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INFINITE CASCADES FOR THE DEFOCUSING CUBIC NONLINEAR SCHRÖDINGER EQUATION ON THE SQUARE TORUS

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ABSTRACT. For each finite $s > 1$ we construct a solution of the defocusing cubic nonlinear Schrödinger equation on the square two-torus with initial data in H^s whose H^s norm tends to infinity as $t \to +\infty$. The growth can be arbitrarily slow. More precisely, the squared norm agrees, up to a bounded error at all positive times, with the number of selected axial frequencies K_j for which $K_j^s \le t$. The central construction couples a finite tree of orthogonal frequency pairs to a finite-dimensional family of one-dimensional solutions of the same equation. Each transfer leaves a nonvanishing Fourier coefficient on the horizontal axis. Control on an entire half-line allows successive transfers to retain the coefficients produced earlier. At and above the index two, we match the second-order normal frequency shift separately on each triangle using harmonics of the axial solution. For every fixed $2 < s < \infty$ we also obtain a solution for which $\log \|u(t)\|_{H^s} \asymp \sqrt{\log t / \log\log t}$ at every sufficiently large time.

1. INTRODUCTION

On the square torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ consider

$$i\partial_t u = -\Delta u + |u|^2 u, \qquad u(0) = u_0. \tag{1.1}$$

Here $u(t, \mathbf{x})$ is complex valued, $\mathbf{x} := (x, y)$, and $\Delta := \partial_x^2 + \partial_y^2$. We write $d\mu := dx\, dy/(2\pi)^2$ for normalized Haar measure. For $k = (k_x, k_y) \in \mathbb{Z}^2$ put $\langle k \rangle := (1 + |k|^2)^{1/2}$, where $|k|$ is the Euclidean length, and use the same bracket for integer indices in one dimension. Our Fourier and Sobolev conventions are

$$\widehat{u}(k,t) := \int_{\mathbb{T}^2} u(t,\mathbf{x}) e^{-ik\cdot\mathbf{x}}\, d\mu, \qquad \|u(t)\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^2} \langle k\rangle^{2s} |\widehat{u}(k,t)|^2.$$

We call $(n, 0)$ an axial frequency and a field independent of y an axial field. The conserved mass $M(u)$ and energy $E(u)$ are the integrals against $d\mu$ of $|u|^2$ and $\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4$, respectively. They bound the H^1 norm along a trajectory. Above H^1, a sequence of smaller and smaller amounts of mass can carry a growing Sobolev norm. The question is whether the autonomous equation can produce this behavior from a single initial datum.

Theorem 1.1. *For every real $1 < s < \infty$ there is $u_0 \in H^s(\mathbb{T}^2)$ such that the global solution of (1.1) satisfies*

$$\lim_{t\to+\infty} \|u(t)\|_{H^s} = +\infty.$$

The datum can be chosen with arbitrarily small H^s norm.

The growth in Theorem 1.1 comes from distinct axial Fourier coefficients that persist once they have been formed. The next result describes how these contributions build up in physical time. The quantity $\langle k\rangle^s |\widehat{u}(k,t)|$ is the normalized Fourier modulus at k. Its square is precisely the contribution of this frequency to $\|u(t)\|_{H^s}^2$. In the formula below, # counts the indices $j \in \mathbb{N} := \{1, 2, \dots\}$ in the indicated set.

Theorem 1.2. *Fix $1 < s < \infty$, and let $G : [0,\infty) \to (0,\infty)$ be nondecreasing with $G(t) \to \infty$. One can choose a datum as in Theorem 1.1 and increasing positive integers $K_j \to \infty$ so that*

$$\sup_{t\geq 0} \Big| \|u(t)\|_{H^s}^2 - \#\{j : K_j^s \leq t\} \Big| < \infty, \qquad \|u(t)\|_{H^s} = o(G(t)). \tag{1.2}$$

For each j, there is a time t_j such that $\langle K_j\rangle^s |\widehat{u}((K_j,0),t)| \geq 1/2$ for every $t \geq t_j$. Given any nondecreasing function $\ell : [1,\infty) \to (0,\infty)$ tending to infinity, the choices can be made with $t_j \leq K_j^s \ell(K_j)$ for all j.

For $s > 2$ a different choice of initial datum gives a quantitative growth rate at all sufficiently large times.

Theorem 1.3. *For each fixed $2 < s < \infty$ there exist $u_0 \in H^s(\mathbb{T}^2)$, $0 < c_s \leq C_s < \infty$ and $t_s > e^e$ such that the solution of (1.1) with initial datum u_0 satisfies*

$$\exp\left(c_s\sqrt{\frac{\log t}{\log\log t}}\right) \leq \|u(t)\|_{H^s} \leq \exp\left(C_s\sqrt{\frac{\log t}{\log\log t}}\right) \qquad (t \geq t_s). \tag{1.3}$$

In Section 5 of his survey of Hamiltonian PDE, Bourgain asked whether a solution of the periodic equation with data in H^s can exhibit unbounded Sobolev growth and how fast the growth can be [6, p. 43]. For the square torus, Giuliani and Guardia [22, Question 1] formulate the existence question as follows. Is there a solution with $\limsup_{t\to+\infty} \|u(t)\|_{H^s} = \infty$ for some $s > 1$? Theorem 1.1 answers this question for every $1 < s < \infty$ by giving arbitrarily small data in H^s whose solutions have $\|u(t)\|_{H^s} \to \infty$ as $t \to +\infty$.

On the square torus, Colliander, Keel, Staffilani, Takaoka and Tao [11] showed that a solution can start arbitrarily small in H^s, $s > 1$, and reach any prescribed size. The geometry of their construction comes from the rectangles formed by four distinct Fourier modes satisfying the cubic resonance relation. By arranging these rectangles in generations, they made the output of one exchange available for the next. Guardia and Kaloshin [31] obtained a transfer time polynomial in the desired amplification factor.[1] In both results the initial datum may depend on the target. Colliander et al. [11, p. 43, after (1.6)] explicitly left open the question of unbounded growth for a single datum. The limit in Theorem 1.1 requires control even between successive transfers, when energy can flow back.

The beating solutions of Grébert and Villegas-Blas [27] and Grébert and Thomann [26] exhibit this return explicitly. A mode cedes energy to its partner and recovers it later in the same orbit. One can also follow the creation of high modes from finitely supported data by multiphase geometric optics [7, 8]. A different picture emerges when the size of the torus tends to infinity. In the weakly nonlinear large-box limit the sum over discrete resonances gives rise to a continuous resonant equation [16], with its own stationary states and Hamiltonian dynamics [20]. An earlier statistical approach to high frequencies is due to Kuksin [41]. For a randomly forced equation on a cube he obtained lower bounds on spatial derivatives averaged over realizations and time as the dispersion and diffusion tend to zero.

For one fixed orbit, a particularly close precedent is Hani's theorem for a family of cubic equations on $\mathbb{T}^2$ that includes the resonant system [32]. On the waveguide $\mathbb{R}\times\mathbb{T}^2$, Hani, Pausader, Tzvetkov and Visciglia carried unbounded growth from the resonant system to the cubic NLS by modified scattering [33]. Partial harmonic trapping leads to another setting in which the long-time dynamics of small solutions are governed by a resonant equation [34]. On the circle, the cubic Szegő equation has unbounded Sobolev orbits accessible through inverse spectral

[1] For large finite growth with quintic or analytic nonlinearities on $\mathbb{T}^2$, see [35, 30], and for a convolution potential, see [28]. On irrational two-tori, rational approximations supply nearly resonant rectangles for the cubic equation [22] and for its perturbation by a convolution potential [21].

theory [17, 19]. Its flow approximates small solutions of the cubic half-wave equation over a long finite interval [18]. For the half-wave equation itself, the author obtained exponential growth for $1/2 < s < 3/2$ [10].

Fourier restriction estimates gave Bourgain a local theory for the periodic cubic equation [3, 5]. The same method also yields bilinear estimates for quadratic Schrödinger equations in two dimensions [49]. On the square torus, estimates based on almost conservation extended the global cubic theory below the energy space [12], and Herr and Kwak proved that the defocusing cubic equation is globally well-posed in $H^s(\mathbb{T}^2)$ for every $s > 0$ [36, 37]. For more regular data one can ask how large a higher Sobolev norm can become by time t. Bourgain and Staffilani obtained polynomial upper bounds [4, 50], and subsequent work on flat tori [9] and compact manifolds [47] showed how the bounds depend on the geometry. For example, Deng and Germain improved the polynomial H^2 bound on generic irrational three-tori for powers $3 < p < 5$ [13]. If the norm does tend to infinity, how slowly can it do so? For every $1 < s < \infty$ and every nondecreasing $G(t) \to \infty$, Theorem 1.2 gives a solution on the square torus whose H^s norm tends to infinity and is $o(G(t))$. Staffilani [51, §2] discusses a growth lower bound along a sequence of times for the resonant system and asks what growth is possible for the full equation. For $s > 2$, Theorem 1.3 gives the two-sided rate (1.3) for the full equation at every sufficiently large time. Both theorems describe rates attained by particular solutions. Bourgain's conjecture [6, p. 44, (5.10)] asks for an $O(t^\varepsilon)$ upper bound for every solution and every $\varepsilon > 0$.

The connection with the equation on the circle is particularly useful on $\mathbb{T}^2$, since any one-dimensional solution remains an exact solution when viewed as a function of (x, y). In particular, the finite-gap tori of the one-dimensional equation sit inside the two-dimensional phase space. Small invariant tori for NLS have also been constructed by KAM methods [42, 14, 48]. Long-time control of motion near small tori goes back to Bambusi [1], and the normal forms of Bambusi and Grébert apply to a broad class of Hamiltonian PDE [2]. For the cubic equation, long-time stability is known near plane waves [15] and near suitable small finite-gap tori [44]. Yet one-dimensional tori can also support large transverse excursions. For $s > 1$, Guardia, Hani, Haus, Maspero and Procesi [29] obtained any prescribed amplification in H^s from data arbitrarily close in L^2 to such a torus. Giuliani and Scandone [23, Theorem 1.3 and §1.2] obtained finite H^s amplification near a plane wave, with closeness measured in Fourier ℓ^1, and asked whether a growing solution can start arbitrarily close to a plane wave in H^s. Corollary 6.2 answers this NLS question for $2 \le s < \infty$. Every H^s neighborhood of every plane wave contains a datum whose H^s norm tends to infinity under the flow.

Here the one-dimensional solution supplies a Fourier harmonic whose phase must agree with that of the transverse interaction. In global Birkhoff coordinates the actions of the one-dimensional solution are constant, and each angle advances at a frequency determined by the actions [25]. Grébert and Kappeler described the symmetries of this flow in action-angle and spectral variables [24]. To use it in a two-dimensional rectangle we need both its frequencies and its ordinary Fourier coefficients. For the latter, the nonlinear part of the Birkhoff map gains one derivative [39]. Quantitative estimates for the map and for the Fourier coefficients of the flow are available in [45, 43, 40]. For the frequencies we use the formulas for the NLS hierarchy [38] and Molnar's local inversion of the renormalized frequency map [46]. Appendix A gives the corrected spectral tail estimate for $p = 4$ and the resulting compactness statement at each fixed small action.

1.1. **Strategy of the proof.** Suppose that, for each j, the normalized Fourier modulus at $(K_j, 0)$ stays above $1/2$ from some time t_j onwards. Once the first m such times have passed, these frequencies alone contribute at least $m/4$ to the squared H^s norm. We build towards this conclusion one frequency at a time. At each stage we perturb the initial datum by an arbitrarily

small amount in H^s. The new mode eventually contributes about one to the squared norm, while the change at each of the finitely many earlier modes stays small for all future times. Both the permanence of the contribution and the small cost at the initial time come from the form of a single transfer.

The starting point is a one-dimensional solution, viewed on $\mathbb{T}^2$ as a field independent of y. Multiplying it by e^{2iMt}, where M is its mass, puts it in the mass gauge. Proposition 2.1 chooses the solution so that in this gauge it is periodic and the coefficient of e^{ix-it} in its space-time Fourier series is nonzero. Why this particular harmonic? Put $e \coloneqq (1,0)$ and $k_v \coloneqq e + Rv$ for integer vectors v and a large integer R. If p, q are nonzero and perpendicular, then

$$k_p + k_q = k_{p+q} + e, \qquad |k_p|^2 + |k_q|^2 = |k_{p+q}|^2 + 1.$$

The cubic product of the waves at k_p, k_q with the conjugate of the chosen harmonic has exactly the spatial index and the free time frequency of the wave at k_{p+q}. The harmonic's temporal frequency is as important here as its spatial index. We obtain it by tuning the actions of the one-dimensional solution.

We arrange these rectangles in a binary tree with root velocity $(P, 0)$, so the root has axial Fourier index $(K, 0)$ with $K \coloneqq 1 + RP$. If the two children carry fractions $\zeta, 1-\zeta$ of the squared length of their parent velocity, their combined homogeneous Sobolev weight is a fraction $\zeta^s + (1-\zeta)^s$ of the parent's weight. We keep ζ away from zero and one, so for $s > 1$ this fraction is bounded by a constant $c_s < 1$ at every split. At depth d the sum of the primary leaf weights is therefore at most c_s^d times the root weight. The geometry also produces auxiliary resonances. Section 2 includes them in the finite system and chooses a trajectory on which the transverse amplitudes decay and the root tends to one in model units. The auxiliary amplitudes can be chosen small after the tree is fixed. This is how a contribution of order one at late times can have arbitrarily small initial cost.

To realize this transfer in the full equation, fix the tree and take the physical amplitude $r \coloneqq (K-1)^{-s}$, so that $r\langle K\rangle^s \to 1$ as $R \to \infty$. Two scale comparisons enter Proposition 3.1. The new axial coordinates give an H^1 perturbation of order Rr, which tends to zero. A transverse frequency correction of order R^{-2} accumulates a phase of order R^{-2}/r over the exchange time r^{-1}. This too tends to zero, since $s < 2$. A surviving axial mode must remain in phase on an entire half-line, so we prescribe its exact frequency on an enlarged periodic axial solution. In solving for the difference, we propagate the rapidly decaying transverse directions forwards and prescribe terminal conditions on the complementary directions. Bounds independent of the terminal time give an exact solution approaching the enlarged axial solution. We then follow it back to time zero.

After the first transfer the previous solution is already two-dimensional. If U is that solution and V its periodic axial limit, inserting a new tree at V accounts only for the limiting background. Proposition 4.2 restores the decaying difference $U - V$ by a correction to the newly constructed solution. Section 4 can then choose the successive changes of initial datum summably small in H^s. Uniform control in a lower Sobolev norm protects each previously chosen Fourier coefficient at all later stages. Continuous dependence at each fixed time transfers those bounds to the solution from the limiting datum, giving the divergence in Theorem 1.1.

For Theorem 1.2 we also keep track of the norm during each exchange. The new frequencies lie where the preceding finite-stage orbit has a uniformly small Fourier tail, so cross terms in the squared norm can be made summable. We space the K_j far enough apart that the time windows for successive transfers are disjoint. At any time, all but at most one transfer have either negligible contribution or a contribution within a summable error of one. Counting the thresholds K_j^s therefore gives (1.2) with bounded error even during an exchange. The counting threshold K_j^s lies inside that window. The time by which the root stays above the required

height has a separate bound from the finite system. At fixed depth this bound is a constant times $(K_j-1)^s$, so we can choose K_j large enough to absorb the constant into the prescribed $\ell(K_j)$. Requiring also $G(K_j^s)\geq j$ gives the arbitrarily slow divergence in (1.2). The two corollaries in Section 4 use the same construction to identify the limiting axial motion in lower Sobolev norms and to allow the contribution of an individual root to increase with depth.

Sections 2–4 give the construction for $1<s<2$. If $s\geq 2$, a normal frequency shift of order R^{-2} accumulates on the transfer time of order R^s. Section 5 computes the shift and assigns a separate harmonic to each triangle. Section 6 matches the higher-order shifts needed for $s>2$. At each new exponent the construction selects an initial datum in the stated Sobolev space.

For Theorem 1.3 the construction must also control the time between successive transfers. In Section 7 two families of triangles with shared children allow the root height to grow with the depth d_j, while the logarithmic frequency scale B_j advances by a fixed factor. Proposition 7.1 bounds the cost of a transfer with this choice of scales. At any time between transfers, one earlier root already supplies the lower bound. Propositions 7.2 and 7.3 bound the high-frequency tails of the later transfers at that same time.

Conventions. We use $\|\cdot\|_b$ for the H^b norm, and $dx/(2\pi)$ for measure on $\mathbb{T}:=\mathbb{R}/(2\pi\mathbb{Z})$. On the underlying real spaces we use the real part of the usual inner product. Bars denote complex conjugates, A^* denotes the adjoint for the unweighted L^2 or ℓ^2 inner product, and $[A,B]:=AB-BA$. In estimates $A\lesssim B$ and $A=O(B)$ mean $|A|\leq CB$ for $B\geq 0$. We write $A\asymp B$ when both inequalities hold for nonnegative A,B, and $A=o(B)$ when $A/B\to 0$ in the indicated limit. A subscript indicates allowed dependence of the constant. At a single insertion the tree and the old solution are fixed before $R\to\infty$. Section 4 specifies the order of choices when the tree also varies.

For a constant $c\in\mathbb{R}$, the field $e^{-ict}u(t)$ solves (1.1) with the additional potential c. The choice $c=-2M(u)$ is the *mass gauge*. In that gauge the same Fourier moduli describe the solution, while a common temporal frequency is removed from the equations for the axial modes.

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2. An axial solution and a tree of right triangles

Write x,y for the two spatial variables. The subspace of functions of x alone is invariant under the cubic equation. Our construction starts with an orbit in this subspace that is periodic in the mass gauge. We shall use one Fourier coefficient of the orbit to couple two short sides of a right triangle of frequencies to its hypotenuse. Both the spatial and the temporal frequency of this coefficient matter.

2.1. **A periodic axial solution.**

Write $U(t,x)$ for an axial solution before making the mass gauge transformation. In Birkhoff coordinates $\xi:=\Phi(U)=(\xi_n)_{n\in\mathbb{Z}}$, the action $I_n:=|\xi_n|^2$ is constant and the frequency $\omega_n(I)$ depends only on the action vector $I:=(I_n)_{n\in\mathbb{Z}}$. Thus $\xi_n(t)=\xi_n(0)e^{-i\omega_n(I)t}$. The inverse map is denoted by $\Psi:=\Phi^{-1}$, the differential of Φ at zero is the Fourier transform, and the mass in normalized Haar measure is $M=\sum_n I_n$. We subtract the common frequency $2M$ by multiplying the solution by e^{2iMt}. In this gauge the frequency at n is $n^2+\chi_n(I)$, where

$$\chi_n(I):=\omega_n(I)-n^2-2M. \tag{2.1}$$

The same canonical chart diagonalizes the Hamiltonians of the NLS hierarchy [25, 38]. Translation by x_0 and multiplication by $e^{i\vartheta}$ act on ξ_n by multiplication by e^{inx_0} and $e^{i\vartheta}$, respectively, as follows from the action formulas for momentum and mass. In the period-one convention of [39, p. 7214], the first component of the Fourier differential sends an input v to $-\widehat{v}(-j)$ at spectral index j. We use $n = -j$ and multiply both coordinate components by -1, which preserves the actions and the canonical brackets. The chart and its inverse are real analytic on L^2 and ℓ^2. For $1 \le q < \infty$, ℓ^q denotes the usual sequence space on the indicated indices, with counting measure, and ℓ^∞ has the supremum norm. Use h^b for the space of sequences with norm $(\sum_{n\in\mathbb{Z}}(1+n^2)^b|\xi_n|^2)^{1/2}$. The nonlinear part of each chart means its difference from its differential at zero. These differences gain one derivative as analytic maps from H^1 to h^2 and from h^1 to H^2, respectively [39, Theorem 1.1 and Corollary 1.1]. This also gives the chart regularity at every real order $1 \le b \le 2$ used below. For example, the nonlinear part of Ψ is analytic from h^b to H^b by the continuous inclusions $h^b \subset h^1$ and $H^2 \subset H^b$. Its linear part is the inverse Fourier transform. The same argument applies to Φ. We call $F := \{n : I_n > 0\}$ the Birkhoff support. For fixed actions with finite support, the axial torus is the image under Ψ of all choices of $\xi_j = \sqrt{I_j}e^{i\varphi_j}$, $j \in F$, with all other coordinates zero. A subscript F on a sequence denotes its restriction to these indices.

Another property we shall need is analyticity of $I \mapsto \chi(I)$ on an open ℓ^2 neighborhood of zero, with $d\chi(0) = -\mathrm{Id}$ [46, Theorem 18.1 and Corollary 18.2, with $p = 4$]. To check the normalization, let v solve the equation of period one with cubic coefficient two, as in these references. Then $U(t,x) = \sqrt{2}\,(2\pi)^{-1}v(t/(2\pi)^2, x/(2\pi))$ solves our equation. For finitely supported period-one actions J, the part of the Hamiltonian quadratic in J is $2(\sum_m J_m)^2 - \sum_m J_m^2$. Its J_n-derivative is $4\sum_m J_m - 2J_n$. Subtracting the common frequency $4\sum_m J_m$ leaves the linear term $-2J_n$, as in [46, (4.3) and the proof of Corollary 18.2]. The actions of v are $(2\pi)^2 I_n/2$, after indexing by the physical Fourier mode, and its frequencies are multiplied by $(2\pi)^{-2}$ when passing to U. Thus the derivative $-2\mathrm{Id}$ of the renormalized frequency map in the cited normalization becomes $-\mathrm{Id}$.

Proposition 2.1. *There are smooth axial solutions U with arbitrarily small H^1 norm and finite Birkhoff support whose mass-gauged fields $V(t,x) := e^{2iMt}U(t,x)$ have period $2\pi Q$ for some positive integer Q and satisfy*

$$g := \frac{1}{2\pi Q}\int_0^{2\pi Q}\int_{\mathbb{T}} e^{it-ix}V(t,x)\,\frac{dx}{2\pi}\,dt \neq 0. \tag{2.2}$$

The initial solution can be chosen with Birkhoff support $F = \{0,4,5,2,6\}$. The Birkhoff coordinate at the spatial frequency 1 vanishes identically on this torus.

Proof. Give F the order $S := (0,4,5,2,6)$ and put $\ell := (1,1,1,-1,-1)$. Write $\mathbf{1} := (1,1,1,1,1)$ and take powers on S componentwise. Then

$$\ell\cdot\mathbf{1} = \ell\cdot S = \ell\cdot S^2 = 1, \qquad \ell\cdot S^3 = -35.$$

We compute the coefficient of $\xi_0\xi_4\xi_5\overline{\xi_2\xi_6}$ in the spatial Fourier coefficient at 1 of $\Psi(\xi)$. Its powers of the five phases are encoded by the character $e^{i\ell\cdot\varphi}$, where $\varphi := (\varphi_j)_{j\in S}$ in the indicated order. This is the unique monomial of lowest degree with that character. For a cubic product with unconjugated inputs at a, c and conjugated input at b, put

$$k := a - b + c, \qquad D(a,b,c) := a^2 - b^2 + c^2 - k^2.$$

When $D \neq 0$, comparison of the cubic terms in the physical equation and its diagonal Birkhoff equation gives $1/D(a,b,c)$ per ordered triple in the inverse map. Indeed, temporal differentiation acts on the monomial at linear order by its signed input energy, whereas the output Laplacian has eigenvalue k^2. The resonant cubic monomials contain an action pair, since $D = -2(a-b)(c-b)$. Such a pair cannot occur in our quintic monomial with five distinct indices.

At fifth order the divisor for the NLS flow vanishes. We compute the same coefficient with the commuting flow $U_\tau = U_{xxx} - 3|U|^2U_x$, obtained from [38, (6)] by the same spatial and amplitude scaling, with the third time of the period-one equation set to $-\tau/(2\pi)^3$. The linear term in the equation for $i\partial_\tau U$ acts on e^{inx} by multiplication by n^3. Symmetrizing the cubic term in its two unconjugated inputs A, C gives the multiplier $3(A+C)/2$ per ordered triple. In the outer cubic $U_A\bar{U}_BU_C$, where U_A denotes a spatial Fourier coefficient, insert the cubic part of Ψ in one slot. The other two slots use its linear part. The inner triple (a, b, c) supplies the index $a - b + c$ for the chosen slot. Inserting it into the conjugated slot reverses all three signs. Summing over the assignments of the three positive indices $0, 4, 5$ and the two negative indices $2, 6$ gives

slot of the inner triple	1	2	3
$\sum 1/D(a,b,c)$	$-1/2$	1	$-1/2$
$\sum 3(A+C)/(2D(a,b,c))$	$-3/4$	$11/2$	$-3/4$

For example, in the middle slot the two negative indices $2, 6$ play the roles of a, c, and b ranges over $0, 4, 5$. The sum of the outer positive indices is $A + C = 9 - b$. The reciprocals $1/D$ for the three choices of b are $-1/24, 1/8, 1/6$. Each occurs four times, since a, c and A, C can be permuted independently. This gives 1 in the first row and $4(-9/16 + 15/16 + 1) = 11/2$ in the second. In the first slot the inner conjugated index is $b = 2$ or 6. For these choices the sums of $1/D$ over the positive assignments are $1/4$ and $-3/4$, respectively, with $A+C = 9-b$. The third slot gives the same sums by symmetry. The forcing for the NLS flow therefore vanishes, whereas the third-flow forcing equals four. Corrections to the linear angle velocities carry at least one action pair and have zero coefficient at this monomial. Its third-flow divisor is $\ell \cdot S^3 - 1^3 = -36$, so its coefficient is $-1/9$. Writing $\xi_j = \sqrt{I_j}e^{i\varphi_j}$, analyticity shows that the full angular coefficient equals

$$\left(-\tfrac{1}{9} + O\Big(\sum_{j\in F} I_j\Big)\right) \prod_{j\in F} \sqrt{I_j}.$$

Every higher monomial with the same angle character is obtained from the lowest monomial by adding action pairs.

We now impose $\ell \cdot \chi_F = 0$ so that this harmonic has time frequency 1. The derivative of the left side at the origin is $-\ell$. Set $I_0 = I_4 = I_5 = I_2 = \epsilon$. The derivative with respect to I_6 is 1, so the implicit function theorem gives $I_6 = 2\epsilon + O(\epsilon^2)$. All five actions are positive for small $\epsilon > 0$. On a relatively open patch about this point the angular coefficient computed above stays nonzero. The local coordinate map $I_F \mapsto \chi_F$ sends the patch to an open subset of $\ell \cdot \chi_F = 0$. This hyperplane has rational coefficients, so we may fix a point in the patch with rational χ_F.

It remains to choose an orbit on this torus. Let a_m, $m \in \mathbb{Z}^5$, be the angular Fourier coefficients of the spatial Fourier mode at 1 on the torus with the chosen actions. We have proved that $a_\ell \neq 0$. From initial angles φ, the mass-gauged angles at time t are $\varphi - (S^2 + \chi_F)t$. Inserting their Fourier series into (2.2) and averaging over the common period $2\pi Q$ gives

$$g(\varphi) = \sum_{m\cdot(S^2+\chi_F)=1} a_m e^{im\cdot\varphi}.$$

Here Q is a common denominator of the frequencies, and the series is absolutely convergent by analyticity in the angles. Several characters can survive this time average. Since ℓ is among them, orthogonality in the initial angles gives $\int_{\mathbb{T}^5} |g(\varphi)|^2\, d\varphi/(2\pi)^5 \geq |a_\ell|^2 > 0$. This supplies a single periodic orbit with $g \neq 0$. □

At each subsequent insertion we shall add finitely many axial coordinates and keep the angle velocities already used. This is why we tuned the frequencies through the actions in the first construction.

Lemma 2.2. *Fix a torus of support F in a sufficiently small ℓ^2 action ball, and let G be a disjoint finite set. Sufficiently small positive actions I_G can be added while preserving χ_F by a change of size at most $\|I_G\|_{\ell^2}/3$ in the old actions. By allowing the new positive radii to vary in arbitrarily small open neighborhoods of their prescribed values, one may in addition take all new χ_j, $j \in G$, rational.*

Proof. Take the action ball so small that $\|d\chi + \mathrm{Id}\| \le 1/4$. The compression of $d\chi$ to F has an inverse of norm at most $4/3$, and the derivative from G to F has norm at most $1/4$. We need the allowed size of I_G to be independent of the locations of the new indices. Identify both coordinate subspaces with their zero extensions and put $x_0 \coloneqq I_F^{\mathrm{old}}$, $y \coloneqq I_G$. For fixed y, the equation for the new old actions is the fixed point equation

$$x = x + \chi_F(x + y) - \chi_F(x_0).$$

The right-hand side has Lipschitz constant at most $1/4$ as a function of x. At $x = x_0$ it differs from x_0 by at most $\|y\|_{\ell^2}/4$, since y is supported off F. It thus maps the closed ball of radius $\|y\|_{\ell^2}/3$ about x_0 into itself. For small enough $\|y\|_{\ell^2}$ this ball keeps all old actions positive and $x + y$ inside the action chart. Both restrictions depend only on the old point and the chart neighborhood. The contraction theorem gives the required old actions. The full derivative on $F \cup G$ is likewise within $1/4$ of minus the identity and hence invertible. In its local frequency coordinates, fixing χ_F therefore leaves an open set of χ_G values. This remains true in any open neighborhood where the prescribed new radii are positive. The new χ_G may therefore be taken rational, with the new positive radii in any prescribed sufficiently small open neighborhood of their nominal values. □

Keep the initial old angles fixed. Their subsequent trajectories in the two mass gauges agree in every old angular coordinate. By continuity of Ψ on the compact old angle torus, the change of the axial field is uniformly small in H^1 whenever the new coordinates are sufficiently small in h^1 and the old radii change sufficiently little. When the old occupied frequencies are rational, as in every application below, the two mass-gauged fields have a common period after the new frequencies are chosen rational. Averaging over this common period shows that their coefficients in (2.2) differ by at most their uniform L^2 distance.

We also need the unoccupied frequencies of a fixed smooth axial torus. Here a bound at an individual mode will be more useful than an asymptotic in a sequence norm.

Lemma 2.3. *For a smooth axial torus of finite Birkhoff support let*

$$\mathcal{H} = \int_{\mathbb{T}} (|U_x|^2 + \tfrac{1}{2}|U|^4)\,\frac{dx}{2\pi}, \qquad \mathcal{Q} = 2\left(\sum_j I_j\omega_j - \mathcal{H}\right), \qquad A = \mathcal{Q} - M^2, \quad B = \mathcal{Q}/2.$$

For empty indices n, as $|n| \to \infty$,

$$\chi_n(I) = \frac{A - B}{n^2} + O(|n|^{-3}).$$

More generally the complexified transverse linearization at vertical frequency m has an approximate column with carrier energy $E + 2M + A/n^2 - B/E$, where $E = n^2 + m^2$. After extracting the carrier its two envelopes are $1 + O(|n|^{-1})$ and $O(E^{-1})$. Its residual in L^2_x is $O(|n|^{-3})$, uniformly in m. The constants are locally uniform on compact families with fixed finite support.

Proof. We work in the ungauged variables, write $\langle\cdot\rangle$ for spatial average, $D = -i\partial_x$ and $\mathcal{L}$ for differentiation along the angle flow. The difference of the physical Liouville form $-\operatorname{Im}\langle\bar{U}\,dU\rangle$ and the pullback of $-\operatorname{Im}\sum_j \bar{\xi}_j\,d\xi_j$ is exact on the Birkhoff chart about zero, by canonicity. For

a general smooth torus the same conclusion follows on the star-shaped preimage in Birkhoff coordinates, by radial integration. Evaluating a primitive G along the flow gives

$$\mathcal{L}G = \langle |U_x|^2 + |U|^4 \rangle - \sum_j I_j \omega_j, \qquad \langle |U|^4 \rangle = \mathcal{Q} + 2\mathcal{L}G.$$

This identity holds on the whole torus, including resonant trajectories. Put $V_0 = 2(|U|^2 - M)$ and $C = U^2$. After removing $e^{inx+imy-i(E+2M)t}$ from both components of the doubled linearization, prescribe a common scalar a. The two residuals, with the second equation multiplied by minus one, are

$$(2nD + K_+ - a)f + Cg, \qquad (2E + 2nD + K_- + a)g + \bar{C}f,$$
$$K_+ = D^2 + V_0 - i\mathcal{L}, \quad K_- = D^2 + 4M + V_0 + i\mathcal{L}.$$

Define $f_1 = -D^{-1}V_0/2$, $f_2 = -D^{-1}K_+ f_1/2$, with the inverse zero on constants. Both arguments have mean zero since $\langle V_0 D^{-1} V_0 \rangle = 0$. Set

$$L_2 = K_+ f_2 - \frac{n^2}{2E}|C|^2, \qquad f_3 = -\tfrac{1}{2}D^{-1}(L_2 - \langle L_2 \rangle), \qquad a = \langle L_2 \rangle / n^2, \qquad f = 1 + \sum_{j=1}^{3} n^{-j} f_j.$$

For the lower row put $g_0 = -\bar{C}f/(2E)$, $S = (2nD + K_- + a)/(2E)$ and $g = (1 - S + S^2)g_0$. Its residual is $2ES^3 g_0 = O(|n|^{-3})$, since $E \geq n^2$ and only finitely many derivatives of the smooth torus occur. In the upper row $g = -\bar{C}/(2E) + O(|n|^{-3})$, and the definitions of f_1, f_2, f_3, a cancel the terms through order n^{-2}. All these estimates hold with any fixed number of envelope derivatives.

For the scalar, bilinear integration by parts gives

$$\langle V_0 f_2 \rangle = -\langle f_1 K_+ f_1 \rangle = \tfrac{1}{4}\langle V_0^2 \rangle + \tfrac{i}{2}\mathcal{L}\langle f_1^2 \rangle.$$

Indeed $Df_1 = -V_0/2$ and $\langle (Df_1) f_1^2 \rangle = 0$. Since f_2 has spatial mean zero, the mean of $K_+ f_2$ is $\langle V_0 f_2 \rangle$. Now $V_0 = 2(|U|^2 - M)$ and $|C|^2 = |U|^4$, so the formula for L_2 gives

$$a = \left(\frac{1}{n^2} - \frac{1}{2E}\right)\langle |U|^4 \rangle - \frac{M^2}{n^2} + \frac{i}{2n^2}\mathcal{L}\langle f_1^2 \rangle.$$

Substituting $\langle |U|^4 \rangle = \mathcal{Q} + 2\mathcal{L}G$ gives

$$a = \frac{A}{n^2} - \frac{B}{E} + \mathcal{L}b, \qquad b = \left(\frac{2}{n^2} - \frac{1}{E}\right)G + \frac{i}{2n^2}\langle f_1^2 \rangle.$$

Multiplication of both envelopes by e^{-ib} replaces a by $A/n^2 - B/E$, with the same residual bounds; $b = O(|n|^{-2})$ on the fixed torus.

For $m = 0$ apply the complexified differential of the Birkhoff map to this column. At an empty index the variation of ω_n does not enter its equation. After factoring out the carrier time phase, the positive nth coefficient c_n is $1 + O(|n|^{-1})$: the input has H^1 norm $O(|n|)$, and the one-smoothing part of the differential gives an $O(|n|)$ bound in h^2, uniformly in the old angles. The transformed residual is $O(|n|^{-3})$ in ℓ^2 by the L^2 chart bound. Thus

$$i\mathcal{L}c_n - (\chi_n - (A - B)/n^2)c_n = O(|n|^{-3}).$$

Integrate over the full torus of old angles. The integral of $\mathcal{L}c_n$ vanishes because $\mathcal{L}$ is a constant linear combination of angle derivatives. The factor $\chi_n - (A - B)/n^2$ is constant on the torus and the average of c_n is $1 + O(|n|^{-1})$. Dividing by this average proves the empty-frequency expansion, including for a resonant angle flow. □

2.2. **The resonance geometry.** For a velocity $v \in \mathbb{Z}^2$ the spatial frequency of a new wave will be $k_v := (1,0) + Rv$, with R a positive integer. If $p + q = h$, then

$$|k_p|^2 + |k_q|^2 - |k_h|^2 = 1 - 2R^2 p \cdot q.$$

Thus the coefficient (2.2) couples the three waves exactly at the level of the free dispersion when $p \cdot q = 0$. We call (p, q, h) a triangle in this case and regard its first two entries as unordered. A pair p, h with $p \cdot (h - p) = 0$ can also produce $q = h - p$. We shall keep track of both operations when choosing the velocities.

Proposition 2.4. *For every integer $d \geq 1$ there is a finite set of integer velocities containing a full binary tree of depth d, with two children at every internal vertex and d edges from the root to each leaf. Its root $(P, 0)$ has $P > 0$, and the remaining primary vertices (the vertices of the binary tree) are off the horizontal axis. Each parent is the sum of its two orthogonal children. There is an absolute $p_0 > 0$ such that the two squared child lengths lie between p_0 and $1 - p_0$ times the squared parent length. All horizontal components are nonzero and have distinct absolute values. The following properties hold.*

(1) *At a split $p + q = m$ with m off the axis, write $v = (v_x, v_y)$ for either child and include the points*

$$J(v) := (v_y^2/v_x, -v_y), \quad A(v) := v + J(v) \quad (v = p, q), \qquad T_m := J(p) + J(q).$$

The four pairs of legs at this split are (p, q), $(p, J(p))$, $(q, J(q))$, and $(J(p), J(q))$, with respective hypotenuses m, $A(p)$, $A(q)$, and T_m. The root split consists of a single triangle. These are all nondegenerate forward and reverse quadratic resonances among pairs of the selected velocities, including those whose output is initially sought outside the set.

(2) *If P_i, P_j are distinct horizontal coordinates of axial vertices, then $P_i(P_j - P_i)$ is never a positive rational square.*

Proof. We construct the tree over the rationals, starting with root $(1, 0)$. For an off-axis parent $m = (X, Y)$, take a rational parameter $t \neq 0$ and put

$$p := \frac{X + tY}{1 + t^2}(1, t), \qquad q := \frac{Xt - Y}{1 + t^2}(t, -1).$$

These are the two orthogonal projections of m. The vectors $p, J(q)$ lie on one of the perpendicular lines and $q, J(p)$ on the other. Their four possible sums are precisely the prescribed hypotenuses. In particular, $J(p)$ is the unique partner of p that is orthogonal to p and has vertical component $-p_y$. Thus both $A(p)$ and $A(q)$ lie on the axis. At the horizontal root $Y = 0$ the projection formulas give $J(p) = q$ and $J(q) = p$, so only p, q are added.

We give the exclusions used at each step. For a previously chosen u and a moving vertex v, a pair can generate a nondegenerate source only on one of the three loci

$$v \cdot u = 0, \qquad v \cdot u = |u|^2, \qquad |v|^2 = v \cdot u. \tag{2.3}$$

The first two are lines perpendicular to u, and the third is the circle with diameter $[0, u]$. Each of p, q traverses the circle with diameter $[0, m]$, so the only persistent relation of this kind for an old point is the prescribed one with $u = m$. Each of $J(p), J(q)$ traces an unbounded curve contained in no line. For example, writing $U := (X + tY)/(1 + t^2)$, one has $J(p) = (t^2 U, -tU)$. As $t \to \infty$ the first coordinate is unbounded and the second has a finite limit and is nonconstant. The calculation for $J(q)$ uses $t \to 0$. The remaining three new points traverse horizontal lines, and $u_x \neq 0$ rules out either line in (2.3). Consequently each unwanted old–new condition eliminates finitely many values of t.

For pairs within a single split the equations, after clearing denominators and scaling by X, are polynomials in $Y/X, t$. The following one rational evaluation tests all of them simultaneously.

For $m = (1, 2)$, $t = 3$, multiply the ordered list $(m, p, q, J(p), J(q), A(p), A(q), T_m)$ by 30 to obtain

$$(30, 60),\ (21, 63),\ (9, -3),\ (189, -63),\ (1, 3),\ (210, 0),\ (10, 0),\ (190, -60).$$

Numbering from zero, the pairs with vanishing dot product are $(1, 2), (1, 3), (2, 4), (3, 4)$. The ordered pairs (i, j), $i \neq j$, with $v_i \cdot (v_j - v_i) = 0$ are $(1, 0), (2, 0), (1, 5), (3, 5), (2, 6), (4, 6), (3, 7), (4, 7)$. All horizontal components in this test have different absolute values. For each unwanted relation, write its cleared polynomial in powers of t, with coefficients polynomial in the parent slope Y/X. The test just given makes at least one coefficient a nonzero polynomial. Its finitely many roots contain all the slopes for which the relation could hold for every t. Taking the union over the finite list of unwanted relations still excludes only finitely many slopes. At the birth of every primary vertex avoid these exceptional slopes, as well as vanishing coordinates and coincident or opposite horizontal components. The horizontal components of the new points are nonconstant rational functions of t, so this also avoids coincidences with older vertices. The two child slopes are t and $-1/t$, which lets us impose in advance the slope exclusions needed when each child later becomes a parent. At the first split the children move on the root circle, and the same finite exclusions on their slopes and coordinates apply.

To ensure the arithmetical condition on axis vertices, note that the two new horizontal outputs have abscissae

$$U_1 \coloneqq X + tY, \qquad U_2 \coloneqq X - Y/t.$$

For a prime π and a nonzero rational number w, write $\operatorname{ord}_\pi(w)$ for the exponent of π in its numerator minus that in its denominator. Rational squares have even valuation at every prime. For each old axial abscissa $C \neq 0$, choose a fresh odd prime π for which Y, C are units (valuation zero) and all fixed coordinates are integral. Require $t \equiv (C - X)/Y + \pi \pmod{\pi^2}$, with rational denominators read modulo π^2. Then $\operatorname{ord}_\pi(U_1 - C) = 1$, while U_1 and C are units. Both $C(U_1 - C)$ and $U_1(C - U_1)$ therefore have odd valuation. At another fresh prime use the same argument for $U_2 = X - Y/t$, with $1/t$ as parameter. Here $C \neq X$ by the distinct horizontal components already imposed. We may therefore choose this prime so that the prescribed residue of $1/t$ is a unit and invert it to obtain a congruence on t. For U_1, U_2 themselves take a fresh prime congruent to 1 mod 4 avoiding $Y(X^2 + Y^2)$. A root of $t^2 + 1$ modulo this prime is simple, so all but one of its lifts modulo the prime squared give valuation one. Both U_i are units, and $U_2 - U_1 = -Y(t^2 + 1)/t$ has valuation one. These finitely many conditions are compatible with prescribing the direction of p in any real open interval. By the Chinese remainder theorem they prescribe a residue t_0 modulo a product M_0 of prime squares. For an integer $b \equiv 1 \bmod M_0$, all rationals a/b with $a \equiv t_0 \bmod M_0$ satisfy the congruences. Their spacing on the real line is M_0/b, so any prescribed open interval contains such a parameter when b is sufficiently large.

Choose the angle between the parent and the line of p in a fixed small interval about $\pi/4$. This gives a common lower bound $p_0 > 0$ for both relative squared lengths by the cosine and sine formulas. Removal of the finitely many forbidden directions leaves an open subinterval. Continuing to depth d proves the construction over the rationals. Multiplication of every velocity by a common denominator preserves all incidence conditions and multiplies each expression $P_i(P_j - P_i)$ by a square. □

2.3. **An orbit of the finite system.** Fix the old coefficient g of (2.2). Write the new wave at k_v, after removal of its free oscillation, as ra_v, where $r > 0$ is the common amplitude scale and a_v is the complex model amplitude. For the quadratic terms induced by the old wave, the two factors r on the right and the one on the left leave the time scale r^{-1}. With a dot denoting differentiation with respect to rt, the contribution to $i\dot{a}_h$ of $p + q = h$ is $2\bar{g}\, a_p a_q$. The contributions to $i\dot{a}_p$ and $i\dot{a}_q$ are $2g\, a_h \bar{a}_q$ and $2g\, a_h \bar{a}_p$. The factor two counts the two orders of

the positive inputs at the output h, and is the coefficient of the mixed density term at the other outputs. Writing $a_v = i(g/|g|)X_v$, the real amplitudes X_v satisfy in the model time $z := 2|g|rt$

$$X_h' = X_pX_q, \qquad X_p' = -X_qX_h, \qquad X_q' = -X_pX_h. \tag{2.4}$$

At a vertex in several triangles, all of the displayed contributions are added. From here to the end of this section a prime denotes differentiation with respect to z.

A primary vertex at height j above the leaves has the same incidences as any other primary vertex at this height. We give these vertices a common amplitude x_j, and denote the root by $H := x_d$. For $0 \le j \le d-2$, let w_j, h_j, t_j denote the common amplitudes of the vertices $J(v), A(v), T_m$, respectively, for the split in which v is a child of height j. Thus t_j is one real amplitude per sibling pair. Equation (2.4) preserves this symmetry. In these variables it reads

$$x_j' = x_{j-1}^2 - x_jx_{j+1} - w_jh_j \quad (0 \le j < d), \qquad H' = x_{d-1}^2, \tag{2.5}$$

$$h_j' = x_jw_j, \qquad t_j' = w_j^2, \qquad w_j' = -x_jh_j - w_jt_j \quad (0 \le j \le d-2). \tag{2.6}$$

Here $x_{-1} := 0$, and the term w_jh_j is omitted at $j = d-1$. An empty sum is zero, so the formulas also cover $d = 1$.

For a single split ($d = 1$) the equations can be solved explicitly. If $x_0(0) = a > 0$ and $H(0) = 0$, then $x_0^2 + H^2 = a^2$, so $H' = a^2 - H^2$. Hence

$$H(z) = a\tanh(az), \qquad x_0(z) = \frac{a}{\cosh(az)}.$$

The two leaves decay and the root increases to a. At larger depth the auxiliary triangles also exchange amplitude with the primary vertices. The next lemma chooses their initial amplitudes so that the same final root value is attained in the limit.

Lemma 2.5. *For every $a > 0$ and $b_0, \dots, b_{d-2} > 0$, the system (2.5)–(2.6) has a solution starting with $x_0 > 0$ and $x_1 = \cdots = x_d = 0$. On it, $0 \le H \le a$ and $-b_j \le t_j \le 0$. The solution converges exponentially to an equilibrium at which $H = a$, $h_j = \pm b_j$, and every other amplitude is zero. Its first two derivatives decay exponentially as well.*

Proof. Direct differentiation gives the conserved quantities

$$w_j^2 + h_j^2 + t_j^2 = b_j^2, \qquad \sum_{j=0}^{d} x_j^2 + \sum_{j=0}^{d-2} h_j^2 = a^2 + \sum_{j=0}^{d-2} b_j^2. \tag{2.7}$$

To see the cancellation across levels, sum only up to height $j < d$. Equations (2.5)–(2.6) give

$$\frac{d}{dz}\left(\sum_{k=0}^{j} x_k^2 + \sum_{k=0}^{\min(j,d-2)} h_k^2\right) = -2x_j^2x_{j+1}.$$

At each level $2h_kh_k'$ cancels the term $-2x_kw_kh_k$. The terms linking consecutive primary levels telescope, leaving only the displayed boundary term. At $j = d-1$ this is $-(H^2)'$. The first invariant follows by differentiating the three squares on one auxiliary sphere. These identities are for the reduced amplitudes, one per level. The multiplicities of the individual Fourier modes enter later in the Sobolev norm.

For each initial point on the product of the indicated spheres, set all primary amplitudes above height zero equal to zero, and set $x_0(0) = (a^2 + \sum_j (b_j^2 - h_j(0)^2))^{1/2}$. This gives a continuous choice of initial conditions with $x_0(0) \ge a$. The trajectories are global by (2.7). Let $\mathcal{P} := \prod_{j=0}^{d-2}\{(w,h,t) \in \mathbb{R}^3 : w^2 + h^2 + t^2 = b_j^2\}$. For $Z > 0$ let $F_Z : \mathcal{P} \to \mathcal{P}$ map these initial sphere coordinates to their values at time Z. The maps F_{uZ}, $0 \le u \le 1$, give a homotopy to the identity. Thus F_Z has degree one and is onto. Indeed a map of a closed oriented manifold

missing a point has degree zero at that point. Take a preimage of the point with coordinates $(0, b_j, 0)$ for every j. At time Z the second invariant reduces to $H(Z)^2 + \sum_{k<d} x_k(Z)^2 = a^2$, where the sum starts at zero. Since $H(0) = 0$ and $H' \geq 0$, we have $0 \leq H \leq a$ on $[0, Z]$. Likewise $t_j' \geq 0$ and $t_j(Z) = 0$, while the sphere bounds t_j below by $-b_j$. For a sequence $Z \to \infty$ these preimages have a convergent subsequence in $\mathcal{P}$. Fix any finite observation time. Continuous dependence passes all the inequalities at that time to the trajectory from the limiting initial point. The same initial point therefore works on the whole half-line. For $d = 1$ the product of spheres consists of a single point and the same argument uses the primary invariant alone.

The omega limit set consists of limits of states at times tending to infinity. It is invariant under this finite-dimensional flow. The bounded monotone coordinates t_j, H each have a limit, so these coordinates are constant on the limit set. Their derivatives w_j^2, x_{d-1}^2 therefore vanish there. Invariance and (2.5) then give $x_{d-2} = \cdots = x_0 = 0$. The two identities in (2.7) give $H^2 = a^2 + \sum_j t_j^2$, so $H = a$, all $t_j = 0$, and $h_j = \pm b_j$. The omega limit set is connected, being the intersection of the nested compact connected closures of trajectory tails. It therefore consists of a single point. If the trajectory left some fixed neighborhood of this point at arbitrarily late times, compactness would supply a second limit point. This proves convergence.

For the decay rate, the vector of the lower x_j's and the w_j's solves a linear equation $Y' = (A + E(z))Y$ along this trajectory, with $E(z) \to 0$. The matrix A consists of the scalar $-a$ on x_{d-1} and the symmetric blocks $\left(\begin{smallmatrix} 0 & -h_j(\infty) \\ -h_j(\infty) & 0 \end{smallmatrix}\right)$ on (x_j, w_j). For example, write x_{j-1}^2 as the unknown x_{j-1} times a coefficient tending to zero. Fix $\lambda > 0$ less than the smallest absolute value of the eigenvalues of A. We use the supremum norm weighted by $e^{\lambda(z-Z)}$ on a late half-line $z \geq Z$. Write P_-, P_+ for the orthogonal projections onto the negative and positive eigenspaces of A, and write $\lambda_0 \coloneqq \min_{\mu \in \mathrm{spec}(A)} |\mu|$. The equation on $z \geq Z$ takes the form

$$Y(z) = e^{(z-Z)A} P_- Y(Z) + \int_Z^z e^{(z-v)A} P_- E(v) Y(v)\, dv - \int_z^\infty e^{(z-v)A} P_+ E(v) Y(v)\, dv.$$

In obtaining the last integral, the boundary term at infinity vanishes because Y is bounded. In the weighted norm the two integrals have combined operator norm at most $2(\lambda_0 - \lambda)^{-1} \sup_{v \geq Z} \|E(v)\|$. They also contract in the unweighted norm when Z is sufficiently large. Uniqueness in the unweighted space identifies the bounded solution with the weighted one. Integrating the equations for H, h_j, t_j from infinity gives decay of the remaining differences. The polynomial equations and their first derivatives give the assertion about derivatives. □

The Sobolev cost of this orbit at its initial time is particularly simple. At each primary split, write $\zeta, 1 - \zeta$ for the two ratios of squared lengths. For $s > 1$ define

$$c_s \coloneqq \max_{p_0 \leq \zeta \leq 1 - p_0} \left(\zeta^s + (1 - \zeta)^s\right) < 1.$$

If the tree has root $(P, 0)$, iteration of the split gives

$$\sum_{\text{primary leaves } v} |v|^{2s} |x_0(0)|^2 \leq |P|^{2s} \Big(a^2 + \sum_j b_j^2\Big) c_s^d. \tag{2.8}$$

All higher primary levels are initially zero. Each auxiliary amplitude at height j is bounded by b_j for all time. Since the geometry is fixed before choosing the positive b_j, the total weighted cost of all the auxiliary vertices can be made arbitrarily small.

The identity $H' = x_{d-1}^2$ gives a quantitative estimate on the time spent below a prescribed fraction of the terminal amplitude. The estimate is uniform over the trajectories selected by the degree argument.

Proposition 2.6. *Take $a = 1$, $0 < \theta < 1$, and put*

$$C_\theta := \left(\frac{100}{1-\theta^2}\right)^4, \qquad Z := C_\theta d^{3/2}.$$

If $\sum_{j=0}^{d-2} b_j^{3/2} \le Z^{-1/2}$, then $H(z) \ge \theta$ for every $z \ge Z$. In physical time the root has thus reached this fraction of its terminal amplitude by $C_\theta d^{3/2}/(2|g|r)$.

Proof. Write $E_j := Z^{-1}\int_0^Z x_j^2\,dz$, including $x_d = H$. From (2.7) and monotonicity,

$$1 - H(Z)^2 \le \sum_{j<d} E_j, \qquad E_{d-1} \le Z^{-1}, \qquad E_d \le 1. \tag{2.9}$$

Indeed at each time $1 - H^2 = \sum_{j<d} x_j^2 - \sum_j (w_j^2 + t_j^2)$, and $H' = x_{d-1}^2$. For $1 \le j < d$, integrate (2.5), using $x_j(0) = 0$, $\int_0^Z w_j^2 \le b_j$, and $|h_j| \le b_j$, to get

$$E_{j-1} \le (E_j E_{j+1})^{1/2} + e_j, \qquad e_j := \frac{|x_j(Z)|}{Z} + \frac{b_j^{3/2}}{\sqrt{Z}}, \tag{2.10}$$

where $b_{d-1} := 0$. For example, the mixed term in that integration is bounded by

$$Z^{-1}\left|\int_0^Z w_j h_j\,dz\right| \le b_j Z^{-1/2}\,(t_j(Z) - t_j(0))^{1/2} \le b_j^{3/2} Z^{-1/2}.$$

At height $d-1$ this term is absent. As $Z \ge 1$, our choice of radii implies $\sum_j b_j^2 \le 1$. Hence

$$e_j \le (\sqrt{2}+1)/Z, \qquad \sum_{j=1}^{d-1} e_j \le (\sqrt{2d}+1)/Z.$$

The second estimate uses Cauchy–Schwarz at the single endpoint Z.

To exploit the geometric mean in (2.10), order the levels downward, putting $m_k := E_{d-1-k}$ for $-1 \le k < d$. The starting bounds are $m_{-1} \le 1$ and $m_0 \le Z^{-1}$. If the two preceding averages are bounded by powers of Z^{-1}, their geometric mean has the average of those exponents. This suggests $\alpha_{-1} := 0$, $\alpha_0 := 1$, $\alpha_k := (\alpha_{k-1} + \alpha_{k-2})/2$, giving $\alpha_k = 2/3 + (-1/2)^k/3$. For $k \ge 1$ they belong to $[1/2, 3/4]$. Multiplying (2.10) by Z^{α_k} shows that $m_k Z^{\alpha_k}$ exceeds the larger of the preceding two scaled averages by at most $(\sqrt{2}+1)Z^{-1/4}$. Let $n := 1 + \lceil \log_2(2 + \log Z)\rceil$, with log the natural logarithm. Since $Z \ge 100^4$, we have $n + 1 \le 4Z^{1/4}$ and $nZ^{-1/4} < 5$ (use $n \le 3 + \log Z$). Induction gives $m_k \le 15 Z^{-\alpha_k}$ up to $k = \min(n, d-1)$. These levels contribute at most $60Z^{-1/4}$ to (2.9).

If there are further levels, the last two exponents already satisfy $|\alpha_k - 2/3|\log Z \le 1/3$, for $k = n-1, n$. Thus both averages are at most $21Z^{-2/3}$. For the remaining levels use $\sqrt{uv} \le \max(u, v)$ in (2.10). Each is bounded by $21Z^{-2/3} + \sum_j e_j$. Substitution into (2.9) yields

$$1 - H(Z)^2 \le 60Z^{-1/4} + 21dZ^{-2/3} + \frac{d(\sqrt{2d}+1)}{Z} \le 60C_\theta^{-1/4} + 21C_\theta^{-2/3} + 3C_\theta^{-1} < 1 - \theta^2.$$

Monotonicity of H proves the proposition. □

There is also a lower bound within the finite system. The moment

$$\sum_{j=0}^{d} j x_j^2 + \sum_{j=0}^{d-2} j h_j^2$$

has derivative $2\sum_{j<d} x_j^2 x_{j+1} \le 2B^3$, where $B^2 := 1 + \sum_j b_j^2$. Its initial value is at most $d\sum_j b_j^2$. Thus, if $\sum_j b_j^2 \le \theta^2/2$, the model time of the first crossing $H = \theta$ is at least $d\theta^2/(4B^3)$. Both estimates concern a single choice of tree, radii, and initial angles.

3. Insertion of a tree into the two-dimensional flow

Let V be any smooth axial solution with finite Birkhoff support whose action vector lies in the ball of Lemma 2.2. Assume its occupied frequencies in the mass gauge are rational and its space-time coefficient (2.2) is nonzero. Proposition 2.1 provides an initial solution with these properties. Throughout this section axial solutions are written in their mass gauge. For mass M the equation has the additional constant potential $-2M$. Write g for the $(1,1)$ space-time Fourier coefficient of V, and F for its finite set of occupied Birkhoff indices. Fix the tree of Proposition 2.4 and an orbit given by Lemma 2.5. Let $\mathcal{V} \subset \mathbb{Z}^2$ be its full set of velocities, including the auxiliary vertices. For $v = (v_x, v_y) \in \mathcal{V}$, write $x_v(z)$ for the real amplitude assigned to that vertex in (2.5)–(2.6). We use three time variables. The equation is solved in physical time t, the error estimates use slow time $\tau := rt$, and the tree uses time $z := 2|g|\tau$, in which its interaction coefficient equals one. The positive amplitude scale r is specified in the next proposition. In complex amplitudes these functions all carry the same constant unit factor $\gamma := ig/|g|$. Enumerate the horizontal vertices as $(P_i, 0)$, and let $c_i := \lim_{z\to\infty} x_{(P_i,0)}(z)$. At the main root $P_i = P$, this limit is $a > 0$. At each other horizontal vertex it is one of the signed auxiliary radii. Both the tree and its positive auxiliary radii are fixed throughout the proof below.

Proposition 3.1. *Let $1 < s_0 < s < 2$ and $a_0 > 0$. Set $r := a_0 R^{-s}$ and $k_v := (1,0) + Rv$. For all sufficiently large integers R there are a periodic axial solution V_R and a solution W_R of the full two-dimensional equation in the mass gauge of V_R, with the following properties.*

(1) *On F, V_R and V have the same mass-gauged frequencies and the same Birkhoff angles at $t = 0$. The newly occupied Birkhoff indices of V_R are $K_i := 1 + RP_i$. At K_i the modulus of the Birkhoff coordinate is $r|c_i + e_i|$ for some real e_i with $|e_i| \le R^{-10}$.*

(2) *$W_R(t) - V_R(t)$ tends to zero exponentially in H^s as $t \to +\infty$. In the respective gauges of W_R and V,*
$$\sup_{t\ge 0} \|W_R(t) - V(t)\|_{H^{s_0}} = o(1).$$

(3) *There are real numbers $\kappa_v = |k_v|^2 + O(R^{-2} + r^2)$ such that*
$$\sup_{t\ge 0} \Big\| W_R(t) - V(t) - r\gamma \sum_{v\in\mathcal{V}} x_v(2|g|rt) e^{ik_v\cdot(x,y) - i\kappa_v t} \Big\|_{H^s} \longrightarrow 0. \tag{3.1}$$
In particular
$$\|W_R(0) - V(0)\|_{H^s} \le a_0 \Big(\sum_v |v|^{2s} |x_v(0)|^2 \Big)^{1/2} + o(1).$$
At the main root $K := 1 + RP$ one has, writing $H := x_{(P,0)}$ and using $\widehat{W}_R(K,0,t)$ for the coefficient at $(K,0)$ at time t,
$$\sup_{t\ge 0} \Big| \, |\widehat{W}_R(K,0,t)| - rH(2|g|rt) \, \Big| = o(r).$$

All limits in this statement are taken after the finite tree and the old orbit have been fixed.

The comparison in (3.1) involves three exact solutions. The old periodic field V supplies the coefficient g. The enlarged periodic field V_R prescribes the final axial amplitudes and phases.

It is W_R that undergoes the two-dimensional transfer and converges to V_R. In particular the reference field in the finite-plane expansion (3.1) is V.

There are two different roles for the linear frequencies in this proof. An error in the frequency of a persistent axial mode accumulates for all future time. We therefore evolve these modes at their exact Birkhoff frequencies. Off the axis, the limiting field vanishes, but its linearized equation has both growing and decaying directions. We identify these directions explicitly and use them to prescribe the boundary conditions for the error. Unless indicated otherwise, constants in this section may depend on the fixed old orbit, tree, and a_0, s_0, s. They are independent of R and of the terminal time.

We prove Proposition 3.1 by first prescribing the axial endpoint and then solving for the remaining Fourier coefficients. Section 3.1 gives the equations for those coefficients. Section 3.2 determines which transverse directions can be integrated from the left and which must be integrated from the right. In Section 3.3 we substitute the tree orbit into the full equations and estimate the errors in the transverse field, the new axial field, and the old actions and angles. Section 3.4 solves for them first on $[T, S]$ in slow time, sends S to infinity with R fixed, and extends the resulting solution back to time zero.

3.1. **The norm and the axial equations.** Use Lemma 2.2 to adjust the old actions by $O(r^2)$ and to choose the new radii in the intervals of part (1). Only finitely many new actions are involved, and the old ones stay positive. Let I^∞ denote the resulting action vector, let $c_R := -2\sum_j I_j^\infty$, and take the new initial phases so that the complex terminal amplitudes are $r\gamma(c_i + e_i)$. All renormalized occupied frequencies are rational, so the resulting axial solution V_R is periodic in this gauge. In particular its constant potential differs from that of V by $O(r^2)$.

For one insertion it suffices to work in the exact invariant subspace with vertical Fourier indices in $R\mathbb{Z}$. These are the functions fixed by translation of y by $2\pi/R$. Write $p := (s+2)/2$ and set

$$\alpha := \tfrac{1}{2}\min(2-s, p+s-2), \qquad L := R^\alpha, \qquad \rho := R^{-3\alpha/4}. \tag{3.2}$$

Here $s < p < 2$, $L \to \infty$, and $L/R \to 0$. For $n, q \in \mathbb{Z}$ the integer q labels the vertical frequency qR. We weight the Fourier coefficient at (n, qR) by

$$w(n,q) := (1 + |n|/R + |q|)^p (1 + L^{-1}\operatorname{dist}(n, R\mathbb{Z})).$$

Thus $\operatorname{dist}(n, R\mathbb{Z}) := \min_{j\in\mathbb{Z}} |n - Rj|$. The first factor measures the frequency in units of R. The second measures its horizontal displacement from a multiple of R. For $f_{n,q} := \widehat{f}(n, qR)$ put

$$\|f\|_Z^2 := \sum_{n,q} w(n,q)^2 |f_{n,q}|^2, \qquad \|f\|_B := \sum_{n,q} w(n,q)|f_{n,q}|.$$

For sequences on the horizontal axis only the terms with $q = 0$ occur. Reflection, conjugation and Fourier coordinate projections are bounded in both norms. Since $\langle (n, qR)\rangle^b \le CR^b w(n,q)$ for $0 \le b \le p$, we have $\|f\|_{H^b} \le CR^b \|f\|_Z$.

Lemma 3.2. *Uniformly in R,*

$$\|fh\|_Z \le C\|f\|_B \|h\|_Z, \qquad \|fh\|_Z \le C\sqrt{L}\,\|f\|_Z \|h\|_Z.$$

A plane wave at k_v times an axial function in a bounded smooth family has bounded B norm.

Proof. In the coordinates (n, q), the scaled length $|n|/R+|q|$ and the distance $\operatorname{dist}(n, R\mathbb{Z})$ are both subadditive. Since $1+a+b \le (1+a)(1+b)$ for $a, b \ge 0$, the weight satisfies $w(k) \le w(l)w(k-l)$. The first estimate now follows from $\ell^1 * \ell^2 \to \ell^2$. For the second, write $k = (n, q)$ for the two

indices of the weight. Cauchy–Schwarz in each convolution output gives

$$\|fh\|_Z^2 \le \left(\sup_k \sum_l \frac{w(k)^2}{w(l)^2 w(k-l)^2}\right) \sum_{k,l} w(l)^2 |f_l|^2 w(k-l)^2 |h_{k-l}|^2.$$

The second factor is $\|f\|_Z^2 \|h\|_Z^2$. To bound the first, write $n = Rj + d$, $-R/2 < d \le R/2$. The first factor in w is comparable to $a(j,q) \coloneqq (1 + |j| + |q|)^p$. If $k = l + (k-l)$, the two input (j,q) indices add to the output up to a carry of $0, 1$, or -1 in j. For each fixed input remainder, the carry is fixed by the output. Consequently the squared ratio of the three a weights is bounded by a constant times the sum of the two reciprocal squared input weights, by $(A+B)^p \le 2^{p-1}(A^p + B^p)$ for $A, B \ge 0$. Summing over the first input (j,q) costs a constant since $\sum_{j,q} a(j,q)^{-2} < \infty$. For the second factor put $b(d) = 1 + |d|/L$ on the centered representatives of $\mathbb{Z}/R\mathbb{Z}$. For an input remainder d_1, the other remainder d_2 is the centered representative of $d - d_1$. The triangle inequality on this cyclic group gives

$$\frac{b(d)^2}{b(d_1)^2 b(d_2)^2} \le 2\big(b(d_1)^{-2} + b(d_2)^{-2}\big).$$

The sum of this right-hand side over d_1 is at most $4\sum_{|h| \le R/2}(1 + |h|/L)^{-2} \le CL$. The preceding sum over (j,q) was uniform in d_1, so the entire squared convolution bound costs CL. Taking its square root gives the stated factor $\sqrt{L}$. Finally the velocity components are integers, so shifting an axial envelope by k_v changes its remainder by one and its large-scale weight by at most a fixed polynomial in the envelope index. □

Let P_0 denote averaging in y, equivalently the Fourier projection onto $q = 0$, and let $P_\perp \coloneqq 1 - P_0$. For a solution u in gauge c_R, write $U \coloneqq P_0 u$, $w_\perp \coloneqq P_\perp u$, and $z_j \coloneqq (\Phi(U))_j$, where Φ is the axial Birkhoff map and $\Psi = \Phi^{-1}$. The index on z_j distinguishes the Birkhoff variables from the model time $z = 2|g|rt$. We keep c_R fixed when projecting the equation. The transverse component can exchange mass with U, so the axial mass alone need not determine the prescribed gauge at intermediate times. In the expansion of $|U + w_\perp|^2(U + w_\perp)$, the two terms linear in $w_\perp$ are $2|U|^2 w_\perp + U^2 \overline{w_\perp}$. Both have zero y-average. Projecting the remaining terms and differentiating the real map Φ on the axial component gives the exact equations

$$\dot z_j = -i(\omega_j(I) + c_R) z_j + D\Phi_j(U)[-if], \tag{3.3}$$

$$f \coloneqq 2UP_0|w_\perp|^2 + \overline{U} P_0(w_\perp^2) + P_0(|w_\perp|^2 w_\perp),$$

$$i\dot w_\perp = (-\Delta + c_R + 2|U|^2) w_\perp + U^2 \overline{w_\perp} + N(U, w_\perp), \tag{3.4}$$

$$N(U,w) \coloneqq P_\perp(2U|w|^2 + \overline{U} w^2 + |w|^2 w).$$

Dots denote physical time derivatives, $I_j = |z_j|^2$, and ω_j are the ungauged frequencies. To justify (3.3) for H^p solutions, first prescribe a continuous axial H^p forcing. Approximate the forcing and initial data by smooth functions and use the integral equation in Birkhoff variables after removing the fixed diagonal j^2. Indeed, if $I = |z|^2$, then

$$\omega_j(I) + c_R - j^2 = 2\|z\|_{\ell^2}^2 + c_R + \chi_j(I), \qquad \big\||z|^2 - |z'|^2\big\|_{\ell^2} \le (\|z\|_{\ell^2} + \|z'\|_{\ell^2})\|z - z'\|_{\ell^2}.$$

The first formula keeps track of the mass of the instantaneous axial field. The second, together with the local analyticity of $I \mapsto \chi(I)$ into $\ell^2 \subset \ell^\infty$ [46], makes the remaining diagonal locally Lipschitz as a multiplier on h^p. The chart and its differential are locally Lipschitz at this regularity [25, 39]. The resulting integral equation therefore has a unique local solution and passes to the limit under the smooth approximations. The same reasoning allows reconstruction by Ψ.

On F write $z_j = \sqrt{I_j}\exp(-i\theta_j)$, and put

$$J \coloneqq (I_F - I_F^\infty)/r, \qquad \Theta \coloneqq \theta_F - \theta_F^\infty(t), \qquad V_* \coloneqq \Psi(z|_F). \tag{3.5}$$

Here $I_F = (I_j)_{j\in F}$, and similarly for the other restrictions. The terminal angles are the continuous real lifts $\theta_j^\infty(t) \coloneqq \theta_j^\infty(0) + (\omega_j(I^\infty) + c_R)t$ on F. We use the lift of the actual angles near these nominal angles, so that Θ measures their displacement. In $z|_F$ we set all coordinates outside F to zero. The profile V_* uses the actual old coordinates and is smooth on a fixed compact parameter neighbourhood.

Two chart estimates will be used repeatedly. Write $\mathcal{A}$ for either chart and $\mathcal{F}_\mathcal{A} \coloneqq D\mathcal{A}(0)$ for its linear Fourier part. We identify spatial functions with their Fourier coefficients in norms on either side of the chart. For a point x_* on the old smooth family and an axial perturbation y_* of Z norm at most Cr, we have

$$\|D\mathcal{A}(x_*+y_*) - D\mathcal{A}(x_*)\|_{Z\to Z} \le Cr(1+R^{2-p}), \tag{3.6}$$

$$\|(D\mathcal{A}(x_*) - \mathcal{F}_\mathcal{A})h\|_Z \le C(R^{-1}+L^{-1}) \tag{3.7}$$

in the second line for $h = e^{iRv_x x}h_0(x)$ with fixed $v_x \in \mathbb{Z}\setminus\{0\}$ and h_0 in a bounded smooth family. The same bound applies to finite sums with summable smooth-envelope bounds, by linearity over the reals.

For clarity, first drop the residue factor. On the axis the resulting norm Z_0 is equivalent to $\|\cdot\|_{L^2} + R^{-p}\|\cdot\|_{H^p}$ (and to the corresponding sum with ℓ^2 and h^p norms for coordinate sequences). Thus $\|h\|_{H^1} \lesssim R\|h\|_{Z_0}$, and $Rr = o(1)$. The full chart is analytic in L^2, while its difference from Fourier is analytic $H^1 \to H^2$ in both directions [39]. Let $\|h\|_Z \le 1$ and write $u_t = x_* + ty_*$, $0 \le t \le 1$. The base points stay in a fixed neighborhood of the compact old family in H^1, since $\|y_*\|_{H^1} \le CRr = o(1)$. The fundamental theorem of calculus gives

$$(D\mathcal{A}(x_*+y_*) - D\mathcal{A}(x_*))h = \int_0^1 D^2\mathcal{A}(u_t)[y_*,h]\,dt.$$

The L^2 bound on the integrand is Cr. The H^p bound uses $H^2 \subset H^p$ and the second differential of the one-smoothing part, since the Fourier part has zero second differential. This gives $C\|y_*\|_{H^1}\|h\|_{H^1} \le CrR^2$. Multiplying by R^{-p} gives the second term in the claimed Z_0 estimate. To restore the residue factor, note that $\|h\|_Z \asymp \|h\|_{Z_0} + \|\delta h\|_{Z_0}$, where $\delta \coloneqq (R/L)(T_{2\pi/R}-1)$ and T_ℓ is spatial translation, or its diagonal action on Birkhoff coordinates. Indeed the modulus of the Fourier multiplier of δ is $2(R/L)|\sin(\pi n/R)|$, comparable to $\operatorname{dist}(n, R\mathbb{Z})/L$. The charts commute with translation. To see the corresponding difference of the integrand, abbreviate $T = T_{2\pi/R}$, $u = u_t$. Equivariance and bilinearity give

$$\begin{aligned}\delta(D^2\mathcal{A}(u)[y_*,h]) &= D^2\mathcal{A}(Tu)[\delta y_*, Th] + D^2\mathcal{A}(Tu)[y_*,\delta h]\\ &\quad + \int_0^1 D^3\mathcal{A}(u+v(Tu-u))[\delta u, y_*, h]\,dv.\end{aligned}$$

In the first two terms $\|\delta y_*\|_{Z_0} \le Cr$ and $\|\delta h\|_{Z_0} \le C$ give the same bounds as before. In the last term $\|\delta u\|_{H^1} \le C(L^{-1}+Rr)$ because x_* is uniformly smooth, and $\|\delta u\|_{L^2} \le C(L^{-1}+r)$. Both are bounded independently of R. Applying the L^2 and the one-smoothing bounds for the third differential completes (3.6). At the smooth base x_*, $D\mathcal{A}(x_*)$ is bounded on L^2 and on H^p, hence also on Z_0. Applying the same translation identity to the first differential adds a term involving $D^2\mathcal{A}$ and $\delta x_* = O(L^{-1})$ in H^1. Its scaled H^p norm is $O(R^{1-p}/L)$ for an input of unit Z norm. This also gives $\|D\mathcal{A}(x_*+y_*)\|_{Z\to Z} \le C$, since $rR^{2-p} = o(1)$.

To prove (3.7), at a smooth base the differential minus its Fourier part maps $H^1 \to H^2$. Its real Hilbert adjoint gains the same derivative. Indeed if S is the full chart differential and

$\mathcal{J}_{\rm in}, \mathcal{J}_{\rm out}$ are the constant symplectic matrices in the two sets of coordinates, then $S^*\mathcal{J}_{\rm out}S = \mathcal{J}_{\rm in}$. Applying this formula also to $S_0 = \mathcal{F}_{\mathcal{A}}$ gives explicitly

$$(S - S_0)^* = \mathcal{J}_{\rm in}(S^{-1} - S_0^{-1})\mathcal{J}_{\rm out}^{-1}.$$

Here S^{-1} is the differential of the inverse chart at the image of the same smooth point. Each $\mathcal{J}$ and its inverse preserve Sobolev norms. Duality first gives $H^{-2} \to H^{-1}$ for the chart differential minus Fourier. Interpolation with $H^1 \to H^2$ gives both $H^{-1} \to L^2$ and $H^{p-1} \to H^p$. A shifted smooth envelope h as in (3.7) has $\|h\|_{H^{-1}} = O(R^{-1})$ and $\|h\|_{H^{p-1}} = O(R^{p-1})$, by rapid Fourier decay of the envelope. In the two terms of Z_0 these bounds both give $O(R^{-1})$. If $A(x_*) \coloneqq D\mathcal{A}(x_*) - \mathcal{F}_{\mathcal{A}}$, equivariance gives

$$\delta(A(x_*)h) = A(T_{2\pi/R}x_*)\delta h + (R/L)(D\mathcal{A}(T_{2\pi/R}x_*) - D\mathcal{A}(x_*))h.$$

Since v_x is an integer, translation by $2\pi/R$ preserves the carrier and $\delta h = e^{iRv_x x}\delta h_0$. The mean value theorem on the smooth envelope gives $\|\delta h_0\|_{H^p} = O(L^{-1})$. Modulation by this carrier is uniformly bounded in Z_0, with a constant depending on v_x. Thus the first term costs $O(L^{-1})$. In the second term $\|\delta x_*\|_{H^1} = O(L^{-1})$. The L^2 and scaled H^p estimates cost $O(L^{-1})$ and $O(R^{1-p}L^{-1})$. This proves the claimed bound.

3.2. **The transverse propagator.** We first retain from the linear part of (3.4) the old coefficients at V_*, together with all terms linear in the terminal axial planes rY_∞. Here

$$Y_\infty(t,x) \coloneqq \gamma\sum_i (c_i + e_i)e^{iK_i x - i(K_i^2 + \chi_{K_i}(I^\infty))t}, \qquad \chi_j(I) = \omega_j(I) - j^2 - 2\sum_k I_k.$$

By Lemma 2.3 and the $O(r^2)$ change of the actions, $\max_i |\chi_{K_i}(I^\infty)| = O(R^{-2} + r^2)$. The scalar $2\sum_{j\in F} I_j + c_R$ goes into the remainder because it equals $2r\sum_{j\in F} J_j - 2\sum_{j\notin F} I_j^\infty$ and is $O(r|J| + r^2)$. For each $m \coloneqq qR > 0$, use the two horizontal sequences

$$\big(\widehat{w_\perp}(n,m),\ \overline{\widehat{w_\perp}(n,-m)}\big)_{n\in\mathbb{Z}}.$$

Thus the label in the second copy is the index *before* conjugation. These two copies, taken over $q = 1, 2, \ldots$, represent all transverse Fourier coefficients. Multiplication by an axial potential acts within a copy. The term $U^2\overline{w_\perp}$ acts between copies and its upper-right entry at (n, n') is $\widehat{U^2}(n + n')$. In particular the blocks for different $m > 0$ evolve independently under the axial linearization.

We make the following three changes to the transverse linear equation. First remove the off-diagonal term formed solely from V_*. Next diagonalize the scalar old potential up to a prescribed order at large horizontal indices. Finally use the space-time harmonics of the old orbit to separate the terms linear in rY_∞. All three steps are performed in a single spatial norm.

We use the following version of the Schur test. Given a positive Hilbert weight v_ι, let the size of a matrix be the larger of the two suprema of its absolute row and column sums, after multiplying entry $(\iota, \jmath)$ by $\max(v_\iota/v_\jmath, v_\jmath/v_\iota)$. These sizes are submultiplicative, are unchanged by adjoints, and bound the operator norms on weighted ℓ^1 and ℓ^2. It is convenient to replace the weight in Lemma 3.2 by the equivalent weight

$$\widetilde{w}(n,q) \coloneqq (1 + |q| + (|n|/R - C_*)_+)^p(1 + (\operatorname{dist}(n, R\mathbb{Z})/L - 2)_+), \tag{3.8}$$

where $a_+ \coloneqq \max(a, 0)$. We take $C_* \coloneqq 2 + \max_{v\in\mathcal{V}} |v_x|$, so all axial indices $1 + RP_i$ lie in the flat part of the first factor for large R. Under a shift or reflection $n \mapsto \pm n + RP_i + b$, its ratios are bounded by $C(1 + |b|)^{p+1}$, and by a constant independent of R when $|b| \le C' \log R$. The same holds with $P_i = 0$. This bound covers both factors of the product weight in all Schur estimates below.

The old axial potential. We begin with the order-one terms from V_*. In the doubled generator the two free blocks are $A \coloneqq D_x^2 + m^2$ and $-A$, $D_x \coloneqq -i\partial_x$. Write $C_{nn'} \coloneqq \widehat{V_*^2}(n+n')$, $D_{\mathrm{free}} \coloneqq \mathrm{diag}(A, -A)$, and put

$$X_{nn'} \coloneqq -\frac{C_{nn'}}{n^2 + (n')^2 + 2m^2}, \qquad J_{\mathrm{off}} \coloneqq \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$

A star denotes the ordinary Hilbert space adjoint on the Fourier sequences. Since $AX + XA = -C$, the off-diagonal entries of $[D_{\mathrm{free}}, J_{\mathrm{off}}]$ are $-C, C^*$, where $[D, J] = DJ - JD$. The denominator is at least $2R^2$, and the smooth old coefficients decay rapidly in $n + n'$. It follows from the Schur test above that J_{off} and its derivatives in the finitely many old angles and actions have size $O(R^{-2})$, uniformly in q. Use $e^{J_{\mathrm{off}}}$ as the map from new to physical variables. In $e^{-J_{\mathrm{off}}} D_{\mathrm{free}} e^{J_{\mathrm{off}}}$ the first commutator cancels the old off-diagonal pair $C, -C^*$. Each further commutator has an additional factor J_{off}. The time derivative costs $O(R^{-2})$ at bounded parameter speeds. The resulting scalar potential, after removal of the scalar specified above, is $B_0 \coloneqq 2(|V_*|^2 - \sum_{j \in F} I_j)$ in the upper copy and its signed conjugate in the lower copy. It is real and has zero spatial mean, by the mass identity for the axial chart.

For this scalar potential we construct an orthonormal basis depending on the old angles and actions. Holding the actions fixed gives the nominal angular derivative $\partial_* \coloneqq \sum_{j \in F} (\omega_j(I^\infty) + c_R)\partial_{\theta_j}$. Fix an integer k_* with $k_*\alpha \geq s + 2\alpha$. For $|n| \geq L$ seek a column $e^{inx} f_n$ satisfying, to order $O(|n|^{-k_*})$,

$$(2nD_x + D_x^2 + B_0 - i\partial_*) f_n = d_n f_n.$$

To construct the column, write $\eta \coloneqq (2n)^{-1}$, $\mathcal{H} \coloneqq D_x^2 + B_0 - i\partial_*$, and use the finite expansions $f_n \coloneqq \sum_{j=0}^{k_*} \eta^j F_j$, $d_n^{\mathrm{app}} \coloneqq \sum_{j=0}^{k_*-1} \eta^j \lambda_j$, with $F_0 = 1$. Let $\langle h \rangle_x$ denote the spatial mean. If the coefficients through F_j have been found, set

$$S_j \coloneqq \mathcal{H} F_j - \sum_{a=0}^{j-1} \lambda_a F_{j-a}, \qquad \lambda_j \coloneqq \langle S_j \rangle_x, \qquad F_{j+1} \coloneqq -D_x^{-1}(S_j - \lambda_j).$$

The sum is empty for $j = 0$. We take the inverse on mean-zero functions, with zero-mean output. Thus the coefficient of η^j in $(\eta^{-1} D_x + \mathcal{H}) f_n - d_n^{\mathrm{app}} f_n$ vanishes for $0 \leq j < k_*$. In particular $\lambda_0 = 0$, $F_1 = -D_x^{-1} B_0$, and $\lambda_1 = -\langle B_0 D_x^{-1} B_0 \rangle_x = 0$. For the last identity pair the Fourier labels $l, -l$, whose denominators have opposite sign. Only finitely many smooth coefficients are needed, so the same bounds hold after any fixed number of old-parameter derivatives. Normalize each column in L^2. Its component of the defect perpendicular to the column is still $O(|n|^{-k_*})$. The parallel component of the normalized $(D_x^2 + B_0 - i\partial_*)$-column is real since the derivative of its squared norm vanishes. Taking this real expectation minus n^2 as d_n gives the asserted residual and $d_n = O(|n|^{-2})$. Indeed the normalizing factor is $1 + O(n^{-2})$ since the first correction to f_n had zero mean. The residual Fourier kernels are bounded by $C_j |n|^{-k_*} (1 + |n' - n|)^{-j}$ for every fixed j. The same argument applies in the conjugate copy.

The approximate columns need not be mutually orthogonal. We correct their Gram matrix, including at $|n| = L$. In the next paragraph P is the projection onto the horizontal indices $|n| \geq L$, $Q \coloneqq 1 - P$, and $T : P\ell^2 \to \ell^2$ maps the n-th unit sequence to the normalized column just constructed. Set $D_h \coloneqq (D_x^2 + \mathrm{diag}\, d_n)|_{P\ell^2}$ and $H_* \coloneqq D_x^2 + B_0$. We have $H_* T - i\partial_* T - T D_h = E_*$ with Schur size $O(L^{-k_*})$. Multiplication by T^* and subtraction of the adjoint identity give

$$[D_h, T^*T] - i\partial_*(T^*T) = T^* E_* - E_*^* T.$$

Put $G \coloneqq T^*T$ on $P\ell^2$, and let $\mathfrak{D}B \coloneqq [D_h, B] - i\partial_* B$. This is a derivation, so $\mathfrak{D}(G-1)^j$ is the sum of j terms with one factor $\mathfrak{D}G$. Since $G - 1 = O(L^{-1})$ in Schur size, the binomial series gives $\mathfrak{D}(G^{-1/2}) = O(L^{-k_*})$. The high-column map $U_h \coloneqq T G^{-1/2}$ is an isometry in unweighted ℓ^2, and

its residual equals $E_* G^{-1/2} + T\mathfrak{D}(G^{-1/2})$. It is still $O(L^{-k_*})$. To extend U_h, put $\widetilde{Q} = 1 - U_h U_h^*$, the projection onto the orthogonal complement of its range. Since U_h differs from the inclusion of the high subspace by $O(L^{-1})$, we have $\widetilde{Q} - Q = O(L^{-1})$. On the low subspace $Q\ell^2$ the operator $Q\widetilde{Q}Q$ is therefore invertible. Put

$$U_l = \widetilde{Q}Q\,(Q\widetilde{Q}Q)^{-1/2}.$$

The identity $\widetilde{Q}^2 = \widetilde{Q}$ gives $U_l^* U_l = 1$ on $Q\ell^2$, and $U_l^* U_h = 0$. The combined isometry (U_h, U_l) differs from the identity by $O(L^{-1})$, so it is onto for large L. In particular, applying U_l^* to the residual equation for U_h shows that the block from high inputs to low outputs is $O(L^{-k_*})$. The reverse block is its adjoint, since the generator in this unitary frame is Hermitian. Commuting D_x^2 through a high column of $T - P$ costs $O(1)$ in Schur size, since the factor $2n(n'-n) + (n'-n)^2$ multiplies an $O(1/n)$ smooth envelope. Product rules and the same binomial series bound the commutator with the full extension by $O(1)$. In the new basis the scalar operator, less D_x^2 and including the nominal connection, is Hermitian. Its high columns, and hence high rows, are the real diagonal d_n plus $O(L^{-k_*})$. The remaining bounded Hermitian matrix acts solely on $|n| < L$.

Denote the full scalar column map in the two copies by $\mathcal{U}$, with the conjugate construction in the second copy. Then $\mathcal{S} \coloneqq e^{J_{\mathrm{off}}}\mathcal{U}$ maps the resulting coordinates back to the original field. In an equation $i\dot{w} = \mathcal{L}w$, the substitution $w = \mathcal{S}a$ gives the generator $\mathcal{S}^{-1}\mathcal{L}\mathcal{S} - i\mathcal{S}^{-1}\dot{\mathcal{S}}$. Both $\mathcal{S} - 1$ and its old-parameter derivatives have Schur size $O(L^{-1})$. Along nonnominal old parameters the difference of the actual and nominal connections has size

$$CL^{-1}(|\dot{I}_F| + |\dot{\theta}_F - \dot{\theta}_F^{\infty}|). \tag{3.9}$$

In particular variations of the order-one low Hermitian matrix may be retained without any loss on the slow time scale.

It remains to decide which terms of order r are resonant. Use the Fourier series $\sum_{b,\nu} v_{b,\nu} e^{ibx - i\nu t}$ of the old axial profile with its retuned old actions and nominal angles. Here $b \in \mathbb{Z}$ is a spatial frequency and $\nu \in Q_F^{-1}\mathbb{Z}$ a time frequency, with Q_F a common denominator of the old occupied frequencies in the mass gauge. Thus $2\pi Q_F$ is an old period. The coefficient $v_{b,\nu}$ is the sum of all angular harmonics with that space-time frequency. A displacement $O(r^2)$ of the actions changes the summed coefficients by $O(r^2)$ in every fixed Fourier moment used here. Indeed the finite-angle parametrization is analytic into each fixed Sobolev space on a complex angular strip of positive width. For a multi-index $l = (l_j)_{j \in F} \in \mathbb{Z}^F$, the angular harmonic $e^{-il\cdot\theta}$ has the single spatial frequency $b = \sum_{j \in F} j l_j$, by translation covariance, and time frequency $\nu = \sum_{j \in F} l_j(\omega_j(I^{\infty}) + c_R)$. Its coefficient decays exponentially in $|l|_1 \coloneqq \sum_j |l_j|$. Truncation at $|l|_1 \le C_A \log R$ leaves an $O(R^{-A})$ tail in Schur and multiplier norms, with $|b| + |\nu| \le C'_A \log R$ in the truncation. Choose A larger than every fixed power needed below. Changing the old phases and actions to their actual values in a term already of order r costs $O(r(|\Theta| + r|J|))$.

Resonances of the new axial planes. Pairing the terminal plane at K_i with $v_{b,\nu}$ gives an opposite-copy entry between n and $n' \coloneqq K_i + b - n$. Here and below a bare divisor uses the free root frequency K_i^2, with its shift $\chi_{K_i}(I^{\infty})$ treated separately. For this entry the divisor is

$$D \coloneqq n^2 + (n')^2 + 2(qR)^2 - K_i^2 - \nu.$$

If $D \neq 0$, then $|D| \ge Q_F^{-1}$. If an endpoint has $|n| < L$, write $n' = K_i + (b - n)$. Then

$$D = 2q^2R^2 + 2K_i(b - n) + (b - n)^2 + n^2 - \nu.$$

Since $|b| + |\nu| = O(\log R)$ and $|K_i| = O(R)$, all terms after $2q^2R^2$ sum to $o(R^2)$, uniformly in q. Hence $|D| \gtrsim R^2$ on every edge touching this low subspace. The same calculation with n, n'

interchanged covers $|n'| < L$. On a zero-divisor edge q is bounded and both $|n|$ and $|n'|$ are between fixed positive multiples of R. Indeed, substituting $n' = K_i + b - n$ gives

$$2\big(n - (K_i + b)/2\big)^2 + 2(qR)^2 = K_i^2 + \nu - (K_i + b)^2/2.$$

The right-hand side is $O(R^2)$, so n/R and q are bounded. Along a subsequence q is a fixed nonzero integer and n/R has a limit X. Dividing by R^2 gives $q^2 = X(P_i - X)$, so X and $P_i - X$ are both nonzero. In particular $P_i = X + q^2/X$. Since the P_i are distinct, a row cannot serve two roots for arbitrarily large R. For the same root, changing b by a nonzero integer h changes $(n')^2$ by $h(2n' + h)$. Its modulus is at least cR since $|n'| \gtrsim R$ and $|h| = O(\log R)$. Two retained temporal labels can differ by only $O(\log R)$. Once b is fixed, $D = 0$ determines ν, and all harmonics with these same labels have already been added into $v_{b,\nu}$. The argument applies at either endpoint. Thus the exact resonances form disjoint pairs.

The same-copy terms and their adjoints take n to $n + K_i - b$. Their bare divisor is the difference

$$(n + K_i - b)^2 - n^2 - (K_i^2 - \nu) = 2(K_i - b)(n - b) + \nu - b^2.$$

It has modulus $\gtrsim R$ unless $n = b$. Keep every entry $b \leftrightarrow K_i$ and its adjoint as a Hermitian coupling, with their actual time oscillations. The vertex b cannot meet an opposite-copy resonance. Intersection at K_i would require $P_i(P_j - P_i) = q^2$ in the limit, excluded by Proposition 2.4. This separates all opposite-copy pairs from the low Hermitian matrix and the retained same-copy terms.

For the discarded entries solve the homological equation using the bare divisors above. More explicitly, an entry $B_{uv}e^{-i\zeta t}$ in the generator with free energies E_u, E_v is cancelled to first order by the entry $-B_{uv}e^{-i\zeta t}/(E_u - E_v - \zeta_{\mathrm{bare}})$ in the exponent of the change of variables. The omitted root shift $\zeta - \zeta_{\mathrm{bare}}$ is $O(R^{-2} + r^2)$. Each Fourier harmonic defines a partial shift or reflection with at most one entry per row and column, so the summed generator has Schur size $O(r)$. Its terms touching the low subspace have size $O(r/R)$ for same-copy entries and $O(r/R^2)$ for opposite-copy entries. Its commutator with the bounded low matrix therefore costs $O(r/R)$, and that with the high real diagonal costs $O(rL^{-2})$. The terminal planes oscillate at frequencies of order R^2. To keep those frequencies out of the error bound, combine the free commutator with the time derivative before estimating either one. Write the change of variables as $\mathcal{M} \coloneqq e^{\mathcal{B}(t)}$, where $\mathcal{B}$ is the exponent just constructed. Differentiating the conjugation with respect to θ gives the exact identity

$$\mathcal{M}^{-1}D_{\mathrm{free}}\mathcal{M} - i\mathcal{M}^{-1}\partial_t\mathcal{M} - D_{\mathrm{free}} = \int_0^1 e^{-\theta\mathcal{B}}([D_{\mathrm{free}}, \mathcal{B}] - i\partial_t\mathcal{B})e^{\theta\mathcal{B}}\,d\theta.$$

On a Fourier entry the factor in parentheses multiplies the exponent's coefficient by its actual divisor. It equals the negative of the discarded entry up to the root shift estimated above, and has total Schur size $O(r)$. Since $\mathcal{B}$ itself has size $O(r)$, conjugation inside the integral changes this factor by $O(r^2)$. This gives $\mathcal{M} = 1 + O(r)$ and the required higher-order bound. Conjugation of the order-r terms by the old frame costs $O(r/L)$.

The mixed boundary conditions. These constructions give a bounded inverse for the transverse inhomogeneous problem with mixed initial and terminal data. On the complement of the opposite-copy pairs, the free diagonal, the old low matrix, the real $d_n(t)$ and the retained same-copy terms generate an isometry in (3.8). Indeed each coupling within a given q and copy touches only indices where the weight equals $(1 + |q|)^p$. This holds on $|n| < L$, at $n = b$ with $|b| = O(\log R)$, and at $n = K_i$ with $\operatorname{dist}(K_i, R\mathbb{Z}) \le 1$. Every other entry is real diagonal. This holds for the old matrix evaluated at the actual, time-dependent parameters. On each opposite-copy pair remove the two free phases. Let $\delta_i \coloneqq \chi_{K_i}(I^\infty)$. Since the bare divisor vanishes, the upper-right entry in the equation for $i\partial_t$ is now $r\eta e^{-i\delta_i t}$, where $\eta \coloneqq 2\gamma(c_i + e_i)v_{b,\nu}$

is the coupling coefficient of this pair. The lower-left entry is its negative conjugate. The coordinate change $\operatorname{diag}(e^{-i\delta_i t/2}, e^{i\delta_i t/2})$ removes these phases and adds a diagonal of size $|\delta_i|/2$ in physical time. Thus in slow time $\tau = rt$ the principal matrix for ∂_τ is

$$\begin{pmatrix} 0 & -i\eta \\ i\bar{\eta} & 0 \end{pmatrix}.$$

It is Hermitian and its square equals $|\eta|^2$ times the identity, so its eigenvalues are $\pm|\eta|$. To compare the two endpoint weights, use $n' = RP_i + 1 + b - n$. The distances of n and n' from $R\mathbb{Z}$ differ by at most $|1 + b| = O(\log R) = o(L)$. Their scaled horizontal frequencies stay bounded and their vertical indices have the same absolute value. Thus the weights in (3.8) have bounded ratio in both directions, and the ordinary orthogonal spectral projections are bounded in Z. Discarding the half-angle diagonal correction and the d_n's on these pairs costs $O((R^{-2} + r^2)/r)$ in slow time.

We collect the losses when the old parameters follow their nominal motion. After division by r, the errors from removing the old off-diagonal pair and constructing the scalar columns are bounded by $C(R^{-2} + L^{-k_*})/r$. The error from conjugating the terms of order r by the old frame costs C/L. The mixed conjugation contributes at most $C(R^{-1} + L^{-2} + r + R^{-2} + R^{-A})$ in the same units. Together with the discarded diagonal on the resonant pairs these are $O(L^{-1})$, since $R^{-2}/r = O(L^{-2})$ and $L^{-k_*}/r = O(L^{-2})$ by (3.2). When the old parameters move away from the nominal motion, one must also keep (3.9) and the changes of the coefficients caused by the old action and angle displacements.

The set of exact pairs can change with R. To choose one time weight for all large R, take the coefficients $v^0_{b,\nu}$ of the original periodic field V, before retuning. For every $a > 0$ the set of numbers

$$2|c_i v^0_{b,\nu}|, \qquad i \text{ an axial vertex}, \quad (b, \nu) \in \mathbb{Z} \times Q_F^{-1}\mathbb{Z},$$

contains only finitely many values above a. This follows from absolute summability and the finiteness of the tree. Choose $\sigma > 0$ below the model decay rate in τ and at positive distance from this set and zero. Each rate on a retained pair differs from its listed value by $O(r^2 + R^{-10} + R^{-A})$, by the retuning and angular tail estimates. Thus the distance of all signed rates from $-\sigma$ has a positive lower bound independent of R. For any $T < S$ measure inhomogeneities in the norm

$$\|f\|_{\sigma,Z} := \sup_{T \le \tau \le S} e^{\sigma(\tau - T)} \|f(\tau)\|_Z.$$

Write $\mathcal{U}_0(\tau, u)$ for the propagator of this principal system, including the changes of phase on the pairs, from slow time u to τ. Denote by $\Pi_f(u)$ the projection at time u onto the decaying pair directions whose exponents are less than $-\sigma$, and put $\Pi_c(u) := 1 - \Pi_f(u)$. These projections commute with propagation in the sense that $\mathcal{U}_0(\tau, u)\Pi_f(u) = \Pi_f(\tau)\mathcal{U}_0(\tau, u)$. For forcing f in the equation for ∂_τ, define

$$(\mathcal{G}_\perp f)(\tau) := \int_T^\tau \mathcal{U}_0(\tau, u)\Pi_f(u) f(u)\, du - \int_\tau^S \mathcal{U}_0(\tau, u)\Pi_c(u) f(u)\, du.$$

This prescribes zero fast stable component at T and zero complementary component at S. A scalar exponent μ contributes the time-weighted factor $e^{(\mu+\sigma)(\tau-u)}$. The first integral has $\mu + \sigma < 0$ and $\tau - u \ge 0$, while the second has $\mu + \sigma > 0$ and $\tau - u \le 0$. The isometric part obeys the second estimate with $\mu = 0$. Since all the exponents stay a fixed distance from $-\sigma$, both weighted kernels are bounded in operator norm by $Ce^{-\delta_{\text{gap}}|\tau-u|}$ for some $\delta_{\text{gap}} > 0$ independent of R, T, S. Here δ_{gap} can be any positive lower bound for σ and the distances of the signed pair rates from $-\sigma$. Taking the Hilbert direct sum of the blocks before time integration gives $\|\mathcal{G}_\perp f\|_{\sigma,Z} \le 2C\delta_{\text{gap}}^{-1}\|f\|_{\sigma,Z}$, uniformly in the length of the interval. The placement of the two traces also gives uniqueness. On a fast direction the homogeneous solution is determined by its

value at T, and on each remaining direction it is determined by its value at S. On the half-line the backward integral converges with the same kernel bound. In that case membership in the exponentially weighted space excludes every homogeneous complementary direction, including the isometric part. On a fixed bounded slow interval the same principal propagator is bounded in both time directions. Large unitary rotations used to obtain the pairwise estimate are needed only to define these propagators and projections.

3.3. **Resonant forcing and the nonlinear terminal problem.** *Carrier phases and the prescribed forcing.* We work first on a finite slow-time interval $[T, S]$, with bounds independent of S. There are three parts to the construction. We choose phases so that each selected interaction has the time phase of its output. We then solve the free inhomogeneous equation for the remaining prescribed oscillations. Finally we estimate the errors in the transverse field, the new axial coordinates, and the old actions and angles. Only the last step uses a boundary value problem.

We first choose the time phase of each carrier. A transverse wave has a decaying amplitude, but a new axial wave survives at positive infinity. We therefore use its exact terminal frequency, including the shift due to all occupied axial modes. Put $\kappa_v := |k_v|^2 + \varepsilon_v$, with $\varepsilon_{(P_i,0)} := \chi_{K_i}(I^\infty)$ at every axial vertex. For each selected triangle $v + v' = u$ impose $\varepsilon_v + \varepsilon_{v'} = \varepsilon_u$. These conditions have a common solution. Starting from the root, split the assigned shift of each primary parent $m = p + q$ between its children in any fixed way, for instance $\varepsilon_p = 0$ and $\varepsilon_q = \varepsilon_m$. When m is off the axis its split contains three additional triangles. Their equations read

$$\varepsilon_{J(p)} = \varepsilon_{A(p)} - \varepsilon_p, \qquad \varepsilon_{J(q)} = \varepsilon_{A(q)} - \varepsilon_q, \qquad \varepsilon_{T_m} = \varepsilon_{J(p)} + \varepsilon_{J(q)}.$$

Here $A(p), A(q)$ are on the axis and their shifts have already been specified. The auxiliary vertices are distinct at different splits by Proposition 2.4. This recursion gives $\varepsilon_v = O(R^{-2} + r^2)$ on the fixed tree.

Write $\mathbf{x} = (x, y)$, $\tau = rt$, and separate the known waves into their transverse and axial parts. In the following formulas we identify the axial polynomial Y with its sequence of Fourier coefficients.

$$W(t, \mathbf{x}) := \gamma \sum_{v_y \neq 0} x_v(2|g|\tau) e^{ik_v \cdot \mathbf{x} - i\kappa_v t},$$
$$Y(t, x) := \gamma \sum_i \big(x_{(P_i,0)}(2|g|\tau) + e_i\big) e^{iK_i x - i\kappa_{(P_i,0)} t}.$$

The first sum is over the transverse vertices of $\mathcal{V}$. Here and below $Y_\infty(t)$ is obtained from $Y(t)$ by replacing each $x_{(P_i,0)}$ by its limit c_i and keeping its phase $e^{-i\kappa_{(P_i,0)} t}$. Thus the axial comparison at infinity is to a time-dependent solution. The retuning errors e_i put the limiting axial amplitudes at their prescribed values in Y_∞. By Lemma 2.5, for some $\lambda > \sigma$ and every $\epsilon > 0$ one can choose T so that the amplitudes of $W, Y - Y_\infty$ and their first two derivatives in τ are bounded by $\epsilon e^{-\lambda(\tau - T)}$ for $\tau \geq T$. Such an amplitude derivative keeps the rapid carrier phase fixed.

In expanding $|V_* + r(Y+W)|^2(V_* + r(Y+W))$, the coefficient of r^2 is $2V_*|Y+W|^2 + \overline{V_*}(Y+W)^2$. Terms involving only Y are axial and are already included in the Birkhoff frequencies. Terms involving just one transverse factor also have zero axial projection. Thus the order-r^2 transverse forcing and the axial forcing after subtracting the mean transverse density are given by

$$Q_w := P_\perp\{2V_*(|Y+W|^2 - |Y|^2) + \overline{V_*}((Y+W)^2 - Y^2)\},$$
$$Q_z := P_0\{2V_*(|W|^2 - S_W) + \overline{V_*}W^2\}, \qquad S_W := \sum_{v:\, v_y \neq 0} |x_v(2|g|\tau)|^2. \tag{3.10}$$

In fact S_W is the spatial mean of $|W|^2$, by orthogonality of its distinct Fourier modes. To handle this mean exactly, take $2r^2 S_W U$ out of the full axial force f in (3.3). Gauge equivariance, differentiated along the curve $a \mapsto e^{-ia}U$ at zero, gives

$$D\Phi(U)[-2ir^2 S_W U] = -2ir^2 S_W z, \qquad 2\,\mathrm{Re}(\bar{z}_j(-2ir^2 S_W z_j)) = 0.$$

Thus the mean changes each angular speed by $2r^2 S_W$. Its contribution on a new axial mode of size $O(r)$ is $O(r^3 S_W)$. The remaining known force has leading term $r^2 Q_z$. Its outputs have nonzero horizontal velocity. Indeed a zero difference in horizontal velocity would require the same vertex, whose diagonal density has just been removed, and a zero sum would require opposite horizontal velocities. Both statements use the distinct absolute horizontal components in Proposition 2.4. This observation permits use of (3.7) before projecting to the old coordinates. In extracting the prescribed harmonics of Q_w, Q_z we use the nominal old angles. Replacing the actual V_* by that profile costs $C(|\Theta| + r|J| + r^2)$ times a decaying amplitude. The low linear operator continues to use the actual old coordinates.

We determine the time frequency ζ and spatial frequency k of each remaining product. Put $e = (1, 0)$ and write v_x for the horizontal component of v. For two positive carriers and a conjugated old harmonic (b, ν), one has $k = (2 - b)e + R(v + v')$ and $\zeta = \kappa_v + \kappa_{v'} - \nu$. Direct expansion gives

$$\zeta - |k|^2 = -2R^2 v \cdot v' + 2R(b - 1)(v_x + v'_x) + 2 - \nu - (2 - b)^2 + \varepsilon_v + \varepsilon_{v'}.$$

For a carrier times a conjugated carrier and a positive old harmonic, $k = be + R(v - v')$, $\zeta = \kappa_v - \kappa_{v'} + \nu$, and

$$\zeta - |k|^2 = 2R^2(v - v') \cdot v' + 2R(1 - b)(v_x - v'_x) + \nu - b^2 + \varepsilon_v - \varepsilon_{v'}.$$

When the coefficient of R^2 is nonzero it is an integer of modulus at least one. If it vanishes, Proposition 2.4 puts the nonconstant output velocity in the tree, with nonzero horizontal component.

When $b \neq 1$ the linear term then dominates the logarithmically bounded remainder. When $b = 1$, the only desired temporal harmonic is $\nu = 1$. All others have separation at least $1/Q_F$ before detuning. Consider the surviving harmonic $(b, \nu) = (1, 1)$ on one triangle $p + q = h$. The assigned frequency corrections give $\kappa_p + \kappa_q - \kappa_h = 1$, so the product at each output has exactly that carrier's phase. After removing r^2 and the carrier, the two orderings at the parent and the density term at a child have coefficients

$$\begin{aligned} 2\bar{g}\,\gamma^2 x_p x_q &= 2i|g|\gamma\, x_p x_q, \\ 2g\,|\gamma|^2 x_h x_q &= -2i|g|\gamma\, x_h x_q, \end{aligned} \qquad \gamma = ig/|g|.$$

On the left of the evolution equation the amplitude derivative is $2i|g|\gamma\, x'_v$, with the prime taken in $z = 2|g|rt$. The first line gives the parent equation and the second gives the child equation in (2.4). Their contributions add at shared vertices. The change in g from old-action retuning is $O(r^2)$, and the omitted angular tail has the bound above. Every occurrence of a new radius error e_i in these products is paired with a decaying transverse amplitude.

There is also a linear term at each surviving axial index K_i. Its carrier has the exact frequency $\kappa_{(P_i,0)} = \omega_{K_i}(I^\infty) + c_R$. When the actual diagonal operator acts on that carrier, the remaining frequency difference is $\omega_{K_i}(I) - \omega_{K_i}(I^\infty)$. It tends to zero with $I - I^\infty$, at the rate estimated below. Using the free frequency K_i^2 here would leave the constant terminal shift multiplying a nondecaying wave.

For each remaining retained harmonic $a(\tau)e^{ik\cdot(x,y)-i\zeta t}$ in the nominal forcing put $E_{k,\zeta} \coloneqq e^{ik\cdot\mathbf{x}-i\zeta t}$ and use

$$(i\partial_t+\Delta)\frac{a(\tau)E_{k,\zeta}}{\zeta-|k|^2}=a(\tau)E_{k,\zeta}+\frac{ir\,(\partial_\tau a)(\tau)E_{k,\zeta}}{\zeta-|k|^2}.$$

For example, an order-r^2 source of this form in the physical equation is canceled by adding $r^2aE_{k,\zeta}/(\zeta-|k|^2)$ to the prescribed field. Its uncanceled derivative has size $r^3|\partial_\tau a|/|\zeta-|k|^2|$. The field correction itself has size r in the original variables per unit of normalized error, and $\partial_t=r\partial_\tau$ on its slow amplitude. The normalized error equation therefore divides the physical residual by r^2, leaving one factor of r in this derivative term. Write P_w, P_z for the resulting sums, where on the axis we only take indices outside F. For large R all retained axial outputs here have $|n|\gtrsim R$. The absolute divisors are at least $1/(2Q_F)$. A transverse output with zero horizontal velocity must have a nonzero R^2 coefficient, so its particular carries $O(R^{-2})$. This pays for applying the entire bounded old low matrix. Every other particular lies at $|n|\gtrsim R$, where the real diagonal shifts are $O(R^{-2})$. On the axis the same bound holds for the empty-site renormalized frequency of the smooth old state. Its further change at the actual actions is $O(r|J|+r^2)$ in the gauged frequency. The scalar mass discrepancy is included in this last estimate. Thus the bounded divisors are used in the dressed transverse coordinates and in the true axial coordinates. The B norms of the particular sums and of the sums with one slow amplitude derivative are bounded by $C\epsilon e^{-\lambda(\tau-T)}$, by absolute old-harmonic summation. This construction leaves errors of size $O(r)$ after division by r^2 when a slow derivative falls on the particular amplitudes.

Equations for the four errors. We now seek the actual field as the sum of these prescribed terms and an error. Write $F^c \coloneqq \mathbb{Z}\setminus F$. Let h be the transverse error after both coordinate changes, and $b=(b_j)_{j\in F^c}$ the axial error in Birkhoff coordinates. Each error is divided by r. With B_w, B_z denoting the prescribed profiles, set

$$w_\perp=\mathcal{S}(B_w+r\mathcal{M}h),\qquad z|_{F^c}=B_z+rb,\qquad B_w\coloneqq rW+r^2P_w,\quad B_z\coloneqq rY+r^2P_z. \tag{3.11}$$

In the first identity the left side denotes the doubled transverse field described above, and W, P_w represent their doubled versions. Every transverse index and every axial index in F^c is retained. In particular $\mathcal{M}$ only acts on the unknown h. The terms of the mixed operator on the known fields were already included in Q_w.

For reference we specify the exact equations that the terminal integral map solves. Reconstruct the old coordinates from (3.5) as $z_j=\sqrt{I_j^\infty+rJ_j}\,\exp[-i(\theta_j^\infty+\Theta_j)]$, and put $U=\Psi(z)$. Let $\Lambda\coloneqq\operatorname{diag}(\omega_j(I)+c_R)$, $D_f\coloneqq D\Phi(U)[-if]$, and let $\mathcal{L}(U)$ be the doubled linear generator in (3.4). Denote the doubled nonlinear term by $\mathcal{N}\coloneqq(N,-\overline{N})$, with the appropriate vertical components taken in each copy. Set $\mathcal{K}\coloneqq\mathcal{S}^{-1}(\mathcal{L}(U)\mathcal{S}-i\dot{\mathcal{S}})$. Write Λ_{F^c} for the restriction of Λ to F^c. All four equations below have the same slow time on the left. The factors of r can be seen, for example, from $\partial_t(B_z+rb)=\dot B_z+r^2\partial_\tau b$. For $j\in F$, $\dot I_j=2\operatorname{Re}(\bar z_j\dot z_j)$ and $\dot\theta_j=-\operatorname{Im}(\dot z_j/z_j)$. Substituting the reconstruction into the equations therefore gives

$$\begin{aligned}
i\partial_\tau h&=r^{-1}\mathcal{M}^{-1}(\mathcal{K}\mathcal{M}-i\dot{\mathcal{M}})h+r^{-2}\mathcal{M}^{-1}(\mathcal{K}B_w-i\dot B_w+\mathcal{S}^{-1}\mathcal{N}),\\
\partial_\tau b&=-ir^{-1}\Lambda_{F^c}b+r^{-2}(-i\Lambda_{F^c}B_z-\dot B_z+(D_f)|_{F^c}),\\
\partial_\tau J_j&=2r^{-2}\operatorname{Re}(\bar z_j(D_f)_j),\qquad j\in F,\\
\partial_\tau\Theta_j&=r^{-1}(\omega_j(I)-\omega_j(I^\infty)-\operatorname{Im}((D_f)_j/z_j)),\qquad j\in F.
\end{aligned} \tag{3.12}$$

On a trial path evaluate $\dot{\mathcal{S}}$ by the chain rule using the instantaneous physical velocities $2\operatorname{Re}(\bar z_j(D_f)_j)$ and $\omega_j(I)+c_R-\operatorname{Im}((D_f)_j/z_j)$. This prescription only uses values of the trial

path. At a fixed point these are its actual velocities, so that (3.12) is equivalent to (3.3)–(3.4). The dots on $\mathcal{M}, B_w, B_z$ are ordinary physical-time derivatives of prescribed functions.

For a normed space D put $\|a\|_{\sigma,D} \coloneqq \sup_{T\le\tau\le S} e^{\sigma(\tau-T)}\|a(\tau)\|_D$. Set

$$H_1 \coloneqq \|h\|_{\sigma,Z}, \quad A_1 \coloneqq \|b\|_{\sigma,Z}, \quad J_1 \coloneqq \|J\|_{\sigma,\mathbb{R}^F}, \quad L_1 \coloneqq \|\Theta\|_{\sigma,\mathbb{R}^F}, \quad X \coloneqq H_1 + A_1 + J_1 + L_1.$$

Here $\mathbb{R}^F$ carries its Euclidean norm. For a vanishing trial error the residual is bounded in these norms by a constant times

$$e_R \coloneqq L^{-1} + R^{-1} + R^{-2}/r + r(1 + R^{2-p}) + L^{-k_*}/r + r = O(R^{-\alpha}). \tag{3.13}$$

Indeed $R^{-2}/r = a_0^{-1}R^{-(2-s)}$, $rR^{2-p} = a_0 R^{-(p+s-2)}$, and $L^{-k_*}/r \le a_0^{-1}R^{-2\alpha}$. The scale $L^{-1} = R^{-\alpha}$ bounds their sum along with the remaining terms.

On $[T, S]$ we define a map from trial errors to new errors. Evaluate the coefficients and remainders in (3.12) at the trial path. Solve for the new h using the transverse Green operator, keeping the Hermitian low matrix at its trial values. For the new b, retain its exact real diagonal and prescribe $b(S) = 0$. Thus a source $f_j(\tau)$ in its equation contributes, at $j \in F^c$, the integral

$$-\int_\tau^S \exp\Big(\frac{i}{r}\int_\tau^u [\omega_j(I(v)) + c_R]\,dv\Big)\, f_j(u)\,du.$$

Every diagonal exponential has modulus one. Taking the sequence norm before integrating and then the time weight gives the factor

$$\int_\tau^S e^{\sigma(\tau-T)}e^{-\sigma(u-T)}\,du = \frac{1 - e^{-\sigma(S-\tau)}}{\sigma} \le \sigma^{-1}.$$

This bound is uniform in S, even if an axial frequency vanishes. The old actions and angles also use zero terminal errors and the same integral bound. For the transverse error the fast stable components instead have zero trace at T, as prescribed by $\mathcal{G}_\perp$. These conditions determine which components are integrated in each time direction. Write H_1', A_1', J_1', L_1' for the norms of the four outputs of this map. For $X \le 1$, $\sqrt{L}X \le 1$, they satisfy

$$\begin{aligned}
H_1' &\le C(e_R + \epsilon X + \sqrt{L}X^2 + o(1)H_1),\\
A_1' &\le C(e_R + J_1 + \epsilon X + \sqrt{L}X^2 + o(1)A_1),\\
J_1' &\le C(e_R + \epsilon H_1 + \epsilon^2 A_1 + \sqrt{L}X^2),\\
L_1' &\le C(e_R + J_1 + r(A_1 + H_1) + \sqrt{L}X^2).
\end{aligned} \tag{3.14}$$

The constants here depend on the fixed background, tree and weight σ. They are uniform in R sufficiently large, in $S > T$, and in the starting time for the small tail. The $o(1)$ coefficients tend to zero as $R \to \infty$ for these fixed choices.

For the proof put $E \coloneqq e^{-\sigma(\tau-T)}$. First isolate the error of the axial chart on the prescribed waves. Define

$$\mathfrak{e}_0 \coloneqq r^{-1}\big[\Psi(z|_F + rY) - V_*\big] - Y, \qquad \mathfrak{e}_1 \coloneqq r^{-1}\big[U - \Psi(z|_F + rY)\big],$$

using Fourier series on the field side. In the transverse variables set $W_1 \coloneqq (\mathcal{S} - 1)W + r\mathcal{S}P_w$ and $h_1 \coloneqq \mathcal{S}\mathcal{M}h$. Then

$$U = V_* + rY + r\mathfrak{e}_0 + r\mathfrak{e}_1, \qquad w_\perp/r = W + W_1 + h_1,$$

where

$$\|\mathfrak{e}_0\|_Z \le Ce_R, \quad \|\mathfrak{e}_1\|_Z \le C(A_1 + r\epsilon)E, \qquad \|W_1\|_B \le C(L^{-1} + r)\epsilon E, \quad \|h_1\|_Z \le CH_1E.$$

Indeed $\mathfrak{e}_0$ is the Taylor error in replacing $\Psi(z|_F + rY)$ by $V_* + rY$, divided by r. Equations (3.6)–(3.7) at the actual old state give its bound. For $\mathfrak{e}_1$ use the bounded full chart differential

on the $O(r)$ tube. For W_1, h_1 use the two Schur bounds in (3.11). All prescribed Fourier polynomials have uniform multiplier norms at their indicated amplitude.

Transverse field. Put $v := U - V_*$. The additional linear terms on a physical field w are exactly

$$2(\overline{V_*}v + V_*\bar{v})w + 2V_*v\bar{w} + 2|v|^2w + v^2\bar{w}.$$

Their order-r^2 contributions on W have already been included in Q_w. Their order-r coefficients on h from Y_∞ are precisely the mixed terms of the preceding subsection. Replacing Y_∞ by Y adds $O(\epsilon)H_1$ in slow units. The factor $\mathfrak{e}_0$ against the known multiplier W costs $O(e_R)$. Against a general error it costs $O(\sqrt{L}\, e_R)H_1 = o(1)H_1$. The factor $\mathfrak{e}_1$ against W costs $O(\epsilon(A_1 + r\epsilon))$, while against h_1 it costs $O(\sqrt{L}(A_1 + r\epsilon)H_1)$. For example, two transverse unknowns and a smooth old factor satisfy

$$r^{-2}\|P_\perp(V_*|rh_1|^2)\|_Z \le C\sqrt{L}\, H_1^2E^2, \qquad \|Wh_1\|_Z \le C\epsilon H_1E^2.$$

In the second inequality the known W is a B multiplier. The different estimates for these two products explain the single loss $\sqrt{L}$ at quadratic order. Terms with two factors of v have an additional r. In $N(U, w)$ the two known transverse factors paired with V_* are included in Q_w. A single unknown pairs with a decaying known factor, and two unknowns cost $O(\sqrt{L}X^2)$. The cubic all-error term is $O(rLX^3)$. These bounds follow by applying Lemma 3.2 once for each additional arbitrary factor, using B for the known ones.

The old-frame residual on a known transverse mode costs $O((R^{-2} + L^{-k_*})/r)$. Its horizontal index is comparable to R, so its own scalar diagonal is only $O(R^{-2})$. The scalar $2\sum_F I_j + c_R$ on the known modes costs $C\epsilon(J_1 + r)$, and on h costs $C(J_1 + r)H_1$. The extra connection (3.9), divided by r, is small. The true velocities at the trial point give $|\dot{I}_F| + |\dot{\theta}_F - \dot{\theta}_F^\infty| \le C(rJ_1E + r^2)$ on our tube. To see the r^2 in this estimate, use $\|w_\perp\|_Z \le Cr(\epsilon + H_1)E$, expand products with known multipliers first, and use $\sqrt{L}X \le 1$ on the remaining pair of arbitrary factors. Terms due to freezing the mixed coefficients at the nominal phases carry $|\Theta| + r|J|$. The discarded nonresonant homogeneous terms have slow operator norm $O((R^{-2} + L^{-k_*})/r + L^{-1} + r)$ in addition to these modulation errors. These observations give the first line of (3.14).

Axial modes outside F. The ℓ^1 difference of the actual and terminal actions is bounded by

$$C\big(rJ_1 + r^2(\epsilon + A_1 + r\epsilon)\big)E.$$

Indeed on F we have $I_j - I_j^\infty = rJ_j$. On F^c put $z^\infty := rY_\infty$, with its true time phases, and use

$$\sum_{j\in F^c} \big||z_j|^2 - |z_j^\infty|^2\big| \le \|z|_{F^c} - z^\infty\|_{\ell^2}\big(\|z|_{F^c}\|_{\ell^2} + \|z^\infty\|_{\ell^2}\big).$$

The last factor is $O(r)$, while the first contains $r(Y - Y_\infty)$, r^2P_z, and rb. To see the effect on one persistent coordinate, write $\Delta\omega_i = \omega_{K_i}(I) - \omega_{K_i}(I^\infty)$. The mass formula and the local ℓ^2-Lipschitz bound for χ give $|\Delta\omega_i| \le C\|I - I^\infty\|_{\ell^1}$, uniformly in K_i. In the equation for b_{K_i} this mismatch contributes

$$\frac{|\Delta\omega_i|\,|rY_{K_i}|}{r^2} \le C\big[J_1 + r(\epsilon + A_1 + r\epsilon)\big]E,$$

since $|Y_{K_i}|$ is bounded on the fixed tree. This explains why the coefficient of J_1 in the second row need not be small even on the far tail. The part involving A_1 carries an additional r. The diagonal on the error b itself is retained. After removing the exact density rotation, f contains two transverse factors or three. One unknown transverse factor times a known one costs $C\epsilon H_1$, two arbitrary ones $C\sqrt{L}X^2$. On the leading known terms the price for replacing $D\Phi(U)$ by $D\Phi(V_*)$ is $O(r(1 + R^{2-p}))$, and that for replacing the latter by Fourier is $O(R^{-1} + L^{-1})$ on the high horizontal outputs. All these bounds apply to the strong norm Z.

Old actions and angles. For each $j \in F$ the density rotation drops out of the action equation by $\mathrm{Re}(-i|z_j|^2) = 0$. The other known order-$r^2$ terms have high horizontal index after the old angular truncation, so their Fourier projection to F vanishes. The same chart estimate gives $O(e_R)$. This works at the actual old angles, hence produces no order-one coefficient of L_1. An axial error in the old-coordinate forcing has to act on a transverse pair. The latter either consists of two known decaying factors, or has an additional unknown. Differentiating the chart with respect to an axial change of size rA_1E costs at most $Cr(1 + R^{2-p})A_1E$ by the proof of (3.6). These estimates give the third line. For the angular equation division is by r, and the old z_j are bounded away from zero for this fixed insertion. The frequency shift just obtained gives $C[J_1 + r(\epsilon + A_1)]E$. The remaining force gains an additional factor r compared with the action equation. Since $r \lesssim e_R$, this proves the last line. Finally, the bounds for the particulars, their differentiated amplitudes, and the omitted angular tails give contributions $O(e_R)E$ in each line. Every pure error evaluated in (3.12) has a decaying factor because the axial state at infinity is exactly V_R. In particular, with $h = b = J = \Theta = 0$, replacing Y by Y_∞ and setting $W = P_w = P_z = 0$ reconstructs precisely the axial Birkhoff solution V_R. Substitution into all four equations then gives zero residual. This verifies cancellation at the terminal state itself, including on every persistent axial mode.

3.4. **Passage to the half-line.** *Closing the four bounds.* The only linear terms without a small coefficient on the right of (3.14) lead from J_1 to A_1' and L_1'. Accordingly give the latter two components more room. Choose constants $M \geq 1$ and $M' \geq M$ with $M' \geq 8CM$, increasing C in (3.14) to at least one. On the set

$$H_1, J_1 \leq M\rho, \qquad A_1, L_1 \leq M'\rho$$

we have $X \leq 4M'\rho$. The terms CJ_1 use at most $M'\rho/8$ of the corresponding bounds. Choose $4C\epsilon M' \leq M/8$, with $\epsilon \leq 1$, and then choose T to ensure the required tail bound for the model. Keep these choices fixed. Since

$$\frac{e_R}{\rho} = O(R^{-\alpha/4}), \qquad \sqrt{L}\,\rho = R^{-\alpha/4},$$

taking R sufficiently large makes $C(e_R + \sqrt{L}X^2) = o(\rho)$, $rX = o(\rho)$, and $\sqrt{L}X \leq 1$. The $o(1)$ coefficients in (3.14) are absorbed at this last step. Thus the terminal map takes the displayed set into itself for every $S > T$.

Existence on a finite interval. To apply the compact fixed point theorem, fix R and S. For this fixed R, the space Z is equivalent to H^p on the chosen lattice, with the corresponding sequence interpretation on the axis. Use continuous paths in H^s, and ordinary continuous paths for the two finite-dimensional variables, as the ambient space. Intersect the four bounds above with a common Lipschitz bound in H^{p-2} for the two field variables and in Euclidean norm for the old variables. This intersection contains zero and is convex. The image of the terminal map satisfies such a bound with a constant depending on R and the tree. Indeed an output derivative in slow time is its principal generator times the output plus its source. The free generators divided by r map H^p to H^{p-2} at fixed R, and the remaining bounded operators and sources have uniform bounds on each compact interval. In particular the local bounds can be taken independently of the right endpoint S.

For completeness, spatial tails beyond Fourier radius N of an H^p-bounded path have H^s norm at most $\langle N\rangle^{s-p}$ times the H^p bound. Time increments have H^s norm $O(|\tau - \tau'|^{(p-s)/2})$ by interpolating the uniform H^p bound and Lipschitz H^{p-2} bound. Thus Arzelà–Ascoli gives relative compactness in the path topology. Lower semicontinuity, applied first to finite Fourier sums, shows that the four bounds and the Lipschitz constraint pass to the limit. We have obtained a compact convex domain for the map.

We check continuity on this domain at fixed R, S. Convergence in $C_\tau H^s$ suffices for all the pointwise products since $s > 1$. The chart and its differential are continuous at this regularity. The principal low matrix and $\mathcal{S}$ depend smoothly on the finitely many old parameters. The differentiated frame uses the instantaneous velocities defined after (3.12). There is one unbounded operator to check in the field equation. After separating D_{free}, it enters the remainder through commutators with the changes of coordinates. The preceding Schur bounds for these commutators hold also with unit weight. With the weight w they give bounded operators on H^p at fixed R. Interpolation gives boundedness on H^s and, by the same argument applied to operator differences, continuity there. For $\mathcal{M}$ the computed commutator includes its time derivative, which is bounded for fixed R by the finite angular truncation. Thus convergence of trial paths in $C([T, S], H^s)$ suffices for the bounded remainder. The free generator is kept in the propagator. The differences of axial diagonal generators are bounded operators tending to zero since the actions converge in ℓ^1 and the map χ is locally Lipschitz on ℓ^2. For two principal evolutions with the same free generator, Duhamel's formula bounds the difference of propagators on the finite interval by the integral of the operator-norm difference of their bounded parts times the two evolution bounds. The boundary projections on the opposite-copy pairs are fixed, while the trial-dependent low matrix acts only on their complement. This proves continuity of the terminal integral map.

The limit as the terminal time tends to infinity. Fix the same R, take $S_n = T + n$, and choose a fixed point on each $[T, S_n]$. On $[T, T + m]$ the bounds just proved hold for every $n \geq m$ with constants independent of n. Successive extraction for integer m gives a single subsequence for which the field errors converge in $C([T, T + m], H^s)$ and the old parameters converge uniformly, for every m. The pointwise Z bounds pass to the limit by lower semicontinuity at fixed R. In particular, for the limiting errors,

$$\|h(\tau)\|_Z + \|b(\tau)\|_Z + |J(\tau)| + |\Theta(\tau)| \leq 2(M + M')\rho\, e^{-\sigma(\tau - T)}, \qquad \tau \geq T.$$

To identify the equation, reconstruct the physical field from each fixed point by (3.11). With R fixed, this reconstruction is continuous in the stated topology, so the fields converge in $C_t H^s$ on every compact physical time interval in $[T/r, \infty)$. Each satisfies Duhamel's formula there for the same constant potential c_R. The cubic term converges in $C_t H^s$ since $s > 1$. Passing to the limit in that formula identifies a solution of the full equation throughout the half-line.

The fast stable trace at T passes to the limit because its projection is bounded. At the right end the exponential estimate above supplies the required decay for every component. The known transverse profile and its particulars also decay, and the known axial profile tends to the prescribed rotating coordinates of V_R. Reconstructing at fixed R therefore gives exponential convergence to V_R in H^s, with the factor $e^{-\sigma(rt - T)}$ in physical time. Both solutions conserve mass. Taking their L^2 distance to zero as $t \to \infty$ gives $M(W_R) = M(V_R)$, so the constant $c_R = -2M(V_R)$ used in the construction is also the mass gauge of W_R.

Backward continuation and reconstruction. At $t = T/r$, lower semicontinuity gives H^p data with normalized errors of size $O(\rho)$. Solving the Cauchy problem from these data gives a path continuous in H^p, which by uniqueness in H^s agrees with the compactness limit on their common interval. Thus at fixed R its Z norm is continuous, as needed in the following continuation argument. Extend the solution backwards by the defocusing Cauchy flow [37]. On the fixed slow interval $[0, T]$ the model coefficients and their derivatives are bounded by a finite constant. The same calculation along this actual trajectory now has normalized source bounded by $C_T(e_R + X + \sqrt{L}X^2)$, where X here denotes the sum of the instantaneous error sizes. The principal transverse propagator is bounded by C_T in both directions, and the axial diagonal is isometric. On any continuation interval on which $\sqrt{L}X \leq 1$, the nonlinear term

obeys $\sqrt{L}X^2 \le X$. Duhamel's formula, integrated backwards, gives

$$X(\tau) \le C_T(\rho + Te_R) + C_T \int_\tau^T X(u)\,du.$$

Gronwall yields $X(\tau) \le C'_T\rho$. For large R this gives $\sqrt{L}X < 1/2$ on the whole continuation interval, so by continuity the interval reaches zero. Here persistence and uniqueness in H^p allow continuation of the same field at the auxiliary order.

To read off the physical field, $\mathcal{S} - 1 = O(L^{-1})$ on each known transverse plane, the particulars cost $O(r^2)$, and each free high error is $O_T(r\rho)$ in Z, uniformly in positive time. The axial chart estimates replace the added finite axial planes by their physical Fourier series with error $O(re_R + r\rho)$ in Z. At old indices the action differs from the original one by $O(r^2 + r|J|)$ and the angle by Θ, uniformly in time because the old terminal angular speeds are locked. On this finite-dimensional torus the radii stay positive and the chart is smooth into every fixed Sobolev space. Consequently $\sup_{t\ge0} \|V_*(t) - V(t)\|_{H^s} = O_T(\rho + r^2)$. The e_i contribute $O(rR^{-10})$ in Z. Let f_R denote the difference between $W_R - V$ and the finite sum with its prefactor in (3.1). Combining the old-coordinate estimate in physical norm with the conversion $\|h\|_{H^s} \le CR^s\|h\|_Z$ for the high errors gives

$$\sup_{t\ge0} \|f_R(t)\|_{H^s} \le C_T(\rho + r^2) + C_T R^s r(L^{-1} + r + e_R + \rho + R^{-10}) = o(1).$$

Here $R^s r = a_0$ and each term in parentheses tends to zero. The finite sum itself has H^{s_0} norm $O(rR^{s_0}) = O(R^{s_0-s})$, so the same decomposition also gives the uniform lower-order comparison. At the main root the old smooth field has Fourier coefficient $o(r)$, uniformly over its old angle torus. For f_R the single-coefficient estimate gives

$$|\widehat{f}_R((K,0),t)| \le \langle K\rangle^{-s}\|f_R(t)\|_{H^s} = o(r)$$

uniformly in t, since $K = 1 + RP$ and $r = a_0R^{-s}$ with P, a_0 fixed. Orthogonality of the finitely many distinct planes at time zero gives the asserted input cost. This completes the proof of Proposition 3.1.

4. Persistence along one trajectory

We now return to all vertical frequencies. To use Proposition 3.1 a second time, we must preserve the part of the previously constructed orbit that led to its periodic asymptote. We first describe the linearized equation at this asymptote.

4.1. **The previous orbit.** Let V be smooth, axial and periodic of period $T_0 > 0$ in the equation with real constant potential c. Linearizing the cubic product at V gives the equation for a perturbation h,

$$i\partial_t h = (-\Delta + c + 2|V|^2)h + V^2\bar{h}. \tag{4.1}$$

Let $L(t,v)$ take the value of h at time v to its value at time t. It is real linear because the equation involves $\bar{h}$. The operator norms below are taken on the indicated Sobolev spaces. The one-period map $P := L(T_0, 0)$ is its monodromy. If Π is a spectral projection of P, its transport to time t is $\Pi(t) := L(t,0)\Pi L(0,t)$. Since Π commutes with P, this transported projection is T_0-periodic.

Lemma 4.1. *On each $H^b(\mathbb{T}^2)$, $b \ge 0$, the spectrum of the complexification of P off the unit circle consists of isolated eigenvalues of finite algebraic multiplicity. These eigenvalues and their generalized eigenspaces are independent of b, and the eigenspaces consist of smooth functions. Finite sums of the associated spectral projections, transported by $L(t,0)$, are bounded from H^{b_0} to H^b uniformly in t, for $0 \le b_0 \le b$.*

Proof. Double the equation using $h, \bar{h}$ as independent fields, both expanded in the usual spatial Fourier basis. The generator for $i\partial_t$ is the sum of the free diagonal $D \coloneqq \operatorname{diag}(1-\Delta, -(1-\Delta))$, a diagonal matrix of real smooth potentials, and the off-diagonal entries $V^2, -\bar{V}^2$. In these doubled coordinates the spectra and projections below are taken over the complex numbers. Set $J \coloneqq \left(\begin{smallmatrix} 0 & X \\ X^* & 0 \end{smallmatrix}\right)$. For $k, l \in \mathbb{Z}^2$ the matrix of X in the Fourier basis is

$$X_{kl} \coloneqq -\frac{\widehat{V^2}(k-l)}{\langle k\rangle^2 + \langle l\rangle^2}.$$

Multiplying X_{kl} by $\langle k\rangle^2 + \langle l\rangle^2$ gives the negative of the upper off-diagonal entry. The lower entry cancels by taking adjoints with the opposite sign. For the smoothing bound, $\langle k\rangle^2/(\langle k\rangle^2+\langle l\rangle^2) \le 1$ and $\langle k\rangle^b/\langle l\rangle^b \lesssim_b \langle k-l\rangle^b$. After inserting the Sobolev weights, every entry is bounded in modulus by a fixed summable function of $k-l$, since V is smooth and periodic. The same calculation with $\partial_t(V^2)$ proves $J, \partial_t J : H^b \to H^{b+2}$ uniformly over a period. In the coordinates given by e^J, the conjugated generator is therefore the scalar diagonal Schrödinger operator plus a remainder gaining two derivatives. Indeed the first commutator with the free diagonal is bounded at every Sobolev order and every further commutator has a factor J. The differentiated coordinate change has the same gain. Duhamel's formula on a period and $e^{\pm J} - 1 : H^b \to H^{b+2}$ now give $P = P_D + K$, where P_D is the scalar diagonal monodromy and $K : H^b \to H^{b+2}$.

Here and below a scalar evolution refers to either of the two diagonal equations. Both scalar evolutions are unitary on L^2. Commuting a spatial derivative of integer order m through the potential produces only derivatives of order at most $m-1$ on the solution. Integration and induction give polynomial bounds in $|t-v|$ on H^m, in either time direction. Interpolation gives the same conclusion for real $b \ge 0$. At integer multiples of T_0 these are polynomial bounds on both P_D^n and P_D^{-n}, $n \ge 1$. The resolvent of P_D thus exists off the unit circle, by the geometric series for P_D or P_D^{-1}. Compactness of K and the analytic Fredholm theorem give the spectral assertion, applied separately inside and outside the unit circle. Indeed, for $|\lambda| \ne 1$,

$$\lambda - P = \big[1 - K(\lambda - P_D)^{-1}\big](\lambda - P_D).$$

The bracket is the identity plus a compact operator. The product is invertible at $\lambda = 0$, since the linear evolution is reversible, and for $|\lambda| > \|P\|$ by a geometric series. These give one invertible point in each of the two connected regions. For an eigenvector the identity $h = (\lambda - P_D)^{-1} K h$ gains two derivatives. Induction gives the same conclusion along a Jordan chain. The resolvents on different Sobolev spaces agree on their common domains wherever both exist, so the corresponding finite-rank projections agree. A bounded finite-rank projection at the lower order with smooth range is bounded into H^b. Transport and periodicity complete the proof. Projections defined by conjugation-invariant spectral sets preserve the physical real subspace. □

Here is the nonlinear consequence we need. Its weak-norm assumption controls the finitely many rapidly expanding old directions. The strong-norm assumption is made only at the initial time.

Proposition 4.2. *Let $1 < s_0 < s$. Suppose U, V are solutions in the same constant gauge c, V is as above, and $\|U(t) - V(t)\|_s \le Ce^{-\gamma t}$ for $t \ge 0$, with $\gamma > 0$. Let W_n be solutions in gauges c_n, and suppose*

$$\varepsilon_n \coloneqq |c_n - c| + \sup_{t\ge 0} \|W_n(t) - V(t)\|_{s_0} \to 0, \qquad d_n \coloneqq \|W_n(0) - V(0)\|_s \to 0.$$

There are solutions $U_n = W_n + b_n$ in the gauges c_n such that

$$\sup_{t\ge 0} \|b_n(t) - (U(t) - V(t))\|_s \to 0, \qquad \|b_n(t)\|_s \le C' e^{-\nu t}$$

for some $\nu > 0$ and all large n. In particular $U_n(0) \to U(0)$ in H^s, and $U_n - U \to 0$ uniformly in H^{s_0} on the positive half-line.

Proof. For an eigenvalue λ call $T_0^{-1} \log|\lambda|$ its growth rate. Choose $0 < \nu < \beta < \gamma$ with $[-\beta, -\nu]$ in a single gap of the rates, using Lemma 4.1. Let $P_f(t)$ be the transported projection onto the generalized eigenspaces of rates below $-\beta$ (the fast stable directions), and $P_c(t) \coloneqq 1 - P_f(t)$. For some $k > 0$, in each of the two Sobolev spaces we have

$$\|L(t,v)P_f(v)\| \le Ce^{-(\beta+k)(t-v)} \quad (v \le t),$$
$$\|L(t,v)P_c(v)\| \le Ce^{(\nu-k)(v-t)} \quad (t \le v).$$

For integer periods this is the spectral radius formula with a strict gap, applied to the monodromy or its inverse on the indicated subspace. The propagators over the remaining fraction of a period are uniformly bounded.

Fix $0 < \eta < \beta - \nu$. We claim that

$$\sup_{t \ge 0} e^{-\eta t}\|W_n(t) - V(t)\|_s \le C_\eta(\varepsilon_n + d_n). \tag{4.2}$$

Writing $e \coloneqq W_n - V$, the remainders about the old linearization are $2V|e|^2 + \bar{V}e^2 + |e|^2 e + (c_n - c)(V + e)$, up to a unit factor. Their H^s norm is at most $C\varepsilon_n(1 + \|e\|_s)$. Here and below we use $\|ab\|_s \le C(\|a\|_{s_0}\|b\|_s + \|a\|_s\|b\|_{s_0})$, which follows from weighted Fourier convolution and $\ell^2(\langle k\rangle^{2s_0}) \subset \ell^1$. Project away the finitely many old rates above a threshold in $(0, \eta/3)$ outside the rate set. The removed projection of e is bounded by $C_\eta\varepsilon_n$ in H^s at every time by Lemma 4.1. The complementary propagator grows at most as $C_\eta e^{\eta(t-v)/2}$ forwards. To see explicitly where smallness enters, let $m(t) \coloneqq \sup_{0\le v\le t} e^{-\eta v}\|e(v)\|_s$. Apply Duhamel's formula to the complementary component, multiply by $e^{-\eta t}$, and use $\int_0^t e^{-\eta(t-v)/2}\,dv \le 2/\eta$. Adding back the removed component yields $m(t) \le C_\eta(d_n + \varepsilon_n) + C_\eta\varepsilon_n m(t)$. Absorb the last term for large n to obtain (4.2).

Write $d \coloneqq U - V$. Starting at a late time A, prescribe $P_f(A)b(A) = P_f(A)d(A)$ and set

$$(\mathcal{G}f)(t) \coloneqq \int_A^t L(t,v)P_f(v)f(v)\,dv - \int_t^\infty L(t,v)P_c(v)f(v)\,dv.$$

For $\xi = \beta, \nu$ denote by $\|h\|_{b,\xi} \coloneqq \sup_{t\ge A} e^{\xi(t-A)}\|h(t)\|_b$. For $\nu \le \xi \le \beta$, multiplying the forward kernel by $e^{\xi(t-v)}$ leaves decay at rate $\beta + k - \xi$. On the backward kernel the corresponding rate is $\xi - \nu + k$. Both are at least k, so integration bounds $\mathcal{G}$ in the two norms $\|\cdot\|_{s_0,\beta}$ and $\|\cdot\|_{s,\nu}$. We seek a fixed point of

$$b \longmapsto L(t,A)P_f(A)d(A) + \mathcal{G}F_n(b),$$

where

$$iF_n(b) = (c_n - c)b + 2(|W_n|^2 - |V|^2)b + (W_n^2 - V^2)\bar{b} + 2W_n|b|^2 + \bar{W}_n b^2 + |b|^2 b.$$

By (4.2), for fixed A the bound $D_* \coloneqq \sup_{t\ge A} e^{-\eta(t-A)}(1 + \|V(t)\|_s + \|W_n(t)\|_s) \le 2 + 2\sup_t\|V(t)\|_s$ holds for all sufficiently large n. On a ball of radius $a \le 1$ in both norms, write $h \coloneqq b - b'$. The product inequality gives

$$\|F_n(b) - F_n(b')\|_{s_0,\beta} \le C(\varepsilon_n + a)\|h\|_{s_0,\beta},$$
$$\|F_n(b) - F_n(b')\|_{s,\nu} \le C(\varepsilon_n + a)\|h\|_{s,\nu} + C(D_* + 1)(1 + a)^2\|h\|_{s_0,\beta}.$$

For instance, put $a_n \coloneqq |W_n|^2 - |V|^2$. Then $\|a_n(t)\|_{s_0} \lesssim \varepsilon_n$ and $\|a_n(t)\|_s \lesssim (D_* + 1)e^{\eta(t-A)}$. The product estimate gives

$$e^{\nu(t-A)}\|a_n h\|_s \lesssim \varepsilon_n\|h\|_{s,\nu} + (D_* + 1)e^{-(\beta-\nu-\eta)(t-A)}\|h\|_{s_0,\beta}.$$

The strict inequality $\eta < \beta - \nu$ bounds the second term at every time. In the quadratic terms of F_n, expanding the difference between b and b' leaves one factor of b or b', of size at most a. The cubic term leaves two such factors. In the combined norm $\|h\|_{s,\nu} + \Lambda\|h\|_{s_0,\beta}$, first choose Λ large compared with the second constant and the Green bound, then take a small ball, A large so the free datum lies in half the ball, and n sufficiently large. This gives a contraction. For the limiting coefficients V, c, the fixed point is $d = U - V$, since the backward boundary term vanishes by the decay $e^{-\gamma t}$. To compare the two maps at this fixed d, keep A fixed. By (4.2),

$$\sup_{t \geq A} e^{-\eta(t-A)} \|W_n(t) - V(t)\|_s \leq C_\eta e^{\eta A}(\varepsilon_n + d_n) \longrightarrow 0.$$

Factor $W_n^2 - V^2 = (W_n - V)(W_n + V)$ and use the same expansion with a conjugate for $|W_n|^2 - |V|^2$. By the product inequality, their H^{s_0} norms tend uniformly to zero and their H^s norms are $o(1)e^{\eta(t-A)}$, uniformly for $t \geq A$. Every coefficient difference in $F_n(d) - F_\infty(d)$, where F_∞ uses V, c, multiplies at least one d or $\bar{d}$. Its decay absorbs the factor $e^{\eta(t-A)}$ since $\nu + \eta < \gamma$, and also allows the weak weight $\beta < \gamma$. In the combined norm $\|\cdot\|_* = \|\cdot\|_{s,\nu} + \Lambda\|\cdot\|_{s_0,\beta}$ the two fixed point maps have the same free term $L(t, A)P_f(A)d(A)$. If $\vartheta < 1$ is their common contraction bound, subtraction gives

$$(1 - \vartheta)\|b_n - d\|_* \leq \|\mathcal{G}(F_n(d) - F_\infty(d))\|_* \longrightarrow 0.$$

In particular $b_n(A) \to U(A) - V(A)$ in H^s. On $[0, A]$ continuous dependence from time zero gives $W_n \to V$ in $C([0, A], H^s)$, since $d_n + |c_n - c| \to 0$. Thus $W_n(A) + b_n(A) \to U(A)$ in H^s. Continuous dependence backwards from A gives $W_n + b_n \to U$ on that same interval. Together with the weighted estimate on $[A, \infty)$ this proves the proposition. If W_n tends to a periodic solution V_n, then U_n tends to the very same V_n. Equality of their masses follows on taking the limit of the difference, so their mass gauges also agree. □

4.2. **Choosing the data and the times.** Two indices enter the construction. The stage number j counts transfers on the final orbit. Within a single stage we vary the tree depth d to reduce the initial cost and choose a frequency spacing $R = R(d)$ to approximate that particular tree. Only one pair $(d, R(d))$ is retained at stage j.

Fix $1 < s < 2$, put $s_0 := (1 + s)/2$, and fix G, ℓ as in Theorem 1.2. Begin with $U_0 = V_0$ as in Proposition 2.1. The initial field $V_0(0)$ may be taken arbitrarily small in H^s because its Birkhoff indices are fixed, the five actions tend to zero, and the inverse chart is continuous in H^s. Write $g_0 \neq 0$ for its coefficient in (2.2). At any finite stage we have a true solution U exponentially asymptotic to a smooth periodic axial solution V, both in the mass gauge of V, whose coefficient g in (2.2) satisfies $|g| \geq |g_0|/2$.

We explain first how to make one choice with arbitrarily small error. For an increasing sequence of depths d, apply Proposition 2.4 and Lemma 2.5 with main terminal amplitude $a = 1$. Once the geometry is fixed, take the radii $b_j > 0$ so small that $\sum_j b_j^2 \leq 1$, that the hypothesis of Proposition 2.6 holds for $\theta = 3/4$, and that the sum of $|v/P|^{2s} b_j^2$ over all auxiliary vertices of the respective levels is at most $1/d$. Here P is the positive root abscissa. Use Proposition 3.1 with $a_0 = P^{-s}$, so $r = (RP)^{-s}$, and call its finite plane-wave sum $C_{d,R}$, including its prefactor r. Orthonormality gives

$$\|C_{d,R}(t)\|_s^2 = \sum_{v \in \mathcal{V}} r^2 \langle e + Rv \rangle^{2s} |x_v(2|g|rt)|^2, \qquad e = (1, 0). \tag{4.3}$$

For each fixed tree the coefficients $r^2\langle e + Rv\rangle^{2s}$ tend to $|v/P|^{2s}$. The summed main weight $\sum_v |v/P|^{2s}$ over any single primary generation is at most one, since each split of a weight w replaces it by $w(\zeta^s + (1 - \zeta)^s) \leq w$. The primary amplitudes are constant within a generation and their squares sum over the generations to at most two by (2.7). Thus $\sup_{t \geq 0} \|C_{d,R}(t)\|_s \leq 3$

for all sufficiently large R at each d. Its initial squared norm is at most $2c_s^d + 1/d + o_{R\to\infty}(1)$, by (2.8).

We now apply Proposition 4.2 with the old pair U, V fixed and with a sequence indexed by the depth d. Write c for the gauge of V and $W_{d,R}$ for the solution of Proposition 3.1 at this depth. For each d choose $R = R(d)$ large enough that

$$|c_R - c| + \sup_{t\ge 0} \|W_{d,R}(t) - V(t)\|_{s_0} \le 2^{-d},$$
$$\sup_{t\ge 0} \|W_{d,R}(t) - V(t) - C_{d,R}(t)\|_s \le 2^{-d},$$
$$\|C_{d,R}(0)\|_s \le (2c_s^d + 1/d)^{1/2} + 2^{-d}.$$

The first two choices use the fixed-tree limits, and the third uses the leaf estimate above. At time zero the last two inequalities give $\|W_{d,R}(0) - V(0)\|_s \le (2c_s^d + 1/d)^{1/2} + 2^{1-d} \to 0$. Thus the global weak comparison and the strong initial comparison hold on the same sequence. The strong norm of the new contribution can still be of order one at late times.

Let $b_{d,R}$ be the correction given by Proposition 4.2 and put $\widetilde{U} \coloneqq W_{d,R} + b_{d,R}$. Subtracting the old solution gives

$$\widetilde{U} - U - C_{d,R} = (W_{d,R} - V - C_{d,R}) + [b_{d,R} - (U - V)].$$

The first term is the error for a single tree, made small by choosing R after d. The second tends uniformly to zero by Proposition 4.2, since the strong initial cost tends to zero with depth and the weak error is arbitrarily small for each fixed tree. We conclude that

$$\sup_{t\ge 0} \|\widetilde{U}(t) - U(t) - C_{d,R}(t)\|_s \longrightarrow 0 \tag{4.4}$$

along the choices. The terms in this sum are compared in their specified gauges. Multiplication by any of these scalar phases preserves each Fourier modulus. We also have $\sup_{t\ge 0} \|\widetilde{U}(t) - U(t)\|_{s_0} \to 0$, and $\widetilde{U}$ is exponentially asymptotic to V_R. The initial distances in H^s tend to zero. All these conclusions hold with any additional lower requirements on $R(d)$, since for each fixed tree the limits in Proposition 3.1 hold as $R \to \infty$.

The forward orbit of U is relatively compact in H^s because its restriction to a bounded interval is compact and its late tail approaches the compact range of V. Let $P_{\ge N}$ retain the Fourier coefficients at $|k| \ge N$. Then $\sup_{t\ge 0} \|P_{\ge N} U(t)\|_s \to 0$ as $N \to \infty$. Indeed, cover the compact orbit closure by finitely many H^s balls of radius ε and take N large enough that the tail of each center has norm below ε. The projection has norm at most one, giving 2ε for every point of the orbit. Each frequency in $C_{d,R}$ has length at least $R - 1$. Thus its cross term with U is bounded by

$$|\langle U(t), C_{d,R}(t)\rangle_{H^s}| \le \|P_{\ge R-1} U(t)\|_s \|C_{d,R}(t)\|_s \le 3\|P_{\ge R-1} U(t)\|_s.$$

Here $\langle \cdot, \cdot \rangle_{H^s}$ is the inner product associated with our Fourier norm. The other error in expanding the squared norm is bounded by the left side of (4.4) times $2\sup_t \|U(t)\|_s + 6 + o(1)$. It follows that

$$\sup_{t\ge 0} \Big| \|\widetilde{U}(t)\|_s^2 - \|U(t)\|_s^2 - \|C_{d,R}(t)\|_s^2 \Big| \longrightarrow 0. \tag{4.5}$$

This estimate includes the period during which the transfer is taking place.

Set $K \coloneqq 1 + RP$, $D \coloneqq K^s$, and consider the window $[D^{3/4}, D^{5/4}]$. With the finite tree fixed, rt tends uniformly to zero on $0 \le t \le D^{3/4}$ as $R \to \infty$. The early value of (4.3) is therefore bounded by $2c_s^d + 1/d + o_{R\to\infty}(1)$. On $t \ge D^{5/4}$ the slow time tends uniformly to infinity. The

terminal amplitudes of Lemma 2.5 then imply

$$\sup_{t\geq D^{5/4}} |\|C_{d,R}(t)\|_s^2 - 1| \leq 1/d + o_{R\to\infty}(1).$$

Only the main root and the finitely many axial auxiliary amplitudes survive. A bound on the time of capture itself comes from the finite system. Put

$$t_* \coloneqq \frac{C_{3/4} d^{3/2} (K-1)^s}{2|g|}. \tag{4.6}$$

By this time the model root has reached $3/4$ and stays there or above, by Proposition 2.6. The old field U has uniformly small normalized Fourier coefficient at $(K,0)$ by compactness. The root coefficient of $C_{d,R}$ has normalized modulus $r\langle K\rangle^s H(2|g|rt)$, where $r\langle K\rangle^s \to 1$. Together with (4.4) this gives $\langle K\rangle^s |\widehat{\widetilde{U}}((K,0),t)| \geq 2/3$ for every $t \geq t_*$, if the choice is sufficiently far along the sequence.

We give the simultaneous choices explicitly. At stage $j \geq 1$ use the preceding procedure with $U = U_{j-1}$, $V = V_{j-1}$. Denote the new solutions by U_j, V_j, the root index by K_j, the time by $D_j \coloneqq K_j^s$, and the plane sum by C_j. Fix any $\delta > 0$. Use the convergence in depth and, at each depth, the freedom to enlarge R, to meet the following conditions.

(1) The initial distance $\|U_j(0) - U_{j-1}(0)\|_s$ is less than $\delta 2^{-j}$. The error in (4.5) is at most 2^{-j}. The early squared norm of C_j and the late error from one in its squared norm are at most 2^{-j}.
(2) Require $G(D_j) \geq j$ at every stage. For $j > 1$, also require $K_j > K_{j-1}$ and $D_j^{3/4} > D_{j-1}^{5/4} + 1$. Take $\ell(K_j) \geq C_{3/4} d^{3/2}/(2|g|)$, which makes (4.6) at most $K_j^s \ell(K_j)$.
(3) At every earlier recorded frequency $(K_i, 0)$, $i < j$, the change in normalized Fourier modulus, uniformly for all $t \geq 0$, is at most $2^{-j}/6$. Indeed the modulus difference is at most $\langle K_i\rangle^{s-s_0} \sup_t \|U_j(t) - U_{j-1}(t)\|_{s_0}$ and only finitely many indices have been specified. At the new frequency impose the $2/3$ bound above.
(4) Keep the successive changes of the periodic targets in uniform H^1 below $|g_0| 2^{-j-2}$, and the actions in the ball of Lemma 2.2. At each fixed tree this follows by taking R large, since the total h^1 norm of the finitely many new coordinates is $O_d(Rr) \to 0$, the old action change is $O_d(r^2)$, and their angle speeds are unchanged. The common-period integral (2.2) then keeps $|g| \geq |g_0|/2$ throughout the construction.

There is no uniformity in the finite-tree constants required by these choices. More formally one can first impose all the lower requirements on $R(d)$ involving the given geometry, including the conditions on the clock. The input errors for Proposition 4.2 still tend to zero as $d \to \infty$. Taking one sufficiently large depth in this sequence imposes the remaining strong and weak error tolerances.

The initial fields converge in H^s to a datum u_0 with $\|u_0 - U_0(0)\|_s \leq \delta$. Global well-posedness and persistence of regularity for the defocusing flow [3, 37] give convergence of the physical trajectories in H^s on every compact time interval. Equivalently, one can use the H^s algebra estimate for local existence, uniqueness and continuous dependence at this regularity and the global defocusing theory to continue the limit. The masses converge, so the convergences on bounded intervals also hold in the respective mass gauges. For fixed i and $t \geq t_i$, where t_i is (4.6) for its tree, condition (3) and convergence at the fixed frequency K_i give

$$\langle K_i\rangle^s |\widehat{u}((K_i,0),t)| \geq \frac{2}{3} - \sum_{j=i+1}^{\infty} \frac{2^{-j}}{6} \geq \frac{1}{2}.$$

In particular, for any $m \geq 1$ the squared norm is at least $m/4$ at every time $t \geq \max_{1\leq i\leq m} t_i$, by summing over those m distinct frequencies. This already proves the divergence in Theorem 1.1. The more precise time dependence follows by also accounting for what happens during a transfer.

Finally sum (4.5) from stage one to stage n. The sum of the absolute errors is at most one. At any fixed t the windows are disjoint, so at most one term in $\sum_{j\leq n} \|C_j(t)\|_s^2$ is in a transition window. That term is between zero and nine. At stage j compare the squared norm to the indicator $\mathbf{1}_{\{D_j\leq t\}}$, which is one if $D_j \leq t$ and zero otherwise. Before the window both are within 2^{-j} of zero, and after the window their difference from each other is at most 2^{-j}. Inside the one possible active window the difference is at most nine. Summing these errors gives ten as a uniform bound for the difference between the sum of the squared carrier norms and the counting function. Adding the error of at most one from (4.5) gives

$$\Big| \|U_n(t)\|_s^2 - \#\{j \leq n : D_j \leq t\} \Big| \leq 11 + \sup_{v\geq 0} \|U_0(v)\|_s^2.$$

The right-hand side is finite by periodicity and independent of n, t. For each t, the left-hand norm converges by continuous dependence, and the count stabilizes since $D_j \to \infty$. This proves (1.2) uniformly in t. On $[D_j, D_{j+1})$ it gives $\|u(t)\|_s \leq (j + O(1))^{1/2}$ and $G(t) \geq j$. Hence $\|u(t)\|_s/G(t) \to 0$. The counting function tends to infinity, and the initial norm can be made arbitrarily small by the choices of U_0 and δ. This proves Theorems 1.1 and 1.2 for $1 < s < 2$.

4.3. **The limiting axial solution and the height of a transfer.** The periodic axial solutions constructed above have a limit in every lower Sobolev norm if the weak tolerances are chosen summably. One can also measure the speed at which the two-dimensional solution approaches this limit. Let α be as in (3.2) and put $k_s := 3\alpha/4$, so the error scale in that section is $\rho = R^{-k_s}$. We use almost periodicity in the following sense. An H^σ-valued function is almost periodic if it is a uniform limit on $\mathbb{R}$ of finite sums $\sum_m f_m e^{i\omega_m t}$, with $f_m \in H^\sigma$ and $\omega_m \in \mathbb{R}$.

Corollary 4.3. *For $1 < s < 2$ the data in Theorem 1.2 can be chosen so that there exists an axial solution v, almost periodic in time in H^σ for each $1 < \sigma < s$, satisfying*

$$\lim_{t\to+\infty} (1+t)^b \|u(t) - v(t)\|_{H^\sigma} = 0 \qquad \left(0 \leq b < \frac{\min(s-\sigma, k_s)}{s}\right).$$

At every t, $\langle K_j\rangle^s |\widehat{v}((K_j, 0), t)| \geq 1/2$ for every j. In particular $v(t) \in \bigcap_{1<\sigma<s} H^\sigma \setminus H^s$.

Proof. Fix a finite tree and one pair (σ, b) as in the statement. In the notation of Section 3, on the terminal interval $rt \geq T$ the difference of the old Birkhoff variables from their terminal values at the same time is $O(\rho e^{-\sigma_*(rt-T)})$ in any fixed sequence norm on F. Here $\sigma_* > 0$ denotes the time weight used for this tree. On F^c the difference has Z norm $O(re^{-\sigma_*(rt-T)})$, by subtracting rY_∞ in (3.11). The transverse part obeys the same bound. On $0 \leq rt \leq T$ the backward estimate gives the respective bounds $O(\rho)$ and $O(r)$. In h^σ the new axial coordinates are $O(R^\sigma r)$, tending to zero. The compact set of old angle values thus has a neighborhood where the differential of the inverse chart $h^\sigma \to H^\sigma$ is uniformly bounded. Applying it to the difference of the actual and terminal coordinates at the same time gives, for R large,

$$\|W_R(t) - V_R(t)\|_{H^\sigma} \leq C(\rho + R^{\sigma-s}) \exp[-\sigma_*(rt - T)_+], \qquad t \geq 0.$$

The constants here may depend on the tree. Writing $\tau = rt$ and using $0 < r \leq 1$, we have $(1+t)^b \leq r^{-b}(1+\tau)^b$. The supremum of $(1+\tau)^b e^{-\sigma_*(\tau-T)_+}$ over $\tau \geq 0$ is finite. Therefore the weighted error is at most $C(R^{sb-k_s} + R^{sb+\sigma-s})$. Both exponents are negative precisely under the two strict upper bounds on b in the statement.

Use a countable dense collection of pairs (σ, b) strictly inside the allowed range, with σ, b rational. In the construction of stage j, impose estimates for its first j pairs, taking $s_0 < s$ above

their spatial exponents if necessary. For each pair the preceding bound can be made arbitrarily small. On the history correction, the convergence of $b_n-(U-V)$ in the exponential strong norm in the proof of Proposition 4.2 and on the fixed initial interval gives convergence in the polynomially weighted H^σ norm. It follows that the successive differences of U_j-V_j are summably small in that norm, each field in its own mass gauge. Indeed $(U_j-V_j)-(U-V)=(W_R-V_R)+(b_n-(U-V))$. For each fixed pair, the weighted differences are bounded, for example, by 2^{-j} at all stages after it enters the list. For any n beyond that stage, the weighted distance from the limit to U_n-V_n is at most $\sum_{j>n}2^{-j}$. Let $t\to\infty$ first, using the exponential decay of this fixed U_n-V_n, and then let $n\to\infty$. This proves the asserted vanishing of the weighted limit.

We also require summable uniform H^σ differences of the targets V_j on the whole real line. The locked old angles and the small h^σ norm of the added coordinates give this by Lemma 2.2 and the bounded chart differential. Each V_j is approximable uniformly in H^σ on its period by its time Fourier means (convolution with the Fejér kernels). The uniform limit of these periodic targets is therefore almost periodic in the sense just defined. Local well-posedness identifies it in its limiting gauge as a solution independent of y. At each stage U_j, V_j have the same mass. Their limits consequently have the same mass, and removing the common gauge preserves the norm of the difference. It preserves almost periodicity too, since multiplication by e^{ict} preserves uniform approximation by trigonometric polynomials in time. The strict inequalities on σ, b allow each real pair in the statement to be dominated by one of the rational pairs. To see why the bound for the axial limit holds even at early times, fix i, $j\ge i$, and $t\in\mathbb{R}$. Denote the period of V_j in its mass gauge by T_j^{ax}. For all sufficiently large integers m, the time $t+mT_j^{\mathrm{ax}}$ is past the capture time of K_i. At these times U_j satisfies the lower bound $1/2$ at that index and $U_j-V_j\to 0$ in H^s. Periodicity gives the same lower bound for $V_j(t)$. Let $j\to\infty$, using continuity of the single Fourier coefficient in H^σ. The limit $v(t)$ has normalized modulus at least $1/2$ at every K_i, and hence the sum defining $\|v(t)\|_{H^s}^2$ diverges. □

There is a second way to normalize the trees. A small increase in the initial cost at each leaf allows the Sobolev contribution of a single root to grow exponentially with its depth.

Corollary 4.4. *Fix $1<s<2$ and $0<\kappa<-\frac{1}{2}\log c_s$, with c_s as in the leaf estimate (2.8). There is a single H^s datum, a strictly increasing sequence of depths d_j, and distinct frequencies $(K_j,0)$, with the following property. Put $B_j:=e^{\kappa d_j}$. At that frequency the normalized Fourier modulus $\langle K_j\rangle^s|\widehat{u}((K_j,0),t)|$ of the solution is at least $B_j/2$ at every time*

$$t\ge\frac{C_{3/4}d_j^{3/2}(K_j-1)^s}{2|g_j|B_j},$$

where g_j is the coefficient (2.2) of the preceding periodic orbit and $\inf_j|g_j|>0$.

Proof. Keep $a=1$ but replace a_0 by $B(d)/P^s$ at depth d, where $B(d):=e^{\kappa d}$. The primary initial cost is now $O(\exp[(\kappa+\frac{1}{2}\log c_s)d])=o(1)$. For each fixed tree, choose the positive auxiliary radii so that their total normalized initial cost is at most 2^{-d} and Proposition 2.6 applies. Then take $R=R(d)$ large enough that the weak distance (including the gauge difference) and the absolute H^s error in (3.1) are each at most 2^{-d}. Thus the strong initial distance tends to zero with d, as required by Proposition 4.2, even though the late root height grows. The same subtraction as in (4.4) gives a uniform absolute $o(1)$ error after the history correction.

At the new root, $r=B(d)/(K-1)^s$, so $r\langle K\rangle^s\ge B(d)$. Once $2|g|rt\ge C_{3/4}d^{3/2}$, the model coefficient therefore has normalized modulus at least $3B(d)/4$. The old field has uniformly vanishing normalized coefficients as $K\to\infty$ by precompactness in H^s. Subtracting this coefficient and the absolute comparison error leaves at least $2B(d)/3$ on the new orbit. At each

later stage j, impose, for every retained root $i < j$,

$$\langle K_i\rangle^{s-s_0}\sup_{t\ge 0}\|U_j(t)-U_{j-1}(t)\|_{s_0}\le \tfrac{1}{6}B_i 2^{-(j-i)}.$$

There are finitely many such requirements at each stage. Their sum in $j > i$ costs at most $B_i/6$ at root i, giving the asserted $B_i/2$ bound in the limit. Convergence of the initial data and the lower bound on $|g_j|$ follow by the same summable choices as above. □

5. The endpoint H^2

We prove the qualitative assertion at $s = 2$. In this section $L > 0$ denotes the height of one transfer. The depth, the axial background and the strength of its selected Fourier coefficients will be fixed before the dilation tends to infinity. The time of detection may depend on all these choices.

5.1. **The second-order normal frequency.** Use the individual mass gauge of Section 3. With $A, B, \mathcal{Q}$ as in Lemma 2.3, put

$$\Xi(v) = A/v_x^2 - B/|v|^2, \qquad d(v,w) = \Xi(v) + \Xi(w) - \Xi(v+w).$$

The Liouville primitive in that lemma is unchanged by the common mass gauge, since the two canonical forms have the same pairing with its phase generator. In particular A is the time mean of the spatial variance of $|V|^2$, also for resonant old frequencies. On a small two-site torus with positive comparable radii its first nonzero term is positive. Start there and retain $A, B > 0$ under sufficiently small perturbations. In the mass gauge the lemma gives high transverse diagonal $n^2 + m^2 + A/n^2 - B/(n^2 + m^2)$, with $O_h(R^{-3})$ residual on $|n| \ge hR$, $|m| \ge R$ and arbitrary off-diagonal spatial decay at fixed smooth background.

At an empty axial index use the same doubled computation with $m = 0$. To see also the asymptotics needed for a finite difference below, continue the spatial recursion to order four, keeping the instantaneous spatial mean as scalar $a(t,n) = \sum_{j=2}^{4} a_j(t)n^{-j}$. At each order the upper equation sets its mean and the zero-mean coefficient at the next order by $2D_x$; the lower leading divisor is $2n^2$ and determines the next lower coefficient directly. Taking one more upper and two more lower coefficients gives residual $O(|n|^{-5})$ in L^2. Apply $D\Phi(V)$ to this complexified normal column. With its scalar phase removed its empty nth coefficient $c_n(t)$ is $1 + O(1/|n|)$ uniformly: apply integer one-smoothing $H^1 \to h^2$ to a wave of H^1 size $O(|n|)$ and then take the nth coefficient [39]. In this coordinate the exact linearized equation is a scalar rotation. Divide its inhomogeneous equation by c_n and average in time. The logarithmic derivative has zero average because a bounded continuous branch of $\log c_n$ exists. The coefficients a_j are smooth functions on one finite torus and thus have time means even at resonant frequencies. Consequently at a fixed axial state

$$\chi_n = (A-B)n^{-2} + c_3 n^{-3} + c_4 n^{-4} + O_V(|n|^{-5}) \qquad (|n| \longrightarrow \infty).$$

Only the first coefficient requires a primitive uniform on the torus.

5.2. **A finite tree and distinct driving frequencies.** Use Proposition 2.4. We can further arrange that the numbers $d(v,w)$ on the primary and auxiliary triangles are nonzero and pairwise distinct, and differ from every mixed triangle defect. To verify this, use the rational split parameter t in that proposition. At a nonhorizontal parent (X, Y) and $t \to 0$, the primary defect has a pole of order two, the auxiliary defect at p a pole of order four, and the one at q has order at most two. Interchanging the children by $t \mapsto -1/t$ proves the remaining separations and nonconstancies. At a root of $Yt^2 + Xt - Y = 0$, the mixed output T_m has zero horizontal coordinate and nonzero vertical coordinate, while the intended defects at this split are finite.

Thus its defect has a pole there. At the horizontal root the single defect is nonconstant by the same projection formula. At each split avoid equality with every previously fixed relevant value, and impose the within-split exclusions just proved. This removes finitely many values and is compatible with the real balance and finite-place conditions in Proposition 2.4. All these inequalities persist under small changes of A, B.

First keep only the primary and auxiliary triangles, with equal absolute couplings. After time and phase normalization the symmetric real equations are (2.5)–(2.6) with all mixed edges omitted. In detail the incoming primary term is x_{j-1}^2, and the two losses are $x_j x_{j+1}$ and $w_j h_j$ when that vertex has a private pair; $w_j' = -x_j h_j,\ h_j' = x_j w_j,\ H' = x_{d-1}^2$. Prescribe $w_j^2 + h_j^2 = b_j^2$ for $j \le d-2$ and $H^2 + \sum_{j<d} x_j^2 + \sum h_j^2 = 1 + \sum b_j^2$. Start with only $x_0 > 0$ among the primaries. At every fixed time S the circle-angle map from launch to time S is homotopic to the identity. Choose a preimage of $h_j(S) = b_j$ for all j. Monotonicity gives $0 \le H \le 1$ up to S; compactness as $S \to \infty$ gives such an orbit on the half-line. On its invariant omega-limit set H is constant, so $H' = x_{d-1}^2$ and the equation for x_{d-1} give $x_{d-1} = x_{d-2} = 0$. Suppose $x_{j+1} = x_j = 0$ on a limit orbit, with $1 \le j \le d-2$. The jth circle is stationary, and $x_{j-1}^2 = w_j h_j$ is constant. If this constant were positive, x_{j-1} would be a nonzero constant and the circle of radius $b_{j-1} > 0$ would rotate at that angular speed. The equation for x_{j-1} gives $w_{j-1} h_{j-1} = x_{j-2}^2 \ge 0$, where the incoming square is zero at the bottom. This contradicts the change of sign of the product on a rotating circle. All primaries there vanish. The invariants then force $H = 1,\ w_j = 0,\ h_j = \pm b_j$. The omega-limit is connected and hence is one point. In local coordinates on these invariant circles and the last energy surface the stable and unstable transverse rates are $-1, \pm b_j$, so convergence is exponential. In the full complex graph the two top donors and the private donor pairs are disjoint real-linear hyperbolic pairs at this endpoint; all axes and the unused mixed heads are centers. Phases of the endpoints may be prescribed: assign sibling phases recursively at each primary triangle, and assign each private partner phase after its endpoint phase is fixed.

For root $(P, 0)$ put $\eta = L/P^2$ and $r = \eta/R^2$. The relative fourth-power weight of two children is $z^2 + (1-z)^2 \le c < 1$ by balance. Thus the physical H^2 cost at launch of the bottom layer is at most $L(1 + \sum b_j^2)^{1/2} c^{d/2} + o_R(1)$. The sum of the fourth weights at level k from the top is at most $P^4 c^k$. The contribution of all primary levels at all times is therefore $O(L)$ in norm. Choose d first and then all $b_j > 0$ small enough to make the cost of every private partner and endpoint as small as desired.

5.3. **Constructing the driving coefficients on one axial torus.** At $s = 2$ a shift of order R^{-2} persists on the transfer time scale. The finite system in the next subsection needs a separate driving harmonic for each primary or private triangle of the fixed tree. We make the following simultaneous choice.

Choice of the axial torus. All previously locked frequencies keep their prescribed values. At the frozen frequencies the group of harmonics selected for each triangle has modulus $\lambda(1 + o(1))$ at the space-time key $(1, 1)$, as $\lambda \downarrow 0$ with the passive data fixed. For each fixed sufficiently small λ, take R large. On the resulting terminal torus the weak rates for row j supply its prescribed shift $d_j/R^2 + O(R^{-4})$, where d_j is the defect of that triangle at the frozen base. The same torus has the prescribed hot axial actions up to $o(r^2)$. The disjoint weak sites allow us to fit these shifts while keeping the strong rates fixed.

Proof. Positive actions and the frozen frequency constraints. Write the old gauged frequencies as $n^2 + \tau c_n$, $c_n \in \mathbb{Q}$, with a common denominator making all old numerators 1 mod 5. Here $0 < \tau \ll 1$ is Diophantine relative to 1; countably many additional excluded values are specified below. Start for example with two distinct occupied sites of slopes $-4, -4$. Positivity follows from $D\chi(0) = -\mathrm{Id}$.

For each desired triangle take disjoint sets of new sites, with words

$$h = e_{1-T} + e_{1+3T} + e_{1+4T} - e_{1+T} - e_{1+5T}, \qquad d_* = e_K - 3e_{K+1} + 3e_{K+2} - e_{K+3}.$$

Their triples (charge, spatial momentum, free frequency) are $(1,1,1)$ and $(0,0,0)$, respectively. At the old torus each of the five weak frequencies contributes $O(T^{-2})$. On the four moderate sites the coefficients of d_* take a third difference. Applied to n^{-j} it is $O(K^{-j-3})$, by three applications of the mean value theorem. The remainder in the empty-site expansion is already $O(K^{-5})$. Thus $|(h+d_*)\cdot\chi| \le C(K^{-5}+T^{-2})$. Use comparable, arbitrarily large K's and choose $T \ge K^3$, subject to the congruences below, with mutually disjoint supports. Let J bound these defects, of the displayed order. Set positive actions $L_0 J$ on the four moderate sites in each row, initially zero at the five weak sites. Lock the old rates, and vary one unit moderate pivot per row to obtain $(h+d_*)\cdot\chi = 0$ simultaneously. Indeed $D\chi + \mathrm{Id}$ is compact on ℓ^2 at the old torus and $D\chi$ is close to $-\mathrm{Id}$ throughout a common small ball [46, Theorem 18.1 and Corollary 18.2]. Eliminating the old coordinates therefore leaves the fixed high pivot matrix close to its diagonal principal part. In fact, let $D = D\chi$ at the old torus and write D_{FE} for its block from the new sites E to the old sites F, with the other blocks defined likewise. The derivative on the new sites after locking the old rates is

$$D_{EE} - D_{EF} D_{FF}^{-1} D_{FE}.$$

The projections onto E converge strongly to zero as the sites recede. Compactness of $D+\mathrm{Id}$ gives $\|D_{EE} + \mathrm{Id}_E\| + \|D_{EF}\| + \|D_{FE}\| = o(1)$. Since D_{FF}^{-1} is bounded, the same $o(1)$ estimate holds for the difference of the locked derivative from $-\mathrm{Id}_E$. The uniform initial increment cancels in each principal moderate row. Choose L_0 large, then the sites large enough that the compressed error is less than $L_0 J/10$ per row. The inverse function theorem, or contraction against the unit pivots, keeps all moderate actions positive. Their H^2 bill is $O(K^2\sqrt{J}) = o(1)$ per site. The independent nonpivot rates may now be chosen in $\tau\mathbb{Q}$ by density.

Nonvanishing of the driving coefficients. We first compute a Taylor coefficient at the vacuum and then follow the corresponding angular coefficient to the occupied background. For disjoint positive and negative supports, with multisets $\{p_1,\dots,p_{m+1}\}, \{q_1,\dots,q_m\}$, the coefficient of minimal degree in the physical inverse chart is

$$(-1/2)^m \sum_{\text{distinct orders}} \prod_{\alpha=1}^{m} \frac{1}{(p_\alpha - q_\alpha)(p_{\alpha+1} - q_\alpha)}. \tag{H1}$$

This formula also applies at a bare kinetic resonance. Put $J_0 = (i/2)\,\mathrm{diag}(1,-1)$, $C_{pq} = 1/(p-q)$, and let a_p, b_q be independent small constants. With $t_1 = x$, the diagonal entries of A_*, B_* are respectively $A_p = a_p \exp(i\sum_\ell t_\ell p^\ell)$ and $B_q = b_q \exp(-i\sum_\ell t_\ell q^\ell)$. The upper field is $\mathbf{1}^\top(1 + A_* C B_* C^\top/2)^{-1} A_* \mathbf{1}$. To verify the identity, introduce the rational matrix

$$D(z) = 1 + \sum_p \frac{r_p e_2^\top}{z-p} + \sum_q \frac{s_q e_1^\top}{z-q}, \quad r_p = \frac{iA_p}{\sqrt{2}}\left(e_1 + \sum_q \frac{s_q}{p-q}\right), \quad s_q = \frac{-iB_q}{\sqrt{2}}\left(e_2 - \sum_p \frac{r_p}{p-q}\right).$$

Multiplication by the inverse elementary nilpotent pole factor removes each pole and shows $\det D = 1$. For $D\exp(\sum_\ell t_\ell z^\ell J_0)$, its ℓth logarithmic time derivative has no pole, by the displayed phases, and equals the polynomial part of $z^\ell D J_0 D^{-1}$. Writing $W = D J_0 D^{-1}$, the identities $W^2 = -1/4$ and $W_x = [zJ_0 + [D_1, J_0], W]$, where D_1 is the coefficient of z^{-1}, recursively determine its diagonal and off-diagonal Laurent coefficients. These are the homogeneous NLS hierarchy matrices. The upper and lower off-diagonal entries of $[D_1, J_0]$ are $u/\sqrt{2}, v/\sqrt{2}$; the reversed second flow is $iu_t = -u_{xx} + u^2 v$. On simultaneous Birkhoff coordinates each hierarchy Hamiltonian depends only on the actions [24, Theorem 1.1(3)]; apply its complexified Taylor identity at zero on the finitely many input variables. Each monomial in any action must have

all signed power sums zero. The Vandermonde matrix on the disjoint supports, including degree zero, excludes every nonconstant monomial. With all actions zero the signed frequencies in every hierarchy time are their vacuum values. The same Vandermonde argument after adjoining an output site leaves only its prescribed linear input in that Birkhoff coordinate. Expanding the upper matrix inverse now gives (H1).

For the congruences $K \equiv 10,\ T \equiv 31 \pmod{1009}$, the vacuum coefficient in (H1) for $h + d_*$ is 54 mod 1009. To compute it, use an initial positive site i, remaining positive multiplicities a and negative multiplicities b in the recursion

$$W(i,0,0) = 1, \quad W(i,a,b) = \sum_{b_j a_k > 0} \frac{W(k, a - e_k, b - e_j)}{(p_i - q_j)(p_k - q_j)}.$$

Use $p = (1 - T, 1 + 3T, 1 + 4T, K, K + 2)$, $q = (1 + T, 1 + 5T, K + 1, K + 3)$, $a = (1, 1, 1, 1, 3)$, $b = (1, 1, 3, 1)$. In the displayed order of the positive sites the five values $W(i, a - e_i, b)$ modulo 1009 are $678, 979, 232, 347, 211$, whose sum is 429 mod 1009. Since $2^{-6} = 741 \bmod 1009$, the coefficient is $429 \cdot 741 = 54 \bmod 1009$. All opposite differences are units. The same congruences allow arbitrarily large groups.

The congruence gives a nonzero starting value for continuation. On the occupied torus different old angular labels can contribute to the same Fourier coefficient, so we must continue their sum at the prescribed phase origins. Fix those origins (zero suffices). For each finite recipe of old sites and rational slopes, new groups and free moderate slopes, replace τ by a scalar ζ. At passive weak sites solve the old and free rate equations and the row equations through zero. Divide the selected angular coefficient by the required products of the new radii and sum the aliases differing by old labels β with $\beta \cdot \mathbf{1} = \beta \cdot n = \beta \cdot n^2 = \beta \cdot c_{\mathrm{old}} = 0$, at the prescribed origins. At zero every nonzero such label has an extra old radius and disappears. The vacuum value is (H1). To justify continuation, in the independent frequency coordinates replace each pivot coordinate by $\chi_{\mathrm{piv}} + h \cdot \chi_{\mathrm{weak}}/(d_*)_{\mathrm{piv}}$. On the subspace of zero weak actions the derivative differs from $-\mathrm{Id}$ by at most $(1 + \sqrt{5})\|D\chi + \mathrm{Id}\|$, by disjointness. The inverse exists on a common small complex ℓ^2 ball: the prescribed vector along the segment is small because its endpoint has small total action and all row operators have uniformly bounded norm. The local inverse chart $\ell^4 \to \mathcal{F}\ell^4$ is analytic near zero [46, Theorems 14.1 and 15.2]. Thus it can be evaluated along any Cartesian square-root branch of this curve; its ℓ^4 size squared is the ℓ^2 action size. The alias projection is Haar averaging over the annihilator of the integer subgroup. Factoring the required radii is analytic by the two-sign Taylor series. Multiplying over the signs of the old square roots gives an even holomorphic function of those radii, hence a single-valued function of the actions, nonzero at zero. The isolated zeros of these functions for all countably many recipes can be excluded in the original choice of τ. At the selected point, restore the moderate radii and denote each nonzero modulus, with the five weak radii still divided out, by m_j.

Separating the rows and fitting the terminal rates. Choose the five weak radii in row j equal to $(\lambda/m_j)^{1/5}$ and restore the old and moderate rate conditions as above; the changes tend to zero at fixed sites when $\lambda \downarrow 0$. The selected grouped coefficients now have modulus $\lambda(1 + o(1))$. On the flat $(h_j + d_{*,j}) \cdot c = 0$ approximate the resulting new rates by τ times rationals. Take the new common denominator to be $Q_{\mathrm{old}} D_0$ with $D_0 = 1 \bmod 5$ divisible by distinct private primes $\pi_j > 30000$ outside the old denominator. Give strong numerators residue 1 mod 5, weak ones zero. Modulo π_j impose residues $(1, 13, 169, 2197, -2014)$ on its five weak numerators, and zero on all others. Solve for a unit weak pivot on the flat. These conditions are compatible by the Chinese remainder theorem and dense with arbitrarily large denominators. All new rates are legitimate coordinates at this positive point.

An integer-time charge-one label of total weak order at most five has weak part exactly one h_j. Indeed its scalar product with the residues is an integer of modulus less than π_j divisible by π_j. Eliminate the last entry; all four base-13 digits have absolute value at most five and hence vanish. The charge modulo five then specifies the positive sign and rules out an empty weak part. For any finite angular cutoff avoid all other aliases with nonzero moderate difference at key $(1,1)$: all moderate entries are free on the flat after eliminating a weak pivot. The remaining normalized strong tails have arbitrarily small absolute sum by angular analyticity after factoring the five radii. Choose their cutoff first and then the rational approximation. Over all integer keys (b,ν) the absolute coefficient sum of these order-five groups is $O(\lambda)$, uniform for small λ at fixed passive data; labels of weak order at least six have sum $O(\lambda^{6/5})$ by Cauchy estimates on a fixed weak polydisc.

At this point all the groups have been selected at the frozen frequency vector Ω^0. To fit the prescribed defects, shift only its weak rates to obtain Ω^∞, keeping all strong rates fixed, so that

$$h_j \cdot (\Omega^\infty - \Omega^0) = d_j/R^2 + O(R^{-4}),$$

where d_j is the intended defect evaluated at the frozen base. Simultaneously impose the hot axial actions $r^2|a_i|^2$ at $K_i = 1 + RP_i$ using the compressed frequency inverse. Here a_i is the prescribed terminal amplitude at K_i in the normalized finite orbit above. Finally approximate all free rates, including the root rates, with enough accuracy to change those actions by $o(r^2)$, to preserve positivity and the displayed accuracy, and to give every numerator residue 1 mod 5 for the next stage. One can multiply the strong denominator by an arbitrarily large integer 1 mod 5. Previously locked strong rates are exact throughout. □

5.4. **Finite dynamics and the full normal operator.** Use the exact Birkhoff frequencies on the terminal axes. On mobile heads begin with $|k_v|^2 + \Xi(v)/R^2$; corrections $O(R^{-3})$ make each edge identity exact for its assigned row. Indeed, distribute each parent's discrepancy between its children, starting at the root, then solve one equation per private partner. The unused mixed-head frequency needs no correction. The empty-axis expansion above gives the stated accuracy on axes; the newly occupied very small actions change the empty frequencies by $O_V(r^2)$ after accounting for the old action variation.

Use all angular harmonics with frozen key $(1,1)$ in the finite equation, on all triangles including the mixed triangles. For order five the extra actual phase is determined by h_j alone: the strong rates were kept exact. In slow time $\theta = r\lambda t$ the intended coupling is constant with magnitude $2 + o(1)$, after the assigned phase rotations. Every other order-five harmonic group on that triangle, and every such group on a mixed triangle, oscillates at frequency of modulus at least $c_{\mathrm{tree}}/(\eta\lambda)$ by the strict defect separations. Divide each of these oscillatory vector fields by its signed frequency to obtain a polynomial change of coordinates $O_{\mathrm{tree}}(\eta\lambda)$ in C^1 on a bounded neighborhood. Substitution leaves a C^1 remainder of this same order; the undivided higher-order bank has size $O(\lambda^{1/5})$. Every term vanishes on the axial plane. The detuning correction and endpoint rounding cost $o_R(1)$ at fixed λ. The resulting equation has a solution converging exponentially to its exact rounded axial endpoint: take a time on the ideal landing where the tail is small, use the linearized endpoint splitting with a positive weight strictly below its hyperbolic rates, prescribe the stable projection there, and integrate the other components from $+\infty$. Small C^1 error and vanishing of the field at the exact endpoint give a contraction in the weighted tube, including the center directions. On the preceding fixed interval the inverse propagates backwards by Gronwall. First small λ, then large R, make the launch as close as prescribed to the ideal one; undo the change, which fixes the endpoint. Here and in the subsequent insertion the exact finite solution with all these terms is used.

On the invariant vertical lattice $m = Rq$ use the norm Z^p with weight $(1 + |n|/R + |q|)^p$ and its weighted Wiener counterpart; $p = 2$ or $1 < p = p_0 < 2$. The sum of reciprocal squared weights is $O_p(R)$. Hence two generic factors cost at most $C_p\sqrt{R}$ in the scaled algebra estimate, while any prescribed plane with smooth envelope acts with its Wiener multiplier bound.

We describe the linear normal operator at the true terminal state. First remove the old axial pairing V^2 from every doubled $m, -m$ block. Its first off-diagonal Sylvester generator has kernel $-C_{n-n'}/(E_n + E_{n'})$ with $E_n = n^2 + m^2$, with conjugate index reflected. Its weighted Schur norm, and that of its old derivatives, is $O(R^{-2})$. Exponentiate in the group preserving $\mathrm{diag}(1, -1)$. In the first diagonal commutator replace the divisor by $2E_{n'}$; the reciprocal error is $O(R^{-3})$ times a fixed polynomial in $|n - n'|$, absorbed by the smooth coefficient. The correction on the positive branch is $-|C|^2/(2E(D_x))$ to this accuracy. The remaining off-diagonal error including the time connection is $O(R^{-2})$; a second Sylvester step of size $O(R^{-4})$ leaves error $O(R^{-4})$. For $|n| \geq hR$ use the spatial columns above in their corresponding Hermitian copy, with exact diagonal to error $O(R^{-3})$. Projection to that copy after the two changes preserves the residual. For the high column matrix T, $G = T^*T = 1 + O(R^{-1})$. The quasimode identity and its adjoint bound its covariant commutator $i\dot{G} - [D, G]$ with the high diagonal by $O(R^{-3})$, with the sign matching the equation for the columns. Leibniz and the power series transfer this estimate to $G^{-1/2}$. Complete $TG^{-1/2}$ unitarily by the polar part of the projection of the low subspace onto its orthogonal complement. Spatial off-diagonal decay and the $O(1/|n|)$ high envelopes bound the commutator with D_x^2 by a bounded operator. The low generator is therefore free kinetic energy plus a bounded Hermitian operator. It is retained at the actual trial actions and angles in the nonlinear problem.

The terminal high axial waves also give terms of size r, both Hermitian and anomalous, in the normal operator. Their leading physical shapes are the planes at $K_i = 1 + RP_i$. This follows from (3.6)–(3.7) at $p = 2, p_0$: the new axial input is $O(r)$ in Z^2, $O(rR)$ in h^1; one-smoothing for both chart differentials and symplectic adjointness give $O(R^{-1})$ on a high normalized plane. Changing the input in the differential changes its scaled operator norm by $O(r + rR^{2-p})$; this uses the unweighted analytic L^2 bound and the difference bound from h^1 to H^2, scaled on input and output. Thus the error on a pure physical wave is $O(r[R^{-1} + r(1 + R^{2-p})]) = o(rR^{-1/2})$.

At angular degree up to $C_1 \log R$ write $b = l \cdot n$, $\nu = l \cdot n^2 + \tau l \cdot c^0$; the discarded tails can cost any fixed negative power by taking C_1 large at fixed clock. Every nonzero bare divisor $\mathbb{Z} - \nu$ is bounded away from zero by an inverse power of $1 + |l|$, by the fixed rational base and the Diophantine choice of τ (when the rational multiple of τ is zero the nonzero integer costs at least one). An anomalous pair has $n + n' = K_i + b$ and divisor

$$n^2 + n'^2 + 2R^2q^2 - K_i^2 - \nu.$$

If either leg is low-horizontal this is $\gtrsim R^2$ in modulus: choose $h > 0$ small, since the limiting polynomial at $x = n/R$ is $2(x^2 - P_i x + q^2)$, $q \neq 0$. Take h also below half the least nonzero absolute horizontal velocity of the finitely many heads and pure sources. The Hermitian divisor is $2(K_i - b)(n - b) + \nu - b^2$, of modulus $\gtrsim R$ for $n \neq b$. Keep every star from b to K_i, and its adjoint, including off-key harmonics there. Divide the other nonresonant terms against the free diagonal and the time derivative together. The residual from the actual slow phase costs $O(R^{-2} \log R)$ times the divided kernel. The commutator with the bounded low body acts only when a leg is low, where the preceding large divisors apply. Thus the remaining Schur error is $O(R^{-3} + r^2(1 + \log R)^C + (r/R)(1 + \log R)^C) = o(r\lambda)$. Each harmonic has one shift or reflection per row; the weight ratios are bounded by fixed powers of its spatial displacement in the scaled coordinates. Constants here may depend on the fixed clock.

At an exact bare anomalous resonance, q is bounded and its limiting x satisfies $x(P_i - x) = q^2$. Both legs are high. The row determines P_i in the limit; at that root changing the integer b

changes the partner's square by order R, for the logarithmic range above. The remaining pairs are disjoint, and do not meet a Hermitian star, by the nonsquare condition of Proposition 2.4. Flatten the horizontal weight for $|n| \le C_{\mathrm{tree}} R$ as in (3.8). The low generator and all stars together are Hermitian on each constant-q space in this equivalent norm.

In a surviving anomalous pair there may be several order-five groups even at an integer key different from $(1,1)$. Its limiting second-order defect is

$$E_* = \Xi((x,q)) + \Xi((P_i - x, -q)) - \Xi((P_i, 0)).$$

In the interaction frame removing the full nominal high diagonal, retain the group with d_j nearest E_*, if present. All other groups differ in the R^{-2} coefficient by a fixed positive amount. Divide their oscillations in this frame; the normalized change is $O(\eta\lambda)$, since the generator amplitudes in this frame are bounded. In physical time the trial change in the second-order correction $A/n^2 - B/(n^2+m^2)$ is $O(R^{-2})(r|J_{\mathrm{old}}| + r^2)$, where $J_{\mathrm{old}} = (I_{\mathrm{old}} - I_{\mathrm{old}}^{\infty})/r$. In the fixed terminal mass gauge, the old-mass scalar on the positive branch changes by $2\sum_{j\in F}(I_j - I_j^{\infty}) = 2r\sum_{j\in F}(J_{\mathrm{old}})_j$, with the opposite sign on the conjugate branch. This term is included in the old-mass estimate below. After this averaging, the unitary half-phase rotation for the retained group gives the two exponents $\pm \operatorname{Re}\sqrt{|g|^2 - \delta^2/4}$, with its actual normalized detuning δ and normalized coupling g. In particular no small residual detuning has been suppressed.

Lemma 5.A (A common Green right inverse)**.** *Fix a finite endpoint preparation as above, $1 < p_0 < 2$, and $0 < a < b$ below the finite orbit's landing rate in slow time $\theta = r\lambda t$. For every sufficiently small admissible clock strength λ and every sufficiently large admissible $R \ge R_0(\lambda)$, there is a closed interval $J \subset (a,b)$ of length bounded below in terms of the fixed preparation and $[a,b]$. On the invariant vertical lattice $m \in R\mathbb{Z}$, the exact linearization at the terminal axial torus has a common Green right inverse for $p \in \{p_0, 2\}$ and $\sigma \in J$. More precisely, on a translated late half-line $[t_0, \infty)$ it maps each continuous forcing of finite norm to a mild solution with a bound in the same norm*

$$\|f\|_{p,\sigma,t_0} = \sup_{t \ge t_0} e^{\sigma r\lambda(t-t_0)} \|f(t)\|_{H^p}.$$

The fast stable trace at t_0 is zero in the reduced coordinates. The inverses agree on common data. At the fixed clock and dilation their physical-norm bounds are uniform in t_0.

Proof. A retained decay rate in $[a,b]$ requires $|g| \ge a$. The preceding absolute sum over integer keys is bounded after division by λ, uniformly as $\lambda \downarrow 0$. At fixed pump and integer key, bare resonance prescribes both $n + n'$ and $n^2 + n'^2 + 2R^2q^2$. The latter bounds $|q|$ in terms of the fixed tree since $K_i = 1 + RP_i$ and $|\nu| = O(\log R)$ at the retained angular cutoff. For each q the two equations leave at most two choices of n. Thus the actual count is bounded by an M depending only on this preparation and a, for every sufficiently large admissible R. After adjoining a, b the longest gap has length at least $(b-a)/(M+1)$. Take its middle third for J; its location may vary with the parameters.

Set $k^2 = |g|^2 - \delta^2/4$. For bounded δ the pair propagator uses $\cosh(k\Delta\theta)$ and $\sinh(k\Delta\theta)/k$, with value $\Delta\theta$ for the second factor at $k = 0$. When the cut separates the pair, its spectral projections are bounded by the clearance. For large $|\delta|$ the oscillatory propagator is uniformly bounded. The Hermitian complement is isometric in the flattened norm. Integrate forward from zero on the fast stable subspace and backward from infinity on its complement. The clearance makes both weighted integrals bounded throughout J; their integration directions are the same at every weight. The transverse remainder $O(\lambda^{1/5} + \eta\lambda) + o_R(1)$ is absorbed by the same Neumann series. The separate axial block in Birkhoff coordinates is a rotation with an occupied action-angle shear. Backward integration here is bounded since a linear polynomial times a decaying exponential is integrable. These formulas commute with inclusion of the

stronger weighted space into the weaker one, first for the reduced inverse and then term by term in the Neumann series. For data with crossed regularities and weights, the two solutions agree by uniqueness at the lower order and smaller weight. The scaled norms are equivalent to the physical norms at fixed R, and the estimates depend only on elapsed time. This proves the asserted compatibility and translation bound. □

5.5. **Insertion and passage to a single initial state.** For completeness we record the endpoint error sizes in the original terminal argument. Write the axial projection of the actual field as U_0, its transverse part as w, and $z = \Phi(U_0)$. Equations (3.3)–(3.5) are exact with common constant mass gauge of the new target. Set $J_{\mathrm{old}} = (I_{\mathrm{old}} - I_{\mathrm{old}}^{\infty})/r$ and let Θ be its angle displacement. At these trial actions and angles take the old field V and its column matrix. Let W, Y denote respectively the normalized mobile transverse coordinates and axial Birkhoff planes of the exact finite comparison. Add an unrestricted transverse error rh, an axial error ra, and the quadratic particular solutions in (3.10) –(3.12). The pure quadratic sources, divided by r^2, are

$$P_{\perp}\{2V(|Y+W|^2 - |Y|^2) + \bar{V}((Y+W)^2 - Y^2)\}, \qquad P_0\{2V(|W|^2 - \textstyle\sum_v |W_v|^2) + \bar{V}W^2\}.$$

The removed scalar density acts on the entire axial field as its phase generator, so it does not change old actions. For a fusion the first two kinetic tests are $-2R^2 p\cdot q$, $2R(b-1)(p_x+q_x)$; for a reverse they are $2R^2(p-q)\cdot q$, $2R(1-b)(p_x-q_x)$, with the common sign convention for the two legs. After these tests precisely the nondegenerate graph edges at $(b,\nu) = (1,1)$ survive. Their entire coefficient cancels with the exact finite equation. All other sources are divided against the bare nonzero divisor at angular degree $\le C_1 \log R$. A pure low-horizontal transverse particular has divisor of order R^2, paying for the bounded low body; the other transverse particulars use the scalar dressed columns. Nonzero axial velocities have exact empty frequencies $|k|^2 + O(R^{-2})$. The differentiated envelopes of the particulars cost only powers of $\log R$: the fast derivative of the carrier itself is taken together with its kinetic divisor. The old actual shifts in that divisor are $O(R^{-2}\log R)$. The discarded tail can cost $O(R^{-N_0})$ for any fixed N_0. In particular a bounded nonzero kinetic divisor contributes a normalized error $o(1)$, while an anomalous homogeneous off-key resonance has been included separately in the normal operator.

Use on a late slow half-line an exponential weight in the stated gap. The mobile waves and the differences from their axial endpoint there are bounded in weighted norm by an arbitrarily small c. Put H_1, A_1, J_1, Θ_1 for the four weighted sup norms of $h, a, J_{\mathrm{old}}, \Theta$, and $X = H_1 + A_1 + J_1 + \Theta_1$. The Green images satisfy

$$\begin{aligned}
H_{\mathrm{new}} &\le C(e_R + cX + \sqrt{R}X^2 + o_R(1)H_1),\\
A_{\mathrm{new}} &\le C(e_R + J_1 + c(H_1 + A_1 + \Theta_1) + \sqrt{R}X^2 + o_R(1)A_1),\\
J_{\mathrm{new}} &\le C(e_R + cH_1 + c^2 A_1 + \sqrt{R}X^2),\\
\Theta_{\mathrm{new}} &\le C(e_R + J_1 + r(A_1 + H_1) + \sqrt{R}X^2), \qquad e_R = CR^{-1}(1+\log R)^C.
\end{aligned}$$

Here the pure column error is $O(R^{-3}/r)$ and the divided sources have the error just estimated. A single prescribed factor acts by its Wiener norm; two free factors use the $\sqrt{R}$ bound, and a third free factor has an additional r. In the first row the full trial-dependent Hermitian low body is on the left. Its order-one angle derivative is never applied as an error to a free low field. Changing the old mass and the nominal chart velocity costs $O(r|J_{\mathrm{old}}| + r^2)$; the derivative of the normal change costs this times $O(R^{-1})$. After division by the slow rate $r\lambda$, the variation of the old-mass scalar from its terminal value is $2\lambda^{-1}\sum_{j\in F}(J_{\mathrm{old}})_j$ on the positive transverse branch. Its product with the prescribed transverse packet and its error has weighted norm at most $C_{F,\lambda}J_1(c + H_1)$. This gives $O(cX + X^2)$ in the first row at the fixed preparation. Every

angle error against a prescribed transverse wave carries its mobile factor. In the second row keep the whole real diagonal at the trial actions. Their distance in ℓ^1 from the terminal actions is $O(r|J_{\mathrm{old}}| + r^2(c + A_1 + o_R(1))e)$, where e is the decaying exponential. Its multiplication by a persistent root accounts for the unsuppressed J_1. For the third row, the equal-wave density changes no action. Every other pure axial forcing on a fixed old coordinate has high horizontal output, and hence a factor $O(R^{-1})$ by the chart differential and its symplectic adjoint. A generic transverse variation there needs a mobile transverse factor; varying the axial field needs two such factors. In the last row the occupied shear gives J_1; external forcing divided by $r\lambda$, in place of $r^2\lambda$, gains r. Pure errors vanish at the exact terminal axial solution.

With $\rho = R^{-3/5}$ take radii $M_1\rho$ in the first and third rows and $M_2\rho$ in the other two. Choose M_1, then M_2 relative to the shear, then c small, and finally R large. Both e_R/ρ and $\sqrt{R}\,\rho$ tend to zero. On a finite interval ending at S, use the stable projection forward and the remaining components backward. The low unitary is evaluated along the trial, using the instantaneous velocity from its coordinate equations in the time connection, as in Section 3. At fixed R, S the image has bounded time derivative in Z^0 by its Z^2 bound. Interpolation, compact embedding on the fixed lattice, and the finite-time Duhamel formula give a compact convex invariant set in the $C_tZ^{p_0}$ topology and continuity of the map there. Schauder's theorem and then local compactness as $S \to \infty$ give an exact solution. Strong H^2 persistence and uniqueness identify the physical limit. The same estimates with bounded coefficients and finite Gronwall on the preceding fixed slow interval give an $O(\rho)$ normalized launch error. This constructs an exact solution W_R exponentially approaching the new axial target V' on its invariant lattice.

At fixed old target V and fixed height L, take first the depth to make the launch arbitrarily small, followed by the other finite choices and finally R. We obtain

$$W_R(0) \to V(0) \text{ in } H^2, \qquad \sup_{t\geq 0} \|W_R(t) - V(t)\|_{H^{p_0}} \to 0, \qquad \sup_{R,t\geq 0} \|W_R(t)\|_{H^2} < \infty$$

along the resulting family in its respective mass gauges. Indeed the old rates and origins agree exactly, the incoming clock cost is arbitrarily small, $rR^{p_0} \to 0$, and the strong bound on all primary levels above is independent of depth at fixed height. Apply one-smoothing of $\Psi - \mathcal{F}^{-1}$ from h^1 to H^2 at the terminal torus. Its old image is compact in H^2 and the terminal high inputs have h^1 size $O(rR)$. The normalized physical modulus at $1 + RP$ is thus $L + o_R(1)$ uniformly in time there. By exponential landing it persists on W_R with arbitrarily small fixed slack at sufficiently late times.

We finish by specifying how the historical differences in Section 4 are reused here. Choose increasing heights $L_j = 4j + 4$ and take each new dilation divisible by every old one. For a saved axial reference V use its normal inverse just proved at two weights $0 < \nu < \beta$ in one gap, with the same boundary conditions in H^{p_0}, H^2. Suppose $Y + d$ is an earlier exact solution in the gauge of Y, with $d = O_{H^2}(e^{-\beta t})$ and Y in its saved uniform weak tube about V. The new inputs Y_n converge to Y strongly initially and uniformly weakly on the half-line, have bounded strong norms at this fixed height, and their gauges converge. Solve the equation for a replacement correction $d + h$ by writing

$$h = \mathcal{G}_V\{F_n(d + h) - F(d)\},$$
$$F_n(b) = -i\big[(c_n - c_V)b + 2(|Y_n|^2 - |V|^2)b + (Y_n^2 - V^2)\bar{b} + 2Y_n|b|^2 + \bar{Y}_n b^2 + |b|^2 b\big].$$

Here F uses Y and its gauge. In the sum of weighted strong norm and Λ times weighted weak norm the tame product estimate bounds the linear coefficient by its small uniform weak size plus $C(1 + \sup_n \|Y_n\|_{H^2})^2/\Lambda$. Choose Λ for this fixed-height family and start the half-line sufficiently late to make the shifted β-norm of d arbitrarily small. This gives a contraction at β. At ν its inhomogeneity tends to zero by finite-time strong continuous dependence and the extra

$e^{-(\beta-\nu)t}$ tail. Compatibility of the inverses gives convergence for the same correction. Propagate backwards over the fixed initial interval. This preserves the original rate β for future reuse.

At each stage first lift the newest saved difference about its axial reference, then lift the finitely many earlier differences in reverse order. All inputs belong to the requisite older lattices. Choose a sufficiently late member of the fixed-height family to make the final initial H^2 differences summable and all older weak tube changes summably small. At each previously attained frequency K_i require the remaining sum of weak changes times $\langle K_i\rangle^{2-p_0}$ to be less than i. Each new full solution then has normalized modulus $> 2i$ at its ith frequency for all $t \geq T_i$, and the limit retains modulus $\geq i$ there on that whole interval. Strong initial convergence and finite-time well-posedness identify the limit with a single global H^2 solution. Undoing its mass gauge gives $\|u(t)\|_{H^2} \to \infty$. An arbitrarily small starting axial torus and summably small initial increments give an arbitrarily small datum.

Unit heights and the counting assertion at the endpoint. Alternatively set $L = 1$ at every insertion. We give the additional uniform-in-time comparison needed for the counting law. Let $\delta > 0$ be prescribed after the previous orbit has been fixed. On the entire half-line the raw change from the old axial target is the sum $P_R(t)$ of its finitely many pure new carrier planes, with their actual comparison coefficients, plus an error of H^2 norm at most δ. To verify this, choose d and the private costs first. The old occupied frequencies and origins agree exactly; the finite old actions and the moderator and weak inputs may change by arbitrarily little in h^2. The physical shape error for the high axial insertions tends to zero by one-smoothing since their h^1 cost is $O(rR)$ at this fixed preparation. The normalized transverse columns differ from their individual planes by $O(R^{-1})$ on the finite set; the physical cost of the scaled remainder is $O(rR^2\rho) = o(1)$. The nonlinear old modulations, of size $O(\rho)$ in the old angles and $O(r\rho)$ in their actions, obey the same conclusion uniformly by compactness of this fixed old torus. The difference of each old history correction from its replacement tends to zero in uniform H^2 by its convergence at the smaller positive weight and finite-time strong convergence. Thus after the finitely many lifts the full change is still $P_R + O_{L_t^\infty H^2}(\delta)$, shrinking the input tolerance if necessary.

Write u_{old} for the old trajectory and let $P_{\geq M}$ be the Fourier projection to $\langle k\rangle \geq M$. The range of u_{old} is precompact in H^2, since it approaches a compact torus. Consequently $\sup_{t\geq 0}\|P_{\geq M}u_{\mathrm{old}}(t)\|_{H^2} \to 0$ by Fourier truncation on a finite net of the range. All modes of P_R lie beyond a threshold $M_R \to \infty$, so

$$\sup_{t\geq 0}|\langle u_{\mathrm{old}}(t), P_R(t)\rangle_{H^2}| \leq \sup_{t\geq 0}\|P_{\geq M_R}u_{\mathrm{old}}(t)\|_{H^2}\sup_{t\geq 0}\|P_R(t)\|_{H^2} \longrightarrow 0.$$

The second factor is bounded at unit height. Expanding the new squared norm now gives the old squared norm plus $\|P_R(t)\|_{H^2}^2$ up to any prescribed uniform error. To obtain that error, first choose δ using the old strong bound and then enlarge R. Take these errors summable over the stages. Set $D_j = K_j^2$. On $t \leq D_j^{3/4}$ the slow time $r\lambda t$ tends uniformly to zero as $R \to \infty$ at fixed preparation, and the bounded original finite vector field keeps P_R within the prescribed cold cost. On $t \geq D_j^{5/4}$ the same slow time tends uniformly to infinity; exponential landing and the small cost of the private terminal axes make the squared norm of P_R differ from one by the prescribed tolerance. On the intervening interval its norm is bounded by an absolute constant, using balance and the unit-height level bounds above. Select R to make these intervening intervals disjoint. Summing the squared increments gives (1.2), since at most one interval is in transition. Taking $D_j^{3/4}$ sufficiently late that $G(D_j^{3/4}) \geq j^2$ gives the prescribed $o(G(t))$ bound. The required slow entrance time at fixed preparation is bounded independently of sufficiently large R; taking R last thus also gives $t_j \leq K_j^2\ell(K_j)$, with the slack in each older Fourier coefficient preserved by the weak tubes.

6. Exponents above two

Fix $2 < s < \infty$. All constants in a single insertion may depend on s, the finite axial background and the depth of the tree. In this section only the final spatial dilation is sent to infinity at that insertion. This order of choices is useful for distinguishing the physical and angular small divisors. We use the notation A, B of Lemma 2.3 and the tree with its private horizontal endpoints and its mixed vertices from Proposition 2.4. An insertion of unit height will mean that the final Fourier amplitude on its main axis has H^s weight $1 + o(1)$.

6.1. **Preparing the finite axial solution.** At a smooth axial torus of fixed finite support, for every fixed $L \geq 2$ the empty-coordinate frequencies admit the expansion

$$\chi_n = \sum_{j=2}^{L} c_j n^{-j} + O(|n|^{-L-1}). \tag{6.1}$$

Indeed continue the two spatial recursions of Lemma 2.3, with an instantaneous scalar selected to be the spatial mean in the upper equation. At each order $2D_x$ is inverted on the spatial mean-zero part and the leading constant in the lower equation is $2n^2$. Only differentiation along the smooth, finite-dimensional angle flow and spatial integration occur. After the scalar time phase is removed from the trial column, the positive coefficient of its image under the complexified $D\Phi$ at the empty n coordinate is $c(t, n) = 1 + O(1/|n|)$ by one-smoothing. Division of the transformed scalar equation by $c(t, n)$ and integration along the orbit identifies each term in (6.1) with the time mean of the corresponding instantaneous scalar coefficient to the stated accuracy. Each coefficient has a time mean on the compact orbit closure and $\partial_t \log c(t, n)$ has zero mean since its primitive is bounded. The constants are uniform for the spatial recursion on a fixed compact family. At the second order its constant coefficient is $c_2 = A - B$, also on resonant tori, by the Liouville identity.

Use the five weak letters h of Section 5. Choose an integer $m > \max(2, 2s - 2)$, a prime $\pi > 20\, 2^{m-1}$ and the neutral moderator word with generating polynomial

$$d_*(X) = X^K (1 - X^\pi) \prod_{\ell=0}^{m-2} (1 - X^{2^\ell}).$$

The subset sums of the powers 2^ℓ, $0 \leq \ell \leq m - 2$, give each integer $0 \leq i < 2^{m-1}$ once, and the factor $1 - X^\pi$ gives its oppositely signed partner at $i + \pi$. Write e_a for the resulting sign at $K + a$. The zero of order m at $X = 1$ gives $\sum_a e_a a^j = 0$ for $0 \leq j < m$, by applying $(X\partial_X)^j$ to $X^{-K} d_*(X)$ at 1. For every $2 \leq l \leq m + 1$, Taylor expansion of $(K + a)^{-l}$ through degree $m - 1$ in a therefore gives $\sum_a e_a (K + a)^{-l} = O(K^{-l-m})$. The shifts are fixed as K increases. The remainder in (6.1) with $L = m + 1$ is $O(K^{-m-2})$, and the five weak sites cost $O(T^{-2})$. Together these estimates give

$$|(h + d_*) \cdot \chi| \leq C(K^{-m-2} + T^{-2}).$$

Here T is the common scale of the five weak letters. At fixed depth use disjoint groups and comparable K's. Take $T \gg K^{1+m/2}$. A positive common action buffer on the moderator sites of size $C_1(K^{-m-2} + T^{-2})$ costs $O(K^{s-1-m/2}) = o(1)$ in H^s. As in the endpoint argument the common buffer is annihilated by each signed moderator, and one unit moderator pivot per row corrects the deficit. Compactness of $D\chi + \mathrm{Id}$ bounds the compressed high-coordinate Jacobian error by an arbitrarily small number after the finitely many sites have been separated from the old ones. The Schur complement that keeps the old frequencies fixed has the same property.

We verify nonvanishing of the divided physical coefficient for this choice of moderator. In the alternating-path formula of Section 5 the five-letter coefficient is $-1/(9T^4)$, by summing its twelve orders. Take $T = 1 \bmod \pi$. The five signed weak sites are distinct modulo π. For a

new positive and negative pair a, b, multiply the path coefficient by $a - b$ and let them coalesce at y distinct from the remaining labels. Only paths traversing that edge remain. Deleting the adjacent pair, its insertions into an internal gap v, w contribute

$$\frac{v - w}{(v - y)(y - w)} = \frac{1}{v - y} + \frac{1}{y - w},$$

and endpoint insertions contribute the corresponding single terms. Both ends of every remaining vertex are counted. Including the common factor $-1/2$, the residue is the old coefficient times $\sum_{v \text{ positive}} (y - v)^{-1} - \sum_{w \text{ negative}} (y - w)^{-1}$, up to the sign from ordering the colliding pair. An already coalesced opposite pair contributes zero to this last expression. The weak signed sum has five poles and numerator of degree four with leading coefficient one. Choose $K \bmod \pi$ so that all 2^{m-1} residues $K + i$ avoid its poles and zeros. There are fewer than π excluded residues. Multiply the full coefficient by the product of the paired differences and reduce modulo π. All other differences are units; a path uses a given paired edge at most once. Iterating the residue identity gives a nonzero result. This proves physical nonvanishing for the required disjoint supports.

For clarity, here the frozen renormalized angular frequencies are τc_n with real algebraic c_n, and we require the old list to be rationally independent. Start with two nearby negative independent algebraic slopes and a sufficiently small $\tau > 0$. Keep every previously occupied slope fixed. On each new flat $(h_e + d_e) \cdot c = 0$ one may use a private unit pivot, independently for each group e. The passive calibration and the continuation along the common scaling parameter in Section 5 apply without the congruence condition on the slopes. More explicitly, while all weak actions are zero use a moderator pivot to impose each row. All other newly positive slopes can be approximated by algebraic slopes, preserving positivity. The normalized coefficient with the specified signed row and zero other angular labels has the computed nonzero vacuum value on this analytic recipe. Division by its indicated radii extends analytically across zero by expansion in the complex Cartesian variables and angular averaging. The bounded change of coordinates in the slope inverse replaces each pivot by its row; the finite row lengths here depend only on s, with disjoint groups. The small action ball of the inverse and the complex Fourier–Lebesgue chart used in the preceding section therefore connect this recipe to zero. Take the original τ outside the exceptional sets of the countably many finite algebraic recipes, including the earlier slopes. Activate the five weak radii in each group by the fifth-root prescription at the actual divided coefficient. Their cost at the fixed sites tends to zero with the common target modulus λ. Retune the pivots, and approximate all free slopes on the activated flat by algebraic values independent over the rationals together with the old list. This time eliminate one weak pivot in each row. Density on the open free-coordinate cube preserves positivity and the required relative coefficient fit. The complete integral-relation lattice on this frozen list is then generated by the rows $h_e + d_e$, since they have private unit pivots. After the physical tilt below, independently approximate all newly occupied slopes again, off this flat, for use in the next insertion.

For every one of the countably many algebraic lists so obtained, the nonzero quantities $b + \tau l \cdot c$, $b \in \mathbb{Z}$, have a polynomial lower bound in $1 + |b| + |l|$ at this same τ. For $b = 0$ use the algebraic norm in the finite number field of the list. For $b \neq 0$, restrict τ first to a compact positive interval. An affine expression that can approach zero there has $|l \cdot c|$ bounded below, so the lengths of its exceptional intervals are summable on taking a sufficiently large fixed exponent. Use Borel–Cantelli and exclude exact zeros. A frozen integer-time angular label is consequently precisely a sum of the private rows. Its charge, space frequency and integer time all equal the sum of its row coefficients. For a charge-one term the order-five weak characters in this lattice are the positive singletons; all others have more weak letters and their absolute coefficient sum is $O(\lambda^{6/5})$ by analyticity on a larger weak polydisc at the fixed passive preparation. Each singleton

has the required modulus $\lambda(1+o(1))$. This construction separates old phase harmonics without assuming statistical independence.

6.2. **Choice of directions and matching of the actual frequencies.** On a normalized real tree with main root $(1,0)$ put

$$\Xi(v) = A/v_x^2 - B/|v|^2, \qquad D(p,q) = \Xi(p) + \Xi(q) - \Xi(p+q).$$

At the frozen positive background the functions $1, D_e$ of all the real split parameters, for primary and private edges, are linearly independent over the constant reals. At the last graft, with parent (X,Y), $XY \neq 0$, the private defect for p has a fourth order pole at zero where the other private defect and primary have poles of order at most two. At infinity interchange the children. The remaining primary has a pole, because $A > 0$. Remove the three coefficients in this order and proceed backwards. The initial horizontal split has nonconstant defect. Exclude also the equality of any mixed defect with a desired singleton, using the horizontal pole of the mixed output in the endpoint construction. On a compact box of admissible split angles, multiplication by their common rational-function denominator makes the map from coefficient vectors to polynomial numerators injective, hence bounded below. The polynomial sublevel bound on a box, $|\{|p| \le h\}| \le C(h/\|p\|_\infty)^\alpha$ for polynomials of bounded degree, follows by iterating the one-variable sublevel inequality (split according to the size of a nonzero coefficient polynomial). Summing over integer coefficients and applying Borel–Cantelli supplies real parameters with

$$|a \cdot D| \ge c(1+|a|)^{-C} \qquad (0 \neq a \in \mathbb{Z}^{\#\{e\}}).$$

All constants may depend on this finite preparation.

Lemma 6.A (Uniform rational approximation of the tree)**.** *Fix $s > 2$, the finite depth, the frozen preparation and a real tree as above with $|a \cdot D| \ge c(1+|a|)^{-\kappa}$ for every nonzero integer coefficient vector a. Given $C_1 > 0$ and $b_0 > \kappa + 1$, for every sufficiently large multiple R of the old vertical spacing there is a rational tree satisfying the congruence exclusions of Proposition 2.4. Its split parameters differ from the chosen parameters by $O((\log R)^{-b_0})$; its common spatial denominator P is at most $C(1+\log R)^{C_0}$. With $N = RP$ and rational-tree defects $D(N)$, one has*

$$|a \cdot D(N)| \ge \tfrac{c}{2}(1+|a|)^{-\kappa} \qquad (0 < |a| \le C_1 \log N).$$

The nonzero horizontal components and the strict geometric exclusions on the prescribed finite list retain positive margins in normalized coordinates. The constants and the threshold for R may depend on the fixed stage, C_1 and b_0, and are uniform in R.

Proof. Use a single rational tree from Proposition 2.4 to witness the finitely many odd valuations. Write its split parameters as integer pairs. At each witness prime the valuation of each tested nonzero rational function is locally constant at this tree. Hence fixing the integer pairs modulo one product M of sufficiently high prime powers preserves all the required valuations. For each desired real parameter t, choose a positive denominator $b \asymp (\log R)^{b_0}$ in its prescribed class and a numerator in its prescribed class nearest to bt. The error is at most M/b. At fixed depth the vertex coordinates are fixed rational functions of these parameters. Clearing their denominators costs at most a fixed power of $1+\log R$. On a compact neighborhood of the real tree all relevant real denominators stay away from zero, so the defect functions are Lipschitz there and the finite strict exclusions persist. In particular $\|D(N) - D\| \le C(\log R)^{-b_0}$. Since $\log N = \log R + O(\log\log R)$, the error in $a \cdot D$ is at most $C|a|(\log R)^{-b_0} \le (c/2)(1+|a|)^{-\kappa}$ on the stated range for sufficiently large R. □

Put $r = N^{-s}$; the physical root is at $K_* = 1 + N$, and the slow time is $\theta = r\lambda t$. Below $\mathcal{L}$ denotes a fixed power of $1+\log N$, allowed to increase finitely many times.

Here are the changes to the normal reduction. Choose an integer $J > s + 4$. For all $m' \in R\mathbb{Z}$, $m' \neq 0$, and $|n| \geq cN/P^2$, expand the doubled columns to spatial order greater than $J+3$, before making any temporal division. With $f = e^w$, $\langle w \rangle_x = 0$, $z = g/f$, the upper equation determines the next mean-zero Dw by $2n$, the next scalar by its spatial mean; the lower equation determines the next coefficient of z by $2E$, $E = n^2+(m')^2$. Equivalently use the undivided envelope recursion of Lemma 2.3 and normalize the spatial mean of f to one. The coefficients are smooth old fields times bounded polynomials in n^2/E. Every lower coefficient has a factor n^2/E, so the residual estimate remains uniform even when $|m'|/|n|$ is unbounded. Increasing the finite order absorbs all factors $P = O(\mathcal{L})$. Before normalizing a positive column (f, g), put $S = \|f\|_{L^2_x}^2 - \|g\|_{L^2_x}^2 > 0$. Its equation with instantaneous scalar d gives $\partial_t S = -2\,\mathrm{Im}(d)S + O(N^{-J-2}\mathcal{L})$. Dividing the column by $\sqrt{S}$ gives unit Krein square and changes the scalar to $d + i(\partial_t S)/(2S)$, real to the residual accuracy. Its averaged leading term is $A/n^2 - B/E$, and its nonconstant part at that order is a bounded time derivative by Lemma 2.3.

Fit the carriers before taking the temporal primitives. At each high off-axis head use its bare energy plus the angle mean of its full instantaneous spatial scalar. At an axial root take its exact frequency on the terminal torus, whose new actions are prescribed as r^2 times the desired squared radii. The off-axis action derivative of the averaged shift is $O(N^{-2}\mathcal{L})$, since this mean has no angular divisor. The high-root frequency derivative with respect to the old actions, after mass subtraction, is $o(1)$ by compactness of $D\chi + \mathrm{Id}$. These bounds are uniform on the shrinking neighborhood of the frozen point. The compressed frequency inverse with all previously occupied rates, all moderator rates, and the nonpivot weak rates held at their frozen values allows a change of one private weak rate per row. These are independent coordinates: prescribe the new terminal root actions first, then solve the compressed positive-action chart for the entire remaining rate vector. Write y_e for the signed rate displacement of row e from the frozen value and $I(y)$ for the actions given by this inverse. For a head v, let $\Delta_v(I)$ be the correction to its bare energy, evaluated as above. Then the equations for all edges are the single fixed point equation

$$y_e = \Delta_v(I(y)) + \Delta_w(I(y)) - \Delta_{v+w}(I(y)), \qquad e = (v, w; v + w).$$

At $y = 0$ the right side is $O(N^{-2}\mathcal{L})$. Each head correction has derivative $o(1)$ in the varying actions by the estimates above. The compressed inverse and the number of edges are bounded at this fixed preparation, so the derivative of the full right side with respect to y is $o(1)$. It is therefore a contraction on a ball of radius $CN^{-2}\mathcal{L}$ for a sufficiently large C. At its fixed point each row changes by $D_e(N)/N^2 + O(N^{-3}\mathcal{L})$ by (6.1). For the next stage keep the previously occupied, moderator and nonpivot weak rates fixed in the final rounding as well. Those fixed rates are already rationally independent together: dropping one private weak pivot per row removes all relations. In the full compressed rate chart round the weak pivots and the new terminal root rates to independent algebraic slopes times τ, to precision smaller than both N^{-J-3} and $o(r^2)$. This preserves the positive root radii up to relative $o(1)$. The finite tree equations can be made exact again by changing the auxiliary and mobile carrier speeds by $O(N^{-J-2})$, starting from the specified axial speeds and splitting errors between children.

Fix the divisors at this terminal frequency vector Ω^∞ when evaluating coefficients along a trial with moving old actions and phases. Every neutral, spatially averaged nonzero angular character $|l| \leq C_1 \log N$ satisfies

$$|l \cdot \Omega^\infty| \geq N^{-2}\mathcal{L}^{-1}.$$

If its frozen divisor is nonzero use the polynomial bound and its $O(N^{-2}\mathcal{L})$ displacement. Otherwise it is a nontrivial neutral combination of the rows, and use the bound for $a \cdot D(N)$; the private pivots bound $|a|$ by $|l|$. Higher terms $O(N^{-3}\mathcal{L})$ preserve the separation. For orders

$k \geq 3$, divide each nonconstant spatial mean at order n^{-k} by $il \cdot \Omega^{\infty}$ through this cutoff. The real phase so obtained and all needed old-parameter derivatives are $O(N^2|n|^{-3}\mathcal{L}) = O(N^{-1}\mathcal{L})$. The tail after the cutoff contributes $O(N^{-J-2})$ by angular analyticity, on increasing C_1. At order two use its bounded primitive instead. The last conjugation is a scalar phase on the individual input columns. It commutes with the bare diagonal; its moving connection costs $O(N^{-1}\mathcal{L})(r|J_o| + r^2)$, where rJ_o is the old action displacement. Indeed the old isolated rates and the full terminal old rates differ by $O(r|J_o| + r^2)$ in this tube. The time derivative in the spatial recursion uses the actual isolated old flow at those actions. Its difference from the trial velocity multiplies old-parameter derivatives of the near-identity column. All denominators in this final scalar operation stay fixed. In particular, a temporal divisor of size $N^{-2}\mathcal{L}^{-1}$ is never fed back into a spatial recursion.

On each doubled pair of nonzero vertical copies, first remove the old off-diagonal pairing to arbitrary finite order as in the proof of Proposition 3.1. In detail every off-diagonal kernel is divided by $n^2 + (n')^2 + 2(m')^2 \geq 2R^2$ with the opposite sign on the other copy. Use its Krein-preserving exponential. The commutator with the free generator is exactly the undivided kernel; successive commutators with the bounded old block and the connection lose at most a fixed old norm and gain R^{-2}. All estimates hold with summable off-diagonal moments. Transform the direct high columns above by this frame and project onto their Hermitian copies. Their residuals keep the stated order. For their matrix T, the covariant commutator of T^*T with the real high diagonal has the same order by the equation and its adjoint. The binomial series thus gives the same order for the commutator of $(T^*T)^{-1/2}$. Complete the high isometry $T(T^*T)^{-1/2}$ on its low complement exactly as in Proposition 3.1. The kinetic commutator of an $O(1/|n|)$ spatial envelope is bounded with Schur moments, and the scalar temporal phase is diagonal on the input. This leaves a full Hermitian low block at the actual trial old angles and actions. It is kept in the principal evolution. Freezing that low block at the terminal phases would introduce an uncontrolled $1/(r\lambda)$ angle derivative.

6.3. **Finite dynamics and the normal time weights.** Use the complete finite equation on the tree and its mixed hypotenuses. The assigned singleton gives a constant coefficient on each intended triad. Singletons assigned to any other defect there, or to a mixed edge, rotate at slow speeds bounded below in magnitude by cN^{s-2} at this fixed preparation. Their bounded oscillatory primitives remove them by a near-identity finite change; its remainders are $o_R(1)$ in C^1 on bounded finite sets, by integration by parts with the original bounded vector field. In particular only its first derivative along an actual trajectory is used. Higher weak terms have normalized size $O(\lambda^{1/5})$. Both finite fields vanish on the axial terminal plane in these rotating coordinates. The common mixed inverse on the exponentially landing finite orbit from the endpoint therefore yields the exact finite orbit with the rounded axial radii and an arbitrarily small launch error: first fix the finite tolerance, take λ small, then R large. Equivalently translate the reference by the axial rounding; the forcing at the new constant endpoint is zero and its difference along the exponentially landing reference decays at the same weight. The early interval has fixed slow length.

There is one extra issue for the full terminal linear equation. After removing noninteger angular harmonics at their frozen divisors, retain the Hermitian stars of Proposition 3.1, with their true oscillatory coefficients. Every charged integer-time old harmonic has space and bare time $(1, 1)$, since it is an integral combination of the private rows with charge one. Thus an anomalous bare pair from an axial root $K_i = 1 + RP_i$ has

$$(n-1) + (n'-1) = RP_i, \qquad (n-1)(n'-1) = R^2q^2, \qquad q \neq 0.$$

If present its legs are bounded below in absolute value by cN/P^2. Its root is unique from the sum and product. An intersection with a Hermitian star would give $P_l(P_i - P_l) = q^2$, excluded by the local nonsquare conditions. Entries touching a low endpoint outside the retained Hermitian stars have anomalous bare divisor $\gtrsim R^2$ or same-copy bare divisor $\gtrsim R$: for example $2(qR)^2$ dominates the $O(|K_i|(|n| + |b|))$ term at $|n| < cN/P^2$, with c small and $|b| \lesssim \log N$. Where both endpoints are high, the bare noninteger separation is $\mathcal{L}^{-1}$, and the scalar shifts are $O(N^{-2}\mathcal{L})$. An entrywise primitive therefore costs $O(r\mathcal{L})$ there and $O(r/R)$ when it meets the nondiagonal low block; its commutator with that block retains the latter gain. Sum the shifts and their adjoints with their summable angle coefficients in the Schur norm. Combine the free kinetic commutator and the time derivative before estimating the remainder. This gives $o(1)$ after division by the slow scale.

On each remaining pair use the full nominal high real diagonals. At most one singleton detuning can be near zero, by separation of distinct $D_e(N)$; primitive division of the others has normalized cost $O(N^{2-s}\mathcal{L})$. Retain this one harmonic with its actual detuning, and absorb the higher weak tails. To see that one can choose a positive mixed time weight uniformly in R and small λ at this fixed tree, put $u = (q/P)^2,\ p_i = P_i/P,\ x = (n-1)/N,\ y = (n'-1)/N$. Then $x + y = p_i,\ xy = u$. The two squared lengths are $x^2 + u = xp_i$ and $y^2 + u = yp_i$. Thus the sum of their reciprocals is $1/u$, whereas $x^{-2} + y^{-2} = (p_i^2 - 2u)/u^2$. Subtracting the parent's correction gives the leading mismatch in N^{-2} units,

$$F_i(u) = Ap_i^2/u^2 - (2A + B)/u - (A - B)/p_i^2.$$

An exponent of the two-by-two block with nonzero real part requires $|F_i(u) - D_e(N)| \le C(N^{2-s} + N^{-1}\mathcal{L})$. On the permitted compact range $u \le p_i^2/4$ the small values of u are excluded by $Ap_i^2 > 0$, uniformly for these rational trees near their fixed real limit. The sublevel set in $1/u$ lies in at most two intervals of length $C(N^{2-s} + N^{-1}\mathcal{L})^{1/2}$, since this is a quadratic with leading coefficient bounded below. The mesh of distinct positive $|q|$ there is $\gtrsim 1/P$ in $1/u$. For fixed $s > 2$, Lemma 6.A and the polylogarithmic bound on $\mathcal{L}$ give $P(N^{2-s} + N^{-1}\mathcal{L})^{1/2} = o(1)$. Each interval therefore contains at most one value of $|q|$ for large R, and $x + y = p_i$, $xy = u$ then determine the unordered pair of horizontal coordinates. Thus at most a fixed number of such rates occurs per root and singleton. The count is independent of λ small at the fixed passive preparation: normalized singleton strengths and their number are bounded, and the geometry can be taken in fixed compact intervals separated from zero. Partition a compact positive interval strictly below the finite orbit's exponential rate into more than twice this number of cells, summing over all roots and singletons. For every sufficiently large R a cell has a smaller interior interval with uniform clearance. Use three strictly ordered weights in that interval for the exact landing and the two subsequent history estimates. A subsequence with the same cell suffices.

The pair itself has constant coefficients after a unitary half-phase rotation, with real growth rates $\pm \operatorname{Re}\sqrt{|g|^2 - \delta^2/4}$ for actual coupling g and detuning δ. Its two-sided Green bound on the cleared weights is uniform: bounded g, δ use the matrix exponential, including the continuous Jordan limit; large $|\delta|$ gives uniformly conditioned imaginary eigenvalues. The disjoint Hermitian complement propagates isometrically in the flattened weights used in Proposition 3.1. The axial root abscissae are $O(N)$; all affected axial–low stars are in a common flat horizontal band of this size. The low old block preserves its own vertical index. The discarded homogeneous operators cost $O(\lambda^{1/5}) + o_R(1)$ at the fixed preparation and are absorbed on this same inverse, choosing λ and R in that order. This argument requires no lower bound on the discriminant at a double root.

6.4. **The full equation and the fixed datum.** Put $p = \lceil s \rceil$, and use the scaled norm Z^p with weight $(1 + |n|/N + |q|/P)^p$, $m' = Rq$, and its weak companion Z^2. Prescribed heads have bounded weighted Wiener norms at fixed tree and normalized height. Two unrestricted factors cost $C\sqrt{NP}$ by summing the reciprocal squared weight over (n, q). A perturbation of size $O(r)$ in Z^p has physical H^{p-1} size $O(rN^{p-1}) = o(1)$. The change of either Birkhoff differential from its smooth old value is $O(r)$ on L^2, and its nonlinear part changes by $O(rN^{p-1})$ from H^{p-1} to H^p, by integer one-smoothing. Applied to a scaled unit vector its Z^p size is consequently $O(r + N^{-p}(rN^{p-1})N^{p-1}) = O(N^{-1})$. The old differential itself on a high single carrier equals its Fourier differential up to $O(N^{-1}\mathcal{L})$ in the scaled norm; use one-smoothing and the negative-order dual estimate as in Proposition 3.1. In particular its multiplication on an arbitrary field gains $o(1)$ even after the $\sqrt{NP}$ loss. The physical terminal Fourier shape follows separately from $h^{p-1} \to H^p$ local Lipschitz continuity of the nonlinear part.

We detail which forcing terms enter the terminal ball. In the exact old-action, old-angle, complementary axial and transverse equations use the same split of the known finite packet and the unknown as in Proposition 3.1. In $u = U + w$, $U = P_{m'=0}u$, the forcing of the axial equation is $2UP_0|w|^2 + \bar{U}P_0w^2 + P_0(|w|^2w)$, with the common mass gauge, and the transverse quadratic forcing is $P_\perp(2U|w|^2 + \bar{U}w^2)$. The equal-head density in the first axial term is a common phase on the whole axial field and has zero action derivative. For other two-head terms the nonzero integer macro dot product gives a divisor at least $R^2 - O(N \log N)$. At a right angle the tree contains the output, with nonzero limiting horizontal velocity. The linear term in N forces the spatial key $b = 1$; the remaining integer-time key is $\nu = 1$. Every such harmonic is already included in the exact finite equation. Other low harmonics with $b = 1$ have the polynomial frozen divisor above. Subtract each nonresonant quadratic particular once, with its fast phase intact. The derivative of a true finite amplitude in physical time costs $O(r\lambda)$; the change of a cutoff old harmonic's speed costs $O(N^{-2}\mathcal{L})$. The $O(1)$ lower-order part of the low Hermitian operator can meet a low particular only after the R^2 kinetic division. The all-order column residual after division by $r\lambda$ is $O(N^{-1}\mathcal{L})$, also for its moving connection on known heads up to a linear factor in the old scaled action. The $O(r^2)$ shifts at the exact terminal axial actions are included in its true frequencies. This gives the total normalized pure error $e_N = O(N^{-1}\mathcal{L})$.

For precision let H, A_1, J_o, Θ measure the decaying weighted normalized transverse error, complementary axial field, old action and old phase, respectively, and X their sum. In the terminal operator leave the whole Hermitian low matrix at the trial state on the left. There is no small parameter in its dependence on the old angles before division by slow time. The scaled fields are normalized by r, the old action by r, and the old angle without an amplitude factor. The four bounds for the Green image are

$$\begin{aligned} H_{\text{out}} &\le C(e_N + cX + \sqrt{NP}\,X^2 + o(1)H), \\ (A_1)_{\text{out}} &\le C(e_N + J_o + c(H + A_1 + \Theta) + \sqrt{NP}\,X^2 + o(1)A_1), \\ (J_o)_{\text{out}} &\le C(e_N + cH + c^2A_1 + \sqrt{NP}\,X^2), \\ \Theta_{\text{out}} &\le C(e_N + J_o + r(A_1 + H) + \sqrt{NP}\,X^2). \end{aligned}$$

Here c bounds the nonterminal heads on a fixed sufficiently late slow-time interval. The transverse one-error terms carry a mobile factor apart from the homogeneous operator. A variation of the old action in a high scalar has its $O(N^{-2}\mathcal{L})$ derivative. Variation of the frequency multiplying a persistent axial root gives the displayed unsuppressed J_o. The old action equation contains two transverse factors and its nondiagonal known source has high horizontal frequency, gaining $O(N^{-1}\mathcal{L})$ on old coordinates by smoothing and duality. A change in the old angle of this source retains two mobile factors (or a quadratic error); its only zero macro term is the common density

just removed. The ordinary axial shear gives J_o in the angular equation. Each two-error product obeys the displayed lattice bound, and a third arbitrary field costs an extra $Cr\sqrt{NPX}$. The constants on the unsuppressed linear terms are uniform for large R: the old smooth degrees of freedom, normalized finite Wiener norms and terminal spectral gap are fixed, and each homological or high-cut factor $\mathcal{L}$ on the remaining linear terms accompanies a fixed negative power of N.

Fix $1/2 < a < 1,\ \rho = N^{-a}$. Then $e_N = o(\rho)$ and $\sqrt{NP}\rho = o(1)$. Give H, J_o radius $M\rho$ and A_1, Θ radius $M'\rho$, choose M' large relative to M, then choose c sufficiently small, and finally R large. The displayed bounds preserve this ball. The finite-strip construction uses old velocities evaluated from the exact vector field at the trial state; its chain rule agrees with the actual one at a fixed point. At fixed R the time derivative is bounded in Z^{p-2}. The top ball and its equicontinuity bound are compact in the weaker path topology Z^2 by interpolation. Continuity of the trial-dependent low propagator here only needs a finite interval at fixed R and follows by Duhamel. Schauder and then a diagonal finite-strip limit give the decaying solution on the late half-line. On the remaining fixed slow-time interval retain the actual Hermitian propagator; the bounded finite coefficients and the same quadratic estimate give $O_{\mathrm{stage}}(\rho)$ backward by Gronwall and continuity. Uniqueness identifies the reconstructed solution with the full NLS flow. At fixed R the homogeneous inverse at the asymptotic axial orbit is also bounded in ordinary physical H^2, H^s at the same two history weights by the pair and Schur bounds just proved. This is the inverse at the final smooth torus used for the earlier history replacements, where constants may depend on that prefix.

For a balanced split $v + w = h$ write $z = |v|^2/|h|^2$. Orthogonality gives $|w|^2/|h|^2 = 1 - z$, so the ratio of the sum of the two homogeneous squared H^s weights to the parent weight is $z^s + (1-z)^s \le c_s < 1$, uniformly on the allowed interval of z. Summing by levels and using the level invariant gives a bound independent of the depth for the full primary contribution and $O(c_s^{d/2})$ for its norm at launch. Given a tolerance, first choose d to make this launch cost small. At that depth choose the auxiliary amplitudes, the axial preparation and the finite comparison accuracy, and then take R large. In H^s the physical reconstruction of the normalized error $O(\rho)$ costs $O(rN^s\rho) = O(\rho)$. In H^2 the prescribed unit-height packet costs $O_{\mathrm{stage}}(rN^2) = O_{\mathrm{stage}}(N^{2-s})$ uniformly in time. Taking tolerances tending to zero thus gives raw insertions that converge initially in H^s to the old target, stay bounded in H^s at fixed new height, and converge uniformly on the half-line in H^2. Up to a uniform $o(1)$ in H^s their increment is the finite plane sum, by the preceding reconstruction and exact locking of all earlier occupied frequencies and origins. Apply Proposition 4.2 in its two-norm form with H^2 and H^s, saving compatible weights as in the endpoint argument. The product bound is tame with $H^2 \hookrightarrow L^\infty$; each saved correction decays at its original larger weight and converges strongly at the smaller weight. Divisibility of the vertical spacings puts every new input on the old lattice. Each old trajectory has precompact H^s range, so its cross term with a finite new sum supported beyond $R/2$ tends uniformly to zero. These preparations give summable strong errors initially, summable weak errors and summable deviations of the squared increment from its finite plane sum as in the counting argument. Writing $D_j = K_j^s$, the new slow time tends uniformly to zero for $t \le D_j^{3/4}$ and to infinity for $t \ge D_j^{5/4}$ at fixed preparation. The middle windows can be taken disjoint, and $\ell(K_j)$ can absorb its fixed entrance time divided by λ. This proves the counting and slow-growth conclusions for this s after passage to one strongly convergent sequence of initial states. For each $p' < s$ uniform lower-norm boundedness follows by imposing summably small increments in a countable sequence of exponents increasing to s, using interpolation with the uniform H^2 smallness at each fixed height.

6.5. **The limiting axial solution and data near a plane wave.**

Corollary 6.1. *For every finite $s \geq 2$, the solution in Theorem 1.2 can be chosen with an axial solution v which is almost periodic in H^σ for every $1 < \sigma < s$, such that*

$$(1+t)^b \|u(t) - v(t)\|_{H^\sigma} \longrightarrow 0 \quad \text{for} \quad 0 \leq b < \frac{\min\{s-\sigma, 1\}}{s}.$$

For every j and every real t, $\langle K_j \rangle^s |\widehat{v}((K_j, 0), t)| \geq 1/2$. In particular $v(t) \notin H^s$ at every time.

Proof. Use unit height and put $N = K - 1$, so $r = N^{-s}$. For any finite list of pairs (σ, b) as in the assertion choose $a \in (1/2, 1)$ with $a > sb$ for every pair on the list. At the endpoint the ball estimate also admits $\rho = N^{-a}$: at fixed tree $N = PR$, and both e_R/ρ and $\sqrt{R}\,\rho$ tend to zero. In Section 6 the same choice absorbs the indicated powers of $\log N$. Take three ordered weights in one gap, with the largest one used for the new solution and the two smaller ones reserved for its history. For the estimate as N increases one can use the same smaller positive decay rate on a subsequence. Denote the actual raw solution by W and its exact rounded axial target by V'. For each fixed preparation there are constants $C, c, T > 0$ such that

$$\|W(t) - V'(t)\|_{H^\sigma} \leq C(1 + \log N)^C (N^{-a} + N^{\sigma - s}) \exp[-c(r\lambda t - T)_+].$$

Indeed the change in the finitely many old angles is bounded by the radius of the correction, and that in the old actions by r times this radius. All other coordinates of the difference have scaled norm at most Cr on the early interval, with exponential decay thereafter. This uses the exact finite trajectory and its own endpoint. Its fixed accuracy relative to the ideal trajectory creates no stationary source. Converting a scaled field to H^σ costs at most CN^σ. The old axial coordinates range over a compact smooth torus. For the inverse chart its nonlinear part is locally Lipschitz from h^{m-1} to H^m with $m = \lceil \sigma \rceil \geq 2$, and the added high coordinates are small in h^{m-1}. These bounds prove the displayed inequality in the mass gauge of the new solution.

Multiplication by $(1+t)^b$ costs at most $C_{b,c,T}(r\lambda)^{-b}$, with λ fixed before the dilation. Both resulting powers $sb - a$ and $sb + \sigma - s$ are negative. Every previously saved correction converges in its smaller exponential weight on a late half-line and strongly on the preceding compact interval. Its change can therefore also be made arbitrarily small in the polynomially weighted norm. Impose summable errors on successively larger finite lists of rational pairs as in Corollary 4.3. The same choice gives summable uniform changes of the axial targets in every H^σ on the real line. Indeed the old angular velocities and origins agree in their respective mass gauges, the new clock cost is arbitrarily small, and the new axial roots cost $O_{\text{stage}}(N^{\sigma - s})$ there. Each target is a continuous function on a finite angle torus evaluated along a linear flow. Approximation by trigonometric polynomials on that torus gives almost periodicity of the uniform limit. The identification of the limit and removal of the common mass gauge are as in Corollary 4.3. For a fixed finite target, there are arbitrarily large translations returning all its angles arbitrarily close to their initial values. They carry the persistent lower bound for each previously selected Fourier coefficient back to any prescribed real time. Passing to the limit in that coefficient gives the last assertion. □

Corollary 6.2. *Fix $2 \leq s < \infty$. Every $H^s(\mathbb{T}^2)$ neighborhood of a plane wave $ae^{ik\cdot x}$, $a \in \mathbb{C}$, $k \in \mathbb{Z}^2$, contains a datum whose solution satisfies $\|u(t)\|_{H^s} \to \infty$.*

Proof. For a small nonzero amplitude start with a single axial Birkhoff coordinate at index one. Translation and constant-phase equivariance force its physical inverse to be a single Fourier wave, and preservation of mass gives its modulus. Open a second sufficiently small positive action, with phase origins chosen in the prescribed neighborhood. For a two-site torus of this kind $A > 0$. Indeed the Liouville identity gives A as the angle mean of the spatial variance of the

density. If it vanished, every orbit on this torus would have spatially constant positive density at all times. The mass continuity equation would give a spatially constant phase derivative and hence a single plane wave, which has only one occupied Birkhoff coordinate. Since $D\chi$ is close to minus the identity, the neighborhood contains a positive two-site open frequency box. At $s = 2$ choose rational slopes with numerators congruent to one modulo five and one common denominator; for $s > 2$ choose two rationally independent algebraic slopes. The allowed lists are dense at either exponent. Choose them for one small positive value of the common factor and then vary that factor in an open interval within the box to avoid the countable exceptional sets of the construction. Every subsequent initial increment can be made summably small in this neighborhood. A constant phase and a Galilean transformation give the same conclusion near a small constant.

For general $a \neq 0$ choose an integer L with a/L in this small range. If v solves (1.1), so does $Lv(L^2t, Lx)$, and its datum is arbitrarily close to a when $v(0)$ is sufficiently close to a/L. Its H^s norm is at least $L\|v(L^2t)\|_{H^s}$ by the Fourier formula. Galilean modulation by the fixed k preserves divergence and continuity in H^s. The case $a = 0$ is Theorem 1.1. □

7. Two-sided growth for $s > 2$

Fix $s > 2$. At stage j we use three solutions of the equation in the autonomous mass gauge. The axial solution V_{j-1} is the starting background. The two-dimensional solution W_j stays close to V_{j-1} in H^2 and approaches a new axial solution V_j in H^s. Comparison with V_{j-1} controls the change at previously occupied frequencies. Comparison with V_j detects the new Fourier coefficient at late times. The proposition states the estimates needed on this one solution. Write

$$\|g\|_{q,\beta} = \sup_{t\geq 0} e^{\beta t}\|g(t)\|_{H^q}.$$

For an inhomogeneous Green operator we use the induced norm from forcing to solution. Constants may depend on s.

Proposition 7.1. *Fix $s > 2$. There are scales $B_j = B_*Q^j$, $d_j = \lfloor \alpha\sqrt{B_j/\log B_j}\rfloor$, axial solutions V_j and complete two-dimensional solutions W_j, for $j \geq 1$, starting from a finite axial V_0. There are fixed constants $b, C, C_0 > 0$ and frequencies $k_j = (1 + N_j, 0)$, $B_j + O(1) \leq \log N_j \leq 2B_j$, with the following estimates.*

1. *The two-dimensional orbit lands at V_j with $\|W_j - V_j\|_{q,\beta_j} \leq e^{C_0B_j}$ for $q = 2, s$ and $\beta_j \geq e^{-C_0B_j}$. On the same invariant lattice and time half-line the linearized physical equation at W_j has a right inverse with zero prescribed stable trace, of norm at most $e^{C_0B_j}$ in both of these spaces, with compatible inverses.*
2. *For $T_j = e^{(C_0+1)B_j}$ the mode k_j satisfies $\langle k_j\rangle^s|\widehat{W}_j(k_j, t)| \geq 0.9e^{bd_j}$ at every $t \geq T_j$. At $S_j = e^{L_0B_{j-1}}$, the entire H^s Fourier tail at $\langle k\rangle \geq K_j = e^{L_1B_{j-1}}$ costs at most $Ce^{-d_j/C}$. One has $T_{j-1} \leq S_j < T_j$ and $\sup_{t\geq 0}\|W_j(t)\|_{H^s} \leq Ce^{Cd_j}$.*
3. *$\sup_{t\geq 0}\|W_j(t) - V_{j-1}(t)\|_{H^2} \leq e^{-D_hB_{j-1}}$. The constants $L_0 > C_0 + 2$, $L_1 > 2L_0 + 2$, and the required historical precision D_h may be selected successively after a coarse C_0 and before Q, α, B_*. In particular D_h can pay every price of the earlier stages in the weak history comparison.*

Sections 7.3–7.12 construct the trajectories in the proposition. To pass to one initial datum, we will use the linearized inverse at W_j to restore the earlier two-dimensional transfers. The next two propositions explain how to control the resulting solution at times between transfers.

7.1. **High-frequency energy of an exact trajectory.** The full flow used here is global in H^s for $s > 1$. Local existence, uniqueness and continuous dependence follow by applying the contraction mapping theorem to Duhamel's formula in the algebra $H^s(\mathbb{T}^2)$. The conserved mass and positive energy bound $\|u\|_{H^1}$ by a constant $M \geq 1$. Splitting the Fourier sum at radius $K \geq 2$ gives $\|\widehat{u}\|_1 \leq C_s(M\sqrt{\log(2+K)} + K^{1-s}\|u\|_{H^s})$ by Cauchy–Schwarz on both pieces. Taking $K = 2 + (1 + \|u\|_{H^s}/M)^{1/(s-1)}$, the high-frequency term is at most M. In the interaction representation the cubic convolution has h^s norm at most $C_s\|\widehat{u}\|_1^2\|u\|_{H^s}$. Squaring the preceding ℓ^1 bound therefore gives, for $Y = 2 + \|u\|_{H^s}$, $|Y'| \leq C_{s,M} Y \log Y$ almost everywhere. This prevents a finite lifespan endpoint in either time direction. Mass, energy and the inequality follow first for smooth solutions and then by approximation on each local existence interval. The scalar mass gauge preserves the norms and has the same continuous dependence. This argument also supplies the Cauchy theory used in the preceding sections.

Fix a smooth nondecreasing cutoff $0 \leq \chi \leq 1$, identically zero on $[0,1]$ and identically one on $[2,\infty)$. Put

$$f_N(k) = \chi(\langle k\rangle/N)\langle k\rangle^{2s}, \qquad E_N(t) = \sum_{k\in\mathbb{Z}^2} f_N(k)|\widehat{u}(k,t)|^2.$$

Proposition 7.2. *Let $s \geq \sigma > 1$, $\rho = 1 - 1/\sigma$, and let $u \in C(I; H^s)$ solve (1.1) on a compact interval I. If $\sup_I \|u\|_{H^\sigma} \leq M$, then for sufficiently large N,*

$$|E_N(t) - E_N(t')| \leq C_{s,\sigma,M}(N^{-2} + |t-t'|N^{-\rho}) \sup_{I_{t,t'}} \|u\|_{H^s}^2, \qquad t, t' \in I.$$

Here $I_{t,t'}$ is the closed interval between the indicated endpoints.

Proof. Write $a_k = e^{it|k|^2}\widehat{u}(k,t)$. Symmetrization writes E_N' as a fixed real multiple of

$$\operatorname{Im} \sum_{k_1-k_2+k_3-k_4=0} [f_N(k_1) - f_N(k_2) + f_N(k_3) - f_N(k_4)] e^{-it\Phi} a_{k_1}\overline{a}_{k_2}a_{k_3}\overline{a}_{k_4}$$

with $\Phi = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2$. In a nonzero term let K be the largest of the four brackets $\langle k_i\rangle$. Then $K \geq N$, and momentum conservation makes the second-largest bracket at least cK. The multiplier in square brackets thus costs at most the product of the two largest brackets raised to s. With these two factors in ℓ^2 and the others in ℓ^1, the quartic sum is bounded by $C_s\|a\|_{h^s}^2\|a\|_1^2$. We gain a power of N in one of the following three ways.

Put $L = N^{1/\sigma}$. If a third bracket is at least L, one of the ℓ^1 factors is bounded by

$$\sum_{\langle k\rangle\geq L} |a_k| \leq \Big(\sum_{\langle k\rangle\geq L} \langle k\rangle^{-2\sigma}\Big)^{1/2} \|a\|_{h^\sigma} \leq C_\sigma M L^{1-\sigma} = C_\sigma M N^{-\rho}.$$

If both smaller brackets are below L, their multipliers vanish. Suppose first that the two large entries have opposite signs in Φ. Their momenta differ by at most $2L$. Since $|\nabla f_N(x)| \leq C_s\langle x\rangle^{2s-1}$, the mean value theorem bounds the remaining difference of multipliers by $C_s K^{2s-1}L$. Relative to the preceding quartic bound this gains $L/K \leq N^{-\rho}$.

It remains to consider two large entries of the same sign and two small entries below L. Here $|\Phi| \geq cK^2$ for N sufficiently large, since $L = o(N)$. Integrate $e^{-it\Phi}$ by parts in time on $I_{t,t'}$. The factor $1/\Phi$ gives $O(N^{-2})$ at each endpoint. In the integral the derivative falls on one of the four amplitudes. The cubic equation gives

$$\|\dot{a}\|_{\ell^1} \leq C_\sigma M^3, \qquad \|\dot{a}\|_{h^s} \leq C_{s,\sigma} M^2 \|u\|_{H^s}.$$

Use the first bound when the differentiated factor is one of the two ℓ^1 factors and the second for either weighted ℓ^2 factor. The integral from this case costs $C_{s,\sigma,M}|t-t'|N^{-2}\sup_{I_{t,t'}}\|u\|_{H^s}^2$.

Combining it with the endpoint terms and the first two cases gives the assertion. Approximation by smooth solutions on this compact time interval justifies the integration by parts. □

7.2. **Passage to one initial datum.** For the remaining results $s > 2$. Write $B_j = B_* Q^j$ with $Q \geq 2$, $B_* > e$, and $d_j = \lfloor \alpha \sqrt{B_j / \log B_j} \rfloor$ for fixed $\alpha > 0$. Choosing scalar phases at time zero, where appropriate, is equivalent to selecting representatives of the gauge-invariant estimates.

Proposition 7.3. *Let (U_j) be global H^s solutions with $M = \sup_{j \geq 0,\, t \geq 0} \|U_j(t)\|_{H^2} < \infty$. Assume that U_0 is bounded in H^s for $t \geq 0$. Let $2K_{j-1} < K_j$ increase and assume*

$$\sup_{t \geq 0} \inf_{\theta} \left\| U_j(t) - e^{i\theta} U_{j-1}(t) \right\|_{H^2} \leq \delta_j,$$

$$E_{K_j}(U_j(S_j))^{1/2} \leq c_j, \qquad \sup_{t \geq S_j} \|U_j(t)\|_{H^s} \leq C e^{A d_j}.$$

Here $\ell_j = (2K_j)^{s-2} \delta_j$, c_j and $\sqrt{\omega_j}\, e^{A' d_{j-1}}$ are summable for some $A' > A$, with $\omega_j = C_{s,2,M}(K_j^{-2} + S_j K_j^{-1/2})$. Let T_j increase with $\log T_j \asymp B_j$ and $T_{j-1} \leq S_j$. Assume also that for distinct frequencies k_j, some $a > 0$ and every $t \geq T_j$,

$$\langle k_j \rangle^s |\widehat{U}_j(k_j, t)| \geq e^{a d_j}, \qquad \langle k_j \rangle^{s-2} \sum_{i > j} \delta_i = o(e^{a d_j}).$$

Then, after alignment at time zero, $U_j(0)$ converge strongly in H^s to one datum, and its trajectory satisfies the two-sided estimate (1.3) for every sufficiently large time.

Proof. Set $M_j = \sup_{t \geq 0} \|U_j(t)\|_{H^s}$ and $M_j^e = \sup_{0 \leq t \leq S_j} \|U_j(t)\|_{H^s}$. The early supremum is finite by continuity and the late supremum by hypothesis. For $0 \leq t \leq S_j$, the multiplier $(1 - \chi(\langle k \rangle / K_j))^{1/2}$ is supported in $\langle k \rangle < 2K_j$. After optimizing over the scalar phase, the H^s norm of this part of $U_j(t)$ is at most $\|U_{j-1}(t)\|_{H^s} + \ell_j$. Proposition 7.2 applied between t and S_j bounds the complementary squared norm by $c_j^2 + \omega_j (M_j^e)^2$. These two squared norms add to $\|U_j(t)\|_{H^s}^2$. Hence

$$\|U_j(t)\|_{H^s}^2 \leq (\|U_{j-1}(t)\|_{H^s} + \ell_j)^2 + c_j^2 + \omega_j (M_j^e)^2. \tag{7.1}$$

Take the early supremum and absorb the last term for large j. Together with the late bound this gives

$$M_j \leq \max\left\{ C e^{A d_j}, (1 - \omega_j)^{-1/2} (M_{j-1} + \ell_j + c_j) \right\}.$$

The product of the factors $(1 - \omega_j)^{-1/2}$ converges after a finite prefix. Since $\ell_j + c_j$ is summable and d_j is eventually nondecreasing, iteration gives $M_j \leq C' e^{A' d_j}$. Applying the early bound once more gives $M_j^e \leq C''(1 + e^{A' d_{j-1}})$. In particular

$$\gamma_j := c_j + \sqrt{\omega_j}\, M_j^e \qquad \text{satisfies} \qquad \sum_j \gamma_j < \infty.$$

At time zero, the H^s tail of U_j on $\langle k \rangle \geq 2K_j$ has norm at most γ_j, since $\chi = 1$ there. The corresponding tail of U_{j-1} has norm at most γ_{j-1} by the same argument at its own, smaller cutoff. Choose the initial phases recursively to realize the H^2 comparison at time zero. On $\langle k \rangle < 2K_j$ the aligned difference then has H^s norm at most ℓ_j. The full distance between successive aligned data is therefore at most $\ell_j + \gamma_j + \gamma_{j-1}$ for large j. It follows that the data converge strongly in H^s to one datum. Phase covariance and finite-time continuous dependence give convergence of the aligned solutions to its solution u at each fixed time. The proof applies in the mass gauge as well, since all norm and Fourier-modulus estimates are invariant under scalar phases.

Fix any t with $T_j \le t < T_{j+1}$. For any phase θ the difference of two Fourier moduli is bounded by the modulus of the corresponding coefficient difference after multiplication by $e^{i\theta}$. Summing the H^2 comparisons over future stages and passing to the limit at this t gives

$$\langle k_j\rangle^s |\widehat{u}(k_j,t)| \ge e^{ad_j} - \langle k_j\rangle^{s-2} \sum_{i>j} \delta_i \ge \tfrac{1}{2} e^{ad_j}$$

for sufficiently large j. For the upper bound we can allow the whole contribution M_{j+1} of stage $j+1$. Every subsequent index $i \ge j+2$ satisfies $t < T_{j+1} \le T_{i-1} \le S_i$. Taking square roots in (7.1) at this same time and summing from $i = j+2$ yields

$$\|u(t)\|_{H^s} \le M_{j+1} + \sum_{i \ge j+2} (\ell_i + \gamma_i) \le C e^{A' d_{j+1}} + O(1).$$

Throughout the interval $\log t \asymp B_j$ and $\log\log t = \log B_j + O(1)$. Since $B_{j+1} = QB_j$ with Q fixed, both d_j and d_{j+1} are comparable to $\sqrt{\log t / \log\log t}$. The two estimates give the conclusion. □

7.3. **A family of right triangles with shared children.** Identify $\mathbb{R}^2$ with $\mathbb{C}$, with horizontal and vertical coordinates x, y. In this section the absolute square is the sum of the squares of the two coordinates. Its complexification is a rational function, and does not involve complex conjugation of the parameters. Put

$$P = (1-im)^{-1}, \quad Q = 1 - P = -imP, \quad u = (1+it)^{-1}, \quad g_{nk} = u^n (it)^k, \quad 0 \le k \le n < d.$$

Use 1 as top and Pg_{nk}, Qg_{nk} as the primary vertices. A primary V at an internal level $0 \le n \le d-2$ has children $uV, (1-u)V$. Every primary at $1 \le n \le d-1$ also has a private companion and an axial vertex

$$J(V) = A(V) - V, \qquad A(V) = |V|^2 / V_x.$$

The designated triples are the top and its children, all primary parents and their children, and the triples $(V, J(V), A(V))$. Every triple has nonzero perpendicular inputs summing to its output. At real parameter values outside finitely many algebraic sets, all vertices are distinct. Distinct primary directions at different levels differ by a nonconstant integer multiple of $\arctan t$ modulo $\pi/2$. Within a level, length ratios between parallel primaries are signed powers of t or such powers times m. The companion of a vector of angle θ is $-i\tan(\theta)V$. Its moving pole cannot identically equal the scale of a primary. If two such companions at the same level, or two axes, have the same pole, their scale ratio is one of the preceding primary scale ratios. Axes and nonaxes are distinct by their vertical components. The two top inputs have no private triples.

Write $D(v,w)$ for the pair of rational functions obtained by taking the signed sums (inputs positive, output negative) of $(x^{-2}, |v|^{-2})$ on $v, w, v+w$. For positive constants A_0, B_0 the intended scalar defect is $A_0 D_1 - B_0 D_2$. In the next lemma polynomial nonidentity is asserted before the slopes are specialized.

Lemma 7.4. *At fixed depth d, the designated component pairs $D_e \in \mathbb{R}(m,t)^2$ are linearly independent over $\mathbb{R}$. Spans and identities of pairs are taken before specialization. For a fixed finite list of tests,* generically *means at admissible real (m,t) outside the poles and the zero sets of the nonzero numerators in those tests. A nontrivial reverse rectangle test with an output and one input marked yields a designated triple whenever it holds identically. Every extra perpendicular pair of marked inputs has a component pair different from every constant linear combination of the designated pairs. For one marked output h and one designated pair D_e, a perpendicular split of h with this same component pair (the inputs may be unmarked) is generically impossible unless h is the output of e. For its own output the inputs are determined up to exchange and, for a horizontal output, reflection across the horizontal line.*

Proof. For the private row at $v = (x, y)$, $S = x^2 + y^2$, the two defects are

$$x^{-2} + x^2/y^4 - x^2/S^2, \qquad y^{-2} - x^2/S^2.$$

At the real moving pole $y = 0$ the first has order four. Fix a level and direction, factoring out its common defect. The coefficient from the two top subtrees is $R(t) + m^{-2}S_1(t)$, with distinct Laurent monomials in t within each polynomial. At generic fixed t all other private rows, as well as all primary rows, have pole order at most two there. Thus both the value and the first m-derivative of this coefficient vanish at a nonzero finite $m_0(t)$, implying $R = S_1 = 0$. Varying t gives zero for every coefficient. After eliminating all private rows, the vertical orientations at the deepest remaining primary layer are double poles of its first component, with all other directions regular. The same argument imposes vanishing of every individual formal vertex coefficient. The children of parent coefficients c_j have coefficients $c_j + c_{j-1}$, with $c_{-1} = 0$. Thus every parent coefficient vanishes. Descending leaves only the nonzero top row.

The reverse test for a nonaxis output h and nonaxis input v is $\mathrm{Re}(h/v) = 1$. At different levels the angle difference depends only on t. A companion contributes a factor $T_n = \tan(\arg(Pu^n))$ or T_n^{-1}, with a distinct real moving pole or zero at each strictly positive level. It cannot cancel with a different level or a power of m. Pure primaries in different top branches also leave a power of m. Within a branch the remaining ratio is $(it)^a(1 + it)^b$. If $b < 0$, a nonzero constant real part would require $a \le 0$ at zero and $a \ge -b$ at infinity. If $b \ge 0$, expanding the binomial, the constant real part one occurs only for $(b, a) = (0, 0), (1, 0), (1, -1)$: for $b = 2$ the sole odd term has coefficient two, and for $b \ge 3$ each parity has at least two nonzero terms. These give coincidence or the two assigned children. Within a single level the directions are parallel or perpendicular, giving no additional reverse. An axial output must be $A(v) = A(J(v))$. An axial input would have to equal the horizontal coordinate of the nonaxis output; the latter has a pole at $m = i$ or $-i$ from its primary part, whereas the axes have only real poles. For P, Q the axial output is 1.

Two marked nonaxis inputs can be generically perpendicular only at the same level. In the basis $G = Pu^n, iG$ their signed relative scale is

$$L = \pm t^{l-k} m^{\gamma} T_n^{\delta}, \qquad \gamma = 0, 1, -1, \quad \delta = 0, 1, 2.$$

Indeed a companion on the first basis direction has a factor T_n^{-1} and a companion on the second direction a factor T_n. If L depends on m, any preimage $L = \pm i$ away from $m = \pm i$ is a new pole of the radial component of the output. All individual lengths and abscissae are regular and nonzero at that preimage: their exceptional points are real or $\pm i$. The output is a nonzero isotropic vector, whose horizontal coordinate cannot vanish. For $l \ne k$ such a preimage exists since T_n fixes $\pm i$ and t^{l-k} is nonconstant. For $l = k$ and degree at least two, the possible preimages at $\pm i$ are simple at generic t; the logarithmic derivative there is proportional to $\gamma + \delta T_n'(i)$, which cannot vanish identically at $n \ge 1$ unless L is constant. Every nonconstant factor has its zeros and poles on the real projective line. At $l = k$ the degree-one possibilities are a primary with its own companion or Pg_{nk}, Qg_{nk}: the four vectors in unit-G coordinates are two primaries on opposite axes and their two perpendicular companions. Opposite branch with a single companion is parallel at equal k, and the two compensating companions involve T_n^2 and a power of m. For $Pg_{nk} + Qg_{nk} = g_{nk}$ the output is independent of m. The fourth- and second-order tests force any representation to have precisely the formal primary incidences of the two inputs at all levels $n \ge 1$. Project a node from each top branch to its leaf polynomial $X^{n-k}Y^k(X + Y)^{d-1-n}$. Every internal incidence maps to zero, and the top incidence maps to a scalar multiple of $(X + Y)^{d-1}$ in each branch. The singled-out node has a different polynomial for $n \ge 1$. In the m-independent case the inputs are pure same-branch primaries, with an odd difference $j = |l - k|$. The case $j = 1$ is designated. For $j \ge 3$ the direction of the output involves

$\arg(1 \pm it^j)$ besides the level direction. Its expansion at zero excludes any equality modulo $\pi/2$ to an integer multiple of $\arctan t$. This gives a new moving horizontal reciprocal-square pole.

For the last assertion, let U, V denote a proposed pair, h its output, $z = h_y/h_x$, $g = (1 + |h|^2 V)^{-1}$, and $p = k_x l_x / h_x^2$, where $k + l = h$, $k \perp l$. The two equations to be satisfied are

$$(p - (1 - z^2)g)^2 = z^2 g(1 - 4g), \qquad (1 + h_x^2 U)p^2 + 2p - 1 = 0. \tag{7.2}$$

To derive the equations, write $a = |k|^2/|h|^2$. Orthogonality gives $|l|^2 = (1 - a)|h|^2$ and hence $1+|h|^2 V = 1/a+1/(1-a) = 1/[a(1-a)]$. Let $x = k_x/h_x$, so that $p = x(1-x)$. The decomposition of k along h and its perpendicular gives $x = a \pm z\sqrt{g}$. Thus $p = (1 - z^2)g \pm z\sqrt{g}\,(1 - 2a)$, with $(1 - 2a)^2 = 1 - 4g$. The second equation follows from $1 + h_x^2 U = x^{-2} + (1 - x)^{-2} = (1 - 2p)/p^2$. For $z \neq 0$ and $g > 1/4$ the first equation has no real root. When $2 + h_x^2 U > 0$ the roots of the second are real (its discriminant is $4(2 + h_x^2 U)$, and the linear case is immediate). There is then no common complex root. For $z = 0$ the test is

$$|h|^4 V^2 - |h|^2 U - 2 = 0.$$

One may use a real orientation to disprove the identically vanishing complex resultant. For $0 < g \leq 1/4$ all roots of the first equation are real and arise from the two possible reflected splits of the prescribed length fractions.

For opposite top branches let the branch of h shrink by varying m, with the target bounded at generic t. This sends h and $h_x^2 U$ to zero. Against the top row itself each strict descendant shrinks as $t \to 0$ for $k > 0$, and as $t \to \infty$ for $k = 0$, at generic fixed m; the limiting direction at finite d stays away from the coordinate axes. For a top child against the top row, shrinking that child gives $|h|^2 V \to 1$ and $U > 0$. These are exclusions by the preceding equations.

Consider a primary target in the same branch. Divide the lengths by the common branch radius so the configuration rotates as m varies. For a primary h, both $|h|$ and V are now constant. If $g > 1/4$ approach a vertical target child to make $U > 0$. Otherwise the finitely many relative splitting angles are fixed. The horizontal defect of each such split has three distinct double poles with leading signs $+, +, -$ under rotation. Its unique negative pole gives the output direction and, from its coefficient, the output length. This requires h to be the target parent up to sign, which is absent for distinct marked primaries at generic parameters. For $h = J(w)$ rotate w to horizontal, so h_x vanishes quadratically, $h \to 0$, V stays bounded and U has at most a double pole. For axial $h = A(w)$ try the two perpendicular target-child vertical orientations. At one of them $|h|$ stays finite and nonzero, V stays bounded, and $U \to +\infty$, violating the horizontal test. This also covers $h = 1$: its normalized value can diverge at at most one of the two orientations.

For a private target based at v, rotate the normalized branch to make v horizontal. Now $V \to \infty$ and $U/V \to \infty$. A primary h at a different level has limiting direction away from both axes for generic t. Each split with the required V has a short leg pointing asymptotically perpendicular to h; its $U/V \to |h|^2/h_y^2$. At the same level, if $k' \neq k$, shrink $|h|/|v| = |t|^{k'-k}$ while holding a generic direction of v fixed. Then $|h|^2 V \to 0$ and $h_x^2 U$ stays bounded and nonnegative. For the same primary, rotate v to vertical to obtain $|h|^2 V \to 1$ and $U > 0$. If $h = J(w)$ make w horizontal; unless w is parallel to v, V stays finite, $U > 0$, and $h \to 0$. Parallel companions have the same power-of-t length ratio just used, or, at the identical node, $|J(v)|^2 V \to 1$ when v becomes horizontal. The horizontal test has just one positive solution for $|h|^2$ when $V > 0$, and it is already $|A(v)|^2$. The axis magnitudes are generically distinct. Finally if a marked nonhorizontal output is the correct one, the two p values for a reflected split can both solve the second equation only if the two candidates coincide or $g = (1 + z^2)^{-2}$: compare the sum and product of the roots using (7.2). The latter equality can be excluded by rotation. For a horizontal output reflection is a real ambiguity, treated by the lattice congruence below. □

Lemma 7.5. *Fix a word cutoff $L = B^{k_0}$ with k_0 fixed, and an integer m_* independent of B. The parameters can be chosen with $1/2 < t < 3/4$, m in a fixed compact positive interval, so that all nonidentities of Lemma 7.4 needed for combinations of at most m_* rows, including repetitions and signs, hold. All primary directions satisfy $|\sin\theta|, |\cos\theta| \geq c/d$. There is a common integer denominator P_{lat} with*

$$\log P_{\mathrm{lat}} \leq C_{k_0,m_*} d^2 \log(2+B) \quad (2 \leq d \leq B).$$

For every sum of at most L signed marked macro vertices, with repetitions allowed, a zero horizontal component implies a zero vertical component. On clearing denominators, all nonaxial heads have nonzero residues on one isotropic line in $\mathbb{F}_5^2$ and all axial heads are zero there.

Proof. The bounded arity tests in Lemma 7.4 use at most $d^{C_{m_*}}$ polynomials of degree at most $C_{m_*} d^C$. For an external pair use the resultant in (7.2) with the horizontal test for $z = 0$; for its own output test the remaining reflection condition. Thus there is no enumeration of external integer points before specializing. For each nonzero component comparison choose just one nonzero polynomial, and include the finitely many nonzero denominators and the determinants distinguishing the $2(d-1)$ moving directions. Choose $t = p/q$, q a prime greater than a sufficiently large polynomial in d, L, the number and the degree of these tests, yet bounded by twice that polynomial. A nonzero bivariate polynomial of degree D specializes identically in m at at most D real values of t. There are therefore enough even $p \equiv q \pmod 5$ in the stated interval, with $p > 10L$. Write S for the sum of $2, d, L$ and the test counts and degrees; $\log q = O(\log S)$.

Write $a_n + ib_n = (q - ip)^n$. Its two integer components are coprime: a common rational prime would divide this Gaussian integer and its conjugate, whereas $q - ip$ and $q + ip$ are coprime in $\mathbb{Z}[i]$ by opposite parity and $\gcd(p, q) = 1$. At a moving axis the denominator in $m = k/M$ is of the form $D_i = A_i M + B_i k$, where (A_i, B_i) is one of the primitive signed permutations of (a_n, b_n); denominators proportional up to sign are counted once. To see the formulas, $A(Pu^n) = q^n/(a_n - mb_n)$ and $A(iPu^n) = -q^n/(b_n + ma_n)$, since $\Re Pu^n$ and $\Im Pu^n$ are the corresponding numerators divided by $(1+m^2)(p^2+q^2)^n/q^n$. Magnitudes from the other branch supply m, and a real dilation rescales A by the same real number. Impose $A_i B_i \neq 0$ by the polynomial exclusions. Let R_0 be the product of $10pq(p^2+q^2)$, all nonzero $|A_i|, |B_i|$, and absolute values of all nonzero pair determinants. Its logarithm is $O(d^3 \log q)$.

Take M prime with $\log M = C_0 d \log S + O(1)$, larger than each individual factor of R_0. Restrict k/M to an interval of width $\gtrsim 1/d$ where all $2d$ direction projections are bounded below by c/d of the vector length. Such an interval exists inside a fixed compact interval of rotations by deleting angular neighborhoods of total length less than half that interval and counting components. Choose a residue of k modulo 25 so that $v_5(1 + (k/M)^2) = 1$. There are $N_0 \gtrsim M/d$ candidates. In this interval $|D_i| \gtrsim M/d^C$ for every i.

For each fixed i, the expected logarithm of the part of D_i supported on primes dividing R_0 is $O(\log S)$. For $\pi \mid B_i$ the contribution is zero. Otherwise at each prime power π^h a single congruence is needed. Its frequency is $\leq \pi^{-h} + O(1/N_0)$, allowing a factor 25 for $\pi = 5$ and its first two powers without restriction. The summed endpoint error is $O[(\log R_0)\log(\max_i |D_i|)/N_0] = o(1)$. Finally $\sum_{\pi \mid R_0} \log\pi/(\pi-1) \lesssim 1 + \log\log(3R_0)$; split at $\log^2(3R_0)$, use $\sum_{\pi \leq x} \log\pi = O(x)$ below and the divisor product above. By Markov's inequality, increasing C_0, a positive fraction of k have summed bad logarithm over all i below $\log M/3$. Omit the boundedly many multiples of the prime M inside the rotation interval, so $\gcd(k, M) = 1$. At most polynomially many remaining k violate a polynomial test. Hence every selected D_i has a factor $h_i > M^{1/2}$ coprime to R_0, all the other D_j, M and k; use the pairwise determinants and primitivity. It is also coprime to $M^2 + k^2$, because a prime common to both would divide $A_i^2 + B_i^2 = (p^2+q^2)^n$.

Rewrite every companion as an axis minus a primary. The combined axis part at D_i of any word of original length L is $(Mr_i + ks_i)/D_i$ with integer $|r_i| + |s_i| \le CL(2q)^d$: the numerators before summing are signed $p^j q^{n-j}$. The common primary denominators divide $(M^2+k^2)(q^2+p^2)^d$. If the horizontal word vanishes, clear denominators and reduce at h_i. This gives $h_i \mid B_i r_i - A_i s_i$, hence equality to zero since $h_i \gg L(2q)^{2d}$. Primitivity implies $r_i = c_i A_i$, $s_i = c_i B_i$, with integer c_i. The actual total $\sum |c_i| \lesssim dL$, since every physical real axis in the chosen angular interval has modulus $O(d)$.

Subtract the resulting integer constant C. The remaining primary part is $F(u)+P(G(u)-F(u))$ for integer polynomials of degree $\le d$, given as sums of at most $2L$ signed Bernstein monomials. The horizontal condition after multiplication by $1+m^2$ gives a real quadratic in m with constant term $C + \Re G$, linear term $-\Im(G - F)$ and quadratic term $C + \Re F$. Its coefficients become integers of size $\le CdL(2q)^{2d}$ after multiplying by $(q^2 + p^2)^d$. The rational root k/M forces all of these integers to vanish successively since M is greater than each. So $G(u) = F(u)$ and $\Re F(u) = -C$. Write $F(u) = a/(q + ip)^d$ with a a Gaussian integer. Then $i\Im F(u)(q + ip)^d$ is also a Gaussian integer. The coprimality of the two components of the power shows $\Im F(u) \in \mathbb{Z}$. Reduction modulo (p) gives $u = 1$ there and thus $p \mid \Im F(u)$. The original word bounds give $|\Im F(u)| \le 2L < p$, proving the claim.

Clearing both linear denominators at each level, as well as the common primary denominators, costs $O(d \log M + d^2 \log q)$ in their logarithm. At the prime five the numerator of P lies on an isotropic line and has nonzero coordinates, while the denominator has valuation one. u, itu are units at both Gaussian factors above five since $1 + t^2 \not\equiv 0$ there; the same holds for $-im$. Hence, after multiplying the whole configuration by five in the localized rational lattice, each primary belongs to the same nonzero line. Its squared length vanishes modulo five and its horizontal component is a unit, so A vanishes modulo five and $J = -V$ modulo five. Choosing a common integer denominator with exactly the needed five-adic valuation proves the lattice statement. □

For later scalar selection, on a lattice with vertical period spacing R take physical marked indices $e_1 + Rv$ where v is an integer vector from the cleared configuration. An exterior bare diameter-circle relation has

$$x^2 - Xx + q(q - Y) = 0, \qquad x = (n - 1)/R, \quad q \in \mathbb{Z}, \quad h = (X, Y) \in \mathbb{Z}^2.$$

The monic equation forces the rational x to be an integer. All such pairs have length $O(dP_{\mathrm{lat}})$. Thus for every remaining nonzero component comparison its rational height is at most $(2dP_{\mathrm{lat}})^{C_{m_*}}$, including all arbitrary exterior diameter-circle points. A reflection of a nonzero isotropic vector at a horizontal output exchanges the two different lines mod five. Only the chosen one belongs to the invariant translated coset $k_x - 1 = lk_y \bmod 5$, $l^2 = -1 \bmod 5$. Choosing $5 \nmid R$ preserves this condition. The further choice of an axial parameter separating the two-component defects in the single physical frequency is described in the next section.

7.4. **A finite trajectory landing at a specified axial point.** All estimates in this section use the ordinary Euclidean norm on the independent complex graph coordinates. The real sector introduced below serves only to find a reference orbit. The boundary inverse is for arbitrary complex perturbations satisfying the usual conjugate-copy reality.

Lemma 7.6. *Number the two top children at level* 1 *and the last rows at level* d*, with* $c_{nk} = \binom{n-1}{k}^{1/2}$*. Let* $D = C_* d \log(d + 2)$*,* $G_n = D^n$*,* $b = e^{-c_1 D}$*,* $\kappa = (2 + d + D)^{-10}$*. There are absolute* $c_1 > 0$*,* $C_* < \infty$ *for which the following holds for sufficiently large* d*. Give the top edge coefficient one, the edges below level* $n \ge 1$ *coefficients* $G_n c_{nk}/(c_{n+1,k} c_{n+1,k+1})$*, and the private triangles coefficients* $\kappa/(bc_{nk})$*. The Hamiltonian of an edge has the form* $ig(\bar{h}vw - h\bar{v}\bar{w})$*, with* h *its output; choose the symplectic convention giving* $\dot{h} = gvw$*. There is an orbit whose initial*

primary amplitudes vanish except on the last rows and whose endpoint has only axial amplitudes: the top equals one and each private axis equals bc_{nk}. It decays toward that point at rate $\kappa/2$ from time D. The propagator on any subinterval of $[0,D]$, in either direction, and the mixed boundary inverse on $[0,\infty)$ with any exponential rate $\kappa/8 \le \sigma \le \kappa/3$, have norms at most $\exp(CdD)$. The stable projection is prescribed at D and all its real quadratures are included.

Proof. On the real sector put $c_{nk}x_n, c_{nk}w_n, c_{nk}h_n$ on a primary, companion and private axis, respectively. The top is X. Both subtrees use the same scalar values. For the edge with parent (n,k) the product of the two children cancels the two denominator factors in its coefficient, leaving $G_n c_{nk} x_{n+1}^2$. When (n,k) is a child, its two possible parents contribute together

$$-\frac{G_{n-1}}{c_{nk}}(c_{n-1,k-1}^2 + c_{n-1,k}^2)x_{n-1}x_n = -G_{n-1}c_{nk}x_{n-1}x_n \qquad (n \ge 2),$$

by Pascal's identity, with nonexistent indices set to zero. At $n = 1$ the only parent is the top. Division by c_{nk} now gives the same equation for every k on a level. The sector is therefore invariant and satisfies

$$X' = x_1^2, \qquad x_n' = G_n x_{n+1}^2 - G_{n-1}x_{n-1}x_n + f_n,$$
$$f_n = -(\kappa/b)w_n h_n, \quad w_n' = -(\kappa/b)x_n h_n, \quad h_n' = (\kappa/b)x_n w_n.$$

Here $x_0 = X$, the last forward input is omitted and there are no private variables at $n = 1$. The identities $w_n^2 + h_n^2 = b^2$ and $X^2 + \sum x_n^2 + \sum_{n\ge 2} h_n^2 = 1 + (d-1)b^2$ hold when imposed initially. Start at $X = x_1 = \cdots = x_{d-1} = 0$ with arbitrary circle phases, and choose the unique positive x_d from this last constraint. The primary norm squared is $X^2 + \sum x_n^2 = 1 + \sum w_n^2$.

Set $T = D$, $a_0 = 1/10$, $Y_m^2 = \sum_{n=m}^d x_n^2$. On $[0,T]$ every $x_n \ge -2\kappa bT$: at a first exit, on a negative excursion the incoming square is nonnegative and a negative parent amplifies the negative part at rate at most $2G_{d-1}\kappa bT$. Integration gives $\kappa bT\exp(2G_{d-1}\kappa bT^2) < 2\kappa bT$. Here and below choose c_1 sufficiently small first and C_* large relative to $1/c_1$, so $G_{d-1}b$ times every fixed polynomial in d, D tends to zero. Similarly integrating the positive part gives $|x_j(t)| \le CG_j t + C\kappa bT$ for $j < d$, and $X(t) \le Ct$. Thus the energy of ancestors up to $m-2$, including X, on $t \le 3/G_{m-1}$ is $O(D^{-2} + d\kappa^2 b^2 T^2)$ for $m \ge 2$, by a geometric sum. Telescoping at any m gives

$$(Y_m^2)' = -2G_{m-1}x_{m-1}x_m^2 + 2\sum_{n=m}^d x_n f_n.$$

Consequently $Y_m' \le 2G_{d-1}\kappa bTY_m + \sqrt{d}\,\kappa b$. Where $x_{m-1} \ge c > 0$ and $Y_{m+1} \le E$ we have the stronger $Y_m' \le -cG_{m-1}(Y_m - E) + \sqrt{d}\,\kappa b$; for $Y_m < E$ use nonnegative dissipation directly. These inequalities hold in the upper Dini sense at zero.

Descend from $m = d$ to 2, with $Y_{m+1} \le E_{m+1}$ on $[a_0/G_{m-1}, T]$, starting at zero for $m = d$. On the shorter interval up to $3/G_{m-1}$ the small ancestors and the mass identity give $x_m^2 \ge 0.7$ while $x_{m-1} < 1/2$ if $E_{m+1} < 1/20$. Hence $x_{m-1}' \ge G_{m-1}/2$ there until the crossing; the same inequality prevents a downward crossing. From the first crossing until a_0/G_{m-2}, $x_{m-1}' \ge -2G_{m-2}x_{m-1} - \kappa b$ gives $x_{m-1} \ge 1/3$. Integrating the improved Y_m inequality from $3/G_{m-1}$ yields

$$Y_m(t) \le E_{m+1} + C\sqrt{d}\kappa b/G_{m-1} + 2e^{-G_{m-1}(t-3/G_{m-1})/3}.$$

At a_0/G_{m-2} the last term is $O(e^{-c_0 D})$ for an absolute $c_0 > 0$. Using the general upper derivative for the rest of the interval permits

$$E_m \le (1 + O(G_{d-1}\kappa bT^2))E_{m+1} + Ce^{-c_0 D} + C\sqrt{d}\,\kappa bT.$$

All $E_m < 1/20$, as the product of the d prefactors is bounded. For the last step Y_2 is small from a_0; while $X < 1/2$ one has $X' = x_1^2 \ge 0.7$. After time 3 the top remains above $1/2$. The

improved inequality on $[3, D]$ gives uniformly in every initial circle phase

$$Y_1(D) \le Cde^{-c_0 D} + Cd^{3/2}\kappa bD = o(b) \quad \text{if } c_1 < c_0/4.$$

These same estimates give

$$\sum_{n<d} \int_0^D G_n(|x_n| + |x_{n+1}|)\, dt \le CdD. \tag{7.3}$$

For x_n, before $3/G_{n-1}$ use the bounded primary amplitudes; on the next plateau integrate the exponential at cost $G_n/G_{n-1} = D$. The small constant terms during and after the plateau cost $O(G_{d-1}D[de^{-c_0 D} + d^{3/2}\kappa bD]) = o(1)$. For x_{n+1} use its own plateau and the smaller factor G_n. At $n = 1$ use the $[3, D]$ bound, and at $n = 0$ use $G_0 = 1$.

Near $X = 1, h_n = b$ put $\xi_n = x_n/b$, $\zeta_n = w_n/b$. The invariants give the positive branches $h_n/b = \sqrt{1 - \zeta_n^2}$ and $X = [1 + b^2(\sum \zeta_n^2 - \sum \xi_n^2)]^{1/2}$. The linear equations are $\dot\xi_1 = -\xi_1$ and $(\dot\xi_n, \dot\zeta_n) = -\kappa(\zeta_n, \xi_n)$ for $n \ge 2$. On a small Euclidean ball of radius δ the derivative of the nonlinear remainder costs $C[bG_{d-1}\delta + (\kappa + b^2)\delta^2]$, by the nearest-neighbor products and the two explicit square roots. Integrate the stable variables forward from the section and the unstable variables backward from infinity at weight $\kappa/2$; the kernel has L^1 norm $O(1/\kappa)$ and both projections have norm one. This gives a stable manifold which projects onto a fixed small ball of all the ξ's: the stable coordinates project isomorphically onto them and the derivative of the graph over the linear subspace is $O(\delta)$ by the same contraction. Write the corresponding angles as $F(\xi)$, continuously lifted about the positive-axis origins. If β is the vector of initial phases, the map taking β to the actual phases at D minus $F(x(D; \beta)/b)$ is homotopic to the identity of the torus. Indeed the first term is homotopic by the actual flow and the second has a continuous real lift. This degree-one map is onto. Choose a preimage of zero. This gives the claimed orbit and exponential decay at the specified time.

In the full nonsymmetric graph every vertex has bounded degree and each primary Hessian entry is bounded by $G_n|x_n|$ or $G_n|x_{n+1}|$, by the binomial ratios. In a private row the large $(\kappa/b)x_n$ multiplies the variations of partner and axis into one another in a skew-Hermitian block, including both complex quadratures; hold the primary fixed in the real edge Hamiltonian to see this. All its other entries cost $C\kappa$. The two-sided early bound follows from (7.3). At the endpoint the top daughters have hyperbolic rates ± 1, the private primary/companion pairs $\pm\kappa$ in each quadrature, and all axial coordinates are centers. The endpoint projections have bounded norm. On the tail the derivative differs from the constant endpoint generator by at most

$$C(G_{d-1}b + \kappa + b)(Y_1(D)/b)e^{-\kappa(t-D)/2} = o(\kappa)e^{-\kappa(t-D)/2}.$$

For example the private difference between axis and partner has its factor c_{nk} canceled by the inverse factor in its coefficient. The constant tail Green with stable data at D and its complement at the right endpoint has norm C/κ. Neumann inversion and then backward propagation on $[0, D]$ give the result, uniformly if the right endpoint is finite. □

Lemma 7.7. *The finite mixed inverse and landing persist, with a correction $e^{-C_2 dD}$ in ordinary Euclidean norm, under a perturbation of the finite field of size $e^{-C_3 dD}$ in C^1, and of the prescribed axial endpoint of size $e^{-C_3 dD}$. Here C_3 may be chosen after any sufficiently large C_2. The actual field in the carrier coordinates must vanish at its one specified perturbed endpoint. Time-dependent perturbations are allowed.*

Proof. The quadratic field F_0 has Hessian norm $L \le e^{CD}$ and vanishes on the axial plane. If z_* tends to a as above, then $\|z_* - a\|_\sigma \le e^{Cd}$ in the early-unweighted, late-exponential norm at any of the indicated weights. Translate z_* by the endpoint displacement Δ. The ideal residual is

$D^2F_0(z_*-a,\Delta)$. The remaining residual is estimated by comparing $F(t,z_*+\Delta)-F_0(z_*+\Delta)$ to its value zero at $a+\Delta$. Its weighted size is thus at most $e^{Cd}(e^{-C_3dD}+L|\Delta|)$. Use the mixed Green of Lemma 7.6 on a correction ball e^{-C_2dD}. Its product with $O(Le^{-C_2dD}+L|\Delta|+e^{-C_3dD})$ is less than $1/2$ in the stated order of choices. This also gives the perturbed variational inverse on the same boundary traces by a Neumann series. The estimate compares the actual field at the one prescribed endpoint; a different axial action generally has a nonzero phase velocity in this frame. □

At the cold end the actual H^s cost of the two ideal leaf layers for Fourier amplitude $r=HN^{-s}$ is at most $C_sH[(4/5)^s+(1/5)^s]^{(d-1)/2}$. At all times the primary cost is C_sH by the same binomial expansion on each level and the invariant sum of x_n^2. The actual physical frequencies are e_1+Nv in normalized graph units: for the primaries the relative error in the homogeneous weight is $O_s(C^d/N)$. The two subtree radii are bounded away from zero. All private frequencies have normalized modulus $O(d)$ by the real angular cutoff, and their entire strong cost is at most $C_sHd^{C_s}2^{d/2}e^{-c_1D}$. A finite Euclidean correction η has physical strong cost at most $C_sHd^{C_s}\eta$, without multiplying this bill by a previous-scale preparation price. With $H=e^{b_*d}$ and $0<b_*<-\log[(4/5)^s+(1/5)^s]/2$, the strong initial price is $O(e^{-c_sd})$ and the prescribed terminal top has height H.

7.5. **The marked axial Fourier coefficient.** Use $x\in\mathbb{R}/(2\pi\mathbb{Z})$, normalized Haar measure and $iu_t=-u_{xx}+|u|^2u$. Write Ψ for the inverse periodic defocusing NLS Birkhoff map, normalized so its linear part is inverse Fourier. Complexify the two conjugate Cartesian variables independently, with actions $I_j=z_j^+z_j^-$. A positive input has physical spatial factor e^{ijx} and a negative input e^{-ijx}.

7.5.1. *Normalizations and auxiliary Hamiltonians.* We use the normalized one-dimensional chart of Section 2. The hierarchy Hamiltonians depend only on the actions [24, Theorem 1.1(3)]. At every integer order $m\ge1$ each nonlinear chart remainder gains one derivative [39, Theorem 1.1 and Corollary 1.1]. In the frequency map of [46, Theorem 18.1 and Corollary 18.2] we label modes by their physical Fourier indices, as in (2.1); in particular $D\chi(0)=-\mathrm{Id}$. For the period-one spectral operator below put $a=2\pi/\sqrt{2}$.

We identify precisely the hierarchy normalization used below. Put

$$J=\tfrac{i}{2}\operatorname{diag}(1,-1),\qquad V=2^{-1/2}\begin{pmatrix}0&u\\v&0\end{pmatrix},\qquad A=\zeta J+V,$$

where now v denotes the independent physical conjugate field. There is a unique traceless formal series $H=J+\sum_{j\ge1}H_j\zeta^{-j}$ with $H_x=[A,H]$, $H^2=-\mathrm{Id}/4$. Indeed $[J,H_{j+1}]=(H_j)_x-[V,H_j]$ fixes the off-diagonal part, and the square fixes the diagonal part. Existence follows recursively by formally diagonalizing ∂_x-A: use at each power an off-diagonal conjugating coefficient to remove that power's off-diagonal part using $\operatorname{ad}J$, and conjugate J back by the resulting near-identity formal series. Each coefficient is a differential polynomial. Give the fields and ∂_x weight one. Then H_j has weight j and the vector field

$$V_{t_p}=[J,H_{p+1}]=(\partial_x-\operatorname{ad}A)(\zeta^pH)_+,\qquad p=0,1,\dots,$$

has weight $p+1$. Here $(\cdot)_+$ retains the nonnegative powers of ζ, including the constant term. The off-diagonal monomials have odd field degree g by charge symmetry, with $p+1-g$ spatial derivatives. The single upper wave has rate $+in^p$ and the lower wave has rate $-in^p$ in these times, as also follows from the dressing below. In particular $u_{t_2}=-iu_{xx}+iu^2v$, the reverse of physical NLS time.

For an independent check that these polynomial generators preserve the actions of the specific chart just cited, let S be the formal diagonalizer, $H = SJS^{-1}$ and $A_d = S^{-1}AS - S^{-1}S_x$. For $Y = S^{-1}\delta S$, $\delta A_d = S^{-1}\delta V S + [A_d, Y] - Y_x$. Hence

$$\delta \int \operatorname{tr}(JA_d)\,dx = \int \operatorname{tr}(H\,\delta V)\,dx.$$

Taking the ζ^{-p-1} coefficient shows that $[J, H_{p+1}]$ is a canonical Hamiltonian vector field, with a fixed constant set by the choice of u, v and symplectic form. Its Hamiltonian density may be chosen by integrating this gradient radially in V; a degree-g Hamiltonian term then has $p + 2 - g$ spatial derivatives. The polynomial zero-curvature identity makes the derivative of the one-period monodromy a commutator, since its time matrix is periodic. Thus the entire Zakharov–Shabat discriminant and all its contour actions are invariant. Explicitly, in the period-one system $L = i\sigma_3\partial_y + \left(\begin{smallmatrix} 0 & au \\ av & 0 \end{smallmatrix}\right)$, with $\sigma_3 = \operatorname{diag}(1, -1)$, set $x = 2\pi y$, $\lambda_{\mathrm{sp}} = -\pi\zeta$ and $f = \operatorname{diag}(i, 1)g$. Then $Lf = \lambda_{\mathrm{sp}} f$ is exactly $g_x = (\zeta J + V)g$. With the period-one fundamental matrix normalized to the identity at $y = 0$, let $\Delta(\lambda_{\mathrm{sp}})$ be the trace of its monodromy. By [46, (2.2)], the period-one action at spectral index j is

$$I_j^{(1)} = \frac{1}{\pi} \oint_{\Gamma_j} \frac{\lambda_{\mathrm{sp}} \Delta'(\lambda_{\mathrm{sp}})}{\sqrt{\Delta(\lambda_{\mathrm{sp}})^2 - 4}}\, d\lambda_{\mathrm{sp}},$$

where Γ_j encloses the jth gap counterclockwise and the square root is the canonical branch. The corresponding physical action is $I_{-j} = 2(2\pi)^{-2} I_j^{(1)}$ by the scaling in Section 2. It follows that the Hamiltonian commutes with every I_j of this chart, so its restriction is invariant under each Cartesian coordinate rotation and depends only on the actions. All arguments can first be made on smooth real potentials, then as analytic identities of finite-parameter complex germs. For an individual p one works initially at sufficiently high Sobolev order. Only finitely many p's will be used in any coefficient estimate.

7.5.2. *Square-free coefficients and a reciprocal-moment template.* For disjoint finite labels P, Q, $|P| = m + 1$, $|Q| = m$, let $C(P, Q)$ be the coefficient of $\prod_P z_p^+ \prod_Q z_q^-$ in the physical upper field, after factoring out its spatial exponential. Then

$$C(P, Q) = (-1/2)^m \sum_{\substack{p_1,\dots,p_{m+1} \\ q_1,\dots,q_m}} \prod_{j=1}^{m} \frac{1}{(p_j - q_j)(p_{j+1} - q_j)}, \tag{7.4}$$

where the sum runs over the permutations of each set. To verify the formula even at kinetic resonance, flow the parameters independently as $A_p = a_p \exp(i \sum t_j p^j)$ and $B_q = b_q \exp(-i \sum t_j q^j)$, with $t_1 = x$ and the zeroth character identically one. Let A_P, B_Q be their diagonal matrices and $C_{pq} = (p - q)^{-1}$. Take

$$D(\zeta) = 1 + \sum_p \frac{r_p e_2^t}{\zeta - p} + \sum_q \frac{s_q e_1^t}{\zeta - q}, \qquad r_p = \frac{iA_p}{\sqrt{2}}\Big(e_1 + \sum_q C_{pq} s_q\Big), \quad s_q = \frac{-iB_q}{\sqrt{2}}\Big(e_2 - \sum_p C_{pq} r_p\Big).$$

This finite linear system is invertible for small parameters. At p the matrix $D(1 - N/(\zeta - p))$ is regular with $N = (iA_p/\sqrt{2}) e_1 e_2^t$, and analogously at q with $N = (-iB_q/\sqrt{2}) e_2 e_1^t$. Consequently $\det D = 1$. The potential pole in $D_{t_j} D^{-1} + \zeta^j D J D^{-1}$ is removable since $N_{t_j} = p^j [J, N]$ at p, $NJN = 0$ and $N^2 = 0$; at q use q^j. The resulting polynomial is exactly $(\zeta^j D J D^{-1})_+$ by expansion at infinity. Its first spatial coefficient is $V = [D_1, J]$, $D_1 = \lim_{\zeta\to\infty} \zeta(D - 1)$. The upper physical field is therefore

$$u = \mathbf{1}^t (1 + A_P C B_Q C^t/2)^{-1} A_P \mathbf{1}.$$

These are smooth periodic potentials for complex parameters near zero. Their spatial fundamental solution away from the finitely many poles is

$$D(x,\zeta)e^{x\zeta J}D(0,\zeta)^{-1}.$$

Periodicity of D shows that its monodromy matrix is conjugate to $e^{2\pi\zeta J}$. By analyticity the entire discriminant coincides with that of the vacuum, including at every pole. The contour action formula with its vacuum normalization therefore gives zero for each action of the actual Birkhoff chart. Equivalently, invariance of an action under every hierarchy time forces every moment of $\sum \alpha_p\delta_p - \sum \beta_q\delta_q$ to vanish on each of its parameter monomials. Vandermonde on the disjoint supports gives the same zero-action conclusion. On this complex zero-action face a particular positive coordinate evolves at its linear rate $+ik^j$, and a negative one at $-ik^j$. For completeness, on each finite Cartesian truncation rotational invariance makes the Hamiltonian a power series in the pair products. The quotient of its z_k^- derivative by z_k^+ on the zero-action face thus has its vacuum value. Truncating in an arbitrarily high Sobolev order and passing to the smooth limit gives the same identity for each coordinate. A second Vandermonde test, subtracting or adding δ_k, leaves only a_k or b_k with coefficient one from the free normalization. Thus the dressing is the actual inverse chart on this disjoint complex subspace, and its Neumann expansion gives (7.4). An action-dependent rotation of coordinate angles with unchanged vacuum differential has no effect on this zero-action computation.

Lemma 7.8. *For each fixed integer $h \geq 2$ there are distinct nonzero integers v_i with signs $e_i = \pm 1$ such that*

$$\sum e_i = 1, \qquad \sum e_i v_i = \sum e_i v_i^2 = 0, \qquad \sum e_i/v_i \neq 0,$$
$$\sum e_i v_i^{-j} = 0 \quad (2 \leq j \leq h), \qquad C(v,e) \neq 0.$$

Five designated entries have the signs and offsets of $(-1,3,4\,;\,1,5)$ up to a common dilation, positives listed first. The other entries contain both signs.

Proof. For this five-site starting set S, direct substitution of the twelve terms in (7.4) gives $C_S = -1/9$ and $\sum e_i/v_i = -97/60$. Suppose the result has been obtained through $j = m-1$, $m \geq 2$, using rationals. Write $x_T = N_0 + \sum_{j\in T} h_j$ for $T \subset \{1,\dots,m\}$ with $m!\prod h_j = 1$. Adjoin to S a copy S/x_T with signs multiplied by $w_T = -(-1)^{m-|T|}$ for each T. The mth divided difference gives $\sum w_T x_T^j = 0$ for $0 \leq j < m$ and -1 for $j = m$. The previous inverse moments are multiplied by $1 + \sum w_T x_T^j$, and in each copy the two positive moments vanish separately.

It remains to preserve the nonzero path coefficient. Put $h_1 = t^{m-1}$, $h_j = \Gamma_j/t$ for $j \geq 2$, $N_0 = \Gamma_0/t$, with positive rationals Γ_j satisfying the product constraint. Avoid coincident subset sums among $\Gamma_2,\dots,\Gamma_m$. For $m > 2$ each unwanted equality is a nonzero Laurent polynomial after eliminating the last variable by the product (the resulting monomials are distinct); the positive rational parameter set is dense. For $m = 2$ distinctness holds directly. Next vary Γ_0 to separate the numbers $v/(\Gamma_0 + \sum_{j\in T}\Gamma_j)$ over $v \in S$ and $T \subset \{2,\dots,m\}$. For different v only finitely many values are excluded. For fixed v use the distinct subset sums. The new points are opposing pairs $(y_i + \delta_i, y_i)$, positive first, with $|y_i| \lesssim t$, mutual center separations $\gtrsim t$ and $0 < |\delta_i| \lesssim t^{m+1}$.

Here are the collision estimates for (7.4). For one opposing pair, $\lim_{\delta\to 0} \delta\, C(S \cup (y+\delta, y)) = C_S L_S(y)$, where $L_S(y) = \sum_{v\in S} e_v/(y-v)$. Only neighboring occurrences of the two added letters contribute. If inserted between an old positive a and old negative b the ratio of edge factors after removing their common pole is $1/(a-y) + 1/(y-b)$, in either order of this old edge. Each end insertion contributes $1/(a-y)$ at its positive endpoint. Counting both incidences of each old vertex and multiplying by the new factor $-1/2$ gives exactly $L_S(y)$. For b

simultaneous pairs put $\Delta = \min_{i\neq j} |y_i - y_j|$ capped by one and a fixed small distance from S, and $\delta_* = \max_i |\delta_i|$. The function $G = (\prod_i \delta_i)C$ is holomorphic on $|\delta_i| \le c\Delta$. On the distinguished boundary it is $O_S(\Delta^{1-b})$, since a path has at most $2b-1$ edges entirely among the new vertices. Cauchy's estimate bounds its sum of terms involving every variable by $O_S(\Delta^{1-b}(\delta_*/\Delta)^b) = o(1)$ if $\delta_* = o(\Delta^2)$. By inclusion-exclusion the remaining terms are the signed sum of proper restrictions setting some $\delta_i = 0$. The single residue formula on each such restriction gives the analogous function on the surviving pairs times, for each collapsed pair,

$$L_S(y_i) + \sum_{\text{surviving } j} \Big[(y_i - y_j - \delta_j)^{-1} - (y_i - y_j)^{-1}\Big] = L_S(y_i) + O_S(\delta_*/\Delta^2).$$

Induction on b therefore gives $G = C_S \prod_i L_S(y_i) + o(1)$, with the case of one pair as above. Since $L_S(0) = -\sum e_v/v \neq 0$ and here $\delta_* = o(\Delta^2)$, a sufficiently small rational $t > 0$ works. Clear denominators at the end. □

7.5.3. *The coefficient on a nonzero occupied background.* Fix the template length q, and set $n_a = 1 + KV_a$ for all new sites with distinct nonzero integers V_a, $\max|V_a| \le Cd^C$. A particular marked row has $V_a = lv_a$, $1 \le l \le Cd^C$. Write F for the old sites, $n_F = (m)_{m\in F}$ and $f = |F|$. The sites are bounded in absolute value by Z, and their Cartesian amplitudes are in a common small h^1 ball. For the physical upper field take its angular coefficient with the row's signs on its marked sites, zero on all others including the old sites, divided by the q required Cartesian radii. Its physical Fourier index is one by translation covariance. The divided coefficient is extended analytically through vanishing new radii. It is a single temporal/angular harmonic of that physical Fourier coefficient. Other harmonics at the same spatial site can occur.

Proposition 7.9. *Suppose the new weak actions vanish and the other new actions have sum $S \le K^{-(h+1)/2}$. Let L_0 be a fixed integer with $(L_0+1)(h+1)/2 > q+1$. There are fixed exponents, depending only on q, h, L_0 and the fixed polynomial in the macro bound, for which $E = (2+Z)^C e^{Cf}(2+d+\log K)^C$ has the following property. For K sufficiently large compared with these factors the divided coefficient is*

$$(Kl)^{-(q-1)}C(v,e) + O\big(EK^{-q} + ESK^{-(q-1)} + (CS)^{L_0+1}\big).$$

In particular its modulus is at least $\frac{1}{2}(Kl)^{-(q-1)}|C(v,e)|$ on increasing K. If only five weak radii are divided out and each remaining marked action is at least K^{-C_0}, the resulting modulus is at least K^{-C_1} for a fixed C_1.

Proof. Set $M_0 = q + 2L_0$. Expand at zero new Cartesian variables and keep up to degree M_0. For a monomial P write $l_a = \alpha_a - \beta_a$ and $w = \sum |l_a|$. Let $u^\sigma_{P,j}$ be its coefficient at old angle $j \in \mathbb{Z}^f$ in copy $\sigma = \pm 1$. The spatial exponent is $\nu_j = \sum l_a n_a + j \cdot n_F$ and the sign-copy spectral label is $\sigma\nu_j$. Cauchy on a fixed old-angle strip and fixed Cartesian discs in the at most M_0 tested variables gives $|u^\sigma_{P,j}| \le Ce^{-c|j|_1}$ by the analytic L^2 chart. Shrinking the strip costs at most C^f in the absolute angular sum. Any prescribed spatial derivative adds a factor at most $(CKd^C + Z|j|_1)$ per derivative. With $L = C' \log K$, the tail $|j|_1 > L$ can thus be made $O(K^{-A})$ for any preassigned finite A, also after fixed derivatives, provided K dominates E. Constants C in E may increase a fixed number of times in the proof. The identities involving high Hamiltonians are initially valid on a sufficiently small higher-regularity ball for the finitely supported arguments. They extend to the common low-norm polydiscs by angular translation covariance and the convergent differentiated Fourier series just described.

For a single new Cartesian argument the derivative of the nonlinear part of Ψ maps h^1 to H^2 uniformly on the old complex h^1 ball. Its norm on the unweighted basis vector is $O(|n_a|)$. For $|j|_1 \le L$, $|\nu_j| \ge |n_a|/2$, so the error from its free plane is at most $C|n_a|^{-1}e^{-c'|j|_1}$ by angular

Cauchy; the tail is absorbed above. In particular the positive old-angle-zero entry on a positive test is $1 + O(E/K)$.

For each $2 \le p \le M_0 + 3$, write $\omega^{(p)}$ for the auxiliary angular speeds, with free speed n^p. Differentiating its chart identity at a single empty positive input and taking the upper, old-angle-zero coefficient gives $|\omega^{(p)}_{n_a} - n_a^p| \le EK^{p-2}$: the coefficient of this frequency difference is $1 + O(E/K)$, and each nonlinear physical term has at most $p - 2$ spatial derivatives. The mean of an old angular derivative vanishes and new action differentiation requires a pair. For the old speeds a bound E suffices. Any fixed number of new action derivatives of the tested old or new speeds costs EK^p. To see this, use the action-only Hamiltonian and Cauchy's estimate in a bounded number of tested action slots on fixed small discs. Realize a tested empty pair as $(\rho, I_a/\rho)$, with a phase and its opposite, using fixed small ρ. For a tested occupied index b, write $\rho_b = \max(\sqrt{I_b}, \rho)$ and use $(\rho_b, (I_b + \eta_b)/\rho_b)$ on $|\eta_b| \le c\rho^2$. Cauchy differentiation has a fixed cost also in this slot. The squared Cartesian size of all tested inputs remains in a common small ℓ^2 ball. The untested old entries keep their old angles. To bound the spatial integral defining the Hamiltonian, take absolute angle sums on a smaller strip for all tested and old angles. Each derivative then costs at most $Kd^C + Z$ times an angular moment, and a density term has at most p derivatives in total. With only old inputs use Z instead of Kd^C. There is no division by a shrinking old action in this estimate.

For $w \ge 2$ and a chosen output copy let $t = \sum l_a V_a$. The measure $\sum l_a \delta_{V_a} - \sigma\delta_{\sigma t}$ has a nonzero restriction to the nonzero integers: equality there would force $w = 0$ or 1 since the V_a are distinct. If b is its number of nonzero support points, the moments of degrees $2, \dots, b + 1$ distinguish it, by the Vandermonde matrix times the diagonal of the squared sites. Some such moment has absolute value at least one, and $b \le M_0 + 1$. For this p, at $|j|_1 \le L$ the signed divisor

$$j \cdot \omega_F^{(p)} + \sum l_a \omega^{(p)}_{n_a} - \sigma(\sigma\nu_j)^p$$

has magnitude at least $K^p/2$: its leading coefficient in K^p is that integer and its remaining terms cost EK^{p-1}. Choose p separately in each output copy and divide its own row before estimating the two coupled rows.

We claim $\sum_j |u^\sigma_{P,j}| \le EK^{-(w-1)_+}$. For $w \le 1$ use Cauchy. For larger w induct on the total degree. A derivative of a Birkhoff frequency removes a positive number of balanced pairs from the input monomial, retaining its net label; its EK^p cost is paid by the divisor, with the old-angle derivative cost at most L inside the cutoff. For a nonlinear physical product of g factors the number of spatial derivatives is $p + 1 - g$, so division by the divisor of size at least $K^p/2$ gains $K^{-(g-1)}$ up to the factor E. Let w_i be the weights of the factors and let k be the number of them with $w_i > 0$. The triangle inequality for their signed multiplicities gives $w \le \sum_i w_i$. Consequently the total power gained satisfies

$$(g - 1) + \sum_i (w_i - 1)_+ = \sum_i w_i - 1 + (g - k) \;\ge\; w - 1.$$

The terms having the unknown of the same total degree have all their other factors old. Their operator norm in absolute angular sums after division is at most EK^{-2}, since $g \ge 3$. This absorbs the truncated unknown jointly in the copies. All high-angle complements of these products were bounded beforehand by Cauchy; the bounded degree induction closes.

When P consists of b distinct signed variables at pairwise distinct sites, a second induction compares its coefficient with the vacuum coefficient concentrated at old angle zero. The difference in absolute angular sum is at most EK^{-b}. The single-input case was proved above. No action derivative can remove a balanced pair in such a monomial. With at least one entirely old factor, the last weight inequality gains a further power K^{-1} because every nonold child has

weight equal to its degree. With no old factor, replace the children one at a time by their vacuum values and use the induction bound. The remaining all-vacuum product has only $j = 0$; the change between the true and vacuum divisors there costs EK^{p-1} times a vacuum coefficient $O(EK^{-(b-1)})$ before division by K^p. The same-degree term with old factors may also be estimated using the already proved first induction bound. This proves the claimed comparison including subwords whose vacuum coefficient vanishes by charge. Thus (7.4) and its homogeneity give $(Kl)^{-(q-1)}C(v,e) + O(EK^{-q})$ for the desired character when all new actions vanish.

To restore the nonweak actions, the Taylor terms through L_0 added pairs still have weight q. Summing their action products at each degree j costs at most S^j with the ordinary Taylor coefficient convention, since the multinomial expansion of S^j dominates this sum. For the remaining tail multiply all new nonweak actions by ξ. At each signed marked site take its selected side to be $\rho e^{\pm i\phi_a}$ and the other to be $\xi I_a \rho^{-1} e^{\mp i\phi_a}$; at unmarked new sites use symmetric square roots of ξI_a. Projecting on the selected character and dividing by ρ^q gives precisely the desired action series and removes all square-root choices. On $|\xi| \le c/S$ the total squared size in the fixed q tested slots and all other slots stays in the common small ℓ^2 ball, and normalized Haar integration bounds the physical coefficient independently of the number of unmarked sites. One-variable Cauchy now bounds the omitted tail by $(CS)^{L_0+1}$. This proves the estimate and both lower bounds. □

For a row with $n_a = 1 + Klv_a$, the spatial and kinetic sums both equal one. In applying Lemma 7.8 to a passive deficit one needs the true empty-frequency expansion to start at order n^{-2} with the same inverse-power coefficients for positive and negative n. Under that expansion through h, expanding $(1 + Klv_a)^{-j}$ for $j \ge 2$ and using the reciprocal moments gives a deficit $O(C_{\mathrm{old}}(Kl)^{-h-1})$. An expansion merely allowing a nonzero n^{-1} term would not suffice, since $\sum e_a/v_a \ne 0$.

7.6. **Scale, geometry and calibration.** The quantifiers in this section are for each fixed finite $s > 2$ separately. The Fourier lattice is measured on $\mathbb{T}^2$ of period 2π and all integrals use normalized measure. In the mass gauge the genuine axial frequencies are $n^2 + \chi_n(I)$, where $I_n = |z_n|^2$ in a choice of Birkhoff normalization with free differential Fourier. The mass is $\sum I_n$. The map χ is analytic on a complex ℓ^2 action ball and $D\chi(0) = -\mathrm{Id}$. This follows by rescaling the cubic coefficient in [46, Theorem 18.1 and Corollary 18.2, $p = 4$]. The Hessian of the action Hamiltonian is symmetric on positive finite-dimensional faces. We use translation and phase covariance, simultaneous Birkhoff coordinates for the one-dimensional NLS hierarchy, and the one-derivative gain of the nonlinear part of the one-dimensional coordinate map and its inverse on the integer Sobolev spaces of order at least one. These statements concern the axial equation.

Use $B = B_* Q^j$, $B_o = B/Q$, $d = \lfloor \alpha\sqrt{B/\log B} \rfloor$, and a vertical spacing R with $\log R = B + O(1)$. It may be chosen divisible by all earlier spacings and prime to five since their product is at most $\exp(CB_o)$. A rational family of normalized heads has a common denominator P and physical indices $(1, 0) + Rv$, where v is the integer macro vector, and the top is $(1 + N, 0)$ with $N = RP$. Thus a normalized direction x corresponds to integer macro $v = Px$. With $H = e^{bd}$, choose

$$r = HN^{-s}, \qquad \lambda = e^{-\epsilon B}, \qquad u = r\lambda, \qquad \tau = ut.$$

Here $b > 0$ is small relative to s and the constant contraction of the chosen split. At fixed s choose $a < \min(s - 2, 1/4)/8$. All fixed derivatives and high spatial truncation orders are specified before ϵ. We distinguish factors λ^{-C_s} from

$$E = \exp\{C_{s,\epsilon}(B_o + d^2 \log(2 + B)) + o(B)\}.$$

Constants in these two classes can increase through a fixed number of operations. Only the second class can depend on the cutoff in the number of weak angular letters. First choose ϵ small compared with a and all the fixed powers in the first class. After the intermediate historical and preparation constants have been chosen, take Q large and α small enough that each needed product of factors in the second class is $\le N^{\gamma}$, with γ also much smaller than ϵ. In particular $\alpha^2 Q$ can be made as small as desired. No lower bound on the final constant in the growth exponent is prescribed.

7.6.1. *A true axial frequency with an affordable interval of values.* On a positive old support F let $M = \sum I_n$, $\mathcal{H}$ be the ungauged one-dimensional action Hamiltonian and $\omega_n = n^2 + 2M + \chi_n$. Set

$$\mathcal{Q} = 2\left(\sum I_n \omega_n - \mathcal{H}\right), \qquad A = \mathcal{Q} - M^2, \qquad D = \mathcal{Q}/2.$$

On a small initial two-site torus, $A > 0$ and $2M^2 - \mathcal{Q} > 0$ with strict fixed margins. Indeed the first quartic normal form gives $\mathcal{Q} = M^2 + 2I_{n_1} I_{n_2} + O(M^3)$ when n_1, n_2 are the two comparably occupied sites. The identities $d\mathcal{H} = \sum \omega_n \, dI_n$ and

$$d\mathcal{Q} = 4M \, dM + 2 \sum I_n \, d\chi_n$$

show that sufficiently small cumulative changes in ℓ^1 preserve these margins, independently of the new frequencies. Suppose $\max_F |n| \le e^{CB_o}$, every positive old $I_n \ge e^{-CB_o}$ and the old size in h^1 remains in the small ball. On an old action ball of radius $h_0 \ge e^{-C'B_o}$ inside positivity and this h^1 chart, the high empty axial expansion gives $|\chi_l(I,0)| \le E_o/l^2$ with $E_o \le e^{C''B_o}$. The order two gain here matters. In the doubled high quasimode the order-zero coefficient is the mean of $B_0 = 2(|V|^2 - M)$, which vanishes. The order-one coefficient is a constant times the mean of $B_0 \partial_x^{-1} B_0$, also zero. Projection of the approximate high column by the actual Birkhoff differential onto its positive free mode leaves a scalar $1 + O(E_o/|l|)$ with a single-valued argument on the angle torus. Its directional derivative integrates to zero against Haar measure, giving the same two vanishing coefficients for the true empty frequency. The uniform cost of any fixed higher order is polynomial in the old maximum, inverse positive floors and $e^{|F|}$ by old-angle analyticity.

For any real unit old direction apply the central difference formula at $h = \sqrt{E_o}/|l| \le h_0$, using the uniform bound for $D^2\chi$ on the common ball. This gives $\|P_F \nabla \chi_l\|_2 \le C\sqrt{E_o}/|l|$. Symmetry gives the corresponding bound on $P_F D\chi \, e_l$. Choose the empty $l \equiv 1 \pmod 5$ with $|l| = e^{C_1 B_o + O(1)}$, increasing C_1 to pay $\sqrt{|F|}$. Lock χ_F by its own old actions while filling $I_l > 0$; their Jacobian inverse has norm at most two on a sufficiently small common ball. The derivative of M with respect to I_l is between $3/4$ and $5/4$ on a short positive interval, since this holds at the empty endpoint and $D^2\chi$ and the compressed inverse are bounded. Shrinking the interval pays any prescribed $e^{-C_h B_o}$ old norm tolerance, including the new cost $\langle l \rangle^s \sqrt{I_l}$, and leaves an interval of length $e^{-C_2 B_o}$. On it

$$(A/D)' = 4M(2M^2 - \mathcal{Q})M'/\mathcal{Q}^2 + O(I_l).$$

Take the spectator action in $[a_\ell, 2a_\ell]$ with $a_\ell = \exp(-C_2 B_o)$, increasing C_2 to pay the prescribed old tolerance. The image under A/D has length comparable to a_ℓ. There are at most $e^{C_4 B_o}$ distinct nonzero rational component pairs (x, y) of height at most $e^{C_3 B_o}$. Arrange first $\log P \le C_{s,\epsilon} d^2 \log B \le C B_o$. The quotients y/x with $x \ne 0$ divide the image interval into at most $e^{C_4 B_o} + 1$ pieces. Choose the middle half of a longest piece. Its preimage has length at least $e^{-C_5 B_o}$ and $|Ax - Dy| \ge e^{-C_5 B_o}$ there, after increasing C_5: D has a fixed positive lower bound and nonzero $|x| \ge e^{-C_3 B_o}$. For $x = 0$ use the corresponding bound on $|y|$.

There is an additional choice on this very interval. Assume the previously accepted direct rates χ_F satisfy

$$|n + l \cdot \chi_F| \geq \kappa_F (2 + |n| + |l|_\infty)^{-2|F|-6}, \quad (n, l) \in \mathbb{Z}^{1+|F|} \setminus \{0\}, \quad \kappa_F \geq e^{-C_{\mathrm{reg}} B_o}. \tag{7.5}$$

At the base choose the two positive actions in an interior cube on which the quartic margins hold; the local direct frequency chart and the slicing argument below give this bound with some positive constant. Along the old lock, $\frac{d}{dI_\ell}\chi_\ell$ is the scalar Schur complement of $D_F\chi_F$ in $D_{F\cup\{\ell\}}\chi_{F\cup\{\ell\}}$. It differs from -1 by at most $\eta + \eta^2/(1-\eta)$ if $\|D\chi + \mathrm{Id}\| \leq \eta$. Hence on the chosen subinterval the direct spectator rate covers an interval of width $\delta_\ell \geq e^{-C_6 B_o}$. Choose it so that

$$|n + l \cdot \chi_F + b\chi_\ell| \geq c\,\delta_\ell\, 2^{-C(|F|+1)} (2 + |n| + |l|_\infty + |b|)^{-2(|F|+1)-6} \quad (b \neq 0).$$

Indeed each bad relative length is bounded by twice its threshold divided by $|b|\delta_\ell$, and summing the integer tuples in dyadic height shells costs less than half the interval for an absolute choice of c, C. Restricting beforehand to a middle subinterval leaves the same spatial margins. For $b = 0$ inherit (7.5). Lock this spectator rate and its chosen angle origin exactly through the rest of this stage and every later stage. We write $F_+ = F \cup \{\ell\}$. Its new prefactor and every price of this preliminary spectator selection are $e^{-O(B_o)}$ and $e^{O(B_o)}$, respectively, with constants allowed to depend on the requested old precision. Only now choose the moderator dilation $K = e^{kB_o + O_s(\log B)}$. Thus a fixed spectator rate resonant with the old rates is never silently included in the frozen relation module.

7.6.2. *Simultaneous positivity and affine frequencies.* Let H_e here denote rows of bounded length with disjoint coordinates, each of charge one, horizontal momentum one and kinetic sum one, containing a positive and a negative unit moderator pivot and five private weak letters. Write H for the matrix with rows H_e. Keep all of F_+ locked. If J denotes the prospective new actions, the old action derivative is $-(D_{F_+}\chi_{F_+})^{-1} D_J \chi_{F_+}$ of norm at most $\eta/(1-\eta)$. The resulting derivative of the new direct rates is their Schur complement, within $\eta + \eta^2/(1-\eta)$ of minus identity in Euclidean operator norm. On each interior ball its local inverse takes direct-rate increments to action increments with norm at most two after shrinking the common vacuum neighborhood. This last assertion holds on a ball of radii comparable to the available Euclidean distance to the action boundary, by solving the near-identity map by contraction. No product of one-coordinate inverse norms occurs.

Let the rows use q fixed signs, and write m for their number. Initially every weak site is empty and all moderators get the same positive floor a_m. For each row choose two moderator pivots with opposite signs. Put v_e^+ on its positive and v_e^- (the magnitude of its negative part) on its negative pivot, obtaining $S(v)$. Then $\mathsf{H}S(v) = v$, $\mathrm{Lip}_{\ell^2} S \leq 1$, and $\|\mathsf{H}\| \leq \sqrt{q}$. The map $v \mapsto v + \mathsf{H}\chi_{\mathrm{new}}(I + S(v))$ on the old locked chart contracts if $\sqrt{q}(\eta + \eta^2/(1-\eta)) < 1/2$. It can be solved on the radius $2|\mathsf{H}\chi_{\mathrm{new}}(I)|_2$ before approaching any previous action floor. Its nonnegative increments preserve all new moderator floors. If each unprepared row deficit is $O(a_m)$, adding the initial moderator floors changes its vector of deficits in ℓ^2 by at most $C_q\sqrt{m}\,a_m$. Thus all moderators have total action $O_q(ma_m)$, and the locked old action variation in ℓ^2 is $O_q(\sqrt{m}\,a_m)$. A smooth chart about this interior point uses one unit moderator pivot per row, with either sign of increment. Its row Jacobian, after the signs are accounted for, is within $\sqrt{q}(\eta + \eta^2/(1-\eta))$ of the identity.

For existence of such rows fix an integer $h > 4s + 5$. The reciprocal-moment construction supplies a finite signed template $v_i \in \mathbb{Z} \setminus \{0\}$, $e_i = \pm 1$, all sites distinct, with

$$\sum e_i = 1, \quad \sum e_i v_i = \sum e_i v_i^2 = 0, \qquad \sum e_i / v_i \neq 0, \qquad \sum e_i v_i^{-m} = 0 \ (2 \leq m \leq h).$$

The actual minimal signed inverse-Birkhoff coefficient at vacuum is nonzero. Its order is $q-1$ in inverse site differences, where q is the fixed number of letters in this template. Use distinct disjoint rows $1+Kl_ev_i$ with l_e positive integers bounded by a fixed multiple of the number of edges, chosen greedily; take $5 \mid K$. The uniform common empty-frequency expansion through h, beginning at inverse order two as above, leaves passive deficits $O(e^{CB_o}K^{-h-1})$. Choose $a_m = C_oK^{-h-1}$ with $1 \le C_o \le e^{CB_o}$ bounding these deficits on the F_+ chart. Increasing k makes the total moderate action cost, including the polynomial factor m, at most $K^{-(h+1)/2}$ and preserves all F_+ floors. Their full h^s cost is $O(K^{s-(h+1)/4}\operatorname{poly}(d))$, arbitrarily small on the scale of the previous tolerance. Both signs of unit moderator pivots lie among the added letters beyond the initial five.

Apply Proposition 7.9 with the positive old data just selected, including the spectator, and the total moderator action bounded above. Lemma 7.8 supplies the nonzero vacuum coefficient, and Lemma 7.10 supplies the passive deficit. By increasing the fixed moderator placement constant k, the true physical coefficient, divided only by its five weak radii, has modulus between K^{-C_s} and K^{C_s}. This estimate holds at the real passive torus with the old rates locked.

Let $\Gamma = B_o + d^2\log(2+B) + \log(2+B)$, and allow constants multiplying Γ to depend on the preceding precision and the fixed orders. Let $D_g = C_*d\log(d+2)$ and fix a finite comparison precision $\xi = \exp(-LdD_g)$, taking L sufficiently large for the actual finite comparison, its Hessian, all coordinate counts, and both quadratures. For each row choose the *product* of the five private radii to be $\lambda|g_e|/|F_e|$, with any fixed coefficient convention of the physical Hamiltonian included in g_e. Equal fifth roots suffice. Here F_e is the actual coefficient after division by just these five radii at the passive torus, and $g_e \neq 0$ is the required graph coefficient. Thus

$$e^{-C\Gamma} \le |F_e| \le e^{C\Gamma}, \qquad |\log|g_e|| \le Cd\log(d+2), \qquad I_w^{1/2} = \lambda^{1/5}e^{O(B_o+d\log(d+2)+\log B)}.$$

Turn on these weak actions keeping their values fixed, lock F_+, and use the interior moderator pivot inverse to restore the flat relations $\mathsf{H}\chi = 0$. The aggregate action change in ℓ^1 from the passive base is at most $\lambda^{2/5}e^{C\Gamma}$, including the dimension factors. This is less than half every moderator, spectator and previous action floor after the scale choices. It does not require a weak site to serve as the pivot of this activation.

For later variations the following concrete checks suffice. Write $F_{\min} = \min_e |F_e|$ and $I_{w,\min}$ for the least current weak action. An action movement ΔI with

$$\|\Delta I\|_\infty \le c_q\xi I_{w,\min}, \qquad \|\Delta I\|_1 \le c_q(\xi F_{\min})^2 \tag{7.6}$$

after activation changes every weak product by relative $O_q(\xi)$ and every divided coefficient by at most $O_q(\xi F_{\min})$. For the latter extract its marked harmonic by integration over all angles, replacing each selected weak Cartesian side by fixed small radius ρ and its opposite by I_w/ρ. The map on the common complex ℓ^2 ball is Lipschitz; all the ordinary square-root coordinates move by at most $\|\Delta I\|_1^{1/2}$ in ℓ^2. This gives $C_q(\|\Delta I\|_1^{1/2} + \|\Delta I\|_1)$ uniformly in the number of rows. For the activation itself use only this second inequality against the prescribed fixed product. The resulting relative error is $\lambda^{1/5}e^{C\Gamma} \ll \xi$. In converting relative into absolute normalized finite-vector error also multiply by $|g_e|$ and the necessary fixed powers of the number and size of heads; the constant L above covers these costs.

Keep a fixed fraction of the spatial separation margin throughout. The exact differential of $\mathcal{Q}$ gives $|\Delta M| + |\Delta\mathcal{Q}| \le C\|\Delta I\|_1$ on the small ball, independently of the largest index. Including the maximum rational component height, a movement $\|\Delta I\|_1 \le e^{-C_7B_o}$ suffices. The initial spectator already pays the old h^s tolerance on F; the moderator cost is $K^{s-(h+1)/4}\operatorname{poly}(d)$. Weak-site, old-adjustment and subsequent rounding costs can be made smaller than any prescribed $e^{-D_{\mathrm{hist}}B_o}$: their indices are at most $e^{O(B_o)}$ and their action changes are bounded by the above powers of λ, respectively by the later small powers of N^{-1}. For an old radius change use

$\|(z'-z)_F\|_{h^s} \le \langle Z\rangle^s |F|^{1/4} \|\Delta I_F\|_2^{1/2}$ with identical time origins and old direct rates. For these bounds one first chooses k large for the moderator bill and then takes Q sufficiently large for the remaining prices depending on D_{hist}, k. The true current hot actions are kept as separate high carriers.

7.6.3. *The two frequency cubes and the terminal fit.* Put $f_1 = |F_+| + \#\{\text{new clock sites}\}$. In the direct-rate flat eliminate one unit new pivot per disjoint row. Independent increments of size δ per coordinate give full Euclidean rate increments at most $q\sqrt{f_1}\delta$, and hence total ℓ^1 action movement at most $Cqf_1\delta$ on the locked chart. For example the cube of halfwidth

$$\delta_0 = \lambda^2 \exp(-C_8\Gamma)$$

is available by choosing C_8 after the fixed protection prices. It preserves positivity, (7.6), the spatial margin, and the old tolerances. The exponent 2 on λ is fixed independently of the weak word cutoff and D_{hist}. Changing C_8 affects the Γ price only.

Let x denote the independent direct rates on this cube. Fix an integer form $n + l\cdot\chi_{F_+} + b\cdot x$ with $b \neq 0$ and choose i with $|b_i| \ge 1$. For each fixed choice of the other coordinates, the set where the form has modulus less than w has length at most $2w/|b_i|$ in an interval of length $2\delta_0$. Integration in the other coordinates gives bad relative volume at most w/δ_0. At dyadic height 2^j the number of tuples is at most $(C2^j)^{f_1+1}$. Summing with thresholds

$$c\,\delta_0\, 2^{-C(f_1+1)}(2+|n|+|l|_\infty+|b|_\infty)^{-2f_1-6}$$

leaves positive relative volume. With $b = 0$ use the stored bound on F_+, including the independently selected spectator. Pivot elimination has integral coefficients bounded in terms of q, so the only zero affine labels are integer combinations of the declared rows (with zero constant when using the renormalized rates). At the physical angular cutoff $|\ell|_1 \le C_J B$, an integer kinetic part of size more than $1 + C|\ell|_1$ cannot be canceled by these bounded renormalized frequencies; all remaining heights in the affine test are $O_q(1+B)$. The dimensional log loss is $O_q(f_1 \log(2+B))$ and belongs to $C_{s,\epsilon}\Gamma$, including the previous dimensions. Thus every relevant nonflat denominator has size at least $w_0 = \lambda^2\exp(-C_9\Gamma)$, after increasing C_9. Only one power of the cube width is lost in this measure estimate.

Turn on the prescribed positive hot actions J_∞ at the current axial heads, keeping all the F_+ and clock direct rates fixed for this initial change. The locked actions move by $O(\|J_\infty\|_2)$, with $\|J_\infty\|_1 \le r^2 e^{Cd}$. Then keep J_∞ and all independent nonpivot rates fixed and write $y_e = H_e\chi$ for the variable row displacements; one unit pivot per row realizes these displacements. Denote by $\mathcal{D}_e(y)$ the sum of the actual carrier frequency corrections on the two inputs minus that on the output, in the rotating branch convention. The kinetic key is one. Use the true one-dimensional frequencies on hot axes, including their own actions. On transverse heads use the full angle means of the required fixed-order instantaneous spatial scalar. These means do not use the temporal divisors about to be selected. At the flat center $\|(\mathcal{D}_e(0))_e\|_2 \le N^{-2}E$ after summing the heads. Indeed (T1) gives the leading terms, all higher terms cost $N^{-3}E$, and $\|J_\infty\|_1$ is smaller.

On a real positive F_+-plus-clock neighborhood of Euclidean radius $\lambda^{C_s}e^{-C\Gamma}$ with the hot actions temporarily zero, a high empty axial site n has $|\chi_n| \le E'/n^2$, $E' \le \lambda^{-C_s}e^{C\Gamma}$. Here $|n| \ge Ne^{-Cd}d^{-C}$ on every marked axis. Apply the central-difference estimate at $\sqrt{E'}/|n|$ within this neighborhood to bound every unit occupied-direction derivative by $C\sqrt{E'}/|n|$. Restoring the fixed hot actions changes this bound by at most $C\|J_\infty\|_2$ using the uniform second differential of χ. The transverse mean derivatives cost at most $N^{-2}\lambda^{-C_s}E$ each. All these bounds are in a fixed number of differentiations. The direct-rate inverse and the passage from individual rows to the row-vector norm therefore give $\|D_y(\mathcal{D}_e)_e\| < 1/2$ on $\|y\|_2 \le 2N^{-2}E$, increasing the constant

in E before defining this ball. Write $\mathcal{D} = (\mathcal{D}_e)_e$. Since $\|\mathcal{D}(0)\|_2 \le N^{-2}E$, the derivative bound implies $\|\mathcal{D}(y)\|_2 \le N^{-2}E + \|y\|_2/2$ on this ball. It follows that the ball is invariant and that $y = \mathcal{D}(y)$ has a unique solution there. This single vector fixes the mismatches of all designated edges. Throughout this adjustment the hot actions J_∞, the rates on F_+ and the independent nonpivot rates retain their prescribed values.

Denote by Ω_* the final, rounded vector of physical rates on the clock support F, and by η_{rnd} a common bound for the rounding error in the tested frequency combinations. For each relevant nonflat label the actual perturbation of its divisor is at most $C_J B N^{-2} E$ (plus the tiny final rounding). This is $o(w_0)$, including the additional real corrections of any fixed number of moving carriers. On the frozen module, charge-zero combinations of bounded weak length obey, in root-normalized coordinates,

$$\sum_e b_e H_e \cdot \Omega_* = N^{-2} \sum_e b_e \mathfrak{d}_e + O_{s,\epsilon}\big(N^{-3}\lambda^{-C_s}E + r^2 E + \eta_{\mathrm{rnd}}\big), \qquad \sum_e b_e = 0.$$

The spatial choice guarantees $|\sum b_e \mathfrak{d}_e| \ge e^{-CB_o}$ for each required nonzero word, after allowing the protected movement of the spectator ratio. Here $\mathfrak{d}_e$ is the rational geometric defect paired with the actual (A, D) from (T1). The total $\sum |b_e|$ is bounded by the weak length since the row weak supports are disjoint. The same estimate, with the tested extra head corrections, applies to required wrong charged coincidences. In all cases the displayed remainder is $o(N^{-2}e^{-CB_o})$ after the scale choices. This explains separately why both the larger nonflat gap and the smaller geometric gap survive. The final data are not required to lie on the original flat.

Finally release all new clock and hot direct rates on a full cube, still pinning F_+. Write f_2 for the dimension including every accepted occupied site. The full inverse has Euclidean norm at most two as before, so $\|\Delta I\|_1 \le C f_2 \delta_1$ for cube halfwidth δ_1. For example one may take $\delta_1 = N^{-A_1}$ with the fixed exponent $A_1 = 4s + 30$. Here are the requirements this choice pays. The preliminary minimum hot floor is at least $N^{-2s}e^{-Cd\log(d+2)}$ from the finite endpoint (root amplitude r, private axis radii $r c_{n,k} b_p$, with $b_p = e^{-c_1 D_g}$ and $c_{n,k} \ge 1$). Let $I_{\min}$ be the minimum of all positive actions at the center of this cube, including the new hot axes. Require

$$C f_2 \delta_1 \le \min\Big\{ \tfrac{1}{4} I_{\min},\ c\, r^2 \xi^2,\ c\, \xi I_{w,\min}, c\,(\xi F_{\min})^2,\ e^{-C_7 B_o} \Big\}.$$

Then the normalized hot endpoint moves by at most ξ, by the square-root inequality after dividing by r, and the weak amplitudes, divided coefficients, old actions and all floors keep their earlier prices. In the full chart the high axial frequency is a direct coordinate and the transverse mean has the preceding small derivative in the clock variables. In the hot directions the pure old transverse means change only through the compensating occupied actions on the direct chart. Hence every fitted discrepancy moves by at most $Cq\sqrt{f_2}\delta_1$ times a bounded derivative factor, and all old-clock nonflat labels by $C_J B \delta_1$. In particular one can take $\eta_{\mathrm{rnd}} \le u\, 2^{-d} \exp(-LdD_g)$ in the finite incidence and divisor estimates. To check the displayed budgets, take Q sufficiently large that every fixed multiple of $C(D_{\mathrm{hist}}, k)B_o$ is smaller than the previously assigned fraction of ϵB, and take α small enough for the same to hold of the finitely many multiples of $d^2 \log B$. The prefactors $\mathrm{poly}(f_2, d, B)$ and e^{Cd} are absorbed by the remaining slack, increasing B_* last. This choice of A_1 does not depend on D_{hist}.

Choose the final point in this cube by the same affine slicing with δ_1, f_2 in place of δ_0, f_1, using the stored F_+ bound for pure-prefix forms. This registers (7.5) on all occupied sites for the next stage. For example the new prefactor is at least $N^{-(A_1+1)}$ once B_* is large: the new factor 2^{-Cf_2} costs less than N, and the two inherited prefactors exceed this bound after enlarging Q. Since $\log N \le (1+\gamma)B + O(1)$ this fixes a common C_{reg} at the outset, e.g. $2(A_1 + 2)$. Its use on the previous scale is not an input to A_1. The actual final least action is at least $N^{-(2s+4)}$ and the greatest site at most N^2. Incremental changes on each existing coordinate may be prescribed

below $e^{-D_{\mathrm{hist}}B_o}$ with D_{hist} greater than these coarse floor and historical exponents. Their later sum at a site born at stage i is $O(e^{-D_{\mathrm{hist}}B_i})$, smaller than half its birth floor after an initial enlargement of B_*. Rates and origins already accepted at previous stages remain exactly fixed. At the final frequencies use one fresh weak angle per disjoint row to set the constant argument of its actual nonzero driving harmonic. These private choices commute, do not move any action or rate, and are stored with all the other newly accepted angle origins.

7.7. **The axial potential on the transverse Fourier space.** We prove the estimates for the part of the transverse Hessian generated solely by the occupied axial field. They concern the operator on arbitrary inputs, including its full low horizontal subspace. No Fourier support or smallness condition on an input in that subspace enters this calculation.

Write F for the occupied sites including the prepared clock, before the current hot Cartesian axes are added, $f = |F|$, $n_F = (m)_{m\in F}$, $Z = 1 + \max_{m\in F}|m|$, and

$$V(I,\phi) = \Psi((\sqrt{I_j}e^{-i\phi_j})_{j\in F}), \quad M = \sum I_j, \quad a = 2(|V|^2 - M), \quad c = V^2, \quad \mathcal{D} = \Omega(I,0)\cdot\partial_\phi.$$

In this sign convention $\dot\phi = \Omega(I,0)$ on the isolated torus in its own mass gauge. Use the stage scales $m = Rq$, $N = RP$, $B_o = B/Q$, $\lambda = e^{-\epsilon B}$, and $E_c = \exp(C_{s,\epsilon}(B_o + d^2\log(2+B)) + o(B))$. A fixed number of changes in the constants of E_c is allowed.

7.7.1. *The Schur estimates and the role of the angles.* On real phases a is real. For all complex angle estimates use the analytic reflection of the conjugate field. The old Cartesian coordinates lie in a smaller complex ℓ^2 chart ball on $|\operatorname{Im}\phi_j| \le a_0$ with an extra $\epsilon B/6$ in each weak angle. Indeed their new weak radii are $\lambda^{1/5}e^{O(B_o + d\log d)}$ and their number is polynomial in d. Old and moderator coordinates use the fixed strip and their real total squared size in the small ball. The same conclusion holds on a sufficiently small relative action neighborhood. Fix the desired final number k of angle and action derivatives and any finite asymptotic orders before choosing ϵ. A fixed reduction of the common strip controls every such angular derivative, leaving, for example, width $\epsilon B/7$ in the weak angles above a smaller fixed width. Each genuine action derivative costs $E_c\lambda^{-C_s}$ from Cartesian square roots and the common action-frequency chart. Previous occupied floors and new moderator floors cost $e^{O_s(B_o)}$. Current hot floors are absent from F.

In reflected physical labels each angular coefficient of the old matrix shifts the horizontal coordinate by a prescribed signed old momentum $h = \pm\ell\cdot n_F$. Thus its absolute sum over angular labels, after the supremum over input frequency, bounds both weighted Schur sums. Use both directions of the weight ratio to include adjoints. For example the packet weights with polynomial factor $(1+|n|/N+|q|/P)^p$ and any of the capped exponential word factors have ratio bounded by $(1+|h|/N)^p\exp(|\ell|_1/B^{K_0})$, up to a fixed constant for the choice of signed generators. Here $K_0 > 5$ is the fixed exponent in the word weight. Multiplicative constants for a fixed change of copy can be included by using the word weight on the reflected copy induced by actual conjugation. Fixed spatial moments and nominal time derivatives cost powers of Z and Z^2, respectively, and powers of $|\ell|_1$. On a smaller strip the absolute sum costs C^f, since $\sum_\ell e^{-\delta|\ell|_1} = C_\delta^f$. These losses lie in E_c by the stage bounds. They also control derivatives on the enlarged weak strip standing behind the weak-label tails. In the following, matrices bounded in this translation algebra act both on ordinary L^2 and on the indicated weighted spaces. The assertion does not use a word-length lower bound on low indices.

7.7.2. *A spatial column, its scalar, and the empty axial frequency.* With $D_x = -i\partial_x$, reflect the lower horizontal index so that the two envelopes have the same spatial carrier. The mass-gauged

old operator on a nonzero vertical slice is

$$L_m = \begin{pmatrix} D_x^2 + m^2 + a & c \\ -\bar{c} & -(D_x^2 + m^2 + a) \end{pmatrix}, \qquad \Sigma = \operatorname{diag}(1, -1).$$

It is selfadjoint for the form defined by Σ. This doubled equation follows by differentiating the physical cubic and adjoining the conjugate equation. In the original, unreflected second index the pairing entry is $\widehat{c}(n + n')$.

Fix $H_{\mathrm{geom}} = C(1 + d)P$, with an absolute $C \geq 1$ chosen so that $|X| + |Y| \leq H_{\mathrm{geom}}$ for every marked head. Throughout this stage use $H_x = c_0 R / H_{\mathrm{geom}}$ with a fixed $0 < c_0 < 1/100$. On each nonzero vertical slice the low input projection is $Q_l = \mathbf{1}_{\{|n| < H_x\}}$. For a high input $|n| \geq H_x$, put $E_n = n^2 + m^2$, $\theta = n^2 / E_n$, $\delta = (2n)^{-1}$. A column $e^{inx}(f, g)$ with instantaneous scalar $E_n + e$ satisfies the two equations, up to the remainder of the truncation,

$$(2nD_x + D_x^2 + a - i\mathcal{D} - e)f + cg = 0, \qquad (2E_n + 2nD_x + D_x^2 + a + i\mathcal{D} + e)g + \bar{c}f = 0.$$

Expand f, g, e in powers of δ, with $f_0 = 1$, $g_0 = g_1 = 0$, and $\langle f_j \rangle_x = 0$ for $j \geq 1$. With missing coefficients zero the recursion is

$$\begin{aligned} e_j &= \langle a f_j + c g_j \rangle_x, \\ D_x f_{j+1} &= -(D_x^2 + a - i\mathcal{D}) f_j - c g_j + \sum_{l+k=j} e_l f_k, \\ g_{j+2} &= -2\theta \Big(\bar{c} f_j + D_x g_{j+1} + (D_x^2 + a + i\mathcal{D}) g_j + \sum_{l+k=j} e_l g_k \Big). \end{aligned}$$

The upper inverse is D_x^{-1} on zero spatial mean and costs at most one in its Fourier norm. The lower recurrence contains θ as a factor in every g_j. Even as $|m| \to \infty$, a nominal $2E_n$ acting on such a term costs at most $2n^2$ times its bounded coefficient with this factor removed. Retaining f through order $M_0 + 1$, g through $M_0 + 2$ and e through M_0 gives residual $E_c |n|^{-M_0}$ with the translation moments and angular derivatives above. Each coefficient uses at most $2(M_0 + 3)$ spatial and $M_0 + 3$ nominal time derivatives on any of its factors. Only a bounded number, depending on M_0, of factors occurs. Supremum in the input column followed by the absolute angle sum gives the same residual as an operator estimate.

For the column $h = e^{inx}(f, g)$ put $\mu = \langle h, \Sigma h \rangle = 1 + O(E_c / |n|)$ on real phases. Taking the signed norm derivative in $(L_m - i\mathcal{D})h = (E_n + e)h + R$ gives $\mathcal{D}\mu = -2 \operatorname{Im}(e)\mu + O(E_c |n|^{-M_0})$. The normalized column $\mu^{-1/2} h$ therefore has scalar $\Re e$ up to the same remainder. The analytic reflection used in μ is its holomorphic continuation on the narrower strip. Here $e_0 = e_1 = 0$, $f_1 = -D_x^{-1} a$ is purely imaginary, and $g_2 = -2\theta \bar{c}$. Bilinear integration by parts gives

$$e_2 = \langle a^2 \rangle - 2\theta \langle |c|^2 \rangle + \tfrac{i}{2} \mathcal{D} \langle f_1^2 \rangle.$$

Indeed $\langle a f_2 \rangle = -\langle f_1 (D_x^2 + a - i\mathcal{D}) f_1 \rangle$ and $\langle a f_1^2 \rangle = 0$. Thus the real order-two term in physical units is $\{(1 - \theta/2)\langle |V|^4 \rangle - M^2\} / n^2$.

Let $\alpha_{\mathrm{phys}} = -\operatorname{Im} \langle \bar{V}\, dV \rangle$ and $\alpha_{\mathrm{act}} = -\operatorname{Im} \sum \bar{z}_j\, dz_j$. Their difference pulled back to Cartesian coordinates is closed by canonicity. On the chart ball the radial formula gives its primitive $G(z) = \int_0^1 (\Psi^* \alpha_{\mathrm{phys}} - \alpha_{\mathrm{act}})_{tz}(z)\, dt$ with $G(0) = 0$. This primitive is invariant under the common phase and spatial translation, since both forms and radial contraction are. Its fixed derivatives have the same chart prices already established. In the ungauged physical flow the two evaluations are $\langle |V_x|^2 + |V|^4 \rangle$ and $\sum I_j \omega_j$. Passing to the mass gauge subtracts the same multiple of M from both. It follows with $\mathcal{H} = \langle |V_x|^2 + |V|^4 / 2 \rangle$ that

$$\mathcal{Q} = 2\Big(\sum I_j \omega_j - \mathcal{H} \Big), \qquad \langle |V|^4 \rangle = \mathcal{Q} + 2\mathcal{D}G, \qquad d\mathcal{Q} = 4M\, dM + 2 \sum I_j\, d\chi_j.$$

For the last identity use $d\mathcal{H} = \sum \omega_j\, dI_j$ and $\omega_j = j^2 + 2M + \chi_j$. Define $b_2 = (2/n^2 - 1/E_n)G$. Right multiplication of the column by e^{-ib_2} subtracts $\mathcal{D}b_2$ from its scalar. The order-two constant and its action derivative along any fixed parameter tangent are precisely

$$d_2(n,m) = \frac{\mathcal{Q} - M^2}{n^2} - \frac{\mathcal{Q}}{2E_n}, \qquad dd_2 = \frac{d\mathcal{Q} - 2M\, dM}{n^2} - \frac{d\mathcal{Q}}{2E_n}. \tag{T1}$$

There is no temporal divisor in these formulas.

Lemma 7.10. *On the isolated smooth occupied torus, the empty one-dimensional mass-gauged frequencies have an expansion, through any fixed order h_0, of the form $\chi_n = \sum_{j=2}^{h_0} c_j(I) n^{-j} + O(E_o|n|^{-h_0-1})$, with common coefficients for positive and negative n. Here E_o is a fixed polynomial in $1+Z$, e^f and the inverse occupied floors when corresponding derivatives are taken. In particular $c_2 = \mathcal{Q}/2 - M^2$.*

Proof. Use the direct envelope recurrence just proved at $m = 0$ on this one large input n, so $\theta = 1$, and truncate two orders past h_0. This step does not use any global off-copy diagonalization at $m = 0$. The scalar coefficients depend on θ and the old field but not on the sign of n. The axial mass derivative in the full physical linearization adds a rank-one term proportional, in the time generator, to the common phase tangent of V. Its forward Birkhoff differential is the corresponding phase tangent at the old Cartesian vector and has zero component at an empty site.

Apply the actual forward Birkhoff differential at V to the column at $m = 0$ and take its upper empty component $c_n(\phi)$. This is $1 + O(E_o/|n|)$: the input has H^1 norm $O(E_o|n|)$ and the nonlinear part of that differential maps H^1 to h^2, while the free upper carrier coefficient is $1+O(E_o/|n|)$. The projected L^2 residual has size $O(E_o|n|^{-h_0-1})$ by the L^2 chart differential. An empty coordinate has zero term from the variation of its frequency multiplying its background value. Therefore the actual linearized equation, with the carrier phase factored out, gives $i\mathcal{D}c_n = (\chi_n - e)c_n + O(E_o|n|^{-h_0-1})$. The periodic angle function c_n has a single-valued logarithm for large $|n|$. Divide by c_n and take the full angular Haar mean. The derivative of its logarithm integrates to zero for any real frequency vector. This gives the stated expansion; (T1) at $m = 0$ gives its leading coefficient. The remainder is uniform on the smaller old real or complex chart with the corresponding derivatives. □

7.7.3. *A low complement with a bounded lower-order part.* Now $m = Rq \neq 0$ throughout. First remove old anomalous entries in all horizontal frequencies. For $D_0 = \operatorname{diag}(D_x^2 + m^2, -D_x^2 - m^2)$ and the present off-copy upper block C_{nj} take

$$K_{+,-}(n,j) = -C_{nj}/(n^2 + j^2 + 2m^2), \qquad K_{-,+} = K_{+,-}^*.$$

The signed adjoint relation gives $K^*\Sigma + \Sigma K = 0$ and $[D_0, K] = -B_{\mathrm{off}}$. Its norm gains $(2R^2)^{-1}$ even on two low horizontal indices. With B the current bounded perturbation, the exact conjugation identity reads

$$\begin{aligned} &e^{-K}(D_0 + B - i\mathcal{D})e^K - (D_0 + B - i\mathcal{D}) \\ &\qquad = \int_0^1 e^{-\theta' K}\big([D_0, K] + [B, K] - i\mathcal{D}K\big)e^{\theta' K}\, d\theta'. \end{aligned}$$

The bounded kinetic commutator preserves its graph domain, first by approximation on finite Fourier inputs. The new off-copy remainder gains $E_c R^{-2}$ per step after a further fixed strip loss. At j_0 steps it is $E_c R^{-2j_0}$. The diagonal addition stays bounded, and each diagonal copy is selfadjoint for ordinary Hilbert space on real phases. The conjugate branch is obtained from the reality involution, and the resulting changes preserve the actual Fourier coset.

Apply the inverse of this change to the direct high columns above and project to the positive Hermitian copy. Write its generator as A, the resulting map from the high input subspace as T_h, and its real instantaneous high scalar as Λ. We can leave the exact phase b_2 and all other temporal phases until after the following completion. From the full column equation,

$$AT_h - i\mathcal{D}T_h - T_h\Lambda = R_h, \qquad \|R_h\| \le E_c(H_x^{-M_0} + R^{-2j_0}).$$

The discarded branch feeds back into this row only through the small off-copy remainder. The high columns are allowed to extend into the low horizontal output. For $G_h = T_h^* T_h$,

$$[\Lambda, G_h] - i\mathcal{D}G_h = T_h^* R_h - R_h^* T_h.$$

This derivation identity is valid with angle-dependent real Λ. The matrix T_h differs from the high inclusion by at most $E_c(H_x^{-1} + R^{-2})$; the convergent inverse-square-root series, with the product rule for the same derivation, proves that $U_h = T_h G_h^{-1/2}$ is isometric with the same residual price. Denote its new residual by R_h'. For the fixed low input projection Q_l put $\Pi = 1 - U_h U_h^*$ and define

$$U_l = \Pi Q_l \, (Q_l \Pi Q_l|_{\operatorname{ran} Q_l})^{-1/2}.$$

Since $\|\Pi - Q_l\| < 1$, $\Pi Q_l \Pi$ is invertible on $\operatorname{ran}\Pi$, so the range of U_l is all of that complement. The completed matrix $U = (U_h, U_l)$ is unitary. Its high-to-low block in $U^*AU - iU^*\mathcal{D}U$ is $U_l^* R_h'$, and its low-to-high block is its Hilbert adjoint. No truncation of a high column at its output is performed.

The compressed low block is free kinetic plus a bounded Hermitian operator of norm at most E_c. In fact a spatial shift by j in a nonconstant upper envelope costs $E_c/|n|$ with enough shift moments to pay $|(n+j)^2 - n^2| \le 2|nj| + j^2$. On individual entries of a Sylvester generator the unsigned horizontal kinetic commutator is bounded since $|n^2 - (n')^2| \le n^2 + (n')^2 + 2m^2$. Its products, adjoints, the Gram series and the low completion inherit this bound; both fixed projections commute with D_x^2. The vertical scalar m^2 commutes within a copy. This proves the assertion also for the indicated angle and action derivatives, with the latter's separate fixed floor prices. No smallness is asserted for the bounded part of this low generator in slow time.

7.7.4. *Evaluation of the diagonal at the final frequencies.* Let Ω^0 be the frozen flat clock, Ω_* the actual tilted terminal rates on F, and H_e its private row labels. In the retained angular range $|\ell|_1 \le C_J B$ nonflat integer combinations have distance at least λ^{C_s}/E_c from zero at the frozen vector. The change on F in passing to the physical rates is $O(N^{-2}E_c)$ and the final rounding is smaller, hence these lower bounds, with a factor $1/2$, persist on this range. A nonzero frozen-flat label that is neutral and spatially constant is $\ell = \sum b_e H_e$, $\sum b_e = 0$. For bounded weak length its coefficients have bounded $\sum |b_e|$ by their disjoint private supports. In the convention where the signed unperturbed time key is 1, the fitted rates satisfy

$$H_e \cdot \Omega_* = 1 + N^{-2}\mathfrak{d}_e + O(E_c\lambda^{-C_s}N^{-3} + E_c r^2 + \eta_{\mathrm{rnd}}).$$

Here $\mathfrak{d}_e$ is the signed sum of $(\mathcal{Q} - M^2)/v_x^2 - \mathcal{Q}/(2|v|^2)$ on the two inputs and output of that triangle, in root-normalized unshifted directions v. The anchor e_1, higher powers in the spatial expansion, and the terminal-axis action corrections are included in the displayed error; the actual finite fit uses the full means, not just (T1). The verified geometry and spectator separation required here are $|\sum b_e \mathfrak{d}_e| \ge E_c^{-1}$ on these nonzero tested words. The tilt and rounding errors above are $o(N^{-2}E_c^{-1})$ after their bounded summation, so $|\ell \cdot \Omega_*| \ge \frac{1}{2}N^{-2}E_c^{-1}$. No such lower bound for all weak lengths is used.

The real higher-order scalar satisfies $|d_{\ge 3}| \le E_c|n|^{-3}$. Its angular harmonics are neutral and have zero horizontal shift by covariance of the recurrence. Separate its full angle mean. Divide its remaining coefficients in the indicated temporal range and weak length $\le m_*$ by $i\ell \cdot \Omega_*$,

always at this one fixed vector, obtaining a real phase b_3. Omitted labels have sufficiently small *undivided* sum: by the two strip widths the two tails are bounded by

$$E_c(e^{-aC_JB} + e^{-\epsilon Bm_*/7})$$

on a smaller fixed strip, with $a > 0$. Choose C_J, m_* for the desired fixed target power, also paying any specified fixed derivatives. Then

$$\|\partial_\phi^j b_3\| \le E_c N^2 H_x^{-3} \le E_c' N^{-1}, \qquad \|\partial_I^k \partial_\phi^j b_3\| \le E_c' \lambda^{-C_s k} N^{-1}.$$

In either translation norm the entire spatial frame, including the two high scalar phases, differs from the identity by E_c'/R, also on free low inputs. Indeed $\|T_h - \text{inclusion}\| \le E_c(H_x^{-1} + R^{-2})$ implies the same estimate for $G_h - 1$, $\Pi - Q_l$ and $U_l - Q_l$ by the convergent square-root formulas. The two high scalar phases add at most $E_c' N^{-1}$, and $R/H_x = O(dP)$ fits the stated price. In particular every extra mixed cubic obtained by flipping the copy of a transverse argument with this transformation costs at most $E_c'/(R\lambda)$ in normalized field units.

Right-multiply the completed frame by $e^{-i(b_2+b_3)}$ on the high inputs, with the identity on the low inputs. The low generator is untouched. The high angle-dependent error under $\mathcal{D}$ is the sum of the spectral/angle tail and $-(\Omega(I,0) - \Omega_*) \cdot \partial_\phi b_3$. The high scalar mean itself and its action derivatives have size $E_c H_x^{-2}$ and $E_c \lambda^{-C_s k} H_x^{-2}$ after its free term has been removed. In particular differentiating the mean requires no temporal division. For truly moving coordinates one must add the chain-rule connection from $\dot I$ and $\dot\phi - \Omega(I,0)$ on the entire frame, including all high phases. The denominators in this differentiation stay fixed.

For use in real Hamiltonian coordinates write the finished symplectic frame as T, the constant complex structure as $\mathcal{J}$ with $\mathcal{J}^2 = -1$, and the real untransformed Hessian as B_{old}. Define $C = T^* B_{\text{old}} T + T^* \mathcal{J} \mathcal{D} T$ and $G_a = T^* \mathcal{J} \partial_{\phi_a} T$. Differentiating along an angle,

$$\partial_{\phi_a} C = T^*(\partial_{\phi_a} B_{\text{old}})T + \mathcal{D} G_a + (\mathcal{J}C)^* G_a + G_a \mathcal{J} C. \tag{T2}$$

Indeed $T^{-1}\partial_{\phi_a} T = -\mathcal{J} G_a$, G_a is symmetric, and ∂_{ϕ_a} commutes with $\mathcal{D}$ at fixed actions. For an action derivative the extra commutator $(\partial_{I_a}\Omega) \cdot \partial_\phi$ must also be included. The angle derivative of the high block of C has only the residual and frequency-difference prices just derived; an angle derivative of the compressed low bounded part can have size E_c.

To obtain a physical residual N^{-J}, one admissible choice is $M_0 = \lceil 2J + 12 \rceil$, $j_0 = \lceil J + 6 \rceil$, and angular tails initially N^{-2J-8}. For small enough macro-height exponent, $H_x, R \ge N^{3/4}$. A spatial derivative reservoir of $2(M_0 + j_0 + 6)$ and a nominal time derivative reservoir of $M_0 + j_0 + 6$ beyond the requested final derivatives suffice in these finite operations. First choose ϵ for the fixed action derivative losses, then Q large and the depth coefficient small to pay the finitely many E_c factors. All kinetic estimates hold uniformly in $|q| \ge 1$. Lemma 7.10 used only a large single horizontal carrier and the true axial Birkhoff differential, independently of this vertical gap.

7.8. **Estimates in the two-dimensional variables.** Throughout a stage put $N = RP_{\text{lat}}$, $r = HN^{-s}$, $\lambda = e^{-\epsilon B}$ and $u = r\lambda$; the slow time is $\tau = ut$. Use $\mathcal{E}$ for any one of the fixed products of prices $\exp(C_{s,\epsilon}[B_o + d^2 \log(2 + B)] + o(B))$ involved below and $\mathcal{P}$ when also a fixed power λ^{-C_s} may be included. A coefficient multiplied by a small power of λ will first be absorbed using only $\mathcal{E}$. In these notations the cutoff in the number of weak letters can enter $C_{s,\epsilon}$, but does not enter C_s in an uncompensated power of λ^{-1}. The latter powers arise from a fixed number of coordinate derivatives, the change of time and the noninteger-frequency gap. All spatial derivative and normal-form orders are fixed before ϵ; they can be taken larger than the specified cap powers as necessary.

7.8.1. *A Fourier algebra on the full lattice.* Write $k = (n, Rq)$ on the lattice with fixed vertical spacing. The charge-one coset inside it is invariant for the cubic flow: $k_1 - k_2 + k_3$ stays in the coset. Use the ambient lattice for products of other charges. Let $\ell(k)$ be the smallest number of signed generators summing to k, with value $+\infty$ outside their subgroup. Take as generators $e_1 = (1, 0)$, the axial sites $(m, 0)$ for $m \in F$ (including the current clock), and every physical marked head. On the signed second copy reflect k if using the same horizontal carrier convention. Put

$$b(k) = 1 + |n|/N + |q|/P_{\mathrm{lat}}, \quad W_A(k) = \min(N^A, e^{\ell(k)/B^{K_0}}), \quad X_{p,A} = \ell^2(b^p W_A).$$

Here $p > 2\lceil s \rceil + 10$, $A > 2s + 12$, $A' > A + 3s + 15$ are fixed, $K_0 > 5$ is fixed, and the geometric word test of Lemma 7.5 is used through $L = B^{K_0+3}$. Both W and b are submultiplicative; the polynomial alternative $b(k + k')^p \le 2^p(b(k)^p + b(k')^p)$ will also be used. There are $g = O_s(d^2 + |F_{\mathrm{old}}|)$ generators, where F_{old} is the support inherited from the preceding stage and the constant includes the fixed template length. The positions of length $\le CB^{K_0+1}$ number at most $(2CB^{K_0+1} + 1)^g \le \mathcal{E}$. The number bound counts integer multiplicities rather than ordered strings.

For either cap, if at least one input lies in these unsaturated positions, put that input in ℓ^1 after weighting using this count, and use submultiplicativity for the output. If both are saturated the ratio of their word weights to the output costs at most N^{-A} in the convolution bound. On ordinary $\ell^2(b^p)$ the product costs $O_p(\sqrt{NP_{\mathrm{lat}}})$, by the polynomial alternative and Cauchy–Schwarz with $\sum b^{-2p} \le C_p N P_{\mathrm{lat}}$. Thus $X_{p,A}$ is a convolution algebra with price $\mathcal{E}$. Its embedding into ordinary $\ell^1(b^q)$, $q \le p - 2$, also costs $\mathcal{E}$: use the count on the unsaturated part, and pay the same square root by N^{-A} on the rest. A signed partial shift by one angular label of the old torus and boundedly many heads has word-weight ratio at most $\exp[(|l|_1 + C)/B^{K_0}]$ in either direction. Its polynomial weight ratio costs a power of $1 + d + (|l|_1 + 1)Z/N$, where $Z = 1 + \max_{m \in F} |m|$. These estimates hold also on the stronger cap. The short-word faithfulness shows that every nonaxial vector with $|n| < H_x$ is saturated at both caps, since an old shift of $O(B^{K_0+3}Z) = o(R)$ cannot cancel a nonzero integer horizontal macro displacement.

Here is the version for analytic Cartesian axial coordinates. The inverse Birkhoff map and its inverse-coordinate counterpart are analytic on small complex h^0 balls with the Fourier linear normalization fixed above. For a derivative of order m divided by $m!$ about the old torus, polarization of the Cauchy estimate bounds its value on unit coordinate vectors in L^2 by C^m. On an old angle strip, a Fourier coefficient in these coordinates is supported at the single spatial frequency specified by the signed inputs plus $l \cdot n_F$, by translation equivariance. Cauchy in the angles supplies $e^{-a_0|l|_1}$ on a smaller strip. Taking absolute sums costs $C^{|F|}$; fixed translation moments are paid by the preceding polynomial weight. The same holds for the reverse signed kernel (or the canonical adjoint) by exchanging an output and an input in the momentum condition. Consequently the absolute kernel on m copies of $X_{p,A}$ has norm at most $C^m \mathcal{E}^{m+1}$; here the extra fixed power incorporates the linear derivative and can be taken once outside the convergent Taylor sum. Each successive input is accompanied by the physical factor r, and $r\mathcal{E}^C \ll 1$. For its derivative on one free argument in ordinary $\ell^2(b^q)$ put all other physical arguments in $\ell^1(b^q)$ using $q \le p - 2$. General symmetry or bounded analyticity alone on an unweighted Hilbert space would not imply this convolution estimate; the single signed output is essential. For a derivative on occupied actions, differentiating the old Cartesian radii may cost the inverse positive floors; for any fixed derivative order this costs $\mathcal{P}$. Current high hot axes have Cartesian, not polar, coordinates.

7.8.2. *The projected reference and its actual coefficients.* In the autonomous mass gauge subtract M^2 from the Hamiltonian. Let z_F have occupied polar coordinates (I, ϕ) and vanish off F, let a

be supported on F^c, and write w for the physical nonaxial part. Put $z = z_F + a$, $V = \Psi(z)$, $V_F = \Psi(z_F)$, $M_a = \|z\|_{\ell^2}^2 = \|V\|_{L^2}^2$ and $M_w = \|w\|_{L^2}^2$. The exact pure axial Hamiltonian is the one-dimensional action Hamiltonian in the same gauge. Apart from the transverse kinetic term, its mixed extension is

$$2\langle(|V|^2 - M_a)|w|^2\rangle + \mathrm{Re}\langle V^2\bar{w}^2\rangle + 2\,\mathrm{Re}\langle\bar{V}w|w|^2\rangle + \tfrac{1}{2}\langle|w|^4\rangle - M_w^2,$$

where the brackets denote spatial integral. Vertical momentum eliminates every term linear in w. In particular $D_a M_a = 0$ at the empty Cartesian origin. At $a = 0$ remove the pure quadratic transverse part by the spatial construction. The leading remaining mixed Hamiltonian, homogeneous of field degree three, is

$$4\langle\mathrm{Re}(\bar{V}_F D\Psi(z_F)a)|w|^2\rangle + 2\,\mathrm{Re}\langle V_F D\Psi(z_F)a\,\bar{w}^2\rangle + 2\,\mathrm{Re}\langle\bar{V}_F w|w|^2\rangle.$$

There are at least two transverse factors in every term. The other terms start at degree four. After substituting the spatial frame its derivative, canonical signed adjoints and small axial coordinate corrections obey the preceding kernel estimates. In particular at the terminal configuration with $w = 0$ the mixed field vanishes identically. The rank-one mass factors are bounded by Cauchy–Schwarz on their scalar input, putting a positive spatial weight on their known output when the free variable is in the scalar. The exact pure axial Cartesian field is diagonal, with coefficient $n^2 + \chi_n(I, |a|^2)$; its nonlinear part has bounded first and second action derivatives as a map on ℓ^2. Multiplying a frequency derivative by a known vector x_n in $\ell^2(b^q)$ costs at most $\|b^q x\|_\infty \|D\chi\|_{2\to 2}\|\delta I\|_2$. No weighted mapping theorem for $D\chi$ is needed.

For the finite reference keep the full projection of this leading degree-three Hamiltonian on the marked finite planes, in the Fourier part of the mixed coefficients with $I = I^*$, the rounded terminal action vector restricted to F, and angles advancing at their accepted terminal rates. Keep the actual pure axial action detuning on the marked axes. Retain the exact real constant spatial means on high transverse carriers to the prescribed order and the true Birkhoff frequencies on terminal axes. Corrections from the high spatial frame and $D\Psi$ on the actual marked inputs are $\mathcal{E}/R$ in the projected coefficients and the corresponding two-known sources, as the marked nonzero horizontal macro is an integer. The general unmarked homogeneous axial approximation above $N^{1/4}$ costs only $\mathcal{E}N^{-1/4}$ and is used solely where applicable. The full one-dimensional projection in low empty modes is treated separately in the exterior estimate. Every assigned triad of distinct heads has coefficient $2\overline{v_e}$ on $z_v z_w \bar{z}_h$, with the charge-one old Fourier harmonic v_e specified in the calibration. This factor follows by permuting the two positive and the two negative entries of the physical quartic with prefactor $1/2$. The same coefficient results from each of the three displayed mixed terms in its appropriate axial/nonaxial case. Fit $2\overline{v_e}/\lambda$ to the imaginary coefficient of Lemma 7.6 using one new angle per row. The target accuracies in the amplitude are absolute relative to the prescribed row strengths, whose logarithms cost $O(D + d\log d)$.

In the common mass gauge, at the rounded terminal actions let ν_j be the true axial Birkhoff frequency on an axial head and the nominal carrier frequency including the retained constant diagonal corrections on a transverse head. For the edge $e = (v, w; h)$ put $\Lambda_e = H_e \cdot \Omega_*$ and $d_e = \nu_v + \nu_w - \nu_h - \Lambda_e$. The old harmonic carries the factor $e^{-i\Lambda_e t}$. Its conjugate in the Hamiltonian term $2\overline{v_e} z_v z_w \bar{z}_h$ therefore contributes $e^{+i\Lambda_e t}$. After the true physical fit in the axial parameters one can insist on $d_0 = \max_e |d_e| \le u2^{-d}e^{-C_4 dD}$. Move only the nonaxial rotating frequencies, writing $z_j = e^{-i(\nu_j + x_j)t} y_j$ with $x_j = 0$ on every axial head. The phase of the displayed term is $e^{-itd_e^{\mathrm{rot}}}$, where

$$d_e^{\mathrm{rot}} = (\nu_v + x_v) + (\nu_w + x_w) - (\nu_h + x_h) - \Lambda_e = d_e + x_v + x_w - x_h.$$

Thus the choice $x_h = x_v + x_w + d_e$ gives $d_e^{\mathrm{rot}} = 0$ on each primary edge, and the conjugate monomial has the opposite, also zero, mismatch. At the fixed top choose shifts on the two top children whose sum is $-d_{\mathrm{top}}$ and whose magnitudes are at most d_0. In each branch give all its leaves the same shift ξ, then recurse upward. A head m levels above the leaves has shift $2^m\xi$ plus a binomial sum of mismatches with coefficient sum at most $2^m - 1$. The branch root condition has a solution $|\xi| \le 2d_0$, and all intermediate shifts are $\le 3 \cdot 2^d d_0$. The degree-one private companion absorbs the remaining mismatch in its own row. For $e = (v, J(v); A(v))$, put $x_{J(v)} = -d_e - x_v$. Since $x_{A(v)} = 0$, this gives $d_e^{\mathrm{rot}} = d_e + x_v + (-d_e - x_v) - 0 = 0$. The diagonal perturbation in slow units is thus $O(e^{-C_4 dD})$. The designated monomials thus have no explicit time phase in these rotations. The action-dependent detuning stays in the nonlinear vector field, and every hot axis uses its actual terminal frequency; all old phases are still advanced in the original physical time.

In slow time write the finite vector field as $F_0 + S + G$ on the Euclidean tube of radius one about the reference and on segments to its prescribed endpoint, contained in a ball e^{Cd}. The small part S contains the fitting errors just described, the true hot action detuning of C^1 norm $\mathcal{E}r/\lambda$, and discarded angular/weak tails when already small. The other time-dependent terms have a primitive P at fixed state. For a noninteger class the original physical old coefficient is bounded by $\mathcal{E}$ and its inverse gap by $\mathcal{P}$; for each nearly resonant wrong charged row its summed physical old coefficient is bounded by $\lambda\mathcal{E}$ and inverse gap by $N^2\mathcal{E}$. The very long words in the projected system may be kept as a small C^1 part. The two gap and tail estimates are justified by the axial choice and the external classifications. Taking two finite state derivatives still costs only $\mathcal{E}$. Hence in the slow equation

$$\|P\|_{C^2} \le \mathcal{E}r\lambda N^2 + \mathcal{P}r, \qquad \|G\|_{C^2} \le \mathcal{E}/\lambda.$$

The explicit substitution $z = y + P(\tau, y)$ gives the exact field $(1 + DP)^{-1}[F_0(y + P) + S(\tau, y + P) + G(\tau, y + P) - G(\tau, y)]$. The error relative to F_0 is bounded in C^1 by the small S plus $\mathcal{E}rN^2 + \mathcal{P}r$; in particular there is no division by λ on the former price. Every G and P vanishes at the all-axial point, by vertical momentum. The true pure axial detuning also vanishes there. Undo the finite change after applying Lemma 7.7 at $\sigma = \kappa/3$. Its landing then supplies the two strictly smaller weights $\kappa/4$ and $\kappa/8$ for the full equation. The conjugacy and its differential have uniformly bounded norm. This gives the true finite reference and its full complex Green in the original rotations. In the latter its slow envelope derivative, after subtracting the endpoint on the decaying tail, can cost $\mathcal{E}/\lambda$. In differentiating each covariant particular or homogeneous primitive below pay this once; never differentiate a fast physical phase separately. The finite reference and its derivative may be used on the entire interval from its cold launch to its prescribed hot endpoint.

7.9. **The exterior field and boundary interactions.** Fix $s > 2$. We give the comparison in physical time before passing to the slow time $\tau = ut$, where $u = r\lambda$, $r = e^{bd}N^{-s}$, $N = RP$, and $\lambda = e^{-\epsilon B}$. Write $X \preceq Y$ for a bound $X \le E^C Y$, where

$$E = \exp\{C_{s,\epsilon}(B_o + d^2 \log(2 + B)) + o(B)\}.$$

Each use below costs a fixed power, independent of the depth. Write $L = \lambda^{-C_s}$ for a fixed power sufficient for the noninteger and differentiated small-angle prices. Fixed powers of L are allowed upon decreasing ϵ first. The spatial word cutoff and the weak angular cutoff enter the exponent of E only. Let F denote the old occupied, spectator, moderator and weak axial sites, excluding current hot axial heads. The field of this torus is $V_F = \Psi(z_F)$. Its complex angular strip gives absolute Fourier sums of size $\preceq 1$, including a fixed number of angle derivatives. We cut at $|j|_1 \le C_J B$ in every mixed principal matrix and primitive. For larger $|j|_1$ the ordinary strip

bound, including $\sum_j e^{-c|j|_1} \le C^{|F|}$, pays any prescribed N^{-J} by increasing C_J. A translated word weight costs at most $e^{O(|j|_1/B^{K_0})}$ and is included in this estimate. In a product of finitely many angular series one may first cut each factor separately, then enlarge the cutoff for their total label by this fixed number of factors. Indeed the convolution with any discarded tail remains small on the narrower strip, even if large opposing labels have small sum. Only the compressed pure-old low body, at each fixed vertical label, will be kept without an angular cut.

We use the following specific inputs from the geometric and calibration arguments. The integer macros (X, Y) of the marked heads obey $X \neq 0$, $|X| + |Y| \le H_{\mathrm{geom}}$. The common physical coset is

$$(k_x - 1) \equiv l k_y \pmod 5, \qquad l^2 \equiv -1 \pmod 5, \qquad k_y \in R\mathbb{Z}, \quad (R, 5) = 1.$$

For every nonaxial marked macro $Y \not\equiv 0 \pmod 5$. A signed macro word of length at most B^{K_0+3} with zero horizontal coordinate has zero vertical coordinate. Old axial translations of this length have size $o(R)$. On a retained charged old angular label, with physical space/time keys b, ν, we have $b \equiv 1 \pmod 5$ and $|b|^2 + |\nu| + 1 \le R^\eta$, with a small fixed η. Before the final $O(N^{-2}E)$ tilt, a nonmodule gap of the form integer plus old frequency has reciprocal $\preceq L$. A charge-one label in the frozen module has bare space and time keys $1, 1$. Its five-weak-letter terms are the designated singletons, of summed size $\preceq \lambda$. All nonsingleton charged module terms have at least fifteen weak letters. After normalization and summation their contribution to a homogeneous field row is $\preceq N^{-h_w}$, where $h_w > 0$ is the weak angular tail exponent. On a source with two prescribed inputs one retains every weak length up to the larger fixed forcing cutoff before using its tail estimate.

For a bare anomalous split of an arbitrary unmarked high leg from a marked head, the two-component test separates every undesignated singleton. For a bare split or fusion involving two marked nodes the test instead separates every undesignated charged label through the stated larger cutoff. These are the two different nonidentity statements required of the geometry. The spectator lies outside the forbidden ratios of the resulting nonzero rational component pairs. Marked temporal carriers have already been calibrated to within $O(u\eta_0)$ of the genuine constant high diagonals, where η_0 is the prescribed finite shadow tolerance. On the marked coordinates we may put their calibrated velocities κ_k in the nominal diagonal D and put the discrepancy in the finite equation. On the unmarked high coordinates D uses the true high diagonal through the fixed order of the spatial reduction. On the transverse low coordinates its comparison value is free kinetic. The actual bounded low matrix is left on the left-hand side. Doubled coordinates have diagonal signs $\varepsilon_k = \pm 1$.

7.9.1. *The two low density stars.* Set $h = (h_x, RY)$, $h_x = 1 + RX$, and write the input (n, Rq). The same-copy density shift goes to $(n + h_x - b, R(q + Y))$, and the opposite-copy pairing goes to $(h_x + b - n, R(Y - q))$. After subtracting the time of the moving head, their exact free mismatches are

$$D_F = 2(n - b)(RX + 1 - b) + 2R^2 qY + \nu - b^2, \tag{7.7}$$

$$D_A = 2(n - b)(n - 1 - RX) + 2R^2 q(q - Y) + b^2 - \nu. \tag{7.8}$$

In particular the old harmonic need not have time b^2. The true constant high corrections in the comparisons made here are $\preceq N^{-2}$, uniformly in q, since the high horizontal abscissae have magnitude at least H_x and $P, H_{\mathrm{geom}} \preceq 1$.

A transverse end is low precisely when $q \neq 0$ and $|n| < H_x$. On a low input to (7.7), a nonzero integer qY dominates the remaining terms, giving a gap $\gtrsim R^2$. Otherwise $Y = 0$. For $n \neq b$ the

gap is $\gtrsim R$, and for $n = b$ retain the entire finite-angular density star

$$(b, q) \longleftrightarrow (1 + RX, q)$$

in coordinates (physical abscissa, vertical integer). For a low output put $\vartheta = n' - (2 - b)$. Direct substitution in (7.7) gives

$$2R^2(q'Y - Y^2 - X^2) + 2(RX + 1 - b)\vartheta + 2(1 - b)^2 + \nu - b^2.$$

If the first parenthesis is nonzero it dominates. If it vanishes, $Y \neq 0$, $q' = Y + X^2/Y$ is an integer, and $\vartheta \neq 0$ gives a gap $\gtrsim R$. Retain the other finite-angular star and its reverse,

$$(1 - RX, X^2/Y) \longleftrightarrow (2 - b, Y + X^2/Y).$$

All retained star entries have their actual time-dependent coefficients and Hermitian adjoints. In (7.8) the two legs can be interchanged. At a low leg only $q = Y$ can remove the R^2 term. Then $n = b$ is impossible, since $(n - 1, Rq) = (0, RY) \bmod 5$ lies outside the indicated line. Every remaining gap is $\gtrsim R$. Thus every mixed primitive touching the true transverse low matrix has a large divisor. The common choice $C_J B$ in the stars is essential to the word estimate below.

7.9.2. *Bounded components at arbitrarily large vertical indices.* At a zero bare same-copy high gap with a module label, put $w = ((n - 1)/R, q)$. The retained high graph has edges $w \leftrightarrow w + v$, $w \cdot v = 0$, $v = (X, Y)$, after the finite graph and its divided boundary incidences are removed. The first coordinate of w can be nonintegral. In a run of parallel directions every edge connects a common perpendicular foot to a point on the line through that foot in the given direction. A simple parallel run thus needs at most two steps. Each switch between nonparallel directions occurs at one of the at most four intersections of the two base/translated line pairs for the specified labels. For g heads a simple shortest high-only path has at most $2(1 + 4g^2)$ steps, also after removing edges.

Join all transverse low coordinates at each fixed q for the purpose of forming components. The pure-old off-copy block has been removed to the prescribed order using $n^2 + (n')^2 + 2R^2q^2 \geq 2R^2$ and the paired symplectic transformation. In the remaining same-copy compressed low potential the kinetic term R^2q^2 cancels from all commutators at fixed q. The bounded low part has operator norm $\preceq 1$ by the old angular moment estimates. It can depend on the actual old parameters throughout.

A reverse nonaxial low star has both ordinate magnitudes bounded by $2H_{\mathrm{geom}}^2$. Hence a low slab with $|q| > 4H_{\mathrm{geom}}^2$ can leave through an axial star only. Its high end has macro abscissa $|x| \leq H_{\mathrm{geom}}$ and the same q. On any nonaxial bare density edge there one would have

$$|qv_y| \leq |xv_x| + |v|^2 \leq 3H_{\mathrm{geom}}^2, \qquad |v_y| \geq 1.$$

Axis-only steps keep q and visit only $x = 0$ or $|x| \leq H_{\mathrm{geom}}$ before another slab. The entire component therefore has this same q. No component containing a low slab with $|q| \leq 4H_{\mathrm{geom}}^2$ can reach a larger low ordinate. On that component every high vertex has a high-only path of at most $2(1 + 4g^2)$ steps to an incident star. Both its macro coordinates are bounded by $C(1 + g^2H_{\mathrm{geom}} + H_{\mathrm{geom}}^2)$, regardless of the number of attached low slabs. This proves that the ratios of $b_k^p = (1 + |k_x|/N + |q|/P)^p$ on any such whole component are $\preceq 1$. On a component disjoint from the low slabs the same bound follows from the length of a shortest path, even when all its vertices have large abscissa or ordinate.

For $W(k) = \min(N^A, e^{\mathrm{length}(k)/B^{K_0}})$, every transverse low vertex has length $> B^{K_0+3}$. Indeed the low horizontal coordinate forces zero macro abscissa by integrality and the $o(R)$ old shifts, contradicting nonzero vertical coordinate by short-word faithfulness. To reach a low slab from any high vertex in its component, at most $O(g^2)$ high-only steps and one star step with at most $C_J B + O(1)$ letters suffice. All vertices of such components are saturated. Elsewhere the ratio

of word weights is bounded by $\exp(O(g^2)/B^{K_0}) \le 2$. In particular neither low star contains a marked high endpoint. In the reverse star, matching a marked abscissa $-X$ would force its ordinate to be $-Y$ by faithfulness, while the actual ordinate is X^2/Y.

Replace Wb^p by its component infimum, preserving an equivalent Hilbert norm up to $\preceq 1$. The retained principal same-copy matrices, including the low body, are Hermitian with their actual phases. Their propagation is unitary in this flattened norm. The kinetic diagonal also commutes with the weight. In the slow-time norm with weight one until T and $e^{\sigma(\tau-T)}$ afterwards, the backward integral has norm $\preceq T + 1/\sigma$. The same proof works when the word weight is omitted, at any fixed order p. No price here grows with an upper cutoff for q.

7.9.3. *The actual near divisors.* In a nonmodule high/high gap use the affine inequality for $a + j \cdot \chi_F$ at the frozen flat. The possibly very large integer $j \cdot (n_F^2)$ is included in a. A dangerous value requires $|a| \le C(1 + |j|_1)$ since χ_F is bounded. The cutoff therefore gives reciprocal $\preceq L$, independent of the two free end momenta. The final tilt, summed over the retained labels, and the high diagonal changes are $\preceq BN^{-2}$, which preserve this gap. On a module label a nonzero bare kinetic integer has absolute value at least one. At a zero bare anomalous gap equation (7.8) becomes

$$x^2 - Xx + q(q - Y) = 0, \qquad x = (n-1)/R.$$

The rational root x is integral by monicity. All these legs lie on the circle of diameter $[0, (X, Y)]$, giving at most CH_{geom}^2 possible integer points per head. On a zero same-copy boundary incidence from a marked head the remaining macro is likewise integral by translation of two given ones. At each high nondegenerate leg its two reciprocal-square components have height at most a fixed power of H_{geom}. For the undesignated pair let U, V be the differences from the prescribed clock components in root-normalized coordinates. With $(\mathcal{A}, \mathcal{D}) = (\mathcal{Q} - M^2, \mathcal{Q}/2)$ evaluated at the final rounded background on F as in (T1), the terminal fit in Section 7.6.3 gives the signed gap

$$\Delta = N^{-2}(\mathcal{A}U - \mathcal{D}V) + O_{s,\epsilon}(N^{-3}\lambda^{-C_s}E + r^2E + \eta_{\mathrm{rnd}}).$$

The separation $|\mathcal{A}U - \mathcal{D}V| \ge e^{-CB_o}$ persists at this background, and the remainder is $o(N^{-2}e^{-CB_o})$ for the same C by Section 7.6.3. Thus $|\Delta| \ge \frac{1}{2}N^{-2}e^{-CB_o}$ at all sufficiently large scales, giving $|\Delta|^{-1} \le 2N^2e^{CB_o} \preceq N^2$. The true high axial frequencies on empty coordinates use the axial expansion at the old torus, with terminal action changes $O(r^2E^C)$ harmless at this precision. No Diophantine bound over arbitrary points of an unbounded annulus is used.

7.9.4. *An empty axial Birkhoff direction.* Let $h = (h_x, RY)$, $Y \ne 0$, be a moving head and let $e_n^{\tau_n}$ be the unit direction at $n \notin F$ in the signed Cartesian copy $\tau_n = \pm 1$. Write $\mathcal{R}(z) = |\Psi(z)|^2 - \|z\|_{\ell^2}^2$. At $a = 0$, hence $z = z_F$, the two differentiated transverse coefficients are

$$2D\mathcal{R}(z_F)[e_n^{\tau_n}], \qquad 2V_F\, D\Psi(z_F)[e_n^{\tau_n}].$$

Here the real differentials are extended complex linearly to the signed copies. For a real variation v supported on F^c, $D\mathcal{R}(z_F)[v] = 2\,\mathrm{Re}(\overline{V_F}D\Psi(z_F)[v])$, since the mass differential vanishes in that direction. By spatial translations their coefficient for old angular label j and input copy τ_n is supported only at $\ell = \tau_n n + j \cdot n_F$. Its modulus is bounded by $Ce^{-c|j|_1}$, uniformly in n, by the analytic L^2 chart differential and the uniform old H^1 bound on the strip. In the density differential all coefficients with $\ell = 0$ vanish exactly, since $\int_{\mathbb{T}} \mathcal{R}(z)\, dx/(2\pi) = 0$ for every nearby z. This uses the full Birkhoff differential. Its old labels need not have charge one. For example on the complex zero-action face with a positive old coordinate z_p and a negative test at $n \ne p$, the dressing formula gives the term $-z_p^3 e^{i(3p-n)x}/(p-n)^2$ in $D_{z_n^-}(\Psi^2)$. The following small-index estimates allow any such charge. Only the free Fourier part of the large-index differential uses the charge-one gap formulas.

Take $M = N^{1/4}$ and ensure $(1 + \max_{m\in F} |m|)^2 C_J B \le N^{1/4}$. For $|n| \le M$ we have $|\ell| \le 2M$, $|j \cdot \Omega| \le CM$ and the true empty axial frequency satisfies $|\omega_n| \le C(1+n^2)$. For density, the two transverse indices differ by $(\ell, 0)$ and their true gap has magnitude $\gtrsim R$: the nonzero bare difference is $|\ell(2h_x + \ell)| \gtrsim R$ and all extra times are $O(N^{1/2}) = o(R)$. For the pairing differential the outgoing index is $(-h_x + \ell, -RY)$, and the sum of the two transverse energies is at least $2R^2$. Both estimates hold also on the reverse rows, as the real Hessian in canonical axial coordinates has the corresponding signed transpose. For each fixed j, τ_n, h the discarded/divided matrix is a partial translation or reflection on n, so its weighted Schur price has no factor counting $|n| \le M$.

When $|n| > M$, the coefficient at $k = \tau_n n + j \cdot n_F$ of the difference between the true chart differential and its free Fourier part has modulus at most

$$Ce^{-c|j|_1} \frac{\langle n\rangle}{\langle k\rangle^2} \le C' e^{-c|j|_1} |n|^{-1} \quad (|j|_1 \le C_J B).$$

This is the one-derivative gain $h^1 \to H^2$ of the differential. Use the basic L^2 differential bound on the discarded angular tail. After summation the true minus free one-head matrix in this range costs $\preceq r/M$, including both signed transposes. On a prescribed high axial input $|n| \ge H_x$ the cost improves to $\preceq r/R$. For $M < |n| < H_x$ the free Fourier part is covered directly by (7.7)–(7.8) with axial endpoint $q = 0$. If the other head is transverse, the forward density divisor has main term $2(n - b)(RX + 1 - b)$, and the pairing has main term $2(n - b)(n - 1 - RX)$, both $\gtrsim MR$. At a low axial destination in the reverse density placement, the integer in the first parentheses of the reverse formula is $-Y^2 - X^2 \ne 0$. Above that slab the high classification applies. Terms with only axial fields are already represented by the exact axial action-frequency equation. Varying the axial chart point from the prescribed old torus by the other physical packet changes these differentiated coefficients by $\preceq r$, by its analytic translated Taylor expansion and the known packet Wiener norm. This gives a physical one-head matrix error $\preceq r^2$.

7.9.5. *The signed boundary Hessian and the divided source.* For an undesignated cubic Hamiltonian monomial in the field coordinates, write the complexified term and its conjugate as

$$c_j e^{-i\nu t} \prod_{m=1}^{3} z_{a_m}^{(\delta_m)} + \text{conjugate}, \qquad z^{(+)} = z, \quad z^{(-)} = \bar z, \quad \delta_m = \pm 1.$$

The momentum shifts use these same signs. Differentiate in slot three to form its field equation. For each of the other two signed inputs write $b_m + \xi_m$, where b_m is its prescribed value and ξ_m its variation in physical units. At a marked slot $b_m = r g_m(\tau) e^{-i\kappa_m t}$, with conjugation when required. After factoring out the common coefficient and the Hamiltonian sign, the two inputs contribute

$$(b_1 + \xi_1)(b_2 + \xi_2) = \underbrace{b_1\xi_2 + \xi_1 b_2}_{\text{one prescribed input}} + \underbrace{b_1 b_2}_{\text{two prescribed inputs}} + \xi_1\xi_2.$$

The first term gives a linear operator on the variation. Differentiating in the other slots and including the conjugate monomial supplies the reverse entries of the real Hessian. The second term is a forcing in slot three that is already present at zero variation. In physical units the matrix coefficient carries one factor r, and the forcing carries two. In the signed rotating variables both have the same scalar mismatch up to the sign of the orientation,

$$\Phi = \nu + \sum_m \delta_m D_{a_m}.$$

Here the signs of ν and of all slots are understood in the displayed Hamiltonian convention. Taking $D_{a_m} = \kappa_{a_m}$ on marked slots makes this statement exact, including when the free variation in one of those slots belongs to the finite block. Using the genuine high diagonal there instead

changes all comparisons by at most $3u\eta_0$, well below the lower high/high gap. A repeated index carries its elementary multiplicity in the Hessian. Every derivative of a mixed finite/exterior edge, including an exterior input and a finite output, is placed in the field matrix. Removing only an inhomogeneous particular would leave one or both of these homogeneous rows.

In doubled variables put $D_{ii} = \varepsilon_i D_i$. If a matrix entry is $B_{ij} = A_{ij}(\tau)e^{-i\zeta t}$, set

$$\Delta_{ij} = \varepsilon_i D_i - \varepsilon_j D_j - \zeta, \qquad K_{ij} = -B_{ij}/\Delta_{ij}.$$

Reverse signed Hessian entries satisfy $B_{ji} = \varepsilon_i\varepsilon_j\overline{B_{ij}}$ with frequency $-\zeta$, so $K_{ji} = -\varepsilon_i\varepsilon_j\overline{K_{ij}}$. Use $\exp K$ including the reality-related entries. Each retained head and angle acts by a partial shift or reflection in either direction, bounded by its coefficient times the ratio of radial weights and $e^{O((1+|j|_1)/B^{K_0})}$. The forward and reverse radial ratios are both bounded by a fixed power of $1 + H_{\mathrm{geom}}/P + |j| \max_{m\in F} |m|/N$. The divided homogeneous operator has accordingly no cardinality cost from free endpoints.

The exact cancellation uses the covariant combination

$$([D, K] - iK_t)_{ij} = -B_{ij} + iu\,\Delta_{ij}^{-1}A'_{ij}(\tau)e^{-i\zeta t}.$$

For B_{ij} here its frozen coefficient c_j and temporal divisor are evaluated at fixed terminal parameters. The evolving prescribed envelopes satisfy $|g| + 1 \preceq 1$, $|g'| \preceq \lambda^{-1}$ even before removing their fast finite harmonics. Variation of the actual old parameters in the original coefficients goes into the old-to-field derivative on a known profile or a small coefficient times an additional free field. In particular no derivative of a nearly zero Δ in an old action is used. For the free part of the conjugation,

$$e^{-K}De^{K} - ie^{-K}\partial_t e^{K} - D = \int_0^1 e^{-aK}([D, K] - iK_t)e^{aK}\,da.$$

Thus subsequent terms of this conjugation use an additional factor $\|K\| = o(1)$. Against a strong physical one-head matrix of size $\preceq r$, a near primitive costs $r(r\lambda N^2)/u = rN^2$ in slow units. The retained bounded low operator commutes exactly with the part of K supported on two high endpoints. Its commutator with the remaining part costs only $\preceq r/R$ in physical units, even if that low operator is time dependent.

For the independent two-known source into a high exterior coordinate, subtract its signed particular divided by the same true mismatch. Its physical numerator is $r^2c_j g_a(\tau)g_b(\tau)e^{-i\zeta t}$ with conjugations as appropriate. In normalization by r the particular is of the same order as K below. Applying a strong one-head matrix and dividing the physical forcing by ru costs at most the corresponding last-column price. Differentiating $g_a g_b$ costs at most one λ^{-1} in slow time. Every independent mixed boundary source contains a decaying transverse input and hence vanishes at the axial limit. Contributions with only axial inputs are included in the integrable axial flow. The operator K can have a nonzero terminal value and is used as a change of variables, not as a decaying independent forcing. Actual endpoint and section coordinates are transformed along with it.

The following table suppresses factors $\preceq 1$. In the middle column the physical primitive of a two-known source is the listed quantity multiplied by r. The last column includes a strong one-head commutator and one slow envelope derivative, in the normalized evolution units u for

matrices and ru for independent sources.

class	$\lvert B_{\mathrm{phys}}\rvert$	$\lvert K\rvert$ or $\lvert p_{\mathrm{phys}}\rvert/r$	normalized error
near high/high	$r\lambda$	$r\lambda N^2$	rN^2
noninteger or fast high/high	r	rL	rL/λ
divided low incident	r	r/R	$r/(R\lambda)$
low body commutator			$1/(R\lambda)$
empty axial differential error ($\lvert n\rvert > M$)	r/M		$1/(M\lambda)$

For the column $|B_{\mathrm{phys}}|$ the corresponding physical two-known source bounds are $r|B_{\mathrm{phys}}|$. Long angular tails cost any prescribed N^{-J} before a fixed finite number of normalizations. Long weak tails in a homogeneous field row and the stricter independent-source tail are estimated on their appropriate separate cutoffs. Since $s > 2$, $rN^2 = e^{bd}N^{2-s}$ has a fixed negative power margin. The remaining displayed prices are small after the stated order of $\epsilon, Q, d/\sqrt{B/\log B}$ and the initial scale. These estimates concern the exterior principal inverse and its interface with the finite graph. The growing branch and its prescribed stable data still require the finite dynamical inverse and the separate nonlinear feedback estimates.

7.9.6. *A separate estimate for the evaluated source.* The linear term and the forcing term in the preceding expansion require different weak angular cutoffs. A homogeneous tail of norm EN^{-h_w} multiplies the unknown variation. Composing it with the field inverse, of norm at most E, costs $E^2N^{-h_w} = o(1)$, so it can be absorbed. The forcing with two prescribed inputs remains when the variation is zero. Its image under the same inverse must fit inside the field correction radius $\rho = N^{-a}$ of Section 7.10. The exponent h_w can be smaller than a. We therefore keep and test the forcing terms through the longer weak angular cutoff estimated below.

Let $m(\tau)$ be the actual finite trajectory in normalized field units r, and Π its full marked projection. In defining m, keep on Π every term for which the only available normalized estimate is EN^{-h_w}. Such terms qualify for the small perturbation in the finite contraction and are included in the exact reference. Only the stronger independent-source bound permits omitting them when the residual is evaluated. Form the normalized inhomogeneous residual by inserting m into the full field *before* changing the unknown error. In these old frame and rotating coordinates the actual substitution is $m + p + e^K x$, with p the divided particular in normalized field units. Thus the matrix conjugation acts on x. Its time derivative does not act on m a second time.

On the homogeneous field block the retained density part is physically compressed to $(1 - \Pi)B_{\mathrm{ext}}(1 - \Pi)$ with its complete Hermitian reverse. It annihilates m. If marking either endpoint would have created a density edge into an unmarked high coordinate, this is precisely a two-known output of the cubic Hamiltonian. Its independent source must be treated using the stronger cutoff, and its two homogeneous Hessian orientations as in the preceding subsection. A connection to a principal low star cannot be such a marked endpoint by the word argument. During the integration of the independent source, applying this exterior matrix to a known particular costs its physical $O(rE^C)$ one-head norm; its large old low potential is supported entirely on the low subspace. The known low part of a transformed profile is estimated in the stronger cap A': on the low slab the $X_{p,A}$ norm is at most $N^{A-A'}$ times the $X_{p,A'}$ norm. This pays the physical $1/u$ on that explicit known piece. The actual low potential on an arbitrary unknown is the principal evolution throughout.

There is a sharper axial dichotomy for this source than the one for the homogeneous field. If two marked transverse modes give an empty axial output n, translation covariance of the full $D\Psi$ tensor yields

$$\pm n + j \cdot n_F = \pm h_{a,x} \pm h_{b,x}.$$

At $|j|_1 \le C_J B$ the right side is R times an integer plus 0 or ± 2. Hence either $|n| \le 2 + C_J B \max_{m\in F} |m| \ll N^{1/4}$, or $|n| \ge R/2$. In the second case the one-smoothing comparison costs $\preceq R^{-1}$ also when the axial coordinate is the *output*, by the signed Hessian transpose above. In the first case the density term requires identical horizontal macros, and equal vertical labels by its projection to the axis. Thus the prescribed transverse plane waves are identical, their density has only spatial harmonic zero, and

$$\int_{\mathbb{T}} D\mathcal{R}(z_F)[e_n^{\tau_n}]\,\frac{dx}{2\pi} = 0$$

for every old angle and every signed empty input. If one prescribed plane has been deformed by the old transverse frame, the corresponding term with $T-1$ instead costs $\preceq R^{-1}$. The pairing term in the low case can have opposite macro heads but its mismatch contains the sum of the transverse kinetic energies and is $\gtrsim R^2$. Thus no source is left with only the homogeneous $N^{-1/4}$ gain. If the output is transverse low, a two-known short angular word cannot reach it at all, by the horizontal word test. A high transverse output with a prescribed axial input already gains $\preceq R^{-1}$ on its chart-differential remainder, since the prescribed axial abscissa has nonzero integral macro.

For any fixed target J_0, choose the source weak cutoff large enough that the enlarged weak strip gives normalized cost $\preceq N^{-J_0}$ after division by λ. For instance the factor $\lambda^{-1}\exp[-\epsilon B m_w/6]$ from weak length m_w pays this for $m_w > 6(2J_0+2)/\epsilon$ when $1 \le \log N/B \le 2$. All stationary words below this cutoff are explicitly tested as two-known words. In particular discarding every nonsingleton on an independent source using its homogeneous EN^{-h_w} estimate alone would not suffice. A retained stationary charged coefficient has summed size $\preceq \lambda$: the exact five-letter labels are precisely the fitted singletons by the disjoint-pivot module, and the others have at least fifteen weak letters. Nonstationary labels may have order-one coefficients but use the fast denominator.

The finite-only block, both types of exterior output, repeated indices, and the angular/weak tails now exhaust the degree-three Hamiltonian evaluated on two known slots. Its noncancelled particular or one-head residual costs $\preceq L^C(rN^2 + R^{-1})$ in the field equation divided by ru. Physical field terms of degree at least three in its vector field give $O(r^3E^C)$, hence $\preceq r/\lambda$ after the same normalization. The prescribed spatial residual on a known field costs $\preceq N^{-J_0}/u$; one can use the larger $\preceq N^{-J_0}/u^2$ common allowance when the old variables are included. At zero unknown old variables the implicit old action displacement is $O(r^2L^CE^C)$, and the mismatch of old isolated and full terminal frequencies is $O(r^2E^C)$. Their field connections on a known wave therefore cost at most $\preceq L^Cr/\lambda$, with a further N^{-1} where the connection differentiates a small spatial column or scalar primitive. Mixed terms decay at the true terminal point by their transverse factor, including the envelope derivative since the actual bounded finite vector field vanishes there and has C^1 bound $\preceq \lambda^{-1}$ on the joining segment. Pure axial terms use the exact diagonal action flow and its prescribed endpoint. In the common time weight the field source is consequently bounded by

$$E^CL^C\big(rN^2 + N^{-1} + r + N^{-J_0}/u^2\big),$$

where $R^{-1} = P/N \preceq N^{-1}$. Choosing J_0 above the fixed normalization prices gives the asserted power for the independent source. This estimate is for the inserted field. The old action and angle equations require their own differentiated Hamiltonian identities.

7.10. **The occupied actions and the full half-line problem.** We work in the autonomous mass gauge. With actions $I_k = |z_k|^2$ and angles $z_k = \sqrt{I_k}e^{-i\phi_k}$, $\dot\phi = \partial_I\mathcal{K}$ and $\dot I = -\partial_\phi\mathcal{K}$. On the physical transverse space use the real scalar product $\langle f, g\rangle = 2\,\mathrm{Re}\sum_k \bar f_k g_k$ and complex

structure $\mathbb{J} = -i$, transported together to its real representation. Then $\mathbb{J}^* = -\mathbb{J}$, $\mathbb{J}^2 = -1$ and $\dot{w} = \mathbb{J}\nabla_w \mathcal{K}$ with precisely this scalar product. Dots denote physical time derivatives. Primes denote derivatives in $\tau = ut$, where $u = r\lambda$.

7.10.1. *Field estimates used in this step.* Here are the precise estimates on the field problem used below. They distinguish the inverse on the finite calibrated graph from the inverse on the full equation at the solution we will select. Fix

$$a = \tfrac{1}{16}\min(s-2, \tfrac{1}{4}), \qquad \rho = N^{-a}, \qquad \rho_o = N^{-3a}.$$

Choose $A > 2s+12$, $A' = A + 3s + 20$, $J_0 > 2s+12$ and $p > 2\lceil s\rceil + 10$. With the same K_0 and the signed word length ℓ, use

$$X_c = \ell^2(b^p W_c), \qquad b(k) = 1 + (|k_x| + |k_y|)/N, \qquad W_c = \min(N^c, \exp(\ell(k)/B^{K_0})).$$

Constants denoted by E have the scale dependence prescribed in the registration of the clocks. Write $\mathcal{P}_0$ for a fixed product of these constants and fixed powers of λ^{-1}, including the fixed occupied derivatives. The constants are chosen so that all the resulting products of type $\mathcal{P}_0$ are at most N^η, $\eta < a/20$. We keep track of the smaller E prices at the field-only inversion: each fixed product there is $o(N^{h_w})$ for the homogeneous weak tail exponent h_w.

Use the same stable section at $\tau = T$ at both time weights. For β equal to either weight put $w_\beta(\tau) = \exp(\beta(\tau - T)_+)$ and $\|f\|_\beta = \sup w_\beta \|f(\tau)\|$. The two weights are strictly smaller than the decay exponent of the actual finite reference. The following are the field estimates to be used, in physical field units r:

(1) The actual finite reference reaches its specified axial endpoint with $\|Z_\perp\|_\beta + \|Z_{\mathrm{ax}} - A_\infty(t)\|_\beta \le E$, and $\sup\|A_\infty(t)\| \le E$. Here $A_\infty(t) = a_\infty(t)/r$ includes the fixed terminal rotations. Its full real finite linearization has a mixed inverse of norm E with arbitrary stable trace at T. The estimate is uniform also on strips $[0, L]$, $L \ge T$, with complementary data prescribed at the right end.

(2) All known profiles, explicit two-known particulars, old frames, frozen mixed changes and their inverses needed in the field equations have their stated translated operator or vector bounds at both caps A, A'. In particular the known transverse profiles and particulars have $X_{A'}$ bounds E with the temporal decay above. The field coordinate change on an error, denoted below by $U(t)$, costs E on X_A and on $\ell^2(b^q)$, $q = 0, 2, s$. The unweighted b^q-Wiener norm of any X_A-bounded reference costs at most E times its X_A norm. The analytic axial chart tensors have the corresponding translated convolution bounds. These assertions also follow at the stronger cap since each translation by old angular index l and a fixed number of heads costs at most a fixed power of $(1 + |l|\max_F |n|/N)\exp((|l|_1 + C)/B^{K_0})$, independently of the cap. The cardinality bound for short words changes only by replacing A by the fixed A'.

(3) In the pure old transverse frame $w = T(I, \phi)y$, put $D = \Omega(I)\cdot\partial_\phi$, $C = T^*BT + T^*\mathbb{J}DT$, where B is the physical transverse Hessian at vanishing new variables and $\Omega(I) = \partial_I \mathcal{K}_{\mathrm{ax}}(I, 0)$. Let $\Omega_* = \partial_I \mathcal{K}_{\mathrm{ax}}(I^*, |a_\infty|^2)$. If Q_L is the compressed low transverse projection, the high-high and high-low parts, including the transpose, obey

$$\|(\partial_\phi C)_{HH,HL,LH}\| \le \mathcal{P}_0(N^{-J_0} + N^{-1}|\Omega(I) - \Omega_*|). \tag{7.9}$$

The low part apart from the free kinetic is bounded and selfadjoint on each signed copy, with fixed derivatives bounded by $\mathcal{P}_0$. The remainder in (7.9) enjoys the same estimate after another angle derivative. On differentiating in I one must also differentiate $\Omega(I) - \Omega_*$, allowing $\mathcal{P}_0 N^{-1}\|D_I\Omega\|$ separately. The high scalar mean is action-only, with first derivative bounded by $\mathcal{P}_0$. The old spatial transformations and their fixed parameter

derivatives are bounded by $\mathcal{P}_0$. For a temporal primitive used in these assertions the divisor is evaluated at the *fixed* terminal Ω_*.

(4) The field-only principal inverse uses the finite graph and the exterior Hermitian components in the flattened norm. It has norm E for every path of admissible real trial I, ϕ in its low coefficient, with the same component supports and the same two boundary conditions. This follows from the finite estimate and unitary backward integration in the exterior, costing $E(T + \beta^{-1})$. After the physical divided rows and explicit particulars have been restored, the additional nominal homogeneous field operators have norm at most

$$EN^{-h_w} + \mathcal{P}_0\big(N^{-1/4} + rN^2 + r + N^{-1}\big).$$

The nominal normalized known field source away from the undifferentiated low principal coefficient costs

$$\mathcal{P}_0\big(rN^2 + N^{-1} + r + N^{-J_0}/u^2\big).$$

The rN^2 term allows a physical strong row $O(rE)$ times a divided near primitive $O(r\lambda N^2 E)$, divided by u. Near primitives have two high ends. A homogeneous primitive meeting the compressed low body has physical size Er/R. Their adjoints obey the same estimates. The stronger cutoff on a two-known source is included in this last source budget.

Items 1–4 follow, respectively, from the perturbed finite landing, the translated Fourier algebra, the transverse spatial reduction, and the exterior-component calculation. We now use them to solve for the occupied actions jointly with the field and then to invert the physical variational equation on the resulting solution.

7.10.2. *Recovery of the actions and differentiation of the weak harmonics.* The occupied list F contains the previous sites and the newly prepared spectator, moderator and weak sites. The *current* high axial endpoints are in the Cartesian complement a_{ax}. In the frame preceding the frozen mixed change write

$$(a_{\mathrm{ax}}/r, Y) = Z_{\mathrm{kn}}(t) + U(t)e, \qquad Y = y/r.$$

This includes the full correction and its frozen rotations. The particulars are included in Z_{kn}. In particular $\|Y\|_{\beta,X_A} + \|a_{\mathrm{ax}}/r - A_\infty(t)\|_{\beta,X_A} \le E$ on $\|e\|_{\beta,X_A} \le \rho$. The prescribed rotations, divisors, and arguments of the known fast phases are fixed at this stage.

Set $G_i = T^*\mathbb{J}\partial_{\phi_i} T$, and prescribe normalized coordinates j, θ by

$$\phi = \phi_*(t) + \theta, \qquad I = I^* + uj + \tfrac{r^2}{2}\big(\langle Y, G_i(I,\phi)Y\rangle\big)_{i\in F}, \qquad \phi_*(t) = \phi_*(0) + t\Omega_*. \tag{7.10}$$

The scalar product is real. The I-Lipschitz constant of the last term is at most $\mathcal{P}_0 r^2\|Y\|_2^2 = o(1)$, with 2 here and in scalar products denoting ordinary L^2. The finite Euclidean summation over F is included in E. The right side stays inside a fixed relative fraction of each positive floor: the previous floors have inverse price $\exp(O_s(B_o))$, and the inverse of the present minimum on F costs at most $\lambda^{-C_s}E$. Thus (7.10) has one solution in this neighborhood and

$$|I - I^*| \le u|j| + \mathcal{P}_0 r^2\|Y\|_2^2, \quad \|I_j\| \le 2u, \quad \|I_\theta\| \le \mathcal{P}_0 r^2\|Y\|_2^2, \quad \|I_e\| \le \mathcal{P}_0 r^2\|Y\|_2. \tag{7.11}$$

The last differential is bounded on ordinary L^2 as well. Replacing Y by the finite graph in this formula would miss the actual low return.

On a complex occupied-action ball of radius $c\min_F I_i^*$, the signed square roots have the same bounds, up to fixed factors, as at I^*. The enlarged strip $|\operatorname{Im}\phi_i| < \epsilon B/6$ on the weak sites still belongs to the small Cartesian chart, by the initial $O(\lambda^{1/5}E)$ radii and the fixed slack in the exponent. An angular coefficient at l therefore has the additional factor $\exp[-(\epsilon B/6)m_w(l)]$, where $m_w(l)$ counts its weak letters, besides ordinary exponential decay on a smaller angular

strip. An angle derivative is paid by this ordinary strip. By Cauchy's formula, each fixed action derivative costs $C(\min_F I_i^*)^{-1}$, independently of $m_w(l)$ and of the cutoff used for the known forcing. The same Cauchy estimate applies along every complex ℓ^2 unit direction. An undivided non-singleton homogeneous weak tail has normalized norm EN^{-h_w} at the frozen parameters. Its action derivative may cost $\mathcal{P}_0 N^{-h_w}$. We absorb only the undifferentiated tail at the field-only inversion. On parameter differences one may use the coarser normalized bound $\mathcal{P}_0(|\theta|+|I-I^*|)$, even for the sum over the entire weak tail. Its contribution to a field derivative through I contains the I_e of (7.11). In the old torque these same mixed Hamiltonian terms are taken before division and keep their physical factor r^3.

7.10.3. *The exact torque and the angular shear.* In physical transverse variables the Hamiltonian takes the form

$$\mathcal{K} = \mathcal{K}_{\rm ax}(I, |a_{\rm ax}|^2) + \tfrac{1}{2}\langle w, B(I,\phi)w\rangle + V(I,\phi,a_{\rm ax},w).$$

Every term of V has at least two transverse factors and degree at least three in $(a_{\rm ax}, w)$. The unbounded kinetic in B is independent of I, ϕ. Define

$$S = T^{-1}\mathbb{J}V_w - T^{-1}\big[(\partial_I T)\dot I + (\partial_\phi T)(\dot\phi - \Omega(I))\big]y.$$

The velocities on the right are the original Hamiltonian vector field evaluated at the instantaneous state. Thus $\dot y = \mathbb{J}Cy + S$. Differentiating $T^*\mathbb{J}T = \mathbb{J}$ at fixed I gives

$$\partial_{\phi_i} C = T^*(\partial_{\phi_i} B)T + DG_i + (\mathbb{J}C)^* G_i + G_i\mathbb{J}C.$$

Here $T^{-1}\partial_{\phi_i}T = -\mathbb{J}G_i$, and D commutes with ∂_{ϕ_i} at fixed I. To see which action to correct, differentiate $q_i = \langle y, G_i y\rangle/2$ along $\dot y = \mathbb{J}Cy + S$. The terms from $\mathbb{J}Cy$ and DG_i sum to

$$\tfrac{1}{2}\langle y, (\partial_{\phi_i}C - T^*(\partial_{\phi_i}B)T)y\rangle.$$

The original action satisfies $\dot I_i = -\langle y, T^*(\partial_{\phi_i}B)Ty\rangle/2 - V_{\phi_i}$. Subtracting $\dot q_i$ therefore cancels the derivative of the physical Hessian. The corrected action is $I_i^\sharp = I_i - q_i$, and its equation in the variables (7.10) is

$$\begin{aligned} u^2 j_i' = &-\tfrac{1}{2}\langle y, (\partial_{\phi_i}C)y\rangle - V_{\phi_i} - \langle S, G_i y\rangle \\ &-\tfrac{1}{2}\langle y, [(\partial_I G_i)\dot I + (\partial_\phi G_i)(\dot\phi - \Omega(I))]y\rangle. \end{aligned} \tag{7.12}$$

The derivatives of V here are taken at fixed physical w. Formula (7.12) is a vector field on state space. No derivative of a trial path appears in it.

The bounded old potential and the degree structure, using the convergent chart expansion with all undifferentiated factors in their packet or Wiener norms, give

$$|\dot I| \le \mathcal{P}_0 r^2 \|Y\|_2^2, \qquad |\dot\phi - \Omega(I)| \le \mathcal{P}_0 r^2, \qquad |V_\phi| \le \mathcal{P}_0 r^3 w_\beta^{-1}, \qquad \|V_w\|_2 + \|S\|_2 \le \mathcal{P}_0 r^2 w_\beta^{-1}.$$

For example V_w has an undifferentiated transverse factor, and its remaining packet factor has physical size rE in Wiener norm. The last line of (7.12) has physical size at most $\mathcal{P}_0 r^4 w_\beta^{-1}$. On the full trial ball, $\|Q_L Y\|_{\beta,2} \le EN^{-A}$, by the original cap on the actual pre-mixed field. From (7.11), $|\Omega(I) - \Omega_*| \le \mathcal{P}_0(u|j| + r^2)$. The nondecaying r^2 in this last bound is multiplied by transverse fields in (7.9). Dividing (7.12) by u^2 consequently bounds its normalized right hand side by

$$\|R_j\|_\beta \le \mathcal{P}_0 \lambda^{-2}(N^{-J_0} + N^{-1}u\|j\|_\beta + r + N^{-2A}). \tag{7.13}$$

In the angular equation retain the matrix $D_I\Omega_*$ evaluated at $(I^*, |a_\infty|^2)$, differentiating the occupied actions at those fixed Cartesian endpoint actions. Then $\theta' = (D_I\Omega_*)j + R_\theta$, where

$$uR_\theta = \partial_I \mathcal{K}_{\rm ax}(I, |a_{\rm ax}|^2) - \Omega_* - u(D_I\Omega_*)j + \tfrac{1}{2}\langle w, B_I w\rangle + V_I.$$

In ℓ^2 one has $\||a_{\mathrm{ax}}|^2 - |a_\infty|^2\| \le Er^2 w_\beta^{-1}$. Taylor expansion of the analytic mass-subtracted frequency, (7.11), and the bounded derivative B_I yield

$$\|R_\theta\|_\beta \le \mathcal{P}_0(r/\lambda + u\|j\|_\beta^2). \tag{7.14}$$

An isolated $\Omega(I^*, 0) - \Omega_*$ in place of this subtraction would give a constant source at infinity.

7.10.4. *A compact self-map with unequal radii.* On the finite strip use the real closed ball

$$\|e\|_{\beta,X_A} \le \rho, \qquad \|j\|_\beta, \|\theta\|_\beta \le \rho_o.$$

For each trial recover (7.10) and evaluate the two preceding right hand sides. With $\mathcal{I}_L f(\tau) = -\int_\tau^L f(s)\,ds$, put

$$j_{\mathrm{new}} = \mathcal{I}_L R_j, \qquad \theta_{\mathrm{new}} = \mathcal{I}_L[(D_I\Omega_*)j_{\mathrm{new}} + R_\theta].$$

Using the new value in this linear shear performs two successive integrations. Since $\|\mathcal{I}_L\| \le T + \beta^{-1} \le E$, (7.13)–(7.14) give $o(\rho_o)$ in both coordinates, uniformly in L.

Invert the field-only principal problem at the *trial* occupied parameters, retaining the entire compressed bounded low operator on the left. The same-copy exterior stars may keep their actual old coefficients and both signed adjoints, so the exterior on each copy is Hermitian in the fixed flattened norm. A star kept at its nominal parameters instead has a coefficient difference bounded by $r\mathcal{P}_0(|\theta| + |I - I^*|)$. The simultaneous field-only Green absorbs the nominal weak tail and the remaining small homogeneous rows by its E price. Denote this inverse by $\mathcal{G}_{\mathrm{tr}}$.

The low factor in the construction norm requires particular care. On all nonzero-vertical low sites both W_A and $W_{A'}$ are saturated, as their word lengths exceed B^{K_0+3}. For each prescribed vector Z one therefore has

$$\|Q_L Z\|_{X_A} \le N^{A-A'}\|Z\|_{X_{A'}}.$$

This pays $1/u$ for the bounded low operator on known vectors, including its moving parameter derivatives. Its restriction preserves the low sector, where the output cap is constant. On a free field the commutator with a high-high localized mixed primitive vanishes at first order, and a primitive meeting the low body has norm Er/R. Thus its normalized first commutator costs $E/(R\lambda)$, also when the bounded low operator is evaluated at the trial. Further Lie commutators contain an extra small primitive. In contrast the low body on the arbitrary trial correction itself is never estimated by N^{-A} in X_A.

We record the remaining parameter-to-field estimates. For the high action-only scalar and the Cartesian axial diagonal, $\delta I/u = O(|j| + \mathcal{P}_0 r/\lambda)$ and $\delta|a_{\mathrm{ax}}|^2/u = O(Erw_\beta^{-1}/\lambda)$, where the latter difference is from the exact endpoint profile. The connection involving $\dot I, \dot\phi - \Omega(I)$ costs $\mathcal{P}_0 r/\lambda$ on a transverse field. The mismatch from the frozen velocity in the high scalar costs $\mathcal{P}_0 N^{-1}(u|j| + r^2)/u$ there. On a persistent axial carrier take the full difference from its true terminal frequency. These terms have combined weighted cost $\mathcal{P}_0(\rho_o + r/\lambda + N^{-J_0}/u)$. The change of the true one-head or two-known cubic coefficient on a known field is bounded in normalized units by $\mathcal{P}_0(|\theta| + |I - I^*|)$, summing the full angle series. When it also acts on a correction its norm has an extra factor ρ. Terms at least quadratic in the corrections cost $\mathcal{P}_0\rho^2$, by the X_A product inequality and the physical r on further input factors. Every source has a decaying transverse factor, an axial action difference, or a decaying occupied variation. The residual vanishes at the exact axial endpoint.

Apply $\mathcal{G}_{\mathrm{tr}}$ to the remaining field source evaluated at the trial to define e_{new}. Combining these estimates and the nominal field budget gives

$$\|e_{\mathrm{new}}\|_{\beta,X_A} \le E\mathcal{P}_0(rN^2 + N^{-1} + r/\lambda + N^{-J_0}/u^2 + N^{A-A'}/u + \rho_o + \rho^2) + o(\rho) \le \rho/2$$

at sufficiently large scales. In particular $rN^2 = N^{-(s-2)+o(1)}$ and $s - 2 \ge 16a$. Also $N^{A-A'}/u = N^{-2s-20}\lambda^{-1}/H$ and $N^{-J_0}/u^2 = N^{-J_0+2s}\lambda^{-2}/H^2$. The choice of ϵ in Section 7.12 absorbs the

fixed powers of λ^{-1}. Among the two terms involving the radii, $\rho_o = \rho^3$ and ρ^2 each leave at least one extra factor ρ before division by the field radius. These bounds are uniform in the right endpoint L.

Fix N and L. Give the closed ball the uniform path topology of spatial order $p - \delta$, where $0 < \delta < 1$. Bounds at order p pass to limits by lower semicontinuity. On this interval Duhamel's formula in the free interaction frame gives continuous dependence in L^2 of the low propagator on its bounded coefficient. The other inverses are given by uniformly convergent Neumann series. Interpolation with the uniform bound at order p gives continuity of the map at order $p - \delta$ as well, since the weight W_A is bounded above and below at this fixed scale.

Its image is relatively compact. At each time the spatial embedding is compact since $b(k) \to \infty$. The equation bounds the time derivatives of the image at order $p - 2$ on the fixed interval. Interpolating differences between two times gives equicontinuity at order $p - \delta$, so Arzelá–Ascoli applies. Schauder's theorem gives a solution on each interval $[0, L]$. As $L \to \infty$, the equation gives the same local time-derivative bound on each $[0, L']$ for all $L \geq L'$, using the uniform top-order estimate. Compactness on these intervals yields a diagonal subsequence. In Duhamel's formula its limit satisfies the full equation, and the stable trace passes to the limit at the fixed time T. The exponential bound passes to each finite time, giving the required axial limit as time tends to infinity. Local Sobolev uniqueness identifies this solution with the full NLS trajectory through its time-zero value.

Support of the undivided low operator. The projection Q_L here is taken in the completed pure old coordinates y. First form the entire transformed Hessian C in those coordinates. Its low same-copy bounded part $L_L = Q_L L_L Q_L$ is Hermitian by compression there. The opposite-copy row and the high-to-low remainder are included with their adjoints in the field budget above. The fixed diagonal rotations commute with Q_L and with the flattened weights, and the finite averaging on the marked graph misses this low sector. Extend Q_L by zero on the axial field coordinates and put $Q_H = \mathrm{Id} - Q_L$ on the field space. For a subsequent bounded mixed primitive write $K = Q_H K Q_H + K_L$, where all the rows *and columns* in K_L have total operator bound Er/R. Irrespective of whether $\exp K$ is unitary, the exact identity

$$e^{-K} L_L e^{K} - L_L = \int_0^1 e^{-\xi K} [L_L, K_L] e^{\xi K} \, d\xi$$

costs $\mathcal{P}_0/(R\lambda)$ after division by u. No time derivative of the trial L_L is involved. For the unbounded free operator one instead uses the combined commutator and time derivative of the primitive: for the equation $i\partial_t v = D_0 v + \cdots$ its exact integrand is $e^{-\xi K}([D_0, K] - i\partial_t K) e^{\xi K}$. These identities first hold on smooth vectors with the prescribed finite or absolutely summable translations, then by closure in the equation. The free commutator alone is never priced as a bounded potential.

A Hermitian star connecting low and high sites has no such support cancellation with $Q_H K Q_H$. Its physical size Er, times the near primitive $Er\lambda N^2$, divided by u, costs $E^2 r N^2$. This is one place requiring $s > 2$. The fast primitive costs at most $r\mathcal{P}_0$ in this product. These terms are errors even if the full conjugating transformation fails to be unitary. Finally the connection $T^{-1}(T_I \dot{I} + T_\phi(\dot{\phi} - \Omega(I)))$ costs $\mathcal{P}_0 r^2$ in physical units even on a free low field. Its actual parameter derivatives retain the small velocities or the quadratic field dependence of their differences from the axial frequencies. The old torque was evaluated before these mixed changes; its two kinetic terms cancel in the exact formula (7.12). The only unscaled derivative of the undivided low body in the field equation itself is therefore $(\delta L_L) Q_L Y/u$. The two-cap estimate above controls it on the known state, and the actual low cap controls it in the ordinary variational norms below.

7.10.5. *The linear inverse at the installed state.* Linearize along the full solution just constructed, in the ordinary scaled spaces $\ell^2(b^q)$, $q = 0, 2, s$, at each of the two time weights. Write h for the field variation and $z = (\delta j, \delta\theta)$ for the occupied action and angle variations. With forcing f, g and zero stable trace, the equations have the form

$$h = \mathcal{G}_f(f + A_o z), \qquad z = \mathcal{G}_o(g + R_f h + R_o z).$$

Here $\mathcal{G}_f$ inverts the field equation with the occupied variables held fixed, and $\mathcal{G}_o$ integrates the old shear backward. The operator A_o maps occupied variations into the field equation. The operators R_f, R_o give the remaining terms in the occupied equations.

Eliminating h leaves the operator $\mathcal{G}_o(R_o + R_f\mathcal{G}_f A_o)$ on z. Thus the estimate we need is

$$\|\mathcal{G}_o(R_o + R_f\mathcal{G}_f A_o)\| < \tfrac{1}{2}.$$

Thus even a large A_o is admissible if the feedback through R_f is sufficiently small.

The field equation and its dependence on the occupied variables. When one input in a chart derivative or a cubic term varies, bound the other inputs in their actual b^q-Wiener norms. The installed field correction contributes at most $E\rho$ to each such multiplier. Write $A_{\mathrm{ax}} = a_{\mathrm{ax}}/r$ on this trajectory. For a frequency derivative known only on ℓ^2, use

$$\|b^q A_{\mathrm{ax}}\, D\chi[\delta I]\|_2 \le \|b^q A_{\mathrm{ax}}\|_\infty \|D\chi\|_{2\to 2}\|\delta I\|_2.$$

The field-field terms induced by the correction are bounded by $\mathcal{P}_0\rho$. Differentiating an action-only diagonal through I_e contributes a further $\mathcal{P}_0 r/\lambda$.

In the low block the actual field before the mixed change has $\ell^2(b^q)$ norm at most EN^{-A}, by its X_A bound. The I_e contribution there is therefore at most $\mathcal{P}_0(r/\lambda)N^{-A}$. Differentiating in θ or j gives bounds $\mathcal{P}_0 N^{-A}/u$ and $\mathcal{P}_0 N^{-A}$, respectively. In the high block the derivative of the frequency mismatch $\Omega(I) - \Omega_*$ costs $\mathcal{P}_0 N^{-1}$ on the scaled j input and $\mathcal{P}_0 N^{-1} r/\lambda$ on a field input. All other terms in A_o cost at most $\mathcal{P}_0$. The rotations and mixed changes use fixed frequencies, so their state derivatives introduce no factor t.

Absorb the nominal small tail first, and then the field perturbations bounded by $\mathcal{P}_0\rho$ and by the estimates for I_e. This gives $\|\mathcal{G}_f\| \le E$, enlarging E within the allowed bounds. The backward inverse for the old shear also has norm at most E. Collecting the occupied derivatives above gives $\|A_o\| \le \mathcal{P}_0(1 + N^{-A}/u)$.

Feedback into the occupied equations. Consider the first term of (7.12). A free low variation pairs with an actual low factor at cost $\mathcal{P}_0\lambda^{-2}N^{-A}$. Indeed, at fixed I, ϕ set $C_{i,L} = Q_L(\partial_{\phi_i} C)Q_L$. Since $y = rY$, $u = r\lambda$ and $C_{i,L}$ is selfadjoint, the field derivative of the normalized low quadratic term in the direction $h_Y = \delta Y$ is

$$-\lambda^{-2}\langle Q_L h_Y, C_{i,L} Q_L Y\rangle.$$

The factor N^{-A} comes from $\|Q_L Y\|_{\beta,2} \le EN^{-A}$ on the installed trajectory. The variation satisfies $\|Q_L h_Y\|_2 \le \|h_Y\|_2$ in the ordinary variational space. The high part costs $\mathcal{P}_0\lambda^{-2}[N^{-J_0} + N^{-1}(u\rho_o + r^2)]$. Differentiating its coefficient in a field direction inserts I_e and costs a further $\mathcal{P}_0\lambda^{-2}N^{-1}r^2$ in the high mismatch. A free j derivative there costs $\mathcal{P}_0\lambda^{-2}N^{-1}u$. An old angle derivative preserves the small high mismatch or the two low factors.

The mixed terms involving V_ϕ or S retain physical size $\mathcal{P}_0 r^3$ under normalized field differentiation. In the last term of (7.12), the velocities $\dot I, \dot\phi - \Omega(I)$ before the Ward correction and their normalized field derivatives have size $\mathcal{P}_0 r^2$. In the angle equation the field and θ derivatives, as well as the error after subtracting the retained j-shear from the j derivative, cost at most $\mathcal{P}_0(r/\lambda + u\rho_o)$. Together these bounds give $\|R_f\|, \|R_o\| \le B_{\mathrm{ret}}$, where

$$B_{\mathrm{ret}} = \mathcal{P}_0\big[\lambda^{-2}(N^{-A} + N^{-J_0} + N^{-1}u + r) + r/\lambda + u\big].$$

Solving the coupled equations. The norm of the operator left after eliminating h is at most

$$\|\mathcal{G}_o\|B_{\mathrm{ret}}(1+\|\mathcal{G}_f\|\,\|A_o\|)<\tfrac{1}{2}.$$

Indeed $N^{-A}/u=N^{s-A}\lambda^{-1}/H$, and each term in B_{ret} retains a negative power of N after the fixed products of $E,\mathcal{P}_0$ and λ^{-1} are accounted for as in Section 7.12. The Neumann series determines z, and the first equation then determines h. A prescribed nonzero stable trace adds its field solution to the first right-hand side and gives the same inverse estimate. Both time weights and the three spatial orders impose the same stable trace. Solutions constructed on their common domain coincide by uniqueness at the weaker weight and order. All coefficients in these equations are evaluated along the full solution obtained above.

To return to physical variables, at each time let $\mathcal{F}_t$ denote the instantaneous coordinate map used above. If $\dot v=X(v)$ and $\dot\xi=DX(v)\xi+f$, then $k=D\mathcal{F}_t(v(t))\xi$ solves the transformed variational equation with forcing $D\mathcal{F}_t(v(t))f$, with the additional constant factor for slow time. This follows by the chain rule on smooth inputs and Duhamel continuation at fixed scale. All coordinate differentials and their inverses have fixed polynomial prices in N on the three scaled spaces by the translated chart estimates, the occupied floors and (7.11). Also $\|f\|_{\ell^2(b^q)}\le C_q\|f\|_{H^q}\le C_qN^q\|f\|_{\ell^2(b^q)}$. Thus the physical bounds remain fixed powers of N. If Π_{st} is the complete finite stable trace in the transformed coordinates, the physical trace is

$$P\xi=\Pi_{\mathrm{st}}D\mathcal{F}_{T/u}(v(T/u))\,\xi(T/u).$$

It includes both real stable quadratures of each complex pair. The unrestricted homogeneous unaugmented physical operator can have a nonzero decaying kernel, for example $\xi=\partial_y v$ when the endpoint is axial and the solution is not. The invertibility just proved concerns precisely the displayed augmented trace.

7.11. **Physical comparison and iteration.** All comparisons before the very last limit use the autonomous mass-gauged equation. The transformation back to the original equation is a scalar phase depending only on the solution's mass. For an exponent q let $\|d\|_{q,\beta}$ denote $\sup_{t\ge0}e^{\beta t}\|d(t)\|_{H^q}$ in physical time.

7.11.1. *The three estimates on each raw input.* Let W_j be the raw field at a stage and V_j its axial endpoint. When comparing V_{j-1} and V_j, all occupied preexisting rates $n^2+\chi_n$ and angular origins are locked in this gauge. Compare actual old angles to $\phi_{j-1,0}+t\Omega_{j-1}$ with the common time orientation. For each earlier site the reference angle is therefore the same on both tori for every t. Along W_j its displacement is precisely $\Theta=\phi-\phi_*(t)$ from (7.10), whose bound also holds on the initial interval. A frequency error, even a small constant one, would instead give an angle error growing linearly in t.

There are constants C_0,C_1 independent of the stage and the requested prefix precision such that, for $q=2,s$,

$$\beta_j\ge e^{-C_0B_j},\qquad \|W_j-V_j\|_{q,\beta_j}+\|G_j\|_{q,\beta_j}\le e^{C_0B_j}. \tag{7.15}$$

Here G_j is the inhomogeneous physical variational right inverse along W_j with the same augmented stable data at the section in both orders. To convert the section estimate to the physical norm, $\|f\|_{b^q}\le C_s\|f\|_{H^q}\le C_sN^q\|f\|_{b^q}$. The instantaneous coordinate differential is composed of the axial Birkhoff coordinate differential and the polar map on the occupied coordinates, the triangular action correction, the spatial frame and the small mixed exponential, fixed unitary coordinate rotations, and the scale multipliers $r^{\pm1},u^{\pm1}$. Both directions in ordinary $\ell^2(b^q)$ cost a fixed power of N. Indeed the known inputs in the forward and the inverse physical chart expansions have scaled Wiener norms bounded by $r\mathcal{E}$ about their pure old values, the old positive floors cost at most N^{C_s}, and no radial derivative of a hot Cartesian coordinate is divided

by its modulus. This conversion applies to the inhomogeneous forcing after differentiating the exact coordinates at W_j; the physical-time source introduces an additional $1/u$. The same stable trace at a single physical time prescribes the boundary condition. Converting the early/late weight costs $\exp(\sigma D)$, and one may take $\beta_j = u\kappa/8$. These losses in total have logarithm at most $C_0 B_j$, once prices $\mathcal{E} \le N^{\gamma}$ and $\mathcal{P} \le N^{a/10}$ have been absorbed. Exponential landing uses the same coordinate conversions for the finitely many parameters and the decaying field components. The stationary Cartesian endpoints have their true terminal phases. In particular one does not integrate a pointwise frequency approximation along the half-line.

Here is the sharper budget for the unweighted raw strong profile. Set $m = \lceil s \rceil - 1 \ge 2$. The one-dimensional inverse chart $\Psi = \mathcal{F}^{-1} + G$ has a uniformly Lipschitz remainder from a small h^m ball to H^{m+1}; take its center at zero and shrink the ball using continuity of the derivative there. This uses only an integer m even for real s. Every retained current carrier satisfies $|k| \le CdN$ and their normalized Euclidean size is e^{Cd}. Thus their h^m cost is at most $HN^{m-s}e^{C_s d}$. The weighted field error of size N^{-a} has physical H^q cost at most $C_s H N^{q-s-a}$, $q \le s$. Each mixed change and particular on the known field adds a normalized $\mathcal{E}(r\lambda N^2) + \mathcal{P}r$ or $\mathcal{E}/R$ in the scaled strong norm, and its action on the unknown costs at most $\mathcal{E}$. Thus the physical reconstruction error from them is at most $HN^{-a'}$ for some fixed $a' > 0$. The finite-only shadow and hot-action rounding cost at most $C_s H d^{C_s} e^{-C_2 dD}$ in physical H^s, since they are supported on the marked planes.

Let F_o be the old support and $Z_o = \max_{n \in F_o} |n|$. At an old site n, split the change in $\sqrt{I_n}e^{-i\phi_n}$ into a change of radius and a change of angle. The bounds $|\sqrt{x} - \sqrt{y}|^2 \le |x - y|$ and $|e^{it} - 1| \le |t|$ show that the ℓ^2 norm of the radial change is at most $\|I - I_o\|^{1/2}_{\ell^1(F_o)}$. Cauchy–Schwarz on the finite set F_o gives the following estimate for the total difference on the old sites in h^q for $q \le s$

$$C\langle Z_o\rangle^q \big[|F_o|^{1/4}\|I - I_o\|_2^{1/2} + \|(\sqrt{I_{o,n}}|\Theta_n|)_{F_o}\|_2\big].$$

The true I here includes the restored quadratic action correction of physical order $r^2\mathcal{P}$; the raw old difference from the retuned endpoint has size at most $uN^{-3a} + r^2\mathcal{P}$. All changes of the old actions during the choice of the endpoint, and the h^s costs of the new clocks and spectator, can be made $\le e^{-D'B_o}$ for any specified D' after allowing the indicated $O_{D'}(B_o)$ expenses in calibration. For the old-coordinate estimate use this adjustable precision before the square root and the Z_o^s factor. The uniform h^m ball follows by summing these bills and $H_i N_i^{m-s} e^{C_s d_i}$, since $m < s$; take the two-site base small first. Consequently

$$\|G(z) - G(z_o)\|_{H^s} \le C_m \|z - z_o\|_{h^m}.$$

In particular one can use the physical strong error for the nonlinear part of the axial reconstruction without any uniform chart assumption on unbounded h^s balls.

Let $\mathcal{C}_j(t)$ denote the actual marked plane waves of the ideal finite reference in their true carrier rotations. In H^s the raw error relative to $V_{j-1} + \mathcal{C}_j$ is bounded uniformly by

$$C\{e^{-D'B_o} + HN^{-a'} + HN^{m-s}e^{C_s d} + Hd^{C_s}e^{-C_2 dD}\}.$$

For the weak comparison keep the finite perturbation on its marked frequencies, where its H^2 cost carries an additional factor N^{2-s}. Alternatively bound all the actual marked coordinates together in H^2 by $HN^{2-s}e^{Cd}$. The one-smoothing comparison in H^2 can likewise use the small h^2 ball and this last h^2 bill. It follows that for arbitrarily prescribed D_h one has

$$\sup_{t \ge 0} \|W_j(t) - V_{j-1}(t)\|_{H^2} \le e^{-D_h B_{j-1}}. \tag{7.16}$$

The old angular terms and the current negative powers of N satisfy this bound once Q is sufficiently large. To compare the two axial solutions themselves, fix t and take $t_n \to +\infty$ such

that the pair of phase vectors at t_n tends to the pair at t. Such times exist by recurrence of a linear flow on the product torus, including when its frequencies satisfy rational relations. By continuity,

$$\|V_j(t) - V_{j-1}(t)\|_{H^2} \le \limsup_{n\to\infty} \left(\|V_j(t_n) - W_j(t_n)\|_{H^2} + \|W_j(t_n) - V_{j-1}(t_n)\|_{H^2}\right) \le e^{-D_h B_{j-1}}.$$

Thus summing the increments gives a common H^2 bound for all V_j, W_j. In H^s the marked waves cost CH_j at all times, where $H_j = e^{bd_j}$, and the preceding reconstruction bound costs $O(e^{-cd_j})$. Applying recurrence once more gives this same increment bound for $V_j - V_{j-1}$. Summing up to $j-1$ and then adding $W_j - V_{j-1}$ yields

$$\sup_{t\ge 0} \|W_j(t)\|_{H^s} \le Ce^{Cd_j}.$$

For a coarse old H^{s+1} cap, the finite-support axial parameterization on a fixed angular strip has a uniformly bounded physical L^2 norm. Its angular Fourier coefficient has spatial index $l \cdot n_F$ and size $Ce^{-a_0|l|_1}$. Summing with its weight gives a bound $C_s^{|F|+1}(1+Z)^{s+1} \le e^{C_0 B_j}$ for V_j, after increasing the coarse constant.

Let $T_j = \exp[(C_0+1)B_j]$. The physical root of V_j has Fourier modulus in the H^s normalization at least $0.95H_j$: the true axial coordinate has modulus $(1+o(1))r$, and its one-smoothing error at that frequency in this normalization is $O((1+N)^{s-m-1}) = O(1)$. The latter includes integer s. By (7.15) the error $W_j - V_j$ at all times $t \ge T_j$ is at most $\exp(C_0 B_j - e^{B_j})$ in H^s. It leaves a persistent root of normalized height $0.9H_j$. Choose $L_0 > C_0 + 2$, $L_1 > \max(2L_0 + 2, C_0 + 1)$ and put $S_j = e^{L_0 B_o}$, $K_j = e^{L_1 B_o}$. For the cold estimate at S_j compare the actual finite profile with its launch in their individual rotating coordinates. The slow derivative costs at most $\mathcal{E}/\lambda$, so the physical H^s price of this comparison is at most $C_s H d^{C_s} \mathcal{E} r S_j$, in addition to its cold launch. The free phases all have modulus one. The old axial tail costs $e^{C_0 B_o}/K_j$. These bounds, the leaf estimate in Lemma 7.6, and the raw strong error above give

$$\|P_{\langle k\rangle \ge K_j} W_j(S_j)\|_{H^s} \le Ce^{-cd_j}, \qquad T_{j-1} \le S_j < T_j. \tag{7.17}$$

Here Q is taken sufficiently large and eventually fixed before choosing the starting scale. In particular $B_o \gg d_j$ at large stages with this fixed Q.

7.11.2. *Composition in the smaller Sobolev space.* At stage j, W_j is close to the preceding axial solution V_{j-1}. We must also restore the decaying parts of the earlier two-dimensional trajectories. Write $X_i^{(j)}$ for the result of restoring the earlier stages down to index i, starting with $X_j^{(j)} = W_j$ and $X_{j+1}^{(j)} = V_j$. For $i = j-1, j-2, \dots, 1$ the three known solutions are

$$U = X_i^{(j-1)}, \qquad Y = X_{i+1}^{(j-1)}, \qquad Y' = X_{i+1}^{(j)}.$$

The old difference $d = U - Y$ decays at rate β_i, and the new change $v = Y' - Y$ is small uniformly in H^2. Seek $X_i^{(j)} = Y' + d + e$. Let F be the mass-gauged cubic vector field, including its factor $-i$. Subtracting the equations for Y', U, Y cancels the free evolution of $Y' + U - Y$. Subtract and add $DF(W_i)e$ in the remaining equation and apply the inverse G_i along W_i. With zero stable trace for the additional correction e, the result is

$$e = G_i\{F(U+v+e) - F(Y+v) - F(U) + F(Y) - DF(W_i)e\}.$$

The lattice at stage j is contained in that at stage i, with the same congruence condition. All these fields therefore lie in the domain of G_i.

Let $\mathcal{T}$ be the real symmetric trilinear polarization of F, including its mass term. The independent source is $6\mathcal{T}(Y,d,v) + 3\mathcal{T}(d,d,v) + 3\mathcal{T}(d,v,v)$. The other three terms are $3\mathcal{T}(U + v - W_i, U+v+W_i, e) + 3\mathcal{T}(U+v,e,e) + \mathcal{T}(e,e,e)$. On a common H^2 ball this gives a weighted weak

source of size $CA_i\delta$, where $A_i = \|d\|_{2,\beta_i}$, $\delta = \sup\|v\|_{H^2}$. Write $k_i = \max(1, \|G_i\|_{2,\beta_i}, \|G_i\|_{s,\beta_i})$. The weak ball of radius $R_e = 2Ck_iA_i\delta$ is invariant and contractive if $Ck_i(\sup\|U - W_i\|_{H^2} + \delta + R_e) < 1/2$. At its first use $A_i \le e^{C_0B_i}$. We allow twice this bound in later stages. The new difference from Y' is $d + e$, so its weighted change from d costs at most R_e. The difference from U is $v + e$, whose uniform H^2 norm is at most $\delta + R_e$. Both are bounded by $e^{C_wB_i}\delta$, with C_w depending only on the coarse bounds at stage i. The first change to be propagated is $W_j - V_{j-1}$, bounded by (7.16). Descending from $j - 1$ to i therefore bounds the change at each row by

$$e^{-D_hB_{j-1}}\prod_{k=i}^{j-1} e^{C_wB_k} \le \varepsilon_j, \qquad \varepsilon_j := e^{-(D_h-2C_w)B_{j-1}},$$

since $\sum_{k<j} B_k \le 2B_{j-1}$. At fixed old i the accumulated change of either $U - W_i$ or the weighted norm of d is at most $\sum_{j>i}\varepsilon_j$. It is much smaller than $e^{-(C_0+1)B_i}$ if D_h is chosen large enough. Induction maintains the ball, its contraction, the common weak bound and the registered A_i. Choose $D_h - 2C_w$ also larger than $(s - 2)\max(L_1, C_0) + 2$.

For the strong order use the same weak Picard iterates. The tame product estimate for $\mathcal{T}$ puts the H^s norm on one factor and H^2 on the others, by Fourier convolution and the mass scalar product. Terms with an e carry the exponential weight on an e. If it is also the strong factor in the term linear in e, its coefficient contains $U + v - W_i$, which is weakly small; if there are two or three e's the extra one is weakly small. If the strong factor is different, the weak weighted e costs R_e. In the independent source put the weight on a d. Thus the coefficient of the previous strong iterate can also be made at most $1/2$, and such iteration gives

$$\|d + e\|_{s,\beta_i} \le e^{C_sB_i}(1 + \|d\|_{s,\beta_i} + \|Y'\|_{L^\infty H^s} + \|Y\|_{L^\infty H^s} + \|U\|_{L^\infty H^s}).$$

Pointwise weak compactness in H^s and convergence in the lower norm give this same bound for the actual weak fixed point. Uniqueness in H^2 and local H^s persistence identify it with the true strong solution on each finite interval. The logarithm L'_j of two plus the maximum of all strong sup norms and weighted pair prefactors through stage j satisfies $L'_j \le \max(L'_{j-1}, CB_j) + Cj + C\sum_{i<j} B_i$. To iterate this estimate, bound every new maximum through stage j by CB_j or by its initial value, and add the accumulated increments. The terms Ck sum to $O(j^2)$ and the double sum is at most $2C\sum_{k\le j} B_{k-1} = O(B_j)$. Thus $L'_j \le C'B_j$. The use of a single strong factor in each product is what gives an additive recurrence for these logarithms.

Write $U_j = X_1^{(j)}$. Telescoping the row pairs gives

$$\|U_j(t) - W_j(t)\|_{H^s} \le je^{C'B_j}\exp[-e^{-C_0B_{j-1}}t].$$

For $t \ge S_j$ the logarithm of the right-hand side is at most

$$\log j + C'QB_{j-1} - \exp\left((L_0 - C_0)B_{j-1}\right) \le -B_{j-1}$$

after increasing B_* with Q fixed. This transfers the tail bound at S_j, the Fourier lower bound for $t \ge T_j$, and the late H^s upper bound from W_j to U_j. The bound $\sup_{t\ge0}\|U_j(t) - U_{j-1}(t)\|_{H^2} \le \varepsilon_j$ above controls the effect of subsequent stages.

7.12. **Order of parameters and construction on successive lattices.** Fix a single real $s > 2$ and the integers m, p above. Use $a = \min(s - 2, 1/4)/16$ for the normalized field error; for comparisons where $s - m$ also occurs keep that strictly positive number as a separate exponent. Choose, for example, $A > 4s + 20$, $A' = A + 3s + 20$ and a spatial and angular remainder order $J_0 > 4(A' + s + p + 10)$. In the spatial recurrence and the angular tail cuts take their finite internal orders higher as indicated there. Choose an integer $h > 4s + 5$ for the axial template, then the template, its length q, the number of hierarchy derivatives and the finite-order coefficient

expansion. All these numbers depend on s alone. The relative range of the two splitting angles and the constants in Lemma 7.6 can be fixed throughout.

Initialize with a two-site torus on, for instance, $1, 6$. The actions range over a small open box of two positive comparable values. The direct rate Jacobian is close to $-\mathrm{Id}$ on this box; shrinking it and the surrounding analytic ball can guarantee the smallness required by the disjoint template rows. Choose the two direct rates inside a small interior cube by the affine slicing argument. All phases can be fixed. This initializes the prefix affine bound, positive margins for $\mathcal{Q} - M^2$ and $2M^2 - \mathcal{Q}$, small integer-order h^m bounds and finite high norms. With B_* large it obeys the first coarse maximum-site, least-action and inverse-price bounds. The sum of all subsequent small action differences prescribed in the preceding sections keeps the two strict mass margins by $d\mathcal{Q} = 4M\, dM + 2\sum I_n\, d\chi_n$.

Here are the choices for the prices involving the current logarithmic scale. All fixed powers of λ^{-1} on the current scale arise from the slow normalization, the fixed current positive-floor/weak-radius derivatives, and the reciprocal of the finite-dimensional direct-rate cube at the frozen module. The cube there has an explicit λ^2 factor with the other prices involving $B_o + d^2 \log B$. There is no current-scale inverse of the much smaller final direct-rate cube used in the nonlinear PDE estimate. That cube is saved for the next stage only. Take a common upper bound for the fixed powers C_s through the finite sequence of estimates, and choose $\epsilon > 0$ so small that the powers λ^{-C_s} used together, including coordinate and derivative prices, fit within $N^{a/100}$ (and any further fixed losses can be reserved by reducing ϵ before the following step). Thereafter choose once the weak letter cut m_* so large that the absolute tail past that length, using width $\epsilon B/6$ in the weak coordinates and ordinary angle width on the rest, is $O(N^{-2J_0})$ after the fixed normalizations. For an unmarked homogeneous charge-one nonsingleton the width bound already pays $15\epsilon B/6$ against a single divisor λ^{-1} from the field normalization. Thus the fixed positive exponent h_w in its bound $\mathcal{E}N^{-h_w}$ may be much smaller than a. That bound is used with a field-only inverse price $\mathcal{E}$ before the upper triangular old-coordinate arrows. The weak cutoff affects the polynomial counts/degrees of geometry exclusions, the required spectator intervals, the fixed numerical constants in the old-and-macro price and the lengths of bounded sums, but never the exponent of an extra unabsorbed inverse weak floor per weak letter.

For clarity about induction, register the following coarse caps at stage j, now taking $I_{\min}$ over the final occupied actions:

$$\log(2 + \max|n|) \le 2\log N_j + O(1), \quad -\log I_{\min} \le (2s+4)\log N_j + O(1), \quad f_j \le C_s \sum_{i \le j} (1+d_i)^2.$$

Register the final affine estimate with exponential prefactor $e^{-C_{\mathrm{reg}} B_j}$ and polynomial exponent $2f_j + 6$, as in the direct-rate selection. If all current designated products of $\mathcal{E}$ fit within N^γ, with $\gamma \ll \min(a, h_w, \epsilon)$, then the conclusions (7.15) and the old H^{s+1} bound hold with a coarse C_0 depending only on these fixed caps, s and the chosen orders. To see this before imposing any prefix precision, all normalized inverse norms are at most N^a after the two successive absorptions; converting physical time and coordinates costs fixed powers of r^{-1}, u^{-1}, N^p, the old positive floors and the frame bounds. Also $\log(1/\kappa) = O(\log d)$, $\log(1/u) \le (s+1)\log N$, $\log N \le 2B$ under these coarse caps, and $f_j \le B_j$ can be imposed below. These give a fixed numerical upper bound on C_0 independent of the price of a future request for old accuracy. The raw times and both physical Green bounds may use the same enlarged C_0.

Choose L_0, L_1, C_w, D_h in the order specified in the physical history proof, and any larger prefix accuracy D' needed before reconstructing the old coordinates and clocks. Fix the coarse accuracy with which the finite graph is perturbed, including the designated phase/amplitude and terminal-action precision $e^{-C_4 dD}$. When selecting the spectator the component heights are at most a fixed power of dP_{lat}. Demand $d^2 \log B \le c_* B_o$ with $c_* > 0$ small enough for

their logarithms to fit in the prescribed old-scale budget. At this stage the initial spectator size, its geometric and affine exclusion widths, and the passive moderator placement constant k in $K = e^{kB_o + O_s(\log B)}$ can be chosen in that order. None depends on Q, α. Each old floor, maximum index, strip count, spectator interval and moderator precision used here costs at most $e^{C(s,\epsilon,D_h)B_o}$ times a polynomial in B; for the angular count one also allows $e^{Cf \log B}$. The latter, including the old sites, is dominated by $\exp(C_s d^2 \log B)$ plus the harmless initial two-site factor. The row arity is fixed throughout.

Finally take $Q \geq 2$ sufficiently large for the $B_o/B = 1/Q$ fractions of all the preceding fixed bills, including the old Sobolev comparisons and $L_0 B_o$ at the cold time. Take $\alpha > 0$ so small that the $d^2 \log B/B \leq \alpha^2$ fractions and $\alpha^2 Q$ obey all the indicated budgets. Increase B_* last. This accommodates any fixed powers of d, B, e^{Cd} and e^{CD} not already included. In particular e^{CdD} times the normalized finite perturbations tends to zero. Uniform finite powers of high index polynomials times a previous-site price are accounted for once at each construction; there is no multiplication of one whole stage inverse at each level inside the finite graph. The only multiplication of old stage inverses across stages is the one estimated by $\sum_{i<j} B_i \leq 2B_{j-1}$ in the history proof.

At each stage a vertical spacing R_j with $\log R_j = B_j + O(1)$, prime to five and a multiple of R_{j-1}, exists by rounding a suitable multiple of R_{j-1} in $[e^{B_j}, 2e^{B_j}]$; initialize with $R_0 = 1$. Choose the same mod-five isotropic line at every step. The signed trilinear equation preserves these nested cosets, and the preceding estimates compare the old and new autonomous flows without external forcing.

Assembly of Proposition 7.1. At stage j the two-tree geometry, the prescribed positive axial radii, and the angles determining the edge signs are chosen on one completed terminal torus. The bounds on the logarithmic prices of this choice are

lattice denominator and counting the short words	$C_{s,\epsilon} d^2 \log(2+B)$,
finite Green and required finite accuracy	$C_s dD, \quad D = C_* d \log(d+2)$,
selection on earlier axes, including the spectator	$C_{s,\epsilon,D_h} B_o$,
inverse powers of the current weak floors or slow scales	$C_s \epsilon B$ for the powers of λ^{-1}.

The additional powers of r^{-1} from the change of units occur in the explicit terms N^{-J_0}/u^2, $N^{A-A'}/u$ and in the conversion of norms. They are not charged as inverse powers of λ. The three other small terms to check are $rN^2 = HN^{2-s}$, $N^{-1/4}$ and HN^{m-s}, with $m = \lceil s \rceil - 1 < s$. The last occurs only in passing from the nonlinear axial chart to physical Fourier coefficients. For all required products the choice of the old and arithmetic fractions can also reserve a fraction of $s - m$. For the homogeneous weak angular tail the inverse used for its first absorption has only the first three prices. This is why its small exponent need not exceed $C_s \epsilon$.

Lemma 7.6 first gives a reference landing at the specified axial radii. With the actual harmonics and the final rounded carrier rates, the divided rows and Lemma 7.7 give the finite solution at weight $\kappa/3$. Section 7.9 divides the independent two-known sources and both entries of their signed boundary Hessians using the same physical mismatch. Its remaining same-copy operator is Hermitian on the exterior components. Section 7.10 then constructs W_j at weight $\kappa/4$ and its compatible physical variational inverses, including the one at weight $u\kappa/8$. The original polar action and the angle shear are both included there. This gives the first assertion by (7.15). The direct physical root estimate and (7.17) give the hot and cold assertions on this very W_j. The same trajectory satisfies (7.16), because the frequencies and angle origins of every earlier occupied coordinate were fixed in the terminal choice. The initial two-site torus and the final direct-rate cube verify the initial and successive induction conditions, respectively. □

Proof of Theorem 1.3. The first restored orbit is $U_1 = W_1$, which is uniformly bounded in H^s by Proposition 7.1. The successive comparisons in H^2 give

$$\sup_{j\ge 1,\, t\ge 0} \|U_j(t)\|_{H^2} \le \sup_{t\ge 0} \|W_1(t)\|_{H^2} + \sum_{i\ge 2} \varepsilon_i < \infty.$$

Apply Proposition 7.3 to the solutions U_j just constructed. In its notation take $\delta_j = \varepsilon_j$, $c_j = O(e^{-cd_j})$, $k_j = (1 + N_j, 0)$, and write $D_* = D_h - 2C_w$. We verify the summability and the persistence conditions explicitly. First,

$$(2K_j)^{s-2}\delta_j = 2^{s-2} \exp\big(((s-2)L_1 - D_*)B_{j-1}\big).$$

This is summable by the choice of D_h. Next, $S_j K_j^{-1/2} = \exp(-(L_1/2 - L_0)B_{j-1})$. Since $L_1 > 2L_0 + 2$ and $d_{j-1} = o(B_{j-1})$, the series $\sum_j \sqrt{\omega_j}\, e^{A' d_{j-1}}$ converges for every fixed A'. Also $\sum_j c_j < \infty$, since B_j grows geometrically.

The root of U_j has normalized Fourier modulus at least $0.9e^{bd_j} - e^{-B_{j-1}}$ for all $t \ge T_j$. As $\log N_j \le 2B_j$, the possible loss at this frequency in all later stages is at most

$$\langle k_j\rangle^{s-2} \sum_{i>j} \delta_i \le C_s e^{2(s-2)B_j} \sum_{i>j} e^{-D_* B_{i-1}} \le 2C_s e^{-(D_* - 2(s-2))B_j}.$$

Here $D_* > 2(s-2)$ by the choice of D_h and $C_0 \ge 3$. The hypothesis of Proposition 7.3 follows with any $0 < a < b$ after omitting finitely many stages. Keeping the actual root height in its lower-bound argument gives, for the single limiting solution and every $T_j \le t < T_{j+1}$,

$$bd_j - O(1) \le \log \|u(t)\|_{H^s} \le C' d_{j+1} + O(1).$$

Throughout that interval, $(C_0+1)B_j \le \log t < (C_0+1)QB_j$. Thus $\log\log t = \log B_j + O(1)$ with a bound independent of j. The definition of d_j now yields (1.3) on the union of all sufficiently late intervals. Returning from the mass gauge multiplies the solution by a scalar of modulus one. □

Appendix A. The local frequency estimate at $p = 4$

We use the period-one normalization of [46]. For finitely supported nonnegative action vectors J, put $\omega_n^* = \omega_n - (2\pi n)^2 - 4\sum_j J_j$. We show that $R(J) = \omega^*(J) + 2J$ extends holomorphically from these data near zero to an open complex ℓ^2 ball, with values in $\ell^{3/2}$.

Spectral tails. Use the period-two Fourier basis for the periodic spectral problem and write $q_j = |\varphi_j^+| + |\varphi_j^-|$ for the two potential coefficients in the signed indexing of [46, (1.15)]. Set $A = \|q\|_4$, $T_N = \|(q_j)_{|j|\ge N}\|_4$, and let $\lambda_n^\pm$ be the two periodic eigenvalues near $n\pi$. For A sufficiently small and $N \ge 1$, we claim

$$\sum_{|n|\ge N,\,\pm} |\lambda_n^\pm - n\pi|^4 \lesssim \sum_{|n|\ge N} q_{2n}^4 + N^{-1}A^8 + T_N^8. \tag{A.1}$$

The first term retains the direct Fourier coefficients in (1.15), which are missing from the quantitative bound in [46, Proposition 6.1].

To see this, first take a trigonometric polynomial potential. Project onto the free double eigenspace at $n\pi$ with projection P, and put $Q = 1 - P$. Write L_0 for the free operator and V for the potential matrix. For $|\lambda - n\pi| \le \pi/4$, let S be a diagonal square root of $(\lambda - QL_0Q)^{-1}$ on the range of Q. Put $h_0 = 0$, $h_j = |j|^{-1}$ for $j \ne 0$. In the free basis the entries coupling indices $n + j, n + k$ are bounded by q_{2n+j+k}. Hence

$$\|SQVQS\|_{\mathrm{HS}}^2 \lesssim \sum_{j,k} h_j h_k q_{2n+j+k}^2 \le \|h * h\|_2 A^2, \qquad \|h * h\|_2 \le \|h\|_{4/3}^2.$$

The inverse of $1 - SQVQS$ is uniformly bounded. Each row of $PVQS$ and column of $SQVP$ has squared norm at most CL_n, where $L_n = \sum_j h_j q_{2n+j}^2$. Thus the Schur correction $PVQS(1 - SQVQS)^{-1}SQVP$ to PVP has norm at most CL_n. At either root the reduced equation gives $|\lambda_n^{\pm} - n\pi| \lesssim q_{2n} + L_n$. The same estimates on the boundary keep both roots in the disc by continuation from zero. Approximation gives the bound for complex Fourier–Lebesgue potentials with coefficients in ℓ^4, by continuity of the isolated spectral pair [46, Sections 3 and 6]. For $|n| \geq N$, split the sum in L_n at $|j| = N$. Its large-j part has ℓ_n^4 norm at most $CN^{-1/4}A^2$ by Young's inequality. In the other part $|2n + j| \geq N$, giving norm at most CT_N^2. This proves (A.1). In particular the gap lengths $\gamma_n = \lambda_n^+ - \lambda_n^-$ are locally bounded in ℓ^4. The bounded reflection extensions in [46, Sections 4–5] give the same qualitative localization for the Dirichlet and auxiliary spectra.

Cancellation in the frequency formula. Let $\tau_n = (\lambda_n^+ + \lambda_n^-)/2$, $G_n = [\lambda_n^-, \lambda_n^+]$, and $w_n(\lambda)^2 = (\tau_n - \lambda)^2 - \gamma_n^2/4$, with $w_n(\lambda) \sim \tau_n - \lambda$ at infinity. Write $\Omega_{nk} = \Omega_{nk}^{(2)}$ for the second contour moment of [46, Section 20]. In its defining integral the normalized differential is multiplied by F_k^2, where F_k is the local Abelian integral of [46, Section 19] vanishing at the endpoints of G_k. The finite-gap frequency formula is $\omega_n^* = -(2/\pi)\sum_k \Omega_{nk}$ [46, Lemma 20.2].

For each m, the critical point $\lambda_m^\bullet$ of the discriminant and the zeros σ_m^n of the numerator ψ_n of the normalized differential (for $m \neq n$) differ from τ_m by $O(|\gamma_m|^2)$, uniformly in the indices on a small complex ball [46, Lemma 10.10 and Theorem 12.1]. Set $U_n = \sum_{m \neq n} |\gamma_m|^2/|m - n|$. On the disc at n, each factor $(s_m - \lambda)/w_m(\lambda)$ in the products (2.18) and (4.9) of [46], with $m \neq n$ and $s_m = \lambda_m^\bullet$ or σ_m^n, equals $1 + O(|\gamma_m|^2/|m - n|)$. Indeed, divide numerator and denominator by $\tau_m - \lambda$ and use $|\tau_m - \lambda| \gtrsim |m - n|$. The products are thus $1 + O(U_n)$, since U is bounded. On either bank of the cut, integration of (2.18) from an endpoint gives

$$F_n = iw_n + O(|\gamma_n|(U_n + |\gamma_n|)).$$

Here $w_n' = -(\tau_n - \lambda)/w_n$ and $\int_{G_n} |d\lambda/w_n| = \pi$ on one bank. On the lower bank, oriented from λ_n^- to λ_n^+, $\int_{G_n^-} w_n\, d\lambda = i\pi\gamma_n^2/8$. The diagonal contour integral reduces to twice the lower-bank integral of $F_n^2\, i(1 + O(U_n))\, d\lambda/w_n$. Its leading term is therefore $\pi\gamma_n^2/4$. In the action formula [46, (2.19)] the leading term is $\gamma_n^2/4$, since the arcsine mean of t^2 is $1/2$ and $2(\lambda_n^\bullet - \tau_n)/\gamma_n = O(|\gamma_n|)$. Consequently

$$|\Omega_{nn} - \pi J_n| \lesssim |\gamma_n|^2(U_n + |\gamma_n|), \qquad |\Omega_{nk}| \lesssim |\gamma_k|^3/|n - k| \quad (k \neq n).$$

For the latter bound, (4.9) on G_k has one factor $(\sigma_k^n - \lambda)/w_k$, with numerator $O(|\gamma_k|)$, and an unpaired $1/w_n = O(1/|n - k|)$. The remaining product is bounded and $F_k = O(|\gamma_k|)$. Closed gaps follow by continuity of the actions and moments. Young's inequality gives $U = h * |\gamma|^2 \in \ell^3$ since $h \in \ell^{6/5}$. The diagonal error thus belongs to $\ell^{4/3}$, and the off-diagonal sum is bounded in $\ell^{3/2}$ by $C\|h\|_{12/11}\||\gamma|^3\|_{4/3}$. In $\omega_n^* + 2J_n$ the term $-(2/\pi)\pi J_n$ cancels. The bounds are locally uniform on the complex potential ball. For fixed n the moment sum is holomorphic there by the contour formula [46, Lemma 20.1] and locally uniform convergence of the sum (the ℓ^4 tail of $(h_{n-k})_k$ tends to zero).

Differentiation at an old action. Use the local holomorphic inverse Birkhoff map on the complex ℓ^4 Cartesian ball [46, Theorems 14.1 and 15.2], with coordinate pairs z_j^+, z_j^- normalized by $J_j = z_j^+ z_j^-$. On each finite coordinate slice, the moment formula on real finite-gap data is invariant under rotations of the individual angles. Holomorphic continuation from the real slice therefore leaves only Taylor monomials with equal powers of z_j^+ and z_j^- at each site. These give compatible holomorphic functions of the finite action vectors. Every such complex J has a lift with $|z_j^+| = |z_j^-| = |J_j|^{1/2}$, so that $\|z^+\|_4^2 = \|z^-\|_4^2 = \|J\|_2$. Thus the $\ell^{3/2}$ bound holds on a common ℓ^2 action ball independently of the slice. Cauchy estimates give a common Lipschitz

bound on each smaller ball in the $\ell^{3/2}$ norm, first on finite output projections and then by taking their supremum. By density the slice functions extend uniquely there, and the one-variable Cauchy formula along lines passes to the limit, giving a holomorphic map R into $\ell^{3/2}$.

For every fixed action J_* in this ball, $DR(J_*) : \ell^2 \to \ell^{3/2}$ is bounded and hence compact by Pitt's theorem, as in [46, proof of Corollary 18.2]. Let $(\mathcal{S}I)_n = I_{-n}$. In the normalization of Section 2, $J_* = (2\pi)^2 \mathcal{S} I_*/2$ and

$$D\chi(I_*) + \mathrm{Id} = \tfrac{1}{2}\mathcal{S}^{-1} DR(J_*)\mathcal{S} \quad \text{on } \ell^2,$$

where the range of DR is included in ℓ^2. This proves compactness also at each fixed nonzero old action.

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