P|^2 \rangle \le C_1^2 + 6$. Set $\alpha = C_1/6 = 57/200$, $t_0 = \arccos \alpha$. Direct integration gives

$$\begin{aligned} \langle |P - V|^2 \rangle &= \tfrac{2}{\pi}((18 + C_1^2)t_0 - 9C_1 \sin t_0) > 7.679, \\ g_0 := \langle W^{-1} \rangle &= 1 - \tfrac{2}{\pi}\big(t_0 - \alpha \log((1 + \sin t_0)/\alpha)\big) \in (7/25, 107/200). \end{aligned}$$

For the rational bounds use $t_0 > 64089/50000$ from the degree-ten cosine lower polynomial, $\sin t_0 < 47927/50000$ by squaring, and $\log((1+\sin t_0)/\alpha) < 193/100$ from the degree-eight exponential sum. Use $\pi < 22/7$ and $g_0 \geq \alpha$. Thus $\langle W|G-V|^2\rangle \leq B_0 = 12451/10000$.

In this weighted inner product the representative of a coefficient at k is $e^{ik\theta}/W$. Its Gram entry against the representative at l is the $k-l$ coefficient of W^{-1}, and the diagonal is g_0. Off-diagonal entries vanish unless $D \mid k-l$. Writing $m = (k-l)/D$, two integrations by parts on $0 \leq \phi \leq 2t_0$ give the off-diagonal bound $\tan(t_0)/(\pi m^2)$. Indeed the loss $1 - W^{-1} = 1 - \alpha\sec(\phi/2)$ on that arc vanishes at $2t_0$, has derivative zero at 0, and its decreasing derivative ends at $-\tan(t_0)/2$. The endpoint and derivative variation have combined magnitude $\tan(t_0)$. We use $\tan t_0 < 7/2$.

For the frequencies $D - jE$, the first five are in distinct residue classes modulo D. The off-diagonal row sum is at most $2\cdot(7/2)/(\pi(E/g)^2)\sum_{k\geq 1}k^{-2} \leq \pi/42 < 2/21$. The row at $D+E$ has nonzero entries against these tail frequencies only for $j+1$ a multiple of D/g, hence $j \geq 5$. Its cross entries have squared sum at most $(7/(2\pi\cdot 7^2))^2\sum_{k\geq 1}k^{-4} < 1/40^2$. Projecting $G - V$ orthogonally off the representative at $D+E$ leaves squared weighted norm at most $B_0 - \beta^2/g_0$ because V vanishes at this frequency. The projected-off part contributes at most $(25\beta/(7\cdot 40))$ times the test's Euclidean tail from $j = 5$.

Finally V has negative coefficients at these positions only from $j = D/g, 2D/g, \ldots$, thus from $j \geq 6$. For $m = jE/D$ at such a position, its coefficient is the negative of the $(1-m)$-th coefficient of $P - V$ in the variable $e^{iD\theta}$. Integrate $3(1+e^{i\phi}) - C_1 e^{i\phi/2}$ over $|\phi| \leq 2t_0$ against $e^{i(m-1)\phi}$, using $C_1 = 6\cos t_0$. This gives the coefficient of $P - V$ as

$$\frac{3}{2\pi(m-1/2)}\left(\frac{\sin(2(m-1)t_0)}{m-1} - \frac{\sin(2mt_0)}{m}\right).$$

For $m \geq 7$ its absolute value is at most $7/(2\pi m^2)$. Consequently this contribution is at most $1/40$ times the test's Euclidean tail from $j = 6$.

Put $G_0 = 107/200$, $L_0 = S(Y,c)$, $L_2 = Y^{12}/(1-Y^2)$, $L_5 = Y^{10}/(1-Y^2)$, and $J_0 = (2/21)(2\sqrt{L_0L_2} + L_2)$. The squared weighted norm of the sum of the tail representatives with weights $\beta^j\cos((j+1/2)\theta)$ is at most $(\beta/Y)^2(g_0L_0 + J_0)$. In fact the first five have diagonal Gram block, and the rest and the cross block are bounded by the row-sum estimate and Cauchy–Schwarz. Combining the three contributions just bounded, for $\beta \leq Y$, $2Y^2 \leq B_0G_0$, gives the upper bound

$$B_{\mathrm{w}}(Y,c) = \sqrt{(B_0G_0 - Y^2)(L_0 + J_0/G_0)} + \frac{25Y}{280}\sqrt{L_5} + \frac{\sqrt{L_2}}{40}. \tag{9.1}$$

For the first root, $(B_0g_0 - \beta^2)(L_0 + J_0/g_0)$ is increasing in g_0, and $\beta^2(B_0G_0 - \beta^2)$ is increasing for $\beta^2 \leq Y^2$. The other terms can use $\beta \leq Y$ directly. All estimates hold first for finite coefficient sums and then for the full sums by geometric decay. Arbitrary phases are allowed in the bound because the Hilbert norm estimates use only test magnitudes and Gram absolute values.

In the coincident case $u + v = 1$ with $u \leq 1/46$, the reduced orders for the right side satisfy $E/g \geq 46$ and $D/g \geq 45$. The middle Parseval inequality is $y^{2u} + \beta^{2v} \leq T_0$. In particular $y \leq T_0^{23} < .15177$. Whenever $y \leq Y$, test an upper ceiling b on β using $2c - B(Y,c;T_0) - B_{\mathrm{w}}(b,c) > C_0$. This forces $\beta > b$, and then $y < (T_0 - b^2)^{23}$ since $v < 1$. The three rational tests, with upward roots in the subtracted terms, are

Y	b	c	10^9 lower excess over C_0	next upper y
.15177	.27	.915	162141	.0228
.0228	.425	.95	309529	.00101
.00101	.46	.955	169068	.00038

The final bound contradicts $y > 1/2500$. Reflection shows that $v > 1/46$ as well.

9.2. **Models for the strict peak.** If $d + D \neq E$, the three residues of n_1, n_0, n_3 modulo E are distinct. In units E relative to n_1 they have frequencies $n, n-u, n+1+v$, $n \in \mathbb{Z}$. Write F, H, G for the components, with known pivot anchors F_0, F_1, H_0, G_0. An angular rotation fixes F_1 relative to F_0, and independent unit component factors fix H_0, G_0. Let s denote time measured in units of E from the first knot. For artificial ceilings $\bar{u} = r/h \geq u$, $\bar{v} = s_1/h \geq v$, multiply H by $e^{-(\bar{u}-u)s}$ and G by $e^{(\bar{v}-v)(s-L-R)}$. These factors are at most one, preserve the unit anchors, and replace the transport frequencies by $n, n-r/h, n+1+s_1/h$. By the norm properties in Section 8.1 this operation bounds the test mean by that for the transported row at the artificial ceilings. The angular fiber still uses the actual residue characters.

Let the individual orders of the nonzero residue characters be $E/\gcd(E,d)$ and $E/\gcd(E,D)$. If both are at most nine, both ratios are at most $8/9$ and the Euclidean norm applies. If one has order two or three and the other at least ten, reverse if needed so the smaller order is on the left. Factor out the character of the right component in the scalar mean, leaving $G + (F + \delta H)\chi$. Conditional on δ, χ has order at least four because taking its second or third power reduces order by at most that divisor. Apply the two-edge norm centered at G, with ceilings $2/3, 1$.

It remains that both orders are at least four and at least one is at least ten. If the difference of the two exterior residues has order two or three, the same conditioning with distinguished component F applies. Otherwise classify the relations $mu_0 + nv_0 = 0 \bmod E$ for the two nonzero residues u_0, v_0, with $\max(|m|, |n|, |m+n|) \leq 3$. Neither multiplier can be zero because both individual orders are at least four. Up to overall sign and interchange, the same-sign pairs are $(1,1), (1,2)$, giving an arithmetic progression. Opposite-sign magnitudes are $(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)$. The first is impossible by distinctness, the second gives an arithmetic progression, the first-equal $(2,2), (3,3)$ give the difference cases just treated, and $(1,3), (2,3)$ give translates or reflections of $0, j, 3j$. For example $2u_0 = 3v_0$ makes $u_0 = 3j, v_0 = 2j$ with $j = u_0 - v_0$. The step in either configuration has order at least the larger individual order, since all three residues differ by multiples of the step. The progression bound works for any midpoint, and reversal exchanges H and G. The sparse and relation-free cases use the all-pair bound. These four models use unit ceilings.

If $d + D = E$, combine the exterior components in the residue J with frequency $n - u$, so the four anchors are F_0, F_1, J_0, J_2. Now $J_0 = 1$ after gauging and $J_2 = \nu$ is a single common unit phase. Multiply the raw J_2 weights by ν while forming the tests. All norms and the real pairing are invariant under whole-profile conjugation with angular reversal, so use $\nu = e^{i\pi t}$, $0 \leq t \leq 1$. This phase is fixed before the choice of row. Finite reversal gives $u = 1/2$ and, if needed, $a \leq b$ for order two. For order three it gives $u = 1/3$. At orders at least four orient so $u \leq 1/2$. The proved lower ratio and radial bounds, with outer Parseval $y^2, \beta^2 \leq T_0$, then put the parameters in

$$a = y^{1/8},\ b = \beta^{1/8} \in [3/8, 199/200], \quad u \in [1/46, 1/2], \quad t \in [0,1].$$

Use $\Psi^{(2)}$ with $(\eta, \zeta) = (0,0), (1/5, 0), (1/4, 0)$ in the three order ranges, respectively.

9.3. **Box comparisons.** The embedded table S (Table 1) gives the raw rows and the closed box partitions for the nine models. For the first six models let the parameters be $a, b \in [0,1]$, and set the scaled frequencies $m = hn, hn - r, hn + h + s_1$. For orders two and three the box is $[0,1]^3$ in a, b, t and $m = hn, h(n-u)$. For the last model use the compact box just displayed with $h = 8$ and the same formulas. There, on the J column the pivot is fixed at each n across the whole box because $0 < u \leq 1/2$. Use C_1 in (8.2).

On an energy box at the first six or the next two models, the four-coefficient middle-knot Parseval bound is contradicted by $a_{\mathrm{lo}}^{2r} + b_{\mathrm{lo}}^{2s_1} > T$, where in the coincident model $r = hu$, $s_1 = h(1-u)$. At the last model use the lower value $a_{\mathrm{lo}}^{16u_{\mathrm{hi}}} + b_{\mathrm{lo}}^{16(1-u_{\mathrm{lo}})} > T$. The energy bound uses the actual exterior coefficients or their smaller artificial values. For a reflection box $a_{\mathrm{lo}} > b_{\mathrm{hi}}$; in the symmetric models

finite reversal permits $a \le b$. Reversal preserves the strict norm cap by using small approximation slack, and the models then use the new actual phases.

For the 180 radial boxes the ceilings on y, β satisfy either $\max(y, \beta) \le .19$ or $\min(y, \beta) \le .15$, $\max(y, \beta) \le .23$. Applying $B(U, .91; T)$ at the indicated ceilings gives a lower two-point value exceeding C_1 by 0.001076624 and 0.000286547, respectively. For the 32 ordinary two-point boxes use $a \le .65, b \le .87, c = .935$ and T; the two adverse bounds are at most $3/200, 143/1000$. For the four weighted boxes use $a \le .72, b \le .87, u \le .08, c = .94$. The last bound on u gives $D/g \ge 6, E/g \ge 7$, so the left bound $B(.72^8, .94; T)$ is at most $7/200$ and the right bound $B_{\mathrm{w}}(.87^8, .94)$ at most $16/125$. Both types give a lower value greater than C_0.

For each remaining box substitute its row into (8.1) and evaluate the directed corner bound and (8.5), with the modes, grid size and norms shown below. For fractional powers use rational endpoint power enclosures (with a fixed-point enclosure at 2^{42} in the indicated fixed-point rows) and charge their radii in the corners. Each comparison gives the positive lower margin shown in the last column, in units of 10^{-9} or larger.

model	(h, r, s_1)	degree, grid	boxes	comparisons	other	floor
Euclidean	$(9, 8, 8)$	$7,512$	143	97	46	49174
midpoint F	$(1, 1, 1)$	$7,512$	144	83	61	738
midpoint H	$(1, 1, 1)$	$6,256$	83	77	6	207092
all pairs	$(1, 1, 1)$	$6,256$	81	48	33	1776421
two edges at F	$(1, 1, 1)$	$6,256$	81	48	33	4485532
two edges at G	$(3, 2, 3)$	$6,256$	81	79	2	971842
coincident order 2	$(2, 1, 1)$	$6,256$	446	329	117	48684
coincident order 3	$(3, 1, 2)$	$6,256$	336	312	24	201452
coincident order ≥ 4	$(8, 8u, 8(1-u))$	$6,256$	1628	1216	412	24754

The volumes and disjoint interiors verified by the box endpoints give coverage, including $t = 0, 1$. Every parameter point for an actual profile therefore belongs to a box with a contradiction to Parseval, the radial bounds, the orientation, or (8.2). This proves Theorem 9.1.

10. Exclusion of a decrease after two equal gaps

Theorem 10.1. *If all three knot norms are at most C_0, the three consecutive gaps cannot be d, d, E with $0 < E < d$.*

Proof. In units of d from the first exponent, the four exposed indices are $0, 1, 2, 2 + \rho$, $\rho = E/d$. The two residue components F, J have frequencies n and $n + \rho - 1$. Rotate so the first shared phase in F is 1, normalize consistently using the positive maximum term, and use a unit factor on J fixing J_3. The known pivot coefficients F_0, F_1, F_2, J_3 are then $1, 1, \xi, 1$, $|\xi| = 1$. As before all raw weights in the F_2 column get the same ξ; conjugation permits use of $\xi = e^{i\pi t}$, $t \in [0, 1]$.

Use the order $o = d/\gcd(d, E)$ to choose the ceiling r/h and the two-residue norm $\Psi^{(2)}_{\eta,\zeta}$ as follows.

o	h	r	η	ζ
2	2	1	0	0
$3 \le o \le 5$	5	4	1/5	0
$o \ge 6$	1	1	1/4	15/64

The inequality $\rho \le r/h$ holds since the reduced numerator is at most $o - 1$. Use the forward attenuation factor about the last knot to replace ρ by r/h for the radial transport, preserving all anchors. Relative to the second exposed index the scaled frequencies in (8.1) are $h(n-1)$ and $h(n-1) + r - h$. Apply (8.2) with C_0.

The boxes of each model cover the closed unit cube in a, b, t. To test the energy prunes use $a_{\mathrm{lo}}^{2h} + b_{\mathrm{lo}}^{2r} > C_0^2 - 2$. For the other prunes put $X = a_{\mathrm{lo}}^{h}$, $Y = b_{\mathrm{lo}}^{r}$. At the middle knot use the known-coefficient test vector $X + z + \xi z^2$, Yz^3. The real pairing is at least $N = 2 + X^2 + Y^2$. Its squared Euclidean norm has mean N, upper bound $M = (2 + X)^2 + Y^2$, and variance $2(1 + 2X^2 + 2X\cos(\pi t))$, by the first two Fourier harmonics. Use the lower endpoint V of this variance obtained from the upper phase endpoint. The square-root tangent identity and the quadratic correction of $\Psi^{(2)}$ bound the angular test mean by

$$\sqrt{N} - \frac{V}{2\sqrt{N}(\sqrt{M} + \sqrt{N})^2} - \eta(2 + X^2)Y^2/M^{3/2}.$$

Thus the prune is valid if N exceeds C_0 times the upper bound. The exact tests use upward roots for the denominators of the subtractions and a lower cosine in V.

In all other boxes the table gives either a fixed raw row or eight vertex rows. Evaluate with modes $-9, \dots, 9$ and 512 grid points as in Section 8.2. Each used vertex lower bound is computed after the coefficient, phase and integration allowances. For a row interpolated from vertices, lower the smallest of its directed floors by the mixed interpolation allowance $w^2MB_2/8 + wDB_1/4$ for each raw entry and its factor, summed as specified there. The raw row for a fixed box uses $D = 0$ and checks all eight corners. The exact cover and margin counts in the supplementary table D (Table 1) are

o	boxes	interpolated	fixed	energy	known test	10^9 minimum margin floor
2	1377	1213	0	81	83	9863
3–5	2084	1859	0	93	132	11901
≥ 6	2345	2154	21	87	83	229

There are 8,329 distinct used vertex rows. Each comparison box gives a strictly positive margin throughout its closure. The other boxes contradict the indicated energy or single-knot inequalities. This gives the contradiction in every box. □

11. Localization on arithmetic windows

11.1. **Bilateral profiles.** Suppose arbitrarily late exposed runs with common gap d have lengths tending to infinity. The value of d may vary with the run. Choose a fixed cap C with $Q(f) < C < A_*$, and choose centers increasingly far from both ends. At each knot project by the normalized average over rotations by d-th roots, after shifting the lower tied frequency to zero. This contracts the norm and keeps every coefficient in the selected residue. Use frequency units of d and rotate and normalize so that both tied coefficients are 1. For each fixed relative knot and frequency, the coefficient eventually belongs to the exposed progression. All normalized coefficients have modulus at most one, so pass to a joint diagonal limit of all the knot coefficients and of all the curvatures

$$w_j = \frac{a_{n_{j-1}} a_{n_{j+1}}}{a_{n_j}^2}.$$

For a finite run $|w_j| = \exp(-d(\tau_{j+1} - \tau_j))$. At the knot between the indices j and $j + 1$, the two normalized tails after the ties are

$$b_j(-n) = \prod_{r=0}^{n-1} w_{j-r}^{n-r}, \qquad b_j(n+1) = \prod_{r=0}^{n-1} w_{j+1+r}^{n-r} \quad (n \geq 1). \tag{11.1}$$

These identities follow at the finite knots from the successive second coefficient ratios, or by dividing by the lower tied term and gauging the upper one to 1. The identities are polynomial in the w's and pass to the limit even through zeros. Parseval at the limiting knot gives $|w_j|^2 + |w_{j+1}|^2 \leq C^2 - 2$, so all $|w_j| \leq \sqrt{C^2 - 2} < 1$. Formula (11.1) therefore gives Gaussian tails and uniform convergence on compact annuli. Every finite Fejér sum of the limiting coefficients has norm at most C by taking the

limit of the capped Fejér sums from the projected knots. Letting the order of these sums tend to infinity proves the cap for the full Laurent profiles $H_j(z) = \sum_n b_j(n)z^n$.

If a positive limiting radius modulus is needed as a maximum, translate the whole sequence toward its supremum and pass to another joint limit. Bounds closed in the curvatures persist, and the Gaussian majorant preserves the cap and (11.1) throughout the translated limit. In the rest of this section we work on such bilateral models, using C_0 as an upper bound unless specified otherwise.

For $x = |w_j|$ consider the knots at the ends of the plateau with exposed index j. Normalize to its common term, gauge the nearest exterior terms to the same phase ξ, $\xi^2 = w_j/x$, and reflect the right knot via $z \mapsto z^{-1}$. Their average on the left-oriented variable is

$$F(z) = 1 + \xi(z^{-1} + xz) + \sum_{n\geq 2} b_n(z^{-n} + x^n z^n). \tag{11.2}$$

Indeed the coefficients at the two ends relative to the shared term differ in absolute scaling by x^{-m} at degree m, and angular reflection interchanges the two sides. The coefficient b_n equals ξ^n times the mean of the two outward triangular products of degree T_{n-1}, one on each side of the plateau. This also follows by expanding (11.1) at the two endpoints. If $x = 0$, the positive side of each left-oriented series vanishes. The phase is then a subsequential limit of the finite phases, giving the same identity. When all moduli are at most $R < 1$, write $b_n = \xi^n R^{T_{n-1}}\alpha_n$. Always $|\alpha_n| \leq 1$. If all curvatures lie in an angular sector with lower radius m, then α_n is a mean of two points between radii $(m/R)^{T_{n-1}}$ and 1 in the product sector. We use the convex hull of this set for a support bound.

11.2. **Virtual radii and positive angular averages.** The adjacent profiles at any positive spacing are obtained from one another by radial scaling, normalization by the shared term, and rotation. This follows either by the pre-limit coefficient identity or by (11.1). On the annulus between their knots, the maximum-modulus principle applied after division by the shared active monomial bounds the profile norm by the cap times that monomial's modulus. If a spacing has infinite logarithmic length because a curvature vanishes, the outward coefficients past that end vanish by (11.1), so the same estimate follows on a disk after inversion if necessary. Therefore radial evaluations on the projected full profiles inherit the cap relative to the maximal term of the bilateral sequence. The adjacent ties mark the envelope of the terms since the radial ratios are positive and adjacent knot spacings are nonnegative, with modulus uniformly below one. In particular this bound is available at the following virtual radii.

For $k \geq 0$ set $G_k(z) = R^{T_k}F(R^k z)$. The first left-oriented term at displacement $n = 1$ has modulus 1 at the original knot. At displacement n the far-side term is bounded by $R^{T_{n-1}}$, so its modulus at the virtual radius, after scaling, is bounded by $R^{T_k+T_{n-1}-kn} \leq 1$. The exponent is the nonnegative integer $(n-k)(n-k-1)/2$. The center and the opposite-side terms are also at most one. It follows that $\|G_k\| \leq C_0$, including $k = 0, 1$ when an adjacent modulus is zero. For $k \geq 2$, if $m^{k-1} \geq R^k$, the term at displacement k is maximal on either virtual circle before averaging, allowing a tie at $k+1$. To see this, ratios from displacement j to $j+1$ are at least $m^j/R^k \geq 1$ for $j < k$, and at most $R^j/R^k \leq 1$ for $j \geq k$. Its scaled amplitude is the modulus of the individual normalized outward product. If the sector of this product has width less than π, with rays l, r, the linear functional

$$J_{l,r}(v) = \frac{\Re((\bar{l} + \bar{r})v)}{1 + \Re(\bar{l}r)}$$

majorizes the modulus there, as it equals $s + t$ for $v = sl + tr$, $s, t \geq 0$. Averaging gives $\|G_k\| \leq C_0 J_{l,r}(\alpha_k)$. There is also a useful version for $k = 2$, $R = .32, m = .10$. Here the maximal scaled term is at most $\max(R, |w_{j\pm1}|/R)$ on each side. The quantity in the maximum is at most $|w_{j\pm1}|/R + .0075$. Consequently $\|G_2\| \leq C_0(J_{l,r}(\alpha_2) + .0075)$ in the single phase sector.

An angular row with unit sample points z, η uses $\Re(\bar{\eta}G_k(z)) \leq C_0$. The corrected row at k uses

$$\Re(\bar{\eta}G_k(z)) - C_0 J_{l,r}(\alpha_k) + C_0(1-\epsilon) \leq C_0,$$

where $\epsilon = 0$ except in the preceding $m = .10$ version. Use positive integer row weights divided by their sum. In a group g of rows define the moments $M_{g,j}$ to be the sums of the normalized weights times $\bar{\eta} z^j$ in that group, and let σ_g be the sum of its normalized weights. The two angular points of each row are given by integer parameters v through

$$z = \chi(v/2^{18}), \qquad \chi(t) = \frac{1 - t^2 + 2it}{1 + t^2},$$

with a null parameter representing -1. Thus the moments are exact Gaussian rationals. Write $\xi = \chi(t)$. A group with depth k contributes constant $R^{T_k} \Re M_{g,0}$, and the n-th coefficient pairing factor for α_n is computed by summing

$$\xi^n R^{T_{n-1}+T_k} \left(R^{-kn} M_{g,-n} + x^n R^{kn} M_{g,n} \right)$$

over the groups. The $n = 1$ product is 1, and its contribution is the real part. A corrected group adds $C_0(1 - \epsilon)\sigma_g$ to the constant. At $n = k$ it adds $-C_0 \sigma_g \, (\bar{l} + \bar{r})/(1 + \Re(\bar{l} r))$ to the pairing factor. This is an additive correction to the factor after the multiplication by ξ^n.

For an unknown coefficient in a product sector with radii $[\rho, 1]$ and rays l, r, define its support in direction u to be

$$\mathcal{S}_{\rho,l,r}(u) = \begin{cases} |u|, & u \text{ lies between the rays,} \\ p, & p \geq 0 \text{ and } u \text{ lies outside,} \\ \rho p, & p < 0 \text{ and } u \text{ lies outside,} \end{cases} \qquad p = \max\{\Re(\bar{u} l), \Re(\bar{u} r)\}.$$

This follows by the angular maximum on the outer arc followed by radial maximization, and is unchanged on the convex hull. A disk uses $|u|$. For a pairing factor D the adverse direction is $u = -\bar{D}$. Each support is convex and 1-Lipschitz since its underlying set lies in the unit disk. On a box $[x_l, x_u] \times [t_l, t_u]$, sum terms with the same power of x before taking endpoint ranges unless specified otherwise. Multiply the resulting complex coordinate intervals by those of ξ^n componentwise and add the correction. For the phase intervals use the endpoint powers and any cardinal points between them on the lifted arc. For sector tests the sampled phase subintervals have width $< \pi$ after multiplying by n, as checked by the oriented arcs. The real part of the $n = 1$ factor uses its lower endpoint. For unknowns subtract the largest support of the four corners of the factor rectangle. For a disk, bound the norm before multiplication by ξ^n, using the largest absolute real and imaginary bounds. Take upper roots at scale 2^{56}. When retaining through N and using largest virtual depth K, subtract the uniform tail allowance

$$2R^{T_{N-K}}/(1 - R^{N+1-K})$$

from the combined lower bound. For $K = 1, 2, N = 6$, the sharper tails used below are $(1+R^{21})R^{15}/(1-R^6)$ and $(1 + R^{35})R^{10}/(1 - R^5)$. At the outermost depth the positive multiplier is at most $R^{2K(N+1)}$ times the negative one. At smaller depths the two multipliers together are dominated by the outermost negative multiplier, using $R^6 + R^{13} \leq 1$ for $K = 1$, and $R^5(1 + R^{21}) \leq 1$, $R^{11}(1 + R^7) \leq 1$ for $K = 2$ in the ranges below. These give the sharper tails by geometric summation. A lower bound above the cap rejects the box.

For a radial floor with upper test boundary y, evaluate the box with $x = y$. If the known real slope in x is B, and B_n is the coefficient of x^n in the unrotated pairing factor, the lower value at $x \leq y$ is bounded by the lower value at y plus

$$(y - x) \left[-B - \sum_{n=2}^{N} n y^{n-1} |B_n| \right]. \tag{11.3}$$

This follows by support Lipschitz continuity and $y^n - x^n \leq n y^{n-1}(y - x)$. At each phase interval use the lower bound on the bracket and, if negative, multiply it by $y - x_l$ for that full floor interval.

11.3. **The arithmetic parameter covers.** We give the sequence of finite inequalities in the supplementary table A (Table 1). Each box contains its own index of the sampled positive angular row list. The compact domain and the cap from the preceding step are used for all neighbor coefficients. The tests eliminate the target closed strips listed below.

For the first cover use the weaker cap C_1. Set $T = C_1^2 - 2$, $P = T/2$, $\rho = \sqrt{T - x^2}$. Parseval gives $|w_{j\pm1}| \le \rho$ and every adjacent product of moduli at most P. In (11.2), $|b_n| \le \rho P^{\lfloor (n-1)^2/4 \rfloor}$, by grouping neighboring factors in each nested product of lengths $1, \dots, n-1$, with the first factor bounded by ρ. Use $0 \le t \le 1$ by conjugation and $z \mapsto -z$. With $G_0 = F$, use

$$\Re[M_0 + \xi(M_{-1} + xM_1)] - \rho \sum_{n=2}^{6} P^{\lfloor (n-1)^2/4 \rfloor} |M_{-n} + x^n M_n| - \frac{2\rho P^9}{1 - P^3}.$$

Use the upper root for ρ at the lower modulus endpoint of the box. This rules out $x \ge .68$, since $\sqrt{T} < .962$.

For the second cover translate to an attained maximum x and take $R = x$ at depths 0, 1. In (11.2) use $B_n = x^{1-n} b_n$, $|B_n| \le x^{(n-1)(n-2)/2}$. The two factors for B_n at the two depths are $x^{n-1} z^{-n} + x^{2n-1} z^n$ and $z^{-n} + x^{3n} z^n$. Use the upper endpoint of x in the coefficient ceiling and the first sharper tail. Enclose each individual power times its moment by its two radial endpoints before summing. The maximum interval $[.32, .68]$ is excluded. For the third cover use $R = .32$ fixed at the same two depths and $B_n = R^{1-n} b_n$, with factors $R^{n-1}(z^{-n} + x^n z^n)$ and $z^{-n} + R^{2n} x^n z^n$. Bound $|B_n| \le R^{(n-1)(n-2)/2}$, using the same interval and tail rule. This proves the union of conjugate sectors $\Re w \le -(161/289)|w|$.

Before using a sector for products, we show that all imaginary parts have the same strict sign. Put $R = .32$, $c = 161/289$, $\delta_0 = \arccos c$. Write the two nearest outer curvatures at a knot as $u = -r_1 e^{\pm i\delta_1}$, $v = -r_2 e^{\pm i\delta_2}$, $0 \le \delta_i \le \delta_0$, and $s_i = r_i \sin\delta_i$. For an adjacent outward curvature u' in either sector, $\Re(u^2 u') \le (R/c) s_1^2$. Indeed $3\delta_0 < \pi$, and $-\cos(2\delta_1 + \delta_0) \le \sin^2\delta_1/c$ by completing the square in $\cot\delta_1$, with the endpoint by continuity. The same bound applies to $v^2 v'$. Use the half-angle profile $e^{-i\theta} H_j(e^{2i\theta})$ at $\theta = \pm\pi/6$, and let E_0, O_0 be its even and odd parts there. The tied contribution to E_0 is $\sqrt{3}$; the nearest outer terms have zero cosine. The next two cosine magnitudes are $\sqrt{3}/2$ with negative sign, and the following cosine is zero. With

$$e = R^6 + R^{15}/(1 - R^6), \qquad h = R^3 + R^6 + 2R^{10}/(1 - R^5),$$

the triangular bounds give $\Re E_0 \ge \sqrt{3}(1 - R(s_1^2 + s_2^2)/(2c) - e) > 0$. If the two imaginary parts have opposite weak signs, the real part of the nearest contribution $i(v - u)$ to O_0 has absolute value $s_1 + s_2$. The next two sine magnitudes are $1/2$, and the remaining sum is bounded by $2R^{10}/(1 - R^5)$. Thus $|O_0| \ge (s_1 + s_2 - h)_+$. Using the mean of the two squared values and $(s - h)_+^2 \ge s^2 - 2sh$, their greater squared value is at least

$$3 - \left[(3R/c - 1)(s_1^2 + s_2^2) - 2s_1 s_2 + 2h(s_1 + s_2) + 6e\right].$$

The bracket is separately convex on $0 \le s_i \le R(240/289)$. At its four corners substitution bounds it by $3 - C_0^2 - .003$. This contradiction excludes opposite weak signs, including zero. Hence after one common conjugation we can use the positive sector with $t \in [.6, 1]$.

For the next cover (A4), use the fixed two circles as in the third cover but apply the sector of product degrees T_{n-1} for $n = 2, 3$, with inner radius zero. In other degrees use disks. This gives $|w| > .10$. For A5 use $k = 0, 1, 2$, with only the corrected group at depth 2, inner radii zero for $n = 2, 3$, disks otherwise, and $\epsilon = .0075$. The coefficient ceiling can be factored as $R^{(n-2)(n-3)/2}$ with multipliers $R^{T_k - 3 + (2\pm k)n}$, as follows by moving this radial power from the pairing factor. Enclose each individual power times its moment before summing, and use the sharper $K = 2$ tail. The conclusion is $|w| > .15$, $t < .76$.

For A6–A8 use $k = 0, 1, 2$ and the corrected group of depth 2 in addition, with $R = .32, m = .15$, phase window $[.6, .76]$, annular supports for $n = 2, 3, 4$ and disk supports otherwise. Use the same

factored multipliers as in A5 and the sharper tail. For A7 apply (11.3). A6, A7 and A8 successively give the lower radius .17, lower radius .19, and then $.62 < t < .72$, $|w| < .31$.

For A9 at depths $0, 1, 2, 3$ use the ordinary groups and corrected groups at 2 and 3. Take $m = .19$, $R = .31$, use annular sectors for $n = 2, 3, 4, 5$, disks otherwise and $N = 8$. For A10 use depths up to 4 and add the corrected group at 4, with $m = .215$, $R = .31$, $N = 9$ and phase window $[.632, .70]$. In these two checks the general tail bound is $2R^{15}/(1 - R^6)$. For A11 keep the depth-4 rule, apply the attained maximum reduction, and take R to be the upper endpoint of the applicable slab delineated by $.308, .3085, .309, .3095, .31$. Use $m = .228$ and phases $[.633, .696]$. Use an annular support precisely when the T_{n-1}-fold sector width is below π, certified by the positive cross products of every intermediate power of the endpoint quotient. The base angular width is in $(0, \pi/2)$, so this detects the first crossing of π. A10 and A11 meet $m^{k-1} \geq R^k$ at each corrected depth. Floors in A9 and A10 use (11.3) with the lower and upper radial boundaries of the strip.

The table reports the leaf totals in the stored subdivisions, each with a directed lower value strictly above the appropriate cap. Phase-only intervals in the slope cases partition the indicated phase interval at the radial boundary. For the rectangular covers, check inclusion, disjoint interiors and volume as in Section 8.2.

	excluded modulus and phase rectangles (x, t)	leaves
A1	$[.68, .962] \times [0, 1]$	69
A2	$[.32, .68] \times [0, 1]$ at the maximum	1564
A3	$[0, .32] \times [0, .6]$	394
A4	$[0, .10] \times [.6, 1]$	171
A5	$[.10, .15] \times [.6, 1]$, $[.15, .32] \times [.76, 1]$	593
A6	$[.15, .17] \times [.6, .76]$	43
A7	$[.17, .19] \times [.6, .76]$	54
A8	$[.19, .32] \times [.6, .62]$, $[.19, .32] \times [.72, .76]$	186, 421
	$[.31, .32] \times [.62, .72]$	1139
A9	$[.19, .215] \times [.62, .72]$	53
	$[.215, .31] \times [.62, .632]$, $[.215, .31] \times [.70, .72]$	331, 591
A10	$[.215, .228] \times [.632, .70]$	63
	$[.228, .31] \times [.632, .633]$, $[.228, .31] \times [.696, .70]$	90, 501
A11	$[.308, .31] \times [.633, .696]$ at the maximum	489

They give $m = .228 \leq |w_j| \leq R = .308$, $.633 \leq t_j \leq .696$ throughout the bilateral model.

11.4. **The final adjacent-pair disk.** Let $H = H_j$ be the full bilateral profile with H's two ties equal to 1 at degrees 0 and 1. Set $m = .228$, $R = .308$ as just proved, and let $u = w_j = x\chi(t)^2$, $v = w_{j+1} = y\chi(s)^2$. In (11.1) write

$$b_j(-n) = u^n R^{T_{n-1}} \alpha_n, \qquad b_j(n+1) = v^n R^{T_{n-1}} \beta_n.$$

Here $\alpha_1 = \beta_1 = 1$, and the outward products have the annular-sector support already described. The products on the two sides can be bounded independently. Use seven plain groups of angular rows, at

$$H(z), \quad L_k(z) = R^{T_k} H(xR^k z), \quad P_k(z) = yR^{T_k+k} H(z/(yR^k)), \qquad k = 0, 1, 2.$$

At $k = 0$ the radial evaluations are the adjacent knots with monomial factors of modulus one on the unit circle. The virtual comparisons apply from there, so all seven profiles have norm at most C_0. Because $m > R^2$, the stronger bound at L_2 is $C_0|\alpha_2|$ and that at P_2 is $C_0|\beta_2|$. Indeed the scaled displacement-one term has modulus $R < m/R$; the scaled displacement-two term has modulus $|\alpha_2|$ on the left or $|\beta_2|$ on the right and is maximal by the ratio argument. Use two more groups to apply the corresponding linear corrections at these radii, exactly as in Section 11.2.

For clarity the nine groups have order H, L_0, L_1, L_2, corrected L_2, P_0, P_1, P_2, corrected P_2. Consider one table of positive weights, with row points and moments as before, now for degrees $-7, \dots, 8$. The known contribution is $\Re(C_b + C_x x + C_y y)$. Form these coefficients by applying the weighted evaluations at the two tied modes $1, z$, and add $C_0\sigma_g$ to C_b in each corrected group. For the left tail use moment index $j_n = -n$ and for the right $j_n = n + 1$, $1 \le n \le 7$. Define

$$\begin{aligned} A_n &= \sum_{\text{same side } g} R^{T_k + T_{n-1} - kn} M_{g, j_n}, \\ B_n &= R^{T_{n-1}} M_{H, j_n}, \\ D_n &= \sum_{\text{opposite side } g} R^{T_k + T_{n-1} + k(n+1)} M_{g, j_n}. \end{aligned}$$

Set $E_n = 0$ except for $n = 2$, when it is $-C_0\sigma_g(\bar{l} + \bar{r})/(1 + \Re(\bar{l}r))$ for the corrected group on the same side. Then the left factor is

$$F_n = \chi(t)^{2n}\left[A_n + x^n(B_n + y^{n+1}D_n)\right] + E_n$$

and the right factor exchanges (x, t) with (y, s). This is obtained by multiplying the coefficient of each z^{j_n} by the scale and radial power at its group. Its contribution is $\Re F_1$ for $n = 1$. For $2 \le n \le 6$ subtract the support for $(m/R)^{T_{n-1}}$ and the rays $l^{T_{n-1}}, r^{T_{n-1}}$ in direction $-\bar{F}_n$. Their product widths are $< \pi$ since $15 \cdot 4(.696 - .633)/(1 + .633^2) < 3$. At $n = 7$ use a disk, and for $n \ge 8$ subtract $2R^{15}/(1 - R^6)$ for the total tail. In fact the same-side exponents are bounded below by T_{n-3}, as are the larger exponents on the base and opposite groups. Every ordinary or corrected row has this common tail bound.

Floor every real and imaginary part of $A_n, B_n, D_n, E_n, C_b, C_x, C_y$ at scale 2^{48} before the box test. Since $x, y \le 1$, charging $118/2^{48}$ for the fourteen factors and three affine terms covers their total error. Take the minimum directed lower value at the 16 corners and deduct the pure-coordinate interpolation error. For each left factor the three bounds, in coordinates x, y, t, are

$$\begin{aligned} &n(n-1)x_{\text{hi}}^{n-2} \max_{\pm} |B_n + y_{\pm}^{n+1} D_n|, \\ &n(n+1)x_{\text{hi}}^{n} y_{\text{hi}}^{n-1} |D_n|, \\ &\left(n^2(4/(1 + t_{\text{lo}}^2))^2 + 3n\right) \max_{\pm,\pm} |A_n + x_{\pm}^n (B_n + y_{\pm}^{n+1} D_n)|. \end{aligned}$$

In the first line use zero at $n = 1$. The angular bounds follow from the first and second derivatives of the phase $4 \arctan t$, whose second derivative has absolute value at most 3. The right side swaps $(x, t), (y, s)$. Multiply each bound by the width squared divided by 8 and sum. Convexity and the unit Lipschitz bound of the supports justify this lower comparison with the corner values.

The subdivision in the supplementary table K (Table 1) starts with $([m, R] \times [.633, .696])^2$, in order x, t, y, s, and is read in preorder (parent before the lower and upper child). Entries $0, 1, 2, 3$ bisect the indicated coordinate. An entry -2 satisfies $x_{\text{lo}} > y_{\text{hi}}$ and is excluded by ordering $x \le y$, which is allowed by reflecting the whole model. An entry -1 has both curvatures in the disk of radius $1/40$ about

$$c_* = (-191206 + 204897i)/10^6.$$

For this check it suffices to compare the four endpoint distances for each curvature. The angle span is $< \pi$, and the dot products of c_* with its endpoint rays are positive. Over the arc the dot product with c_* thus attains its minimum at an endpoint, and squared distance is convex in radius. At all other leaves entry $j + 4$ uses weight list j (numbering from zero). Each comparison exceeds C_0 after deducting the displayed errors. There are respectively $69, 103, 935$ leaves of the three types. In the four subtrees specified by the first two splits, the lower comparison excesses in units of 10^{-9} are $2272, 2463, 15313, 9381$. The exact tree covers the ordered box. Its reflection gives the other order with

the same curvature orientation, since the normalized reversed profiles exchange the two tail products. This proves $|w_j - c_*| < 1/40$ for all j.

12. The sharp arithmetic comparison

Theorem 12.1. *If a transcendental entire function has arbitrarily late equal-gap runs of unbounded length, then $Q(f) \geq A_*$.*

Proof. If $Q(f) < A_*$, Section 11 gives the bilateral profiles with cap $C < A_*$. Let $w_* = q_* e(t_*)$, $e_* = 101/4000$ and $R_* = 153/500$. Substitution of the q, t box of (1.2) into $qe(t)$ gives $|w_* - c_*| < 1/4000$; also $|c_*| + 1/40 < R_*$. These bounds follow using the rational interval evaluation of the squared distances. Therefore $|w_j - w_*| \leq e_*$ and $|w_j| \leq R_*$. We will prove that for any periodic sequence within these closed bounds, some knot norm is at least A_*.

This suffices for the bilateral sequence. The Gaussian tails in (11.1) are uniform. Retain finitely many terms such that the uniform tail is smaller than a fixed fraction of the strict margin $A_* - C$. Quantize the curvatures using a finite net of representative values within the two closed bounds. Take the net sufficiently fine that all truncated profiles change by less than the same fraction of the margin, by uniform continuity. The finite directed graph whose edges are the consecutive truncation windows of this discretized sequence (prefix and suffix are the vertices) has a directed cycle, since the sequence gives an infinite directed walk. Repeating the cycle gives a periodic sequence whose windows all occur in the discretized sequence. The truncated profile bounds and the uniform tail still give cap strictly below A_*, which contradicts the periodic comparison.

Write $w = w_*$ and $w_k = w + v_k$ for the periodic sequence, with indices cyclic and average $\langle \cdot \rangle_k$. Denote $\Delta^2 = \langle |v_k|^2 \rangle_k$. Shift the index of the profile so that the two first exterior coefficients are w_{k-1} and w_k. With $J_n = 2n + 1$, the half-angle profile is

$$H_k(x) = 2\cos x + \sum_{n \geq 1} \left[e^{iJ_n x} \prod_{r=0}^{n-1} (w + v_{k+r})^{n-r} + e^{-iJ_n x} \prod_{r=0}^{n-1} (w + v_{k-1-r})^{n-r} \right].$$

At zero deviations it is $H_*(x) = 2\sum_{n \geq 0} w^{T_n} \cos(J_n x)$. Its squared norm equals A_*^2, attained at $0, a, -a$, where $\cos^2 a = y_*$, $0 < a < \pi/2$. At each of these angles $(|H_*|^2)' = 0$. Let g_i be the derivative of this squared value in the direction of the real part of the curvature, with $i = 0$ at zero and $i = 1$ at a or $-a$. Put $\lambda = -g_1/(g_0 - g_1)$. The parameter map $(q, t) \mapsto qe(t)$ has nonzero Jacobian at the minimizer. Equations (1.1) give collinear parameter gradients of the active values, with opposite t-derivatives, and thus a unique $\lambda \in (0, 1)$ canceling the linear variation under constant curvature perturbations in both real directions. It agrees with the displayed real-direction formula by $g_0 > 0 > g_1$. We use weights $\lambda, (1 - \lambda)/2, (1 - \lambda)/2$ at $0, a, -a$, respectively.

At any one of these angles x_i, put $F = |H_k(x_i)|^2$, $F' = (|H_k|^2)'(x_i)$. We use the test $F + K_i(F - A_*^2)^2 + c_i(F')^2$. All constants of the comparison are given below.

	$i = 0$	$i = 1$
K_i	$14/5$	$1/5$
c_i	$31/1000$	$17/500$
d_i (window width)	$3/20$	$7/100$
s_i (step)	$1/6$	$7/100$
L_i (curvature bound)	6	14
b_i (bound for $\lvert H_*'(x_i)\rvert$)	0	$6/5$
p_i (value retention)	$5/8$	$7/10$
q_i (derivative retention)	$1/4$	$3/4$

Put $A = 171/100$. Expanding the products along the segment from w to w_k bounds $|H_k - H_*|, |H_k' - H_*'|, |H_k'' - H_*''|$ by $e_*A_1, e_*A_x, e_*A_{xx}$, respectively, where $A_1 = 13/5,\ A_x = 91/10,\ A_{xx} = 34$. For the derivative of order $j = 0, 1, 2$, the sum for the constant is $2\sum_{n\ge1} J_n^j T_n R_*^{T_n-1}$. All segment points have modulus at most R_*. On $|x - x_i| \le d_i$, the lower bound for the second derivative of $|H_k|^2$ is

$$(|H_*|^2)'' - 4e_*A_x|H_*'| - 2e_*(AA_{xx} + A_1|H_*''| + e_*A_1A_{xx}) > -L_i.$$

The last inequalities and $7/25 < \lambda < 3/10,\ |H_*'(a)| < b_1$ are checked below. At x_i the derivative $|F'|$ is at most $2e_*(AA_x + b_iA_1 + e_*A_1A_x)$. The constants give $|s_iF'| < d_i$, so a Taylor bound at $x_i + s_iF'$ gives a squared value at least $F + \gamma_i(F')^2$, where $\gamma_i = s_i - L_is_i^2/2 > 0$. Suppose all knot norms are at most A_*. Then $D = A_*^2 - F \ge 0$ and $(F')^2 \le D/\gamma_i$. The deviation bound gives $D \le D_{\max} = 111/500$, using $2AA_1e_* - (A_1e_*)^2 < D_{\max}$. The rational inequalities $K_iD_{\max} + c_i/\gamma_i < 1$ show that the test is at most A_*^2 at every knot and angle.

For a lower bound on its mean, write $H_k = H_* + U_k + E_k$, with U_k the first variation and E_k of degree at least two, and let $E_{2,k}$ be the degree-two part. Define $\ell_k = 2\Re(\bar H_*U_k)$ and let ℓ_k' be its angular derivative. All norms in the next estimates are normalized over the knots. Expanding the products with integral Taylor remainders gives

$$\|E\|_2 \le e_*B_2\Delta, \qquad \|E'\|_2 \le e_*B_x\Delta, \qquad \langle|E - E_2|\rangle_k \le e_*B_3\Delta^2,$$

where $B_2 = 211/100,\ B_x = 111/10,\ B_3 = 161/50$. The first two bounds come from the sums of $2\binom{T_n}{2}R_*^{T_n-2}$, with the extra factor J_n for the second. The third uses the sum of $2\binom{T_n}{3}R_*^{T_n-3}$. This uses $\|v_iv_j\|_2 \le e_*\Delta$ and $\langle|v_iv_jv_l|\rangle_k \le e_*\Delta^2$ by Cauchy–Schwarz, with indices translated by k. First-order expansion similarly gives the bounds $A_1\Delta$ and $A_x\Delta$ for the L^2 norms of $H - H_*$ and $H' - H_*'$, respectively. Consequently

$$\|F - A_*^2 - \ell\|_2 \le e_*(2AB_2 + A_1^2)\Delta =: B_F\Delta,$$
$$\|F' - \ell'\|_2 \le 2e_*(AB_x + b_iB_2 + A_1A_x)\Delta =: B_{D,i}\Delta.$$

Use $\|X + Y\|_2^2 \ge h\|X\|_2^2 - h\|Y\|_2^2/(1 - h)$ for $0 < h < 1$, by completing a square. For the central value square $\|U + E\|_2^2$ take $h = 24/25$, and for the test's other two squares take p_i, q_i. Expanding the central value and replacing E by E_2 in its pairing with H_* costs at most $2Ae_*B_3\Delta^2$. The test's mean is thus bounded below by $A_*^2 + \langle\ell\rangle_k$ plus

$$\left\langle 2\Re(\bar H_*E_2) + h|U|^2 + K_ip_i|\ell|^2 + c_iq_i|\ell'|^2\right\rangle_k$$

minus $b_{\mathrm{err},i}\Delta^2$, where

$$b_{\mathrm{err},i} = 2Ae_*B_3 + \frac{h}{1-h}(e_*B_2)^2 + K_i\frac{p_i}{1-p_i}B_F^2 + c_i\frac{q_i}{1-q_i}B_{D,i}^2 < 1.$$

The two lower margins $1 - b_{\mathrm{err},i}$ exceed $0.024721, 0.064883$ by substitution. Averaging over knots makes $\langle\ell\rangle_k$ the linear variation under the constant deviation $\langle v_k\rangle_k$, so its angle-weighted mean vanishes.

For the quadratic forms, use cyclic Fourier frequency ω and write $X = \cos\omega$. Define two polynomial sequences C_r°, S_r°, both initially 1, with first successors $2X-1, 2X+1$ and recurrence $P_{r+1} = 2XP_r - P_{r-1}$. They are the polynomial quotients $\cos((r + 1/2)\omega)/\cos(\omega/2)$ and $\sin((r + 1/2)\omega)/\sin(\omega/2)$ with continuous endpoint values. Set

$$P_n = \sum_{r=0}^{n-1}(n-r)C_r^\circ, \qquad S_n^\sharp = \sum_{r=0}^{n-1}(n-r)S_r^\circ, \qquad K_n^\sharp = \sum_{r=0}^{n-1}\binom{n-r}{2} + \sum_{0\le r<s<n}(n-r)(n-s)\mathcal{T}_{s-r}(X).$$

For any active angle x, define

$$U_\omega = 2\sum_{n\ge1} w^{T_n-1}\cos(J_n x)P_n, \qquad V_\omega = 2\sum_{n\ge1} w^{T_n-1}\sin(J_n x)S_n^\sharp,$$
$$Z_\omega = 4\bar H_*\sum_{n\ge2} w^{T_n-2}\cos(J_n x)K_n^\sharp.$$

In U_k, a shift by r in the positive term and by $-1-r$ in the negative term has Fourier multipliers $e^{ir\omega}$ and $e^{-i(r+1)\omega}$. Factoring $e^{-i\omega/2}$ gives complex linear symbol $\cos(\omega/2)U_\omega - \sin(\omega/2)V_\omega$. Its squared energy averaged over $x, -x$ is $\mathcal{E}I$ on the two real components of v, where $\mathcal{E} = (1+X)|U_\omega|^2/2 + (1-X)|V_\omega|^2/2$. Indeed the mixed product cancels in the reflected average, and the resulting scalar weights are even in frequency so the cross terms of the two real component Fourier series cancel between opposite frequencies. The quadratic product mean pairs opposite frequencies of $v_k v_{k+r}$, effectively giving $\cos(r\omega)$ in the mean. Thus the $2\Re(\bar H_* E_2)$ form is the real symmetric matrix with entries $\Re Z_\omega, -\Im Z_\omega, -\Re Z_\omega$ at $00, 01, 11$. The computation accounts for repeated shifts in a period by the same Fourier identity.

For a complex number z, let $T(z) = (\Re z, -\Im z)$. The real linear variation ℓ has symbol, after the same unit factor, $\cos(\omega/2)T(2\bar H_* U_\omega) + i\sin(\omega/2)T(2i\bar H_* V_\omega)$. This follows by combining the coefficients at ω and $-\omega$ when taking a real part. In the squared energy the mixed term cancels in the reflected angular average. Therefore its matrix is

$$\frac{1+X}{2}T(2\bar H_* U_\omega)^{\mathsf T}T(2\bar H_* U_\omega) + \frac{1-X}{2}T(2i\bar H_* V_\omega)^{\mathsf T}T(2i\bar H_* V_\omega).$$

Differentiate the two row vectors in x to get the matrix for ℓ' by the same formula. At zero angle $V_\omega = 0$ and $\partial_x(\bar H_* U_\omega) = 0$, so each mixed term vanishes there separately. These matrices also apply at zero and self-opposite frequencies by their polynomial expressions and the endpoint trigonometric values.

Here are the exact interval tests for these estimates. Use the parameter box (1.2), $X \in [-1,1]$, and fixed-point scale 2^{60}. Retain $n \le 8$ and enclose the omitted part of each raw complex sum, before products, by adding $[-10^{-12}, 10^{-12}]$ to each coordinate. An upper majorant for every needed raw tail is

$$2\sum_{n\ge9}(n+1)^6(2n+1)^2 R_*^{T_n-3} < 10^{-12}.$$

Here $|C_r^\circ|, |S_r^\circ| \le 2r+1$ from the finite geometric sums for the quotients, and the bound for $K_n^\sharp$ follows from $\binom{T_n}{2}$. The ratio of consecutive majorant terms is at most $16R_*^{10} < 1/2$, and twice the initial term is below 10^{-12}. It also bounds the tails for the six derivative and remainder constants. Compute w from $qe(t)$ first, then take powers by successive multiplication, and enclose the trigonometric terms using $y = \cos^2 x$. For $\cos(J_n x)$ use $\sqrt{y}\,S_n(y)$, and for $\sin(J_n x)$ use $\sqrt{1-y}$ times the sequence with initial terms $1, 4y-1$ and recurrence $P_{n+1} = (4y-2)P_n - P_{n-1}$. Take directed roots at the interval endpoints.

For the weight g_i, evaluate $2\Re(\bar H_* \partial_w H_*)$, using the n-th summand $2T_n w^{T_n-1}\cos(J_n x)$ for the derivative. At $y = 1$, $g_0 > 0$. In the interior angle range test $-g_1 > 0$, $(1-7/25)(-g_1) - (7/25)g_0 > 0$, $(3/10)g_0 - (1-3/10)(-g_1) > 0$, and $6/5 - |\partial_x H_*| > 0$. Midpoint bisection of the y range gives six leaves and lower margin exceeding 0.001503. For the angular second-derivative inequalities use $y \in [391/400, 1]$ about zero and $y \in [563/1000, 713/1000]$ about either interior angle. These contain the respective windows by $|\sin x| \le |x|$ and $|(\cos^2 x)'| \le 1$. Substitute $(|H_*|^2)'' = 2\Re(\bar H_* H_*'') + 2|H_*'|^2$ in the bound with $-L_i$ and bisect in y. There are six and four leaves with lower margins exceeding $0.012456, 0.238432$, respectively.

For the Fourier test of the quadratic matrices, at zero angle use $y = 1$ and at the two interior angles take $y \in [.633, .643]$. The interval matrix is the sum of $h\mathcal{E}I$, the matrix for Z_ω, $K_i p_i$ times the value row matrix and $c_i q_i$ times the derivative row matrix. At each respective endpoint $\lambda = 7/25, 3/10$ form the weighted sum, subtract $(4/3)I$, and test for positive lower endpoints of both diagonals and their product

minus the off-diagonal square. Bisect any unaccepted rectangle in X if its X-width is at least three times its y-width, and in y otherwise. The resulting cover has 46 leaves, depth at most 10, and minimum lower endpoint among these three tests greater than 0.357326. Convexity of positive definiteness gives the bound for every intermediate weight. The finite test uses the polynomial frequency recurrences above throughout $[-1, 1]$.

By Fourier orthogonality, the angle-weighted mean of the test is at least $A_*^2 + (4/3 - 1)\Delta^2$. If $\Delta > 0$, it exceeds the pointwise cap A_*^2. If $\Delta = 0$, all curvatures are w_* and the norm is A_*. In both cases some knot norm is at least A_*. The periodic closing argument now proves the theorem. □

Appendix A. Tables for the exact comparisons

The numerical data of the proof are provided as JSON in the `anc/` folder of the source archive. Table 1 identifies the five tables and the corresponding formulas. Their entries are rational numbers, integer test weights and subdivision data. Table D is provided in five numbered JSON parts. Run `python anc/rebuild_d.py` from the extracted source directory to recreate the original `anc/D-equal-gaps.json`. A rational endpoint is recorded as an integer or a fraction, and endpoints are in consecutive lower–upper pairs. Numbering of rows and group indices starts at zero unless a row identifier is given explicitly.

Table	Data file	Comparison
P	P-plateau.json	Radial Schur comparison in Section 6
S	S-peaks.json	Strict peaks in Section 9
D	D-equal-gaps-01.json through -05.json	Equal gaps in Section 10
A	A-arithmetic-windows.json	Covers in Section 11.3
K	K-adjacent-pairs.json	Adjacent-pair disk in Section 11

Table 1. Exact comparison data.

Table P.. Each entry lists the endpoints of a rectangle in c, u. Use the four rejection tests following the definition of $P = (41/100)^2$.

Tables S and D.. For each model, rows gives a row identifier and an array weights. Read the weights in order of knot, component and increasing integer mode. For distinct-residue models of S these are real integers. In the two-component models each entry consists of two consecutive integers giving real and imaginary parts. Divide all weight coordinates by 2^{30}. For vertex-dependent rows, point identifies the corner and vertexFloor is a lower bound for the corner margin in units of 10^{-9}. The box records have either a single row identifier, eight vertex row identifiers in lexicographic order of the endpoints, or a reason for exclusion. In the variable ratio model S the fixed-point power enclosures are indicated by the arithmetic field fixed. A comparison stores the lower margin floor after the interpolation and integration allowances, in the same units.

Table A.. The groups A1 through A11 follow the comparison order in Section 11.3. Each part lists the row lists weights and the cover records. In A1 a row lists its weight followed by the two angular parameters. Otherwise a row begins with its angular group, then the weight and the two angular parameters. In A1–A4 the group number is the virtual depth. In A5 the depth-2 rows are corrected. In A6–A8 the order is $0, 1, 2,$ corrected 2. At greater depth the ordinary group is followed immediately by the corrected group at each depth $k \geq 2$. Ordinary cover records list four endpoints followed by a row-list index. A radial boundary part lists the lower and upper radial boundary in its root box, and each record lists its two phase endpoints and row-list index. Use (11.3) on the radial interval.

Table K.. Use the nine fixed angular groups and the midpoint tree described in Section 11. The angular row format is the same as in table A. A tree entry referring to a row list is offset by 4 to distinguish it from a coordinate split.

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