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GPF-TRIBONACCI SEQUENCES ADMIT NO UNIFORM LINEAR BOUND

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Abstract. A GPF-Tribonacci sequence is a sequence of primes in which every term is the greatest prime factor of the sum of the three preceding terms. Back and Caragiu asked whether there is a constant L such that every term of every such sequence is at most L times the largest of its three initial terms. We show that the answer is no. For every $C > 0$ we show that there are GPF-Tribonacci sequences, starting from three distinct and arbitrarily large primes, in which three consecutive terms are all larger than C times the largest initial term. This happens after $O(\log C)$ steps. In the proof we prescribe the quotient of the sum of three consecutive terms by the next term. These quotients follow the periodic pattern $1, 3, 5$. Then we apply the theorem of Green, Tao and Ziegler on linear equations in primes. The conjecture of Back and Caragiu that every GPF-Tribonacci sequence is ultimately periodic remains open.

1. Introduction

1.1. **GPF sequences.** For an integer $m \geq 2$, let $\mathrm{gpf}(m)$ be the greatest prime factor of m. A *GPF sequence* is a sequence of primes $(p_j)_{j\geq 0}$ that satisfies a recurrence of the form

$$p_j = \mathrm{gpf}(c_1 p_{j-1} + c_2 p_{j-2} + \cdots + c_d p_{j-d} + c_0)$$

with fixed non-negative integers $c_0, \ldots, c_d$. Sequences of this type were introduced by Caragiu and Scheckelhoff [7] in the case $d = 1$, and the general recurrence was introduced by Back and Caragiu [3]; see also [6, 12] and the monograph [5, Chapter 3]. Back and Caragiu conjectured that every sequence of this type is ultimately periodic [3, Conjecture 2].

The simplest case is the *GPF-Fibonacci* recurrence $p_n = \mathrm{gpf}(p_{n-1} + p_{n-2})$. Here the conjecture is a theorem: every GPF-Fibonacci sequence with $p_0 \neq p_1$ enters the cycle $7, 3, 5, 2$ [3, Theorem 3]. The proof uses a bound. All terms satisfy $p_n \leq \max(p_0, p_1) + 4$ [3, Remark after Proposition 2]. The main reason is parity. If $r < s$ are odd primes, then $r + s$ is even, so $\mathrm{gpf}(r + s) \leq (r + s)/2 < s$. Only the prime 2 needs a separate argument.

1.2. **The question of Back and Caragiu.** The next case is the following.

Definition 1.1. A *GPF-Tribonacci sequence* is a sequence of primes $(a_k)_{k\geq 0}$ such that

$$a_{k+3} = \mathrm{gpf}(a_k + a_{k+1} + a_{k+2}) \qquad \text{for all } k \geq 0.$$

A GPF-Tribonacci sequence is determined by its three initial terms a_0, a_1, a_2, which can be any primes. Now the parity argument is lost. The sum of three odd primes is odd, and it can even be a prime. Still, the computations of Back and Caragiu suggest that the behaviour is as regular as before. They ran the recurrence from 100,000 random triples of primes below 1000. Apart from the constant sequences, all of these sequences ended in one of only four cycles, of lengths 100, 212, 28 and 6 [3, Section 3]. They conjectured that every GPF-Tribonacci sequence is ultimately periodic [3, Conjecture 1]. They also observed that the terms can become large compared with the initial terms. For example, the sequence

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that starts with $5, 13, 7$ is periodic with period 100 and contains the term 1321. They then asked the following question [3, Section 3, (8)].

Question 1.2 (Back and Caragiu)**.** Is there a constant L such that every GPF-Tribonacci sequence $(a_k)_{k\geq 0}$ satisfies

$$a_k \leq L \cdot \max(a_0, a_1, a_2) \qquad \text{for all } k \geq 0\,?$$

As they remark, a positive answer would prove that every GPF-Tribonacci sequence is ultimately periodic. Indeed, a bounded sequence of primes takes only finitely many values, and a sequence with values in a finite set that satisfies a recurrence of order three is ultimately periodic.

1.3. **Main result.** We show that the answer to Question 1.2 is negative.

Theorem 1.3. *Let $C > 0$ and $H > 0$ be real numbers, and let N be a positive integer with $(17/15)^N > C$. Then there are pairwise distinct primes $p_0, p_1, p_2 > H$ with the following properties. Let $(a_k)_{k\geq 0}$ be the GPF-Tribonacci sequence with $a_0 = p_0$, $a_1 = p_1$ and $a_2 = p_2$, and let $M = \max(p_0, p_1, p_2)$.*

(i) *For $0 \leq k < 3N$ we have $a_k + a_{k+1} + a_{k+2} = q_k\, a_{k+3}$, where q_k is 1, 3 or 5 according to whether k is congruent to 0, 1 or 2 modulo 3.*

(ii) *The three consecutive terms a_{3N}, a_{3N+1} and a_{3N+2} are all larger than CM.*

Corollary 1.4. *There are no real numbers L and D such that*

$$a_k \leq L \cdot \max(a_0, a_1, a_2) + D$$

holds for every GPF-Tribonacci sequence $(a_k)_{k\geq 0}$ and every $k \geq 0$. In particular, the answer to Question 1.2 is negative.

We make three comments on what Theorem 1.3 says.

First, the initial primes depend on C. So the theorem does not give one GPF-Tribonacci sequence that is unbounded. It does not decide the conjecture of Back and Caragiu that every GPF-Tribonacci sequence is ultimately periodic. That conjecture remains open. In fact, the conjecture is equivalent to the existence of some function F such that $a_k \leq F(\max(a_0, a_1, a_2))$ for all GPF-Tribonacci sequences and all k, because there are only finitely many triples of primes below a given bound. Our theorem says that such a function F cannot be of the form $F(x) = Lx + D$. We also note that the proof gives no bound for the size of the initial primes in terms of C.

Second, the initial primes can be taken larger than any given H. So the large ratios are not an effect of small initial terms.

Third, the ratio C is reached after $3N$ steps, where we can take for N the smallest positive integer with $(17/15)^N > C$. The number $17/15$ is the smallest row sum of a matrix P that we define in Section 3; see also Remark 4.1. Up to a constant factor, this is the smallest possible number of steps. Indeed, each term of a GPF-Tribonacci sequence is at most the sum of the three preceding terms. By induction this gives

$$a_k \leq 3^k \max(a_0, a_1, a_2) \qquad \text{for all } k \geq 0.$$

So if a term a_k is larger than $C \max(a_0, a_1, a_2)$, then $k > \log C / \log 3$.

Example 1.5. The GPF-Tribonacci sequence with initial terms 149, 4549, 883 begins with

$$149,\ 4549,\ 883,\ 5581,\ 3671,\ 2027,\ 11279,\ 5659,\ 3793,\ 20731,\ 10061,\ 6917.$$

These twelve numbers are primes. In the first nine steps the quotients are $1, 3, 5$, then again $1, 3, 5$, and again $1, 3, 5$. This is the pattern of Theorem 1.3(i) with $N = 3$. For instance, the first three steps are

$$149 + 4549 + 883 = 5581, \qquad 4549 + 883 + 5581 = 3 \cdot 3671, \qquad 883 + 5581 + 3671 = 5 \cdot 2027.$$

The last three terms are larger than 1.5 times the largest initial term 4549. We found this example by a computer search. It only illustrates the pattern, and its ratios are still small.

1.4. **Idea of the proof.** We do not study the function gpf directly. Instead we decide in advance what the quotients

$$q_k = \frac{a_k + a_{k+1} + a_{k+2}}{a_{k+3}}$$

should be. Once the quotients are fixed, the recurrence becomes linear, namely $a_{k+3} = (a_k + a_{k+1} + a_{k+2})/q_k$. Conversely, suppose that a finite sequence of primes larger than 5 satisfies these linear relations with all q_k in $\{1, 3, 5\}$. Then a_{k+3} is the greatest prime factor of $q_k a_{k+3} = a_k + a_{k+1} + a_{k+2}$, so the sequence is the beginning of a GPF-Tribonacci sequence.

So we need long finite sequences of primes that satisfy a system of linear equations. If we run the linear recurrence backwards, every term becomes a linear form with integer coefficients in the last three terms. The theorem of Green, Tao and Ziegler [13, 14, 15] says that finitely many such forms are simultaneously prime infinitely often, if no two of the forms are proportional, if there is no obstruction modulo any prime, and if the forms can be positive at the same time. The pattern $1, 3, 5$ is chosen so that two things hold at the same time, for every length N. The linear recurrence expands, and there is no obstruction modulo any prime. The constant patterns do not have both properties (see Remark 4.2).

1.5. **Related work.** Other recurrences with the function gpf are presented in Chapter 3 of the monograph of Caragiu [5], with references to the original papers. We mention the results that are closest to ours, following that chapter. For the recurrence $x_n = \mathrm{gpf}(x_{n-1} + c)$ of order one, with a fixed positive integer c, every sequence is ultimately periodic [7, 4]. Recurrences of order one in which the coefficients and the terms are matrices are studied in [8]. For the recurrence $u_n = \mathrm{gpf}(u_1 + \cdots + u_{n-1})$, in which each term depends on all the preceding terms, one has $u_n = n/2 + O(n^{0.525})$, so these sequences are not bounded [10]. There are also results on an analogue of the Proth–Gilbreath conjecture for the map $(x, y) \mapsto \mathrm{gpf}(x + y)$ [11], and on the binary operations $(p, q) \mapsto \mathrm{gpf}(ap + bq)$ on the set of primes [9]. For the GPF-Tribonacci recurrence, the chapter repeats the computations of [3] on a larger scale, and it does not discuss Question 1.2. The OEIS [17] contains the sequence that follows the GPF-Tribonacci recurrence from the initial terms $1, 1, 1$ (A177904) and the cycle of length 212 that it enters (A177923).

There are similar recurrences in which the sum is divided in another way. In Conway's *subprime Fibonacci sequences*, one adds the last two terms and, if the sum is composite, divides it by its smallest prime factor [16]. Alm and Herald took instead the smallest odd prime factor of the sum of the last two terms [1]. They proved that every such sequence terminates in a power of 2. They also proved that there are arbitrarily long sequences of this kind, in one way with the theorem of Dirichlet on primes in arithmetic progressions, and in another way with the theorem of Green and Tao on arithmetic progressions of primes. Alm, Herald, Miller and Sexton considered the sum of the last four terms, divided by 2 until the result is odd [2]. For every N they construct, by a backward recursion, a sequence in which each of the first N sums is divided by 2 only once. They also give a probabilistic model which suggests that most of these sequences are unbounded. Our proof also prescribes the divisions on an initial segment and goes backward, but for the function gpf and with all terms prime.

1.6. **Organization.** In Section 2 we state the theorem on linear equations in primes in the form that we use. In Section 3 we define the linear forms and prove three lemmas about them. In Section 4 we prove Theorem 1.3 and Corollary 1.4.

2. Linear equations in primes

Let $d \geq 1$. A *linear form* on $\mathbb{Z}^d$ is a map $\psi\colon \mathbb{Z}^d \to \mathbb{Z}$ of the form $\psi(n) = c_1 n_1 + \cdots + c_d n_d$ with integers $c_1, \ldots, c_d$. We use the same letter for its extension to a map $\mathbb{R}^d \to \mathbb{R}$. An *open convex cone* in $\mathbb{R}^d$ is an open convex subset $K \subset \mathbb{R}^d$ such that $\lambda x \in K$ for all $x \in K$ and all real $\lambda > 0$.

Theorem 2.1 (Green–Tao–Ziegler)**.** *Let $\psi_1, \ldots, \psi_t$ be non-zero linear forms on $\mathbb{Z}^d$ such that no two of them are rational multiples of each other. Let $K \subset \mathbb{R}^d$ be an open convex cone. Assume the following two conditions.*

(a) *For every prime p there is $n \in \mathbb{Z}^d$ such that none of the integers $\psi_1(n), \ldots, \psi_t(n)$ is divisible by p.*
(b) *There is $n \in K \cap \mathbb{Z}^d$ such that $\psi_1(n), \ldots, \psi_t(n)$ are all positive.*

Then there are infinitely many $n \in K \cap \mathbb{Z}^d$ such that $\psi_1(n), \ldots, \psi_t(n)$ are all prime.

This is [13, Corollary 1.9] for linear forms without constant term. In [13] the statement is made for systems of finite complexity, and it is conditional on two conjectures, called GI(s) and MN(s) there. A system of linear forms without constant terms has finite complexity if and only if no two of the forms are rational multiples of each other [13, Lemma 1.6]. The conjecture MN(s) was proved in [14] and the conjecture GI(s) was proved in [15], for all s. So Theorem 2.1 is unconditional.

3. The linear recurrence with quotients $1, 3, 5$

3.1. **One step forward and one step backward.** We write vectors in $\mathbb{R}^3$ as columns. For $q \in \{1, 3, 5\}$ let

$$T_q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/q & 1/q & 1/q \end{pmatrix}, \qquad S_q = \begin{pmatrix} -1 & -1 & q \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So

$$T_q \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ (x+y+z)/q \end{pmatrix}, \qquad S_q \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} qz - x - y \\ x \\ y \end{pmatrix}.$$

The matrix T_q moves a window of three consecutive terms one step forward, when the quotient of this step is q. The matrix S_q moves the window one step backward, and $S_q T_q$ is the identity matrix. Three steps forward with the quotients 1, 3, 5, in this order, are given by the matrix

$$P = T_5 T_3 T_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1/3 & 2/3 & 2/3 \\ 4/15 & 1/3 & 8/15 \end{pmatrix},$$

and the three steps backward are given by

$$B = P^{-1} = S_1 S_3 S_5 = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 4 & -5 \\ -1 & -1 & 5 \end{pmatrix}.$$

We will use two facts about these matrices. All entries of P are positive, and all entries of B are integers.

3.2. **The linear forms.** Fix a positive integer N. We take the last three terms of the finite sequence as variables, and we express all the other terms by going backward.

Definition 3.1. For $V \in \mathbb{R}^3$ and $0 \le j \le N$ let $s_j(V) = B^{N-j}V$. We write the coordinates of this vector as

$$s_j(V) = \begin{pmatrix} \psi_{3j}(V) \\ \psi_{3j+1}(V) \\ \psi_{3j+2}(V) \end{pmatrix}.$$

This defines $3N+3$ linear forms $\psi_0, \psi_1, \dots, \psi_{3N+2}$ on $\mathbb{Z}^3$. Their coefficients are integers, because the entries of B are integers. The forms depend on N, but we do not show this in the notation.

So $s_N(V) = V$ is the last window, and $s_0(V) = B^N V$ is the first window. Since $B = P^{-1}$, we have

$$s_j(V) = P^j s_0(V) \qquad \text{for } 0 \le j \le N. \tag{1}$$

Lemma 3.2. *Let q_k be 1, 3 or 5 according to whether k is congruent to 0, 1 or 2 modulo 3. Then for every $V \in \mathbb{R}^3$ and every k with $0 \le k < 3N$ we have*

$$\psi_k(V) + \psi_{k+1}(V) + \psi_{k+2}(V) = q_k\, \psi_{k+3}(V).$$

Proof. Let $0 \le j < N$ and write $s_j(V) = (x, y, z)^\mathsf{T}$. By (1) we have $s_{j+1}(V) = T_5 T_3 T_1\, s_j(V)$. We apply the three matrices one after the other. We get

$$T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x' \end{pmatrix}, \qquad T_3 \begin{pmatrix} y \\ z \\ x' \end{pmatrix} = \begin{pmatrix} z \\ x' \\ y' \end{pmatrix}, \qquad T_5 \begin{pmatrix} z \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

where $x' = x + y + z$, $y' = (y + z + x')/3$ and $z' = (z + x' + y')/5$. So $s_{j+1}(V) = (x', y', z')^\mathsf{T}$, and the three equations that define x', y' and z' are the claim for $k = 3j$, $k = 3j+1$ and $k = 3j+2$. □

3.3. **The forms are not proportional.**

Lemma 3.3. *The linear forms $\psi_0, \dots, \psi_{3N+2}$ are non-zero, they have integer coefficients, and no two of them are rational multiples of each other.*

Proof. The coefficients of $\psi_{3j}, \psi_{3j+1}, \psi_{3j+2}$ are the rows of the matrix B^{N-j}. This matrix has integer entries and it is invertible. So the three forms have integer coefficients and they are non-zero.

We say that two linear forms are *proportional* if one of them is a real multiple of the other. We show that no two of our forms are proportional. To compare two forms we change the variable. We put $u = B^N V$, so that $V = P^N u$. By (1), in the variable u the forms $\psi_{3j}, \psi_{3j+1}, \psi_{3j+2}$ are given by the three rows of P^j. An invertible linear change of the variable does not change whether two forms are proportional. So it is enough to show that two different rows, taken from the matrices $P^0, P^1, \dots, P^N$, are never proportional.

Two different rows of the same matrix P^j are not proportional, because P^j is invertible. Now let $j < j'$, and suppose that a row r of P^j is proportional to a row r' of $P^{j'}$. We multiply both rows from the right by P^{-j}. Then rP^{-j} is a row of the identity matrix, and $r'P^{-j}$ is a row of $P^{j'-j}$, and these two rows are proportional. A row of the identity matrix has two entries equal to 0. But all entries of P are positive, so all entries of $P^{j'-j}$ are positive. This is a contradiction. □

3.4. **No obstruction modulo a prime.**

Lemma 3.4. *For every prime p there is a vector $V \in \mathbb{Z}^3$ such that none of the $3N+3$ integers $\psi_0(V), \dots, \psi_{3N+2}(V)$ is divisible by p.*

Proof. First let $p \ne 5$. We take $V = (5, -5, 1)^\mathsf{T}$. A direct computation gives $BV = (25, -25, 5)^\mathsf{T} = 5V$. So $s_j(V) = B^{N-j}V = 5^{N-j}V$ for all j. Hence every $\psi_k(V)$ is, up to sign, a power of 5, and it is not divisible by p.

Now let $p = 5$. We take $V = (1, 1, 2)^\mathsf{T}$. A direct computation gives $BV = (-1, -6, 8)^\mathsf{T}$, and this vector is congruent to $-V$ modulo 5. So $s_j(V)$ is congruent to $(-1)^{N-j}V$ modulo 5 for all j. Hence every $\psi_k(V)$ is congruent to ± 1 or ± 2 modulo 5, and it is not divisible by 5. □

Some of the integers $\psi_k(V)$ in this proof are negative. This is not a problem, because condition (a) of Theorem 2.1 is only about divisibility. Positivity is the subject of the next lemma.

3.5. **A cone on which the forms are positive and grow.** For $x \in \mathbb{R}^3$ we write $\min x$ and $\max x$ for the smallest and the largest coordinate of x, and $\|x\|$ for the Euclidean norm. We write $\mathbf{1} = (1, 1, 1)^\mathsf{T}$. For two vectors $x, y \in \mathbb{R}^3$ we write $x \geq y$ if this inequality holds in every coordinate.

Lemma 3.5. *Let $C > 0$ be a real number and let N be a positive integer with $(17/15)^N > C$. Then there are an open convex cone $K \subset \mathbb{R}^3$ and a real number $\delta > 0$ such that every $V \in K$ has the following properties.*

(1) *We have $\psi_k(V) \geq \delta\|V\| > 0$ for $0 \leq k \leq 3N + 2$.*
(2) *The numbers $\psi_0(V)$, $\psi_1(V)$ and $\psi_2(V)$ are pairwise distinct.*
(3) *We have $\min\bigl(\psi_{3N}(V), \psi_{3N+1}(V), \psi_{3N+2}(V)\bigr) > C \cdot \max\bigl(\psi_0(V), \psi_1(V), \psi_2(V)\bigr)$.*

Proof. We first look at the vector $\mathbf{1}$. The row sums of P are 3, 5/3 and 17/15, so $P\mathbf{1} \geq (17/15)\mathbf{1}$. The entries of P are positive, so multiplication by P preserves the relation $\geq$ between vectors. By induction we get $P^N\mathbf{1} \geq (17/15)^N\mathbf{1}$, and therefore

$$\min(P^N\mathbf{1}) \geq (17/15)^N > C = C \cdot \max \mathbf{1}.$$

We cannot use the vector $\mathbf{1}$ itself, because its coordinates are equal and because Theorem 2.1 needs an open cone. So we move away from $\mathbf{1}$ a little. The function $f(u) = \min(P^N u) - C \cdot \max u$ is continuous on $\mathbb{R}^3$, and $f(\mathbf{1}) > 0$. So there is a neighbourhood of $\mathbf{1}$ on which f is positive and on which all coordinates are positive. In this neighbourhood we choose a point u_* whose three coordinates are pairwise distinct. Then we choose $r > 0$ so small that every point u of the closed ball

$$E = \{u \in \mathbb{R}^3 : \ \|u - u_*\| \leq r\}$$

has positive and pairwise distinct coordinates and satisfies $f(u) > 0$.

Now we define

$$U = \{\lambda u : \ \lambda > 0, \ \|u - u_*\| < r\} \qquad \text{and} \qquad K = P^N U.$$

The set U is the set of positive multiples of the points of an open ball. A ball is open and convex, and both properties pass to the set of positive multiples. So U is an open convex cone. The set K is its image under an invertible linear map, so K is an open convex cone as well.

Let $V \in K$. We write $V = P^N(\lambda u)$ with $\lambda > 0$ and $u \in E$. In particular $V \neq 0$. Then $s_0(V) = B^N V = \lambda u$, and by (1) we have $s_j(V) = \lambda P^j u$ for $0 \leq j \leq N$.

Property (2) holds because $\psi_0(V), \psi_1(V), \psi_2(V)$ are the coordinates of λu, and the coordinates of u are pairwise distinct.

Property (3) holds because $\psi_{3N}(V), \psi_{3N+1}(V), \psi_{3N+2}(V)$ are the coordinates of $s_N(V) = \lambda P^N u$, and $f(u) > 0$ gives

$$\min(\lambda P^N u) > C \cdot \max(\lambda u).$$

For property (1), consider the $3N + 3$ functions $u \mapsto \psi_k(P^N u)$ on the compact set E. By (1) they are the coordinates of the vectors $P^j u$ with $0 \leq j \leq N$. These coordinates are positive, because the coordinates of u are positive and the entries of P are positive. The functions are continuous. So there is $m > 0$ such that $\psi_k(P^N u) \geq m$ for all k and all

$u \in E$. There is also $R > 0$ such that $\|P^N u\| \le R$ for all $u \in E$. For $V = P^N(\lambda u)$ as above we get

$$\psi_k(V) = \lambda\, \psi_k(P^N u) \ge \lambda m \ge \frac{m}{R}\, \|V\|.$$

So property (1) holds with $\delta = m/R$. □

4. Proofs of the main results

Proof of Theorem 1.3. Let C, H and N be as in the theorem. We take the linear forms $\psi_0, \dots, \psi_{3N+2}$ of Definition 3.1, and the cone K and the number δ of Lemma 3.5.

We check the assumptions of Theorem 2.1 for these forms and this cone. The forms are non-zero and no two of them are rational multiples of each other, by Lemma 3.3. Condition (a) holds by Lemma 3.4. For condition (b), note that K is open and not empty, so it contains a point with rational coordinates. A positive integer multiple of this point lies in $K \cap \mathbb{Z}^3$, and all forms are positive on K by Lemma 3.5(1).

So Theorem 2.1 gives infinitely many $V \in K \cap \mathbb{Z}^3$ such that $\psi_0(V), \dots, \psi_{3N+2}(V)$ are all prime. Only finitely many points of $\mathbb{Z}^3$ lie in a given ball. So we can choose such a V with $\delta\|V\| > \max(H, 5)$. We put

$$b_k = \psi_k(V) \qquad \text{for } 0 \le k \le 3N+2.$$

By Lemma 3.5(1), every b_k is a prime larger than $\max(H, 5)$.

We claim that $b_0, \dots, b_{3N+2}$ are the first $3N+3$ terms of the GPF-Tribonacci sequence $(a_k)_{k\ge 0}$ with initial terms b_0, b_1, b_2. Let $0 \le k < 3N$. By Lemma 3.2 we have $b_k + b_{k+1} + b_{k+2} = q_k b_{k+3}$ with $q_k \in \{1, 3, 5\}$. The prime factors of $q_k b_{k+3}$ are the prime b_{k+3} and possibly 3 or 5. Since $b_{k+3} > 5$, we get

$$\operatorname{gpf}(b_k + b_{k+1} + b_{k+2}) = b_{k+3}.$$

This proves the claim by induction on k, and it also proves (i).

Finally let $p_0 = b_0$, $p_1 = b_1$ and $p_2 = b_2$. These are primes larger than H. They are pairwise distinct by Lemma 3.5(2). Statement (ii) is Lemma 3.5(3). □

Proof of Corollary 1.4. Suppose that such numbers L and D exist. We may replace L by $\max(L, 1)$ and D by $\max(D, 1)$, so we may assume that $L \ge 1$ and $D \ge 1$. We apply Theorem 1.3 with $C = L + 1$ and $H = D$, and with any positive integer N such that $(17/15)^N > C$. For the sequence $(a_k)_{k\ge 0}$ that we get, the number $M = \max(a_0, a_1, a_2)$ is larger than D, and

$$a_{3N} > (L+1)M = LM + M > LM + D.$$

This contradicts the assumption. A constant L as in Question 1.2 would give such numbers with $D = 0$. So the answer to Question 1.2 is negative. □

We end with two remarks on the construction.

Remark 4.1 (The growth rate). The characteristic polynomial of P is $\frac{1}{15}(5x-1)(3x^2-6x+1)$. So the eigenvalues of P are $1/5$ and $1 \pm \sqrt{6}/3$, and the largest one is $\rho = 1 + \sqrt{6}/3 \approx 1.8165$. Since all entries of P are positive, the Perron–Frobenius theorem shows that the entries of P^N grow like ρ^N. The number $17/15$ in Theorem 1.3 is therefore not the best possible, but it makes the statement and the proof short. For example, the vector $w = (6, 3, 2)^{\mathsf{T}}$ satisfies $Pw = (11, 16/3, 11/3)^{\mathsf{T}} \ge (16/9)\, w$. If we use w instead of $\mathbf{1}$ in the proof of Lemma 3.5, we get Theorem 1.3 under the weaker assumption $(16/9)^N > 3C$. The eigenvalue $1/5$ of P is the eigenvalue 5 of $B = P^{-1}$, and the vector $(5, -5, 1)^{\mathsf{T}}$ in the proof of Lemma 3.4 is an eigenvector for it.

Remark 4.2 (Why the quotients are not all equal). One may ask why we do not use simpler patterns of quotients.

If all quotients are equal to the same number $q \ge 3$, then each new term is at most the average of the three preceding terms, and the terms do not grow.

If all quotients are equal to 1, then the terms grow fast, but such a sequence of primes larger than 3 has at most six terms. The reason is the prime 3. Let $x_0, \dots, x_6$ be integers with $x_{k+3} = x_k + x_{k+1} + x_{k+2}$ for $0 \le k \le 3$. We claim that at least one of them is divisible by 3.

Suppose that none of them is divisible by 3. Then each x_k is congruent to 1 or to -1 modulo 3. Consider three such numbers. If their residues are all equal, then their sum is divisible by 3. If not, then two of the residues are equal, and the sum is congruent to this residue. So the residue of each new term x_{k+3} is the residue that appears twice among x_k, x_{k+1}, x_{k+2}. Let σ be the residue that appears twice among x_0, x_1, x_2. There are three cases for the residues of x_0, x_1, x_2, and in each case the rule determines the next residues:

x_0	x_1	x_2	x_3	x_4	x_5	
$-\sigma$	σ	σ	σ			so 3 divides $x_1 + x_2 + x_3 = x_4$,
σ	$-\sigma$	σ	σ	σ		so 3 divides $x_2 + x_3 + x_4 = x_5$,
σ	σ	$-\sigma$	σ	σ	σ	so 3 divides $x_3 + x_4 + x_5 = x_6$.

In each case we get a contradiction. This proves the claim. In particular, if seven consecutive terms of a GPF-Tribonacci sequence are larger than 3, then the four quotients between them are not all equal to 1. Six terms are possible, for example $5, 5, 7, 17, 29, 53$.

The order of the quotients also matters. For the pattern $1, 5, 3$, the relations $a_1+a_2+a_3 = 5a_4$ and $a_2 + a_3 + a_4 = 3a_5$ give $a_1 = 6a_4 - 3a_5$. So a_1 is divisible by 3.

The pattern $1, 3, 5$ avoids both problems. The terms still grow, by Lemma 3.5, and there is no obstruction modulo any prime, by Lemma 3.4.

Contributions and use of AI tools

The construction and the proof in this paper were produced by AI systems. This happened on 14 November 2026, during the mathematics workshop at Zushi Fes of ZEN University, where the question of Back and Caragiu was one of the problems. OpenAI models, with the identifiers `gpt-6-astra` and `gpt-6.1-sol` and run in OpenAI Codex, found the pattern $1, 3, 5$ and wrote a first proof in Japanese. On the same day, a report written by Claude Fable 5.2 (Anthropic) reached the same conclusion with the quotients in the order $3, 5, 1$. The English text of this paper was written on 17 November 2026 by Claude Fable 5.2, run in Claude Code, on the basis of the Japanese proof. The same system found Example 1.5 by a computer search, and it checked again the matrix identities, the two remarks in Section 4 and the bibliographic data. Codex and Claude Code are commercial products of OpenAI and Anthropic.

The corresponding contributor, Ryuya Hora, has read the proof carefully. Other participants of the workshop also contributed to this work. They are not responsible for the content of this paper.

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