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SHARP GROWTH AND GAP EXPONENTS FOR ADMISSIBLE SEQUENCES

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Abstract. An increasing sequence of positive integers is admissible if any two finite subsets with the same sum have the same cardinality. We solve the growth and consecutive-gap questions of Deshouillers and Erdős. The least possible exponent in an eventual polynomial bound on the n-th term is $2 + 2\sqrt{2}$. The least possible consecutive-gap exponent is $(5 + \sqrt{17})/2$. Both endpoints are attained with little-o bounds, and the endpoint gap bound holds with coefficient one at every index. We also determine the tradeoff when both bounds are imposed on the same sequence. The lower bounds use subset-sum counts, a restricted-sum progression theorem, and an elementary large-gap lemma. The upper bounds use an induction on congruences of signed subset sums.

1. Introduction

Following Deshouillers and Erdős [4, 1], call $A = \{a_1 < a_2 < \cdots\}$ admissible if the sets of sums of r distinct elements are disjoint as $r \geq 1$ varies. If zero were allowed, comparing a positive singleton with that singleton together with zero would rule it out. Positivity then handles the empty subset as well. Equivalently, every finitely supported relation $\sum_i \varepsilon_i a_i = 0$ with $\varepsilon_i \in \{-1, 0, 1\}$ has $\sum_i \varepsilon_i = 0$. We call the latter sum the signed cardinality. Write $\Delta_n = a_{n+1} - a_n$, and set

$$c_0 = \frac{5 + \sqrt{17}}{2}, \qquad p_* = 2 + 2\sqrt{2}, \qquad P(c) = \frac{8c}{(c-2)^2}.$$

The finite admissible-set problem has a different scale. For example, Deshouillers and Freiman [2] showed that for large integer x the largest admissible subsets of $[1, x]$ have cardinality of order $\sqrt{x}$, attained by a consecutive interval at its upper end.

Erdős [3] constructed infinite admissible sets with polynomial counting growth. Erdős, Nicolas, and Sárközy [5, Theorem 2] obtained the bound $\#(A \cap [1, x]) \gg x^{5-2\sqrt{6}}$. For finite sets, Nicolas [8] studied the related condition that the cardinality layers are separated in increasing numerical order. Wang and Yang [10, Theorem 1.3 and Corollary 5.2] proved that every infinite admissible set has $\#(A \cap [1, x]) = o(\sqrt{x})$ and that $\limsup_{n\to\infty} \Delta_n/n = \infty$. Their Remark 5.3 leaves open which polynomial gap bounds can hold.

GPT-5.5 Pro and Lech Mazur [7] constructed an infinite admissible set with $\Delta_n \leq n^{3+2\sqrt{2}}$ for every $n \geq 1$, using a carry invariant for signed subset sums. Their construction also gives little-o gaps at the same exponent. The erdosSweep note [6] sharpens local carry bounds and gives a quadratic restriction on the scalar carry method. The construction in Section 6 uses an affine congruence and two successive blocks to attain the joint term and gap bounds.

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Theorem 1.1. *There is an infinite admissible sequence with $a_n = O(n^p)$ and $\Delta_n = O(n^c)$ if and only if*

$$c \geq c_0, \qquad p \geq \begin{cases} P(c), & c_0 \leq c \leq p_*, \\ p_*, & c \geq p_*. \end{cases}$$

For every admissible pair of exponents, the sequence may be chosen with $a_n = o(n^p)$, $\Delta_n = o(n^c)$, and $\Delta_n \leq n^c$ for every $n \geq 1$.

In particular p_* is the least term exponent, and c_0 is the least exponent for the all-index gap inequality in Erdős Problem 875. The largest exponent α for which one can have $\#(A \cap [1,x]) \gg x^\alpha$ is $1/p_* = (\sqrt{2}-1)/2$, by evaluating at $x = a_n$ in one direction and between consecutive terms in the other. Arbitrarily fast growth is also possible. Given prescribed targets, take each successive term larger than its target and the sum of its predecessors. The largest term in the symmetric difference of two subsets determines which sum is larger.

2. Large gaps at actual indices

Lemma 2.1. *Every infinite admissible sequence satisfies*

$$\Delta_j > \frac{1}{4} \sum_{i \leq j-3} a_i$$

at infinitely many indices j. In addition, a_{n+1}/a_n cannot tend to 1.

Proof. First consider two unbounded sets X, Y of nonnegative integers containing zero, with gaps bounded by P_0, Q_0, respectively. Suppose functions f, g on these sets vanish at zero and $f(x) + g(y)$ depends only on $x + y$. Then $k(y-x) = g(y) - f(x)$ is well defined on $Y - X$. For adjacent $x < x'$ in X, choose $y \in Y \cap [x, x+Q_0]$. The difference $f(x') - f(x)$ is the difference of two k-values on $[-P_0, Q_0]$. The same bound holds for adjacent jumps of g, by choosing $x \in X \cap [y, y+P_0]$. The piecewise linear interpolations F, G are therefore L-Lipschitz for some finite L.

For each real $d \geq 0$, the image of $H_d(t) = G(t+d) - F(t)$, $t \geq 0$, lies within distance $L(P_0 + Q_0)$ of the k-values at integers in $[d - P_0, d + Q_0]$. This follows by choosing $x \in X \cap [t, t+P_0]$ and $y \in Y \cap [t+d, t+d+Q_0]$. The image is connected and the number and lengths of the covering intervals are bounded uniformly in d. Hence every H_d has oscillation at most a fixed T. From

$$F(u+d) - F(u) - F(d) = H_d(u) - H_d(0) - H_0(u+d) + H_0(d)$$

we get a Cauchy defect bounded by $2T$. Iterating gives $|F(md) - mF(d)| \leq 2T(m-1)$. If $f(x) = o(x)$, interpolation gives $F(t) = o(t)$, and then $|F(d)| \leq 2T$ for every d. This makes f bounded.

For an increasing sequence (b_i), its finite subset-sum set, including zero, has bounded gaps whenever $\sup_j (b_j - \sum_{i<j} b_i) < \infty$. Indeed, adjoining b_j takes the union of the previous sum set and its translate by b_j. Inside the two hulls the old gap bound suffices. If the hulls are disjoint, their separation is $b_j - \sum_{i<j} b_i$. This proves the assertion for each prefix and then for their union.

Apply these facts to the even and odd subsequences of A. Their sum sets X, Y carry the cardinality functions f, g. Admissibility on the disjoint even and odd supports gives the required relation on $x + y$. These functions are unbounded, and $f(x) = o(x)$ because a subset summing to x has cardinality at most $k + x/b_{k+1}$ for any k. If both parity sequences had bounded deficits, the preceding argument would make f bounded. Thus for one fixed parity there are arbitrarily late j with $b_j > \sum_{i<j} b_i$. With $b_j = a_\ell$ this gives

$$a_\ell - a_{\ell-2} > \sum_{i \leq j-2} b_i \;\geq\; \frac{1}{2} \sum_{i \leq \ell-4} a_i.$$

For the second inequality, pair each term of the other parity with the next term of the chosen parity up to $\ell - 4$. One of the two adjacent gaps is at least half of this two-step increment. Its left index is $\ell - 2$

or $\ell - 1$, which proves the claimed strict bound. If $a_{n+1}/a_n \to 1$, both parity ratios tend to one, so the two preceding terms of each parity eventually have sum greater than its next term. This would give bounded deficits for both parities. □

3. Counts in the early part of the sequence

Suppose $a_n \ll n^p$, $\Delta_n \ll n^c$, where

$$4.55 \le c \le p_*, \qquad c \le p \le c+1, \qquad p \le 5.562. \tag{3.1}$$

Set $B = p + 1$, $S_n = \sum_{i \le n} a_i$, $S_0 = 0$, and let $T(x)$ be the largest n with $S_n \le x$. Write $Z(n)$ for the number of all subset-sum values on the first n terms, $F(n)$ for the largest size of a single cardinality layer there, and $W(n)$ for the size of the layer with cardinality $\lfloor n/2 \rfloor$. We have $Z(n) \le S_n + 1$ and $F(n) \ge Z(n)/(n+1)$. The caps imply $S_n \ll n^B$. They also imply

$$a_n^2 \ll S_n n^c, \qquad S_{n+1} \ll S_n + (n+1)^c. \tag{3.2}$$

For the first inequality, $\gg \max\{1, a_n/n^c\}$ of the last terms up to n are at least $a_n/2$, by summing the gap bound backwards and using $a_n \ll n^{c+1}$. The second follows from $a_{n+1} \le a_n + O(n^c)$.

Lemma 3.1. *If $W(n) \gg (n+1)^u$ for $2 \le u \le c-1$, then $Z(n) \gg (n+1)^{u+1}$. For every fixed $v < u/c$ with $v > 0$, we have $F(T(x)) \gg x^v$ for $x \ge 1$.*

Proof. Append one subset of each available cardinality from the second half of the first n terms to the middle layer of the first half. These families are disjoint by admissibility, proving the assertion about Z. Put $U = u + 1$. The estimate for S_n gives $Z(T(x)) \gg x^{U/B}$.

Suppose $Z(T(x)) \gg x^w$, where $w < 1$. For $8 \le H \le a_n$, let $k(H)$ count the terms at least H among the first n. The sum set through $T(H/8)$ lies in $[0, H/8]$ and uses none of these $k(H)$ terms. Shift it by their $k(H) + 1$ cumulative sums. The shifts are spaced by at least H, so $Z(n) \gg (k(H)+1)H^w$. Integration yields $S_n \ll Z(n)a_n^{1-w}$. The integral below $H = 8$ is $O(n)$, absorbed by $Z(n) \ge n + 1$. By (3.2) it follows that

$$S_n \ll Z(n)^{2/(1+w)} n^{c(1-w)/(1+w)}.$$

At $n = T(x)$ we have $x < S_{n+1} \ll S_n + (n+1)^c$. Using $n + 1 \ll Z(n)^{1/U}$ gives the new exponent

$$\min\left\{\frac{1+w}{2 + c(1-w)/U}, \frac{U}{c}\right\} \quad \text{in } Z(T(x)).$$

The gain is positive for $w < U/c$, since $1 + w - w[2 + c(1-w)/U] = (1-w)(1 - cw/U)$. Starting at U/B, finite iteration gives every exponent strictly below U/c.

For $F(T(x))$, retain the factor involving n. If $x \ll (n+1)^c$, use $F(n) \ge W(n) \gg x^{u/c}$. In the other branch $x \ll S_n$, so

$$F(n) \gg \max\{(n+1)^u, x^{(1+w)/2}(n+1)^{-1-c(1-w)/2}\}.$$

Balancing the two terms yields $F(n) \gg x^{u(1+w)/(2U+c(1-w))}$. The exponent tends to u/c as $w \uparrow U/c$. □

4. Progressions in one cardinality layer

Pair opposite indices in the interval from $\lfloor n/2 \rfloor + 1$ to $\lfloor n/2 \rfloor + 2\lfloor n/16 \rfloor$. These disjoint pairs have distinct positive differences, forming a set E of order n in an interval of length $O(n^p)$. Let $\ell = \lfloor |E|/2 \rfloor$. Szemerédi and Vu [9, Theorem 7.1] give a proper generalized arithmetic progression of rank r with $\gg n^{r+1}$ points in the sums of ℓ distinct elements of E. Their hypothesis with dimension parameter 5 holds because $\ell^5|E| \asymp n^6$ and $p < 6$. Translate by the sum of the smaller endpoints of all pairs. The resulting progression $\mathcal{P}$ has one fixed-cardinality representation per value using one endpoint from every pair. Differences of its values thus have signed cardinality zero on the paired support.

Assume the hypothesis of Lemma 3.1, choose $0 < v < u/c$, and put $\mu = p - Bv > 0$. An ordinary subprogression of $\mathcal{P}$ of step g and length $L \geq 2$ protects the preceding prefix through $m \asymp (gL)^{1/B}$, where a fixed small multiplier ensures $m \leq n/4$ and $S_m \leq g(L-1)$. If two subset sums from this prefix agree modulo g, their difference is realized by subtracting two values of the ordinary progression. The signed relation on the disjoint supports forces equal cardinalities. Take

$$t = \min\{T(g/4), \lfloor m/4 \rfloor\}.$$

An early layer through t has values distinct modulo g. Padding it with one subset of each size from the next $m - t$ terms gives disjoint residue families. In the uncapped case Lemma 3.1 gives $g \gg mg^v$. In the capped case $W(t) \gg m^u$ gives $g \gg m^{u+1}$. In particular

$$L \ll g^\mu, \qquad \mu \geq B/(u+1) - 1. \tag{4.1}$$

The last comparison follows from $p - Bu/c - [B/(u+1) - 1] = Bu(c-u-1)/(c(u+1)) \geq 0$. Bounded spans are covered by increasing the constant.

If the rank is one, use a progression of length $L \asymp n^2$; a rank-two box with a side of length one also provides it. Its step obeys $g \ll n^{p-1}$, so $m \gg g^{1/(p-1)}$. Because $p - 1 \leq c$, the capped case still supplies an early layer of size $\gg g^v$. Equation (4.1) now gives $g \gg n^{2/\mu}$. Adding that early layer to the progression gives disjoint translates. Padding to the middle layer uses available indices outside both supports. This proves

$$W(n) \gg n^{F_1(v)}, \qquad F_1(v) = 2 + \frac{2v}{\mu}. \tag{4.2}$$

Rank at least three gives $W(n) \gg n^4$ directly. Rank two gives $W(n) \gg n^3$, which suffices for $u < 2.99$.

4.1. **Separation of rank-two chains.** Suppose $u \geq 2.99$ and $\mathcal{P}$ has two nontrivial sides of sizes L_1, L_2. Write $f(x) = g_1x_1 + g_2x_2$ for the progression map on the coordinate box, up to translation. Put $M = L_1L_2 \gg n^3$, $V = \operatorname{diam}\mathcal{P}$, and $R = V/M$. Choose $m \asymp V^{1/B}$ with $m \leq n/4$ and $S_m \leq V/2$. The $m + 1$ translates supplied by different cardinalities in this prefix are disjoint by admissibility. Hence $R \gg m$.

At least one $|g_i|/L_{3-i}$ is at least $R/2$. All nonzero coordinate differences with numerical image of absolute value at most $R/10$ are collinear. Two linearly independent differences would have nonzero integral determinant. Solving for g_i would give both $|g_i| \leq 2(R/10)L_{3-i}$, a contradiction. If such a direction exists, choose its primitive integer vector $z = (a, b)$ and put $d = |f(z)|$. Properness implies $1 \leq d \leq R/10$. All nonzero numerical spacings are at least d. Let

$$\lambda = \min_{z_i \neq 0} \frac{L_i - 1}{|z_i|}, \qquad L = \lfloor \lambda \rfloor + 1.$$

Here $\lambda \geq 1$, and L is the largest length of a chain in direction z, achieved from a suitable corner. By (4.1), $L \ll d^\mu$.

Relabel so that $\lambda = (L_1 - 1)/|a|$. The integer $k = ax_2 - bx_1$ indexes the chains because z is primitive. We have

$$f(x) = \frac{g_2}{a}k + \frac{f(z)}{a}x_1, \qquad V \leq 2|g_2|(L_2 - 1) + d\lambda.$$

The second estimate follows from $|ag_1| \leq |bg_2| + d$. Since $d\lambda \leq V/10$ and $|a|(L_2 - 1) \leq M/\lambda$, we get $|g_2/a| \geq 0.45R\lambda$. Points on distinct chains are separated numerically by at least $0.35R\lambda \geq 0.175RL$.

Take $\delta = 3/25$ and $H = \min(RL, R^{1+\delta})/10$. A length-$H$ interval meets at most one chain. Adding a layer through $T(H)$ therefore has multiplicity at most L. A layer through $T(d/4)$ gives disjoint translates. The larger of these two fixed-cardinality families has size at least a constant times

$$M \max\left\{d^v,\ R^v \frac{\min(L, R^\delta)^v}{L}\right\}.$$

Because the quotient is decreasing in L, use $L \ll d^{\mu}$ in the second term. If $d^{\mu} \geq R^{\delta}$, balancing the terms gives a factor R^{σ}, where $\sigma = (1+\delta)v^2/(\mu+v)$. If $d^{\mu} < R^{\delta}$, the second term gives a factor at least of order $R^{v-\delta(1-v)}$. Choose v sufficiently close to u/c, so $v \geq 0.60$. We have $\sigma/v \leq 1.12(c-1)/p < 0.89$ and $v-\delta(1-v) \geq 0.92v$. Thus the latter case gives the same bound MR^{σ}. If there is no short direction, a layer through $T(R/10)$ gives disjoint translates and an even larger lower estimate MR^{v}.

The early support in each case lies within $T(R^{1+\delta}) = o(m)$. Indeed $S_j \gg (j+1)^{u+1}$ by Lemma 3.1, and

$$\frac{T(R^{1+\delta})}{m} \ll n^{(B-3)(1+\delta)/(u+1)-1} = o(1),$$

using $V \ll n^B$, $M \gg n^3$, and $(B-3)(1+\delta) \leq 3.562 \cdot 1.12 < 3.99$. Pad the family by one subset of each size from the unused part of the prefix through m. Their span is at most $V + S_m$, and their cardinalities are distinct. Therefore $V \gg mMR^{\sigma}$, which gives $R \gg M^{1/(p-B\sigma)}$. Padding the original one-layer family to the middle cardinality instead proves

$$W(n) \gg n^{F_2(v)}, \qquad F_2(v) = 3 + \frac{3\sigma}{p - B\sigma}. \tag{4.3}$$

All middle-layer paddings are possible because the early support and the paired support together use fewer than $n/2$ indices. The denominator is positive since $\sigma < v$ and $\mu > 0$.

5. Excluding all smaller exponents

Under (3.1), assume $p < P(c)$. At $v = u/c$, the inequality $F_1(v) > u$ is equivalent to

$$Bu^2 - (cp + 2B - 2)u + 2cp > 0.$$

The discriminant is $p[p(c-2)^2 - 8c] < 0$. For the refined rank-two argument, $F_2(u/c) > 3$. When $u \geq 3$, writing $s = u - 3$ gives the equivalent inequality

$$1.12(s+3)^2(3 + (p+1)s) - cp^2(c-3-s)s > 0.$$

Here $0 \leq s \leq c-4$. For $s > 0$ the expression decreases in p throughout the range, since $c - 3 - s \geq 1$ and $1.12(s+3)^2 < 2cp(c-3-s)$. Increase p to $\min(5.562, P(c))$. On each rational interval $[l, h]$ below put $B_- = 1 + P(h)$ and $K = h \min(5.562, P(l))^2$. Bounding the positive term with B_-, using K and h in the negative term, and discarding the positive cubic yield a quadratic with coefficients

$$27(1.12), \qquad 1.12(18 + 9B_-) - K(h-3), \qquad 1.12(3 + 6B_-) + K.$$

The leading coefficient is positive, and direct substitution of the rational endpoints gives the strict upper bounds on its discriminant displayed below. In each interval $P(h) < 5.562$, so the lower bound for the positive term applies.

$[l, h]$	discriminant	$[l, h]$	discriminant
$[4.55, 4.562]$	< -4811	$[4.562, 4.59]$	< -3175
$[4.59, 4.62]$	< -2810	$[4.62, 4.66]$	< -1987
$[4.66, 4.70]$	< -1712	$[4.70, 4.75]$	< -920
$[4.75, 4.80]$	< -667	$[4.80, 4.829]$	< -1469

Thus the rank-two gain is also strict. For $u \leq 2.99$ the direct exponent 3 is already larger than u. The exponent 4 for higher ranks is larger than $c - 1$.

For fixed c, p, compactness on the two subintervals of $2 \leq u \leq c-1$ lets us take $v = u/c - \varepsilon$, with one sufficiently small $\varepsilon > 0$, and gives a uniform positive gain in u. Initially $W(n) \gg n^2$, since moving the chosen middle-layer indices one place at a time from their least to their greatest possible positions gives a strictly increasing path of $1 + \lfloor n/2 \rfloor \lceil n/2 \rceil$ sums. Finitely many applications of Lemma 3.1 and the progression estimates give $W(n) \gg n^{c-1+\eta}$ for some $\eta > 0$. The cardinality padding gives $S_n \geq Z(n) - 1 \gg n^{c+\eta}$. Lemma 2.1 contradicts the gap cap.

To see that this excludes every claimed range, first suppose a gap bound holds for an exponent less than c_0. Increase it if needed to $c \in [4.55, c_0)$, and sum the gaps to obtain the term exponent $p = c + 1$. Then $p < P(c)$ because $c^2 - 5c + 2 < 0$. Second, a term bound below p_* implies a term bound for some $p \in [4.8, p_*)$ and the gap bound $c = p$. Here $p < P(p)$ because $(p-2)^2 < 8$. Finally if $c_0 \le c \le p_*$ and $p_* \le p < P(c)$, the parameters satisfy (3.1) since $P(c) \le P(c_0) = c_0 + 1$. This proves the necessity in Theorem 1.1.

6. Construction attaining the joint boundary

Fix $c \in [c_0, p_*]$ and set

$$p = P(c), \quad \rho = \frac{4c}{c+2}, \quad s = c - \rho, \quad t = \frac{\rho}{\rho - 2} = \frac{2c}{c-2}.$$

Then $\rho > 2$, $1 < s \le 2$, $t > 3$, $c \le p \le c + 1$, and

$$\rho + s = c, \qquad t(\rho - 2) = \rho, \qquad sp = \rho + 2t. \tag{6.1}$$

At one stage let $A \subset [1, B)$ have size $N \ge 2$. Let h be a positive integer, $d = h/\gcd(B, h)$, $r = \lceil N^s \rceil$, and $D = \sum_{a \in A}(B - a)$. Maintain

$$d \le r, \qquad D + 10hN^2 < Bd. \tag{6.2}$$

For any signed sum u on A with signed cardinality J_A and any integer q, maintain

$$u + Bq \equiv 0 \pmod{h} \implies J_A + q \equiv 0 \pmod{d}. \tag{6.3}$$

These conditions imply admissibility. If $u = 0$, then $d \mid J_A$, and $|BJ_A| \le D < Bd$, hence $J_A = 0$.

Set $K = d\lceil 16r/d \rceil$ and $g = BK$, so $16r \le K < 17r$ and $h \mid g$. Fix $C \ge 100$ with $65C^{-(\rho-2)} < 1/2$, and set

$$k = \lceil CN^t \rceil, \qquad R = \lceil 64k^2/K \rceil, \qquad B' = B + gR.$$

We have $k \ge 2K$, $64Bk^2 \le B' \le 65Bk^2$, and $R - k \ge R/2 > (r/N)^3$. The last step uses $r \le N^2$ and $R/2 \ge (32/17)C^2N^{2t-2} > N^3$.

Put $H = R - k - (r/N)^3 > 0$. Define the row indices by

$$j_i = \left\lfloor (i/N)^3 + H(i/r)^7 \right\rfloor \qquad (0 \le i \le r).$$

For $0 \le i < r$, let ℓ_i count preceding indices in its own row. Append r values $e_i = B + gj_i + h\ell_i$, in order of increasing i. Equal rows have fewer than N indices preceding any given index, since their cubic values differ by less than one. Thus $h\ell_i < g$ and the values increase. They begin at B and end below $B + g(R - k)$.

We need $\sum_i \ell_i < 10N^2$. Equal-row pairs have cubic values in the same or consecutive unit intervals. Let b_j count indices with $\lfloor (i/N)^3 \rfloor = j$, and let $w_j = N((j+1)^{1/3} - j^{1/3})$. Then $b_j \le \lceil w_j \rceil$, so $\binom{b_j}{2} \le 2w_j^2$. Since $w_0 = N$ and $w_j \le Nj^{-2/3}/3$ for $j \ge 1$, we have $\sum_j \binom{b_j}{2} \le (26/9)N^2$ by the integral test. Also $\sum_j b_j b_{j+1} \le \sum_j b_j^2$. The resulting pair count is at most $3(26/9)N^2 + r < 10N^2$.

For a signed sum u on the old set and the fillers, with signed cardinality J, suppose $u + Bq \equiv 0 \pmod{g}$. Each filler is congruent to B modulo h. Reducing modulo h in (6.3) gives $d \mid J + q$. Write $u + Bq = B(J + q) + gI + \Delta$, with $|\Delta| \le D + h\sum_i \ell_i < Bd$. Since $d \mid K$, divisibility by $g = BK$ forces $\Delta = 0$ and $K \mid J + q$.

Now append the k terms $B' - gj$, $1 \le j \le k$, in increasing order. Their smallest is at row $j_r = R - k$, beyond the fillers. They are all congruent to $B' \equiv B \pmod{g}$. The preceding conclusion gives (6.3) for the new set A', with reference B', modulus $h' = g$, and period $d' = K$. Its size $N' = N + r + k$ is at most $2k$, and $d' \le \lceil (N')^s \rceil$. Since $N + r \le K/8$, its deficits satisfy

$$D(A', B') + 10g(N')^2 \le B'(N + r) + 41gk^2 \le (1/8 + 41/64)B'K < B'd'.$$

Thus the induction closes. Start with $A = \{B - N, \dots, B - 1\}$, $B = 12N^2$, and $h = d = 1$. The two invariants hold, so all finite relations in the infinite sequence have signed cardinality zero.

The normalized reference contracts by

$$\frac{B'}{(N')^\rho} \le 65Bk^{2-\rho} \le 65C^{-(\rho-2)}\frac{B}{N^\rho} < \frac{1}{2}\frac{B}{N^\rho}.$$

A filler at index $n = N + i + 1$ has value at most $B + g + g(i/N)^3 + B'(i/r)^7$. Because $p > 3$ and $7 > p$, its normalized value a_n/n^p is $O(g/N^p + B'/r^p) = O(B/N^\rho)$, by (6.1). The dense group begins beyond $N + r$ and satisfies the same bound using B'/r^p.

For an adjacent pair within one row the gap is $h \le g$. At a row change, including the first dense term at $i = r$, the next within-row index is zero. Thus a gap with left index $n = N + i + 1$ in the filler part is at most

$$g\left(1 + 3(i + 1)^2/N^3 + 7H(i + 1)^6/r^7\right).$$

Dividing by n^c yields $O(g/N^c + B'/r^{c+1}) = O(B/N^\rho)$, since $2 < c < 6$ and $p \le c + 1$. The other gaps in the stage, including the gap to the next reference, equal g and satisfy the same bound. Contraction proves both little-o assertions, with constants depending only on c. Taking the initial N large makes the normalized bound at most one throughout. The seed gaps and the first bridge each equal one. Finally, the construction for $c = p_*$ also works for every larger c. This completes Theorem 1.1.

7. Pointwise growth

Every admissible sequence has $a_n/n^2 \to \infty$ and $\Delta_n \to \infty$. Wang and Yang [10, Corollary 5.2] proved the former conclusion under a more general hypothesis. The following tail argument proves both assertions. Suppose a fixed integer $d > 0$ were represented by signed sums arbitrarily far out. Disjoint representations of d have a common signed cardinality h. Take d distinct terms larger than d^2. Two of their $d + 1$ cumulative sums agree modulo d, so between them is a nonempty subset of size $r \le d$ with sum kd and $k > d$. Choose k pairwise disjoint later signed representations of d. Admissibility would give $r = kh$, which is impossible. Hence for every fixed H all distinct subset-sum values on a sufficiently late tail are spaced by more than H. This proves gap divergence. On m tail terms the layer of size j has at least $1 + j(m - j)$ values by a monotone path, and the layers are disjoint. Summing and comparing with the total span ma_n gives $a_n \ge Hm^2/6$. Take $m = n - O_H(1)$, then let H increase.

Given an admissible prefix A, set $M = 1 + \sum A$ and append $M\{k^2, k^2 + 1, \dots, k^2 + k - 1\}$. In a signed zero relation the old sum is divisible by M, hence vanishes and has signed cardinality zero. The normalized new sum is $k^2 j + I$ with $|I| \le k(k - 1)/2$, so $j = 0$ as well. At stage j choose $k \ge \max\{N^2, (2M)^j\}$, where N is the old length. This gives $a_{N+k} \le 2Mk^2$ and $\Delta_{N+k-1} = M$ on the resulting subsequences. In conjunction with the universal divergence, the lower logarithmic orders are 2 and 0.

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