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# Schwinger–Strebel measures for planar four-point correlators

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November 17, 2026

**Abstract**

The inverse-Schwinger construction associates finite positive measures on decorated moduli space to connected planar correlators of a free Hermitian matrix scalar. We determine their Lebesgue decomposition after integrating the perimeter variables in the four-point case. The reduced normal-ordered skeletons comprise ten multigraph types and 115 oriented cells with fixed vertex labels. The path and four-cycle types have projection rank one; all other types have rank two almost everywhere. For a nonzero correlator, a singular component exists precisely when the operator degrees admit a two-against-two balance. The only purely singular degree multisets are $\{1,1,2,2\}$ and $\{2,2,2,2\}$. At equal degree $k \geq 2$, exact mass formulas give the sharp lower bound $3/(k+1)$ for the singular fraction, a quantitative stability estimate, and fixed-ratio and near-symmetric large-degree limits. We derive square and path marginal densities, including a uniform inverse-cross-ratio law at unit square shapes and a common rate, and the exact path endpoint exponents. The square density is finite and jointly smooth for all positive shapes and rates. For maximal $n$-point cells, a nondegenerate period system on the canonical cover of genus $n-3$ proves that the projections are submersions everywhere and yields a perimeter-simplex density formula. Admissible gamma-shape contractions converge weakly and admit a compensated first variation under a local Hölder condition on test-function pullbacks.

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# 1 Introduction and main results

The Schwinger–Strebel construction sends a planar free-field diagram to a metric ribbon graph and hence to a punctured Riemann surface with positive perimeters. Even when the original Schwinger density is smooth and strictly positive, its image on undecorated moduli space can have different dimensions. Determining which diagrams retain singular support after perimeter integration is therefore a concrete question about the projection of Strebel cells. The answer is sensitive to ribbon topology: two cells with the same dimension and zero orders can have different projection ranks.

We study the normal-ordered single-trace operators

$$\mathcal{O}_J(x) = (JN^J)^{-1/2} :\mathrm{tr}\,\phi^J(x):$$

of a free Hermitian $U(N)$ matrix scalar in four-dimensional Euclidean space, with propagator $(4\pi^2|x-y|^2)^{-1}$. All insertion points are pairwise distinct, all degrees are positive integers, and the connected planar coefficient is extracted with the factor $N^{n-2}$. The Strebel critical graph

is the ribbon *dual* of the reduced field-theory skeleton. This duality determines the perimeter constraints, zero orders, and dimension counts used below.

For four labelled punctures we identify

$$\mathcal{M}_{0,4} = \mathbb{C} \setminus \{0, 1\}, \qquad (z_1, z_2, z_3, z_4) = (0, 1, \eta, \infty),$$

and write $\nu_{\mathbf{J}}$ for the finite positive measure obtained by integrating all perimeters. Its total mass is exactly the connected planar field-theory correlator. The main conclusions are as follows.

1. **Complete rank classification.** There are ten reduced multigraph types and 115 oriented ribbon cells with the four vertex labels fixed. Of these, twelve path cells and three four-cycle cells project with rank one onto real cross-ratio intervals. The other one hundred cells have rank two almost everywhere and give absolutely continuous measures on $\mathcal{M}_{0,4}$ (Theorems 4.1 and 6.5). We also classify the exact preimage of the real cross-ratio locus using orientation-reversing metric ribbon automorphisms (Proposition 6.7).

2. **A degree criterion for the Lebesgue decomposition.** For a nonzero connected planar correlator, the singular component is nonzero if and only if $J_a + J_c = J_b + J_d$ for some permutation of the four labels. Pure singularity occurs exactly for the degree multisets $\{1, 1, 2, 2\}$ and $\{2, 2, 2, 2\}$. Every resulting measure is atomless (Theorem 6.11).

3. **Exact singular mass, stability, and large-degree limits.** The singular mass for arbitrary degrees is a sum over twelve paths and three cycles, with the cycle sums evaluated by finite geometric series. For equal degree $k \geq 2$, write $A = \Delta_{12}\Delta_{34}$, $B = \Delta_{13}\Delta_{24}$, and $C = \Delta_{14}\Delta_{23}$. Then

$$\frac{\nu^{s}_{kkkk}(\mathcal{M}_{0,4})}{G^{(0)}_{kkkk}} \geq \frac{3}{k+1}.$$

   For $k \geq 3$, equality holds exactly when $A = B = C$, a condition realized by four equidistant Euclidean insertion points. At those points the singular fraction is $3/(k+1)$ and the absolutely continuous fraction is $(k-2)/(k+1)$ (Theorem 7.2). A global remainder in squared logarithmic channel ratios controls near equality and has the exact quadratic coefficient at the symmetric configuration (Theorem 7.4). At fixed positive $A, B, C$, with $A = \max(A, B, C)$, the singular fraction satisfies

$$\lim_{k\to\infty} \frac{\nu^{s}_{kkkk}(\mathcal{M}_{0,4})}{G^{(0)}_{kkkk}} = 1 - \frac{2BC}{A(B+C)}.$$

   This limit is positive unless $A = B = C$. A simultaneous scaling of the channel ratios as $e^{u_i/k}$ gives a uniform transition formula (Theorem 7.5 and Proposition 7.6).

4. **Explicit marginal measures.** We integrate the scale and perimeter constraints for the square and path cells. The square marginal is finite and jointly smooth in the interior for all positive shapes and rates, with explicit tail bounds for shapes at least one. Unit shapes and a common rate give $\eta^{-1} \sim \mathrm{Uniform}(0, 1)$ on a square cell. For the complete equal-distance 2222 correlator this produces the anharmonic-invariant measure

$$4\Delta_0^4 \delta(\Im\eta) \max\{1, |\Re\eta|, |1 - \Re\eta|\}^{-2}\, d^2\eta.$$

   The path marginal has a closed expression at unit shapes and explicit endpoint asymptotics for every positive shape (Section 8).

5. **Maximal-cell geometry and controlled contractions.** For every $n \geq 4$, an explicit real period Jacobian on the canonical cover of genus $n-3$ proves that maximal-cell projections have rank $2n-6$ everywhere. After integrating the scale, their densities have an exact integral representation over $n-1$ normalized perimeter variables (Section 9). Admissible gamma-shape contractions converge weakly, and locally Hölder pullbacks have a compensated first variation. When the four-point limit changes from an area measure to a path or cycle measure, its total variation distance from every approximating probability measure remains one (Section 10).

The path and cycle formulas determine the mass carried by the real cross-ratio locus. The remaining cells contribute area measures, whose maximal-cell densities can be expressed in period coordinates. Mirror cells contribute conjugate measures and are both retained, including when their total masses agree.

## 1.1 Relation to earlier work

The Schwinger–Strebel construction gives a geometric organization of free large-$N$ matrix correlators. Gopakumar reorganized Schwinger-parametrized correlators into skeleton graphs and moduli-space integrals [Gop04], and identified inverse Schwinger parameters with the metric lengths of the dual Strebel critical graph [Gop05]. The underlying correspondence between metric ribbon graphs and decorated Riemann surfaces is furnished by Strebel's theorem [Str84]; its relation to matrix integrals and intersection theory was developed by Kontsevich [Kon92]. We use this established dictionary to study the measures obtained after forgetting the perimeter variables. Cellwise regularity is justified using homological coordinates for meromorphic quadratic differentials and the real-period description of the Strebel locus [BK20].

Explicit four-point calculations already reveal both extended and localized support. Aharony, Komargodski, and Razamat computed several families of Strebel differentials, analyzed their worldsheet operator product expansions, and exhibited correlators supported on proper subsets of moduli space [AKR06]. David and Gopakumar obtained an explicit worldsheet four-point function in the $Y$-diagram family and expressed the general four-point problem in elliptic coordinates [DG07]. The distributional interpretation of localized correlators and explicit square and whale differentials were developed further in [Aha+07]. Large-charge localization for extremal correlators, including the lollipop family, was studied in [DGM08]. An independent combinatorial approach expresses general multi-trace correlators through permutations and reduced graphs at arbitrary genus [Suz19]. These results provide benchmarks for the present calculations. In particular, the existence of the Schwinger–Strebel pushforward, distributional support, and explicit $Y$-diagram solutions are established parts of the construction.

The present results determine the Lebesgue decomposition for every nonzero connected planar four-point correlator at fixed operator degrees. The graph classification and projection-rank analysis identify all singular sectors, and their finite-charge weights give the singular mass without solving a period problem. At equal degrees, these weights also give the sharp mass bound, its stability estimate, and the two large-degree limits. For the square family, we integrate the known local differential formula over all perimeters and analyze the resulting marginal for arbitrary positive shapes. For maximal cells, we make the real-period characterization explicit through a nonzero Jacobian determinant and a scale-integrated density formula.

The detailed positive-integer multiplicity solutions appear in Appendix A; independent finite enumeration and numerical checks are described in Appendix B.

## 2 The connected planar Wick expansion

Let $x_{ij} = x_i - x_j$ and $\Delta_{ij} = (4\pi^2|x_{ij}|^2)^{-1}$. We use the action and propagator

$$S_0 = \frac{1}{2}\int_{\mathbb{R}^4} \operatorname{tr}(\partial_\mu\phi\,\partial^\mu\phi)\,d^4x, \qquad \langle\phi^a{}_b(x_i)\phi^c{}_d(x_j)\rangle = \delta^a{}_d\delta^c{}_b\Delta_{ij}. \tag{1}$$

The normalized planar coefficient is

$$G^{(0)}_{\mathbf{J}}(x) = \lim_{N\to\infty} N^{n-2}\left\langle \prod_{i=1}^{n}\mathcal{O}_{J_i}(x_i)\right\rangle_{\text{conn}}. \tag{2}$$

Normal ordering excludes contractions within an individual trace. The chosen normalization gives $\lim_{N\to\infty}\langle\mathcal{O}_J(x)\mathcal{O}_K(y)\rangle = \delta_{JK}\Delta(x-y)^J$: the planar two-vertex pairings reverse the cyclic order and admit $J$ cyclic shifts.

Put $2M = \sum_i J_i$. If this sum is odd, the correlator is zero. Otherwise let $\mathcal{H}$ be a set of $2M$ trace slots and let $\alpha$ be the permutation whose $n$ cycles record the fixed cyclic orders on the traces. Let $\iota(h)$ be the insertion label containing slot $h$. A Wick pairing is a fixed-point-free involution $\beta$. It is normal-ordered when $\iota(h) \neq \iota(\beta h)$ for every $h$, and connected exactly when $\langle\alpha,\beta\rangle$ acts transitively on $\mathcal{H}$. On such a connected pairing the genus is defined by

$$2 - 2g = n - M + c(\alpha\beta), \tag{3}$$

where $c$ denotes the number of permutation cycles. Let $\mathcal{P}^{(0)}_{\mathbf{J}}$ be the set satisfying all three conditions: normal ordering, transitivity, and genus zero.

**Proposition 2.1** (Exact connected planar coefficient)**.** *For positive integer degrees and pairwise distinct insertion points,*

$$G^{(0)}_{\mathbf{J}}(x) = \frac{1}{\sqrt{\prod_i J_i}}\sum_{\beta\in\mathcal{P}^{(0)}_{\mathbf{J}}}\prod_{\{h,\beta h\}}\Delta_{\iota(h),\iota(\beta h)}, \tag{4}$$

*where the product takes each unordered pair once.*

*Proof.* The double-line contraction produces one colour loop for every cycle of $\alpha\beta$, so its colour factor is $N^{c(\alpha\beta)}$. External normalization contributes $N^{-M}/\sqrt{\prod_i J_i}$, and the prefactor in (2) leaves $N^{-2g}$ by (3). Connected cumulants select precisely the transitive contraction graphs. Normal ordering excludes the indicated self-pairings. Taking the planar limit gives the formula. □

The inverse-Schwinger representation and its bundle convolution are

$$\Delta_{ij} = \frac{1}{16\pi^2}\int_0^\infty e^{-\ell|x_{ij}|^2/4}\,d\ell, \tag{5}$$

$$\int_{(0,\infty)^m} F(\ell_1+\cdots+\ell_m)\,d\ell_1\cdots d\ell_m = \frac{1}{\Gamma(m)}\int_0^\infty F(\ell)\ell^{m-1}\,d\ell. \tag{6}$$

The second identity follows from the volume of the positive simplex; it holds first for nonnegative measurable $F$ and then for integrable $F$. Its use requires a geometric definition of a bundle, supplied by the reduction below.

# 3 Reduced ribbon maps and exact Wick multiplicities

All ribbon maps in this section are connected and oriented. Their vertices are individually labelled by $1, \dots, n$, and an isomorphism preserves every vertex label and every cyclic order. We assume $n \geq 3$. A loop is an edge whose two ends lie at the same vertex. A face degree counts boundary darts, so that a bridge, when present, occurs twice on the boundary of its incident face. The normal-ordering condition is precisely the exclusion of loops in the original Wick map.

**Definition 3.1.** A reduced skeleton is a loopless planar ribbon map in which every face has degree at least three. A multiplicity on a skeleton $G$ is a family $m = (m_e)_{e \in E(G)}$ of positive integers. The parallel expansion $M(G, m)$ replaces the ribbon corresponding to $e$ by $m_e$ consecutive parallel ribbons, with the induced cyclic orders at its ends.

The skeleton $G$ and its dual $G^\vee$ have different roles. The vertices of $G$ are the field-theory insertions, whereas the faces of $G^\vee$ are the labelled Strebel punctures. In particular, the critical graph of the Strebel differential is $G^\vee$, not $G$. We use the edge bijection $E(G) \simeq E(G^\vee)$ without changing edge names.

**Proposition 3.2** (Unique reduction)**.** *Every connected loopless planar ribbon map with at least three vertices has a unique reduced skeleton and a unique positive multiplicity vector, up to the preceding notion of isomorphism. Equivalently, deleting empty digons in any order produces the same skeleton and the same bundle multiplicities. Parallel expansion is inverse to this reduction.*

*Proof.* Remove a small open disc about each vertex. The ribbons become disjoint properly embedded arcs in an oriented sphere with $n$ boundary components. Their endpoints lie on distinct boundary components. Two such arcs are parallel when they are isotopic with endpoints allowed to move along their assigned boundary components. This is an equivalence relation defined on the punctured surface, independently of the other arcs.

Disjoint arcs in one equivalence class cobound strips containing no other boundary component. Since $n \geq 3$, the arcs in a class have a linear order across a maximal such strip; the annular exceptional case in which all arcs join the only two boundary components does not occur. An arc lying between consecutive members of the class would itself belong to the same class. Consequently consecutive members bound empty digons. Conversely, the two sides of an empty digon are parallel. Retaining one representative of each class therefore removes precisely the empty digons. Distinct classes cannot merge during this operation, since their isotopy classes were defined relative to all the fixed boundary components. The multiplicity of the retained edge is the cardinality of its class.

The remaining map has no face of degree one, since it has no loop. A degree-two face would either be an empty digon or the doubled boundary of the sole edge of a two-vertex graph. Both alternatives are excluded. Thus the resulting skeleton is reduced. Inserting the prescribed number of consecutive parallel arcs into each strip reverses the construction uniquely and proves the assertion. □

**Proposition 3.3** (Automorphisms and labelled pairings)**.** *Let $G$ be a reduced skeleton and let $m_e \geq 1$. Put*

$$J_i = \sum_{e \ni i} m_e, \qquad M = \sum_e m_e = \frac{1}{2} \sum_i J_i. \tag{7}$$

*The number of labelled Wick pairings whose reduced data are $(G, m)$ is*

$$N_{G,m} = \frac{\prod_i J_i}{|\operatorname{Aut}(G, m)|}. \tag{8}$$

*For planar maps with the $n \geq 3$ vertices individually labelled,* $\mathrm{Aut}(G,m)$ *is trivial. Hence the contribution of one oriented skeleton and one multiplicity vector to the normalized planar correlator is*

$$G^{(0)}_{G,m}(x) = \sqrt{\prod_i J_i} \prod_{e=ij} \Delta_{ij}(x)^{m_e}, \qquad \Delta_{ij}(x) = \frac{1}{4\pi^2 x_{ij}^2}. \tag{9}$$

*Its inverse-Schwinger measure is*

$$\frac{\sqrt{\prod_i J_i}}{(16\pi^2)^M} \prod_{e=ij} \left[ \frac{\sigma_e^{m_e-1}}{(m_e-1)!} e^{-\sigma_e x_{ij}^2/4}\, d\sigma_e \right]. \tag{10}$$

*Proof.* Fix the trace permutation with cycles of lengths $J_1, \ldots, J_n$. Independent cyclic changes of the starting slot act on Wick involutions by conjugation. The acting group has order $\prod_i J_i$, and its orbits are precisely the isomorphism classes of ribbon maps with individually labelled vertices and the prescribed valences. Its stabilizer is the corresponding orientation-preserving map automorphism group.

Unique reduction induces an isomorphism

$$\mathrm{Aut}(M(G,m)) \simeq \mathrm{Aut}(G,m). \tag{11}$$

Indeed, an automorphism induces one of the reduced map. If it acts trivially on the reduced map, consider any parallel bundle. At least one end of its skeleton edge has skeleton degree at least two, since $G$ is connected and has at least three vertices. The neighbouring skeleton edges distinguish the ends of the linear order in that bundle, forcing the automorphism to fix all of its ribbons. Thus the kernel is trivial. Conversely, a multiplicity-preserving skeleton automorphism lifts by mapping the ribbons of each bundle in their induced order. Orbit–stabilizer now gives (8).

For completeness, triviality in the planar labelled case follows directly from the metric ribbon-graph form of the Strebel correspondence. Give every edge of $G^\vee$ length one. A ribbon automorphism of $G$ preserves this metric and every face label of $G^\vee$. It therefore extends, through the canonical flat-cylinder construction, to a holomorphic automorphism of the corresponding punctured Riemann sphere fixing all $n$ punctures. A holomorphic automorphism of the sphere fixing three distinct points is the identity. Its restriction to the critical graph is the original automorphism, which is therefore trivial.

Dividing the number $\prod_i J_i$ by the external normalization $\sqrt{\prod_i J_i}$ gives (9). The convolution identity for $m_e$ positive inverse Schwinger parameters then gives (10). □

*Remark* 3.4. The triviality assertion uses the planar, individually labelled setting. If vertex labels are quotiented or higher genus is considered, the automorphism factor in (8) must be retained. For a fixed representative of an unlabelled skeleton, the equivalent summation conventions are

$$\sum_{[m]} \frac{F(m)}{|\,\mathrm{Aut}(G,m)|} = \frac{1}{|\,\mathrm{Aut}(G)|} \sum_m F(m), \tag{12}$$

where the left sum is over multiplicity orbits and the right sum is over all assignments to the temporarily distinguished edges. One must choose one of these conventions, rather than apply both quotients.

**Proposition 3.5** (Dimension and perimeter restrictions)**.** *Let $G$ be a reduced planar skeleton with $n \geq 3$ vertices, $E$ edges, and face degrees $(d_f)_{f\in F(G)}$. Then*

$$n-1 \leq E \leq 3n-6, \qquad 3n-6-E = \sum_{f\in F(G)} (d_f - 3). \tag{13}$$

*The corresponding Strebel differential has one zero of order $d_f - 2$ for each face $f$ of $G$. Its decorated cell has real dimension $E$. In particular, the cell is top-dimensional exactly when every face of $G$ is triangular, or equivalently when $G^\vee$ is trivalent.*

*Writing $B_G$ for the unsigned vertex–edge incidence matrix, the perimeters and charges satisfy*

$$p = B_G\sigma, \qquad J = B_G m. \tag{14}$$

*For every positive edge-length vector,*

$$\sum_{j\neq i} p_j - p_i = 2\sum_{e\not\ni i} \sigma_e \geq 0. \tag{15}$$

*Equality holds if and only if every edge is incident to $i$.*

*Proof.* Euler's identity gives $|F(G)| = E - n + 2$, and counting face darts gives $\sum_f d_f = 2E$. Subtracting three times Euler's expression proves (13); connectedness gives the lower bound on $E$. The statements about zeros and dimensions follow from duality and the positive edge-length coordinates. The incidence identities follow by traversing the boundary of the face of $G^\vee$ corresponding to vertex $i$. Each edge has two distinct endpoints, so summing the resulting perimeter identities gives (15). □

# 4 Complete classification of reduced four-point skeletons

We classify the skeletons before imposing the external charges. Throughout this section the four insertion labels are fixed. Repeated edge names such as $12_a, 12_b$ denote distinct skeleton edges; their non-homotopy is part of the ribbon embedding. Edges in different homotopy classes remain distinct when their endpoints agree.

The following edge lists choose representatives of the ten abstract multigraph types:

$$\begin{aligned}
S &: \{12,13,14\}, & P &: \{12,23,34\},\\
C &: \{12,23,34,14\}, & T &: \{12,13,23,14\},\\
D_s &: \{12_a,12_b,13,14\}, & D_o &: \{12_a,12_b,13,24\},\\
H &: \{12,13,23,14,24\}, & L &: \{12_a,12_b,13,23,14\},\\
K &: \{12,13,14,23,24,34\}, & M &: \{12_a,12_b,13,23,14,24\}.
\end{aligned} \tag{16}$$

Here $S$ is a star, $P$ is a path, $C$ is a four-cycle, $T$ is a triangle with a pendant edge, and $H$ is $K_4$ with one edge removed. In $D_s$ the two pendant edges meet the same endpoint of the double edge; in $D_o$ they meet different endpoints. For $D_s$ and $D_o$, the pendant vertices lie on opposite sides of the cycle formed by the two parallel edges. The same convention puts the triangle vertex and the pendant vertex of $L$ on opposite sides of its double-edge cycle. For $M$, one of the remaining vertices lies on each side.

**Theorem 4.1** (Four-point classification)**.** *Every reduced four-point planar skeleton has exactly one of the ten abstract types in* (16)*. The complete count of oriented cells with individually labelled vertices is given in Table 1. There are* 115 *cells in all, with*

$$(N_3, N_4, N_5, N_6) = (20, 51, 36, 8). \tag{17}$$

*Every cell has trivial label-preserving ribbon automorphism group.*

| Type | $E$ | $\deg G$ | $(d_f)$ | Zero orders | $\operatorname{rank} B_G$ | $\operatorname{rank} df_G$ | Cells |
|---|---|---|---|---|---|---|---|
| $S$ | 3 | $(3,1,1,1)$ | $(6)$ | $(4)$ | 3 | 2 | 8 |
| $P$ | 3 | $(2,2,1,1)$ | $(6)$ | $(4)$ | 3 | 1 | 12 |
| $C$ | 4 | $(2,2,2,2)$ | $(4,4)$ | $(2,2)$ | 3 | 1 | 3 |
| $T$ | 4 | $(3,2,2,1)$ | $(3,5)$ | $(1,3)$ | 4 | 2 | 24 |
| $D_s$ | 4 | $(4,2,1,1)$ | $(4,4)$ | $(2,2)$ | 3 | 2 | 12 |
| $D_o$ | 4 | $(3,3,1,1)$ | $(4,4)$ | $(2,2)$ | 3 | 2 | 12 |
| $H$ | 5 | $(3,3,2,2)$ | $(3,3,4)$ | $(1,1,2)$ | 4 | 2 | 12 |
| $L$ | 5 | $(4,3,2,1)$ | $(3,3,4)$ | $(1,1,2)$ | 4 | 2 | 24 |
| $K$ | 6 | $(3,3,3,3)$ | $(3,3,3,3)$ | $(1,1,1,1)$ | 4 | 2 | 2 |
| $M$ | 6 | $(4,4,2,2)$ | $(3,3,3,3)$ | $(1,1,1,1)$ | 4 | 2 | 6 |

Table 1: All reduced normal-ordered four-point skeletons. Vertex degrees are listed as unordered partitions. The cell count includes all assignments of the fixed labels $1, 2, 3, 4$ and all inequivalent oriented ribbon embeddings. Zero orders refer to the dual Strebel differential. The generic projection ranks are proved in Theorem 6.5; for $K$ and $M$ the stated rank holds everywhere.

*Proof.* By (13), $3 \le E \le 6$.

For $E = 3$, the graph is a tree on four vertices. Its only abstract types are $S$ and $P$. A star has four choices of centre and two cyclic orders of its three distinct leaves, giving eight cells. A path has $4!/2 = 12$ labelings and a unique cyclic order at every vertex. Both types have one face of degree six.

For $E = 4$, the connected graph is unicyclic. Since loops are excluded, its unique cycle has length two, three, or four. A length-four cycle gives $C$, with $(4-1)!/2 = 3$ labelings. A length-three cycle gives $T$. There are four choices of the pendant vertex, three choices of its attachment vertex, and two cyclic orders at the degree-three vertex, giving 24 cells. The face degrees are three and five.

For a length-two cycle, each complementary disc must contain another vertex; otherwise the cycle bounds an empty digon. Hence each disc contains exactly one of the two remaining vertices. Each must be attached directly to one of the endpoints of the cycle. This gives $D_s$ or $D_o$. In either case there are six choices of the double-edge endpoint pair and two choices of the attachment data. For each such choice the reduced oriented embedding is unique: exchanging the two temporarily distinguished parallel edges identifies the two apparent mirror choices. Thus each type gives 12 cells. Both faces have degree four.

For $E = 6$, every face is triangular. If the graph is simple, it is $K_4$, whose labelled planar rotation systems are the two opposite orientations. Otherwise choose two parallel edges with endpoints $u, v$. Their union is a Jordan curve, and each side must contain at least one other vertex, because an empty side would contain a digon. The two other vertices therefore lie on opposite sides. A third parallel edge is impossible: three parallel arcs would require a further

vertex in each of three complementary regions. Triangulating each of the two discs forces its interior vertex to be joined to both $u$ and $v$. This is precisely type $M$. Its embedding is unique for each of the six choices of the double-edge endpoint pair. Thus there are two cells of type $K$ and six of type $M$.

For $E = 5$, the face degrees must be $(3, 3, 4)$. The degree-four face has at least three distinct boundary vertices. Indeed, a boundary walk alternating between only two vertices would require four distinct parallel edges: an edge appearing twice on the same face boundary is a bridge, and cannot have a parallel companion. Four such parallel edges leave only one additional edge, which cannot connect both remaining vertices. This contradicts connectedness.

A quadrilateral boundary with three or four distinct vertices admits a diagonal joining distinct opposite vertices and splitting it into two triangles. Inserting this edge produces one of the already classified six-edge maps. Deleting an edge of $K$ produces $H$. Deleting one of the parallel edges of $M$ also produces $H$, while deleting any of its other edges produces $L$. Thus these are the only five-edge types. For $H$, the missing edge has six choices, and the three distinguishable paths between its degree-three vertices have two planar cyclic orders, giving 12 cells. For $L$, choose the double-edge pair in six ways, the triangle vertex among the other two vertices in two ways, and the endpoint carrying the pendant edge in two ways. The reduced embedding is then unique, giving 24 cells.

The stated face partitions follow either from these constructions or from Euler's identity. Zero orders are the face degrees minus two. Automorphism triviality follows from Proposition 3.3. The incidence ranks follow from Proposition A.1 below. Adding the cell counts gives (17). □

The classification supplies a useful distinction between graph type and spacetime monomial. For example, type $M$ has two non-homotopic edges joining vertices $1, 2$. Their individual Schwinger variables remain independent even when both edges have multiplicity one. Replacing them by a single edge of multiplicity two changes the ribbon topology and the dimension of its Strebel cell.

## 4.1 Exact low-charge correlators

Set

$$A = \Delta_{12}\Delta_{34}, \qquad B = \Delta_{13}\Delta_{24}, \qquad C = \Delta_{14}\Delta_{23}. \tag{18}$$

The classification and the weight formula give

$$G^{(0)}_{2222} = 4(AB + AC + BC) \tag{19}$$

and

$$\begin{aligned} G^{(0)}_{3333} &= 18ABC \\ &\quad + 9\big(A^2B + A^2C + B^2A + B^2C + C^2A + C^2B\big). \end{aligned} \tag{20}$$

For 2222, the only possible skeletons are the three labelled four-cycles with every multiplicity equal to one. Each has $2^4 = 16$ labelled pairings and normalized coefficient four. For 3333, the two oriented $K_4$ cells each have multiplicity one on every edge and $3^4 = 81$ labelled pairings. Their combined normalized coefficient is eighteen. The other six terms come from the three four-cycles, each admitting the two alternating multiplicity assignments $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$. Each has normalized coefficient nine.

In particular, the coefficient of a single oriented $K_4$ cell is $9/(16\pi^2)^6$ in inverse-Schwinger coordinates. If the two cells are integrated separately, this coefficient is used on each. The coefficient $18/(16\pi^2)^6$ applies to their combined total mass, or to a conjugation-invariant test function integrated on one representative cell. For general test functions the two measures are conjugate images; both terms are retained as in Remark 6.8.

### 4.2 A generating function over all charges

Let $\mathfrak{G}_n$ be the finite set of reduced oriented planar skeletons with the $n \geq 3$ vertices individually labelled. Finiteness follows from $E \leq 3n-6$. As an identity of formal power series in $t_1, \dots, t_n$,

$$\sum_{J_1,\dots,J_n \geq 1} \frac{G^{(0)}_{\vec{J}}(x)}{\sqrt{\prod_i J_i}} \prod_i t_i^{J_i} = \sum_{G \in \mathfrak{G}_n} \prod_{e=ij \in E(G)} \frac{t_i t_j \Delta_{ij}(x)}{1 - t_i t_j \Delta_{ij}(x)}. \tag{21}$$

Indeed, unique reduction and (9) reduce the coefficient on the left to a sum over positive multiplicities with $B_G m = J$. Each edge then contributes the independent geometric series $\sum_{m_e \geq 1} (t_i t_j \Delta_{ij})^{m_e}$. Distinct non-homotopic edges with the same endpoints give distinct factors on the right. For $n = 4$, the right side is a finite sum over the 115 cells of Theorem 4.1.

## 5 Finite Strebel measures and perimeter integration

Fix $n \geq 3$, and let $\mathfrak{G}_n$ be the reduced labelled skeletons from Section 3. The Strebel correspondence [Str84, Kon92] associates to the dual metric graph $G^\vee$ a unique marked sphere and positive perimeters. We write

$$\mathfrak{S}_G : (0,\infty)^{E(G)} \longrightarrow \mathcal{M}^{\mathrm{dec}}_{0,n} := \mathcal{M}_{0,n} \times (0,\infty)^n, \qquad f_G = \pi \circ \mathfrak{S}_G, \tag{22}$$

where $\pi$ forgets the perimeters and the length of the dual edge is $\ell_e = \sigma_e$. At puncture $i$ the differential has principal part $-p_i^2\, dw^2/(4\pi^2 w^2)$ in a local coordinate. Zero orders and perimeters are those in Proposition 3.5.

**Theorem 5.1** (Finite positive pushforward)**.** *For fixed positive integer degrees and pairwise distinct Euclidean insertion points, define*

$$d\mu_{G,m}(\ell; x) = \frac{\sqrt{\prod_i J_i}}{(16\pi^2)^M} \prod_{e=ij} \frac{\ell_e^{m_e - 1}}{\Gamma(m_e)} e^{-\lambda_e \ell_e}\, d\ell_e, \qquad \lambda_e = |x_{ij}|^2/4, \tag{23}$$

$$\mathcal{W}_{\mathbf{J}} = \sum_{\substack{G \in \mathfrak{G}_n,\ m \in \mathbb{Z}^{E(G)}_{>0} \\ B_G m = J}} (\mathfrak{S}_G)_* \mu_{G,m}, \qquad \nu_{\mathbf{J}} = \pi_* \mathcal{W}_{\mathbf{J}}. \tag{24}$$

*Both sums define finite positive Radon measures, and*

$$\mathcal{W}_{\mathbf{J}}(\mathcal{M}^{\mathrm{dec}}_{0,n}) = \nu_{\mathbf{J}}(\mathcal{M}_{0,n}) = G^{(0)}_{\mathbf{J}}(x). \tag{25}$$

*They therefore also define distributions of order zero. Pairing them with the constant function* 1 *is legitimate as integration of a bounded function against a finite measure.*

*Proof.* There are finitely many reduced skeletons since $E \le 3n-6$, and for fixed degrees there are finitely many positive multiplicity assignments. Each $\lambda_e$ is strictly positive. The gamma integral gives

$$\mu_{G,m}((0,\infty)^E) = \frac{\sqrt{\prod_i J_i}}{(16\pi^2)^M} \prod_e \lambda_e^{-m_e} = \sqrt{\prod_i J_i} \prod_{e=ij} \Delta_{ij}^{m_e}.$$

Continuity of the Strebel map makes its pushforward a finite Borel measure; on these locally compact, second-countable moduli spaces such measures are Radon. Summing the exact skeleton weights proves (25). Finite total mass allows every bounded measurable test function, including 1. □

The geometry of $f_G$ determines the Lebesgue type of this measure after perimeter integration. The source measure can be studied at arbitrary positive real shapes $m_e$ as well; the integer constraints $B_G m = J$ are imposed when a field-theory correlator is asserted.

## 5.1 Removing the common scale

The map $f_G$ is invariant under common rescaling of all edge lengths. Write $\ell_e = r a_e$, where $r > 0$, $a_e > 0$, and $\sum_e a_e = 1$. Let $d^{E-1}a$ denote the coordinate volume obtained by eliminating one $a_e$ on this open simplex. For a general source $C \prod_e \ell_e^{m_e-1} e^{-\lambda_e \ell_e} d\ell_e/\Gamma(m_e)$, set $M := \sum_e m_e$. The exact scale integral gives

$$\nu_{G,m,\lambda} = C\Gamma(M)(f_G)_* \left[ \frac{\prod_e a_e^{m_e-1}}{\prod_e \Gamma(m_e)(\sum_e \lambda_e a_e)^M} \, d^{E-1}a \right]. \tag{26}$$

For the physical parameters this becomes

$$\nu_{G,m} = \frac{\sqrt{\prod_i J_i}\,\Gamma(M)}{(4\pi^2)^M} (f_G)_* \left[ \frac{\prod_e a_e^{m_e-1}}{\prod_e \Gamma(m_e)(\sum_{e=ij} a_e |x_{ij}|^2)^M} \, d^{E-1}a \right]. \tag{27}$$

The Jacobian is $r^{E-1}$, and the remaining power is $r^{M-1}$, which proves both formulas. In particular,

$$\operatorname{rank} df_G \le \min\{E-1, 2n-6\}. \tag{28}$$

Thus $E \ge 2n-5$ is necessary for an individual cell to give an absolutely continuous measure on $\mathcal{M}_{0,n}$. It is not a sufficient condition, as the four-cycle example will show.

## 5.2 Relabelling and conjugation

**Proposition 5.2** (Relabelling and conjugation)**.** *Let $c$ denote complex conjugation on the marked genus-zero moduli space. Reversing every cyclic order of a skeleton $G$ produces its mirror $G^{\mathrm{op}}$. Under the natural identification of their edges, with the same shapes and rates,*

$$\nu_{G^{\mathrm{op}},m,\lambda} = c_* \nu_{G,m,\lambda}. \tag{29}$$

*In particular, the complete physical measure satisfies $c_* \nu_{\mathbf{J}}(x) = \nu_{\mathbf{J}}(x)$.*

*For $\sigma \in S_n$, put $(\sigma J)_i = J_{\sigma^{-1}(i)}$ and $(\sigma x)_i = x_{\sigma^{-1}(i)}$, and let $R_\sigma$ be the corresponding relabelling of marked points on moduli space. Then*

$$\nu_{\sigma\mathbf{J}}(\sigma x) = (R_\sigma)_* \nu_{\mathbf{J}}(x). \tag{30}$$

*Proof.* Conjugating the flat-cylinder construction reverses the cyclic orders of the critical graph and leaves all metric lengths and face labels unchanged. Strebel uniqueness gives $f_{G^{\mathrm{op}}} = c \circ f_G$. Their source measures agree under the edge identification, proving (29). The set of physical cell–multiplicity data is closed under this mirror operation, with unchanged coefficients, so summation gives conjugation invariance.

Relabelling gives a bijection of compatible skeletons and multiplicities. The product of external normalization factors is unchanged, and each edge retains its rate and multiplicity under simultaneous relabelling of $x$ and $J$. Naturality of the Strebel construction gives $f_{\sigma G} = R_\sigma \circ f_G$. Pushing forward and summing proves (30). □

### 5.3 Bipartiteness and square differentials

**Proposition 5.3.** *For a reduced planar skeleton $G$, the following conditions are equivalent: $G$ is bipartite; every face degree of $G$ is even; every zero order of its Strebel differential is even; the differential is a global square of a meromorphic one-form on the sphere. If $V(G) = A \sqcup B$ is the bipartition, then*

$$\sum_{i \in A} p_i = \sum_{i \in B} p_i, \qquad E \le 2n - 4. \tag{31}$$

*Proof.* The facial walks of a bipartite graph have even length. Conversely, the facial boundaries generate the cycle space of a connected plane graph over $\mathbb{F}_2$; if all face degrees are even, every cycle has even length and the graph is bipartite. The zero orders are the face degrees minus two. On the sphere choose a nonzero meromorphic one-form $\omega_0$, for example $dw$ in an affine coordinate. If the divisor of $q$ is even, then $q/\omega_0^2$ is a meromorphic function with an even divisor of degree zero. Since every degree-zero divisor on the sphere is principal, $q/\omega_0^2 = cf^2$ for a meromorphic function $f$ and a nonzero constant $c$. Thus $q = (\sqrt{c}\, f\omega_0)^2$. The poles here all have order two. The perimeter identity follows by summing $p = B_G \ell$ over either part. Finally, all face degrees are at least four, so $2E \ge 4(E - n + 2)$, giving the bound. □

The residue signs of a square root alternate across the bipartition. These perimeter relations do not determine the projection rank. In particular, the star, path, and four-cycle are all bipartite, while the star has a two-dimensional image and the other two have one-dimensional images. Since $G$ has no loops, its planar dual has no bridges; this also restricts the portion of the full Strebel decomposition used by normal-ordered correlators.

## 6 Projection to undecorated moduli space

Let $\Gamma$ be a connected, loop-free, reduced planar ribbon graph on $n \ge 3$ labelled vertices. Write $E = E(\Gamma)$ and let $\mathfrak{S}_\Gamma : (0,\infty)^E \longrightarrow \mathcal{M}_{0,n} \times (0,\infty)^n$ be the inverse Strebel map for the *dual* metric ribbon graph. The real-analytic regularity needed below is recorded explicitly. In particular, no differentiability across a change of combinatorial type is assumed.

**Lemma 6.1** (Analyticity on a fixed metric ribbon cell)**.** *On the positive length cone of a fixed stable ribbon type, the map to the marked Riemann surface and its perimeter vector is real analytic. This includes cells with multiple zeros of the Strebel differential and variable perimeters.*

*Proof.* Label the zeros locally and work on their fixed-multiplicity stratum. For a nonsquare quadratic differential $q$, let $\widehat{\Sigma} \to \Sigma$ be its canonical double cover and let $v^2 = q$. Remove the

preimages of the double poles and use the anti-invariant relative homology, with the preimages of the zeros as relative endpoints. Integrals of $v$ along a basis give holomorphic local coordinates on this meromorphic differential stratum. This is the homological-coordinate construction for differentials with double poles in [BK20, Section 3.3, equations (3.22)–(3.24), and Definition 2.1]. Cycles around the poles are included; consequently their periods, and thus the perimeters, are allowed to vary.

The punctured cover retracts onto its lifted critical graph. Relative to all critical vertices, its cellular chain complex has only the edge group in degree one. The anti-invariant differences of the two lifts of each edge therefore give a basis over $\mathbb{R}$; branch vertices add no anti-invariant relative degree-zero term. Their periods are twice the signed critical edge lengths. The edge-to-period assignment is therefore an invertible linear map. Locally in the chosen ribbon type, positive real critical-edge periods form the Jenkins–Strebel slice. Composing this linear assignment with the inverse holomorphic period chart and then forgetting the differential proves real analyticity.

When $q$ is a global square, the canonical cover is disconnected and one instead uses the ordinary relative period coordinates of the meromorphic one-form on one component. On the sphere, with $n$ simple poles and $k$ distinct zeros, these consist of $n-1$ independent pole-loop periods and $k-1$ zero-to-zero periods. Their number is $n+k-2=|E|$, as required by Euler's formula. The local period-coordinate argument is unchanged: cut the punctured surface along a relative homology basis, use a primitive of the one-form on the cut pieces, and vary their complex translation identifications. Fixed zero multiplicities retain the prescribed cone angles, while the simple-pole ends vary through their residue periods. The resulting marked surface varies holomorphically with these periods; restoring real edge periods gives the asserted real-analytic map. In the four-point square cases used below, the explicit rational-square parametrizations also supply these charts directly. □

Set

$$f_\Gamma = \pi \circ \mathfrak{S}_\Gamma, \qquad \pi : \mathcal{M}_{0,n} \times (0,\infty)^n \longrightarrow \mathcal{M}_{0,n}.$$

For $m_e > 0$, $\lambda_e > 0$, and $C > 0$, consider the finite measure

$$d\mu_{\Gamma,m,\lambda}(\ell) = C \prod_{e\in E} \frac{\ell_e^{m_e-1} e^{-\lambda_e \ell_e}}{\Gamma(m_e)}\, d\ell_e, \qquad \nu_{\Gamma,m,\lambda} = (f_\Gamma)_* \mu_{\Gamma,m,\lambda}. \tag{32}$$

Its total mass is $C\prod_e \lambda_e^{-m_e}$. The physical weights are $m_e \in \mathbb{Z}_{>0}$ and $\lambda_e = x^2_{s(e)t(e)}/4$ at pairwise distinct Euclidean insertion points. The undecorated measure $\nu_{\Gamma,m,\lambda}$ integrates all perimeter variables, including their linear constraints.

## 6.1 A rank criterion and an inheritance principle

**Proposition 6.2** (Analytic rank and absolute continuity)**.** *Let $U \subset \mathbb{R}^d$ be connected and open, let $M$ be a $q$-dimensional real-analytic manifold, and let $f : U \to M$ be real analytic. Suppose that $\rho\, du$ is a finite measure with a locally integrable density. If $\operatorname{rank} df = q$ at one point, then $f_*(\rho\, du)$ is absolutely continuous with respect to every smooth positive density on $M$.*

*In local Euclidean coordinates on the target, a version of its density is*

$$h(y) = \int_{f^{-1}(y)\cap U_{\mathrm{reg}}} \frac{\rho(u)}{J_q f(u)}\, d\mathcal{H}^{d-q}(u) \quad \text{for almost every } y, \tag{33}$$

*where $U_{\mathrm{reg}} = \{u : \operatorname{rank} df(u) = q\}$ and $J_q f = \sqrt{\det(df\, df^{\mathsf{T}})}$. This assertion makes no claim that $h$ is bounded or smooth at critical values.*

*Proof.* The analytic section $\bigwedge^q df$ is nonzero at one point and hence cannot vanish on any nonempty open set of the connected domain. On each connected source coordinate ball mapped into a target chart, at least one maximal minor is therefore a nonzero real-analytic function. Its zero set has Lebesgue measure zero. A countable chart cover shows that the complement of $U_{\mathrm{reg}}$ has source measure zero. On $U_{\mathrm{reg}}$, the submersion theorem gives local product coordinates. Fubini's theorem in those coordinates proves absolute continuity, and a countable cover proves the global assertion. The coarea formula, applied on a countable compact exhaustion, gives (33). Finiteness follows from the assumed finite total mass. The argument is local on $M$ and hence is independent of the chosen smooth positive reference density. □

**Lemma 6.3** (Inheritance under admissible edge contraction)**.** *Let $\Gamma_0$ be obtained from $\Gamma$ by deleting a set $A$ of field-theory edges, with the same labelled vertices. Assume that the dual edges form a forest and that the resulting metric ribbon graph is stable. Equivalently, in the present planar situation it suffices that $\Gamma \setminus A$ remain connected and reduced. If $f_{\Gamma_0}$ has full target rank at one point, then $f_\Gamma$ has full target rank at some interior point and hence almost everywhere.*

*Proof.* Write the length variables as $(s, y)$, where $s$ belongs to the deleted edges. By the metric ribbon-graph form of the Strebel correspondence [Str84, Kon92], the inverse map has a continuous extension to the admissible face $s = 0$, and this extension is $\mathfrak{S}_{\Gamma_0}(y)$. Let $q = \dim_{\mathbb{R}} \mathcal{M}_{0,n}$. At a full-rank point of $f_{\Gamma_0}$, choose a smooth $q$-dimensional disk of length vectors $y = y(v)$ on which $f_{\Gamma_0}$ is a local diffeomorphism. Shrinking the disk and choosing target coordinates, its restriction to a closed disk has nonzero Brouwer degree at every point of a smaller target disk.

Fix $s_* > 0$ and put $g_\varepsilon(v) = f_\Gamma(\varepsilon s_*, y(v))$. Continuity on the compact disk gives uniform convergence $g_\varepsilon \to g_0$. Stability of degree implies that, for all sufficiently small positive $\varepsilon$, the image of $g_\varepsilon$ contains a nonempty open target disk. Sard's theorem therefore gives a point at which $dg_\varepsilon$ has rank $q$. At that point $df_\Gamma$ has rank $q$. Cellwise analyticity and the argument in Proposition 6.2 prove the final assertion. Only continuity at the contracted face was used. □

## 6.2 The rank of every four-point skeleton

Hereafter $n = 4$ and $\mathcal{M}_{0,4} = \mathbb{C} \setminus \{0, 1\}$, with the four labelled punctures at $(0, 1, \eta, \infty)$. Overall rescaling of all lengths leaves $\eta$ unchanged.

The two tree maps are computed explicitly in Propositions 8.6 and 8.7 below. The normalized star simplex maps real-analytically onto a half-plane, whereas the path cross ratio is $\cosh^2(u + v)$ for $u, v > 0$. These have ranks two and one, respectively. The cycle calculation in Corollary 8.2 has rank one everywhere.

**Lemma 6.4** (A square differential with a monogon face)**.** *Suppose that the critical graph has two vertices of valence four and four labelled faces. If its dual field-theory skeleton has a leaf, then its projection to $\mathcal{M}_{0,4}$ is a submersion.*

*Proof.* All zero and pole orders are even, so on the sphere the quadratic differential is a global square. Locally write it as

$$q = -\frac{1}{4\pi^2}\omega^2, \qquad \omega = \left(\frac{r_0}{z} + \frac{r_1}{z-1} + \frac{r_\eta}{z-\eta}\right) dz, \qquad r_\infty = -r_0 - r_1 - r_\eta.$$

The residues are nonzero real numbers of locally fixed signs, and their absolute values are the perimeters. Let $c_+$ and $c_-$ be the two distinct simple zeros of $\omega$. All its absolute periods are

purely imaginary. Thus, within this zero stratum, the only relative real-period condition is

$$H(\eta, r) = \sum_{j \in \{0,1,\eta\}} r_j \log \left| \frac{c_+ - z_j}{c_- - z_j} \right| = 0. \tag{34}$$

Here $z_0 = 0$, $z_1 = 1$, and $z_\eta = \eta$. Equivalently, $H = \operatorname{Re} \int_{c_-}^{c_+} \omega$; its value is independent of the path because the residues are real. Positivity of the critical edge lengths and the fixed ribbon type are open conditions in this real-period locus. The period-coordinate chart in Lemma 6.1 identifies this locus locally and real analytically with the positive edge-length cell.

The endpoint terms disappear when the residues vary, because $\omega(c_\pm) = 0$. Hence

$$\frac{\partial H}{\partial r_j} = \log \left| \frac{c_+ - z_j}{c_- - z_j} \right|. \tag{35}$$

If all three derivatives vanished, the punctures $0, 1, \eta$ would lie on the perpendicular bisector of the segment joining $c_+$ and $c_-$. The first two punctures force this line to be the real axis; therefore $\eta$ is real and $c_- = \overline{c_+}$, with the zeros nonreal. Complex conjugation would preserve the differential and all four labelled faces while interchanging its two critical vertices.

A leaf of the dual skeleton is a monogon face of the critical graph. Its boundary is a single loop based at one critical vertex. Conjugation cannot preserve that labelled face while sending its unique boundary vertex to the other vertex. This contradiction proves $d_r H \neq 0$. The implicit function theorem now solves (34) for one residue under arbitrary small variations of the complex modulus. The projection is consequently a submersion. This proof works at real as well as nonreal values of $\eta$. □

**Theorem 6.5** (Complete four-point projection-rank classification)**.** *Among connected, loop-free, reduced planar ribbon graphs on four labelled vertices, the path $P_4$ and the cycle $C_4$ have projection rank one. Every other type has projection rank two almost everywhere. For the six-edge types $K$ and $M$, the rank is two at every point. In particular, every positive Schwinger measure on a type other than $P_4$ or $C_4$ has an absolutely continuous undecorated pushforward. The path and cycle pushforwards are absolutely continuous with respect to one-dimensional Lebesgue measure on their respective real cross-ratio intervals and have no atoms.*

*Proof.* Use the ten types and counts in Table 1. The tree and cycle statements follow from Propositions 8.6, 8.7, and Corollary 8.2. Both $D_s$ and $D_o$ have a leaf and critical zero orders $(2, 2)$, so Lemma 6.4 applies. Types $T, H, L, K, M$ contain a star as a connected spanning subgraph. Deleting their other edges contracts a forest in the dual graph. Their full generic rank follows from Lemma 6.3. Proposition 6.2 proves absolute continuity in rank two. In rank one, the scalar coarea formula on the real interval gives absolute continuity there and rules out atoms. The everywhere rank assertion for the maximal types $K$ and $M$ follows from the independent period argument in Corollary 9.4. □

*Remark* 6.6. Critical zero orders and bipartiteness do not determine projection rank. The star and path both have zero partition (4) and bipartite field-theory skeletons, but their ranks differ. Likewise, the cycle and the two double-edge types have partition $(2, 2)$ and are bipartite, while only the cycle has rank one. The relevant information includes the ribbon topology. These examples also distinguish codimension in decorated moduli space from singularity after perimeter integration.

## 6.3 The real cross-ratio locus

**Proposition 6.7** (The real locus of every four-point cell)**.** *Let $G$ be one of the reduced oriented four-point skeletons, with fixed vertex labels and positive lengths $\ell$. An* orientation-reversing ribbon automorphism *fixes the vertex labels, preserves edge incidence, and reverses every cyclic order. Then $f_G(\ell) \in \mathbb{R} \setminus \{0,1\}$ if and only if $(G,\ell)$ admits such an automorphism preserving all lengths. Consequently the complete classification is*

$$f_G^{-1}(\mathbb{R} \setminus \{0,1\}) = \begin{cases} (0,\infty)^{E(G)}, & G \text{ has type } P \text{ or } C, \\ \varnothing, & G \text{ has type } S, T, H, \text{ or } K, \\ \{\ell_{12_a} = \ell_{12_b}\}, & G \text{ has type } D_s, D_o, L, \text{ or } M, \end{cases} \tag{36}$$

*where the last line uses the representatives in (16). Each oriented cell of type $S, T, H, K$ therefore projects into one open half-plane. For each of $D_s, D_o, L, M$, the two length half-cones $\ell_{12_a} > \ell_{12_b}$ and $\ell_{12_a} < \ell_{12_b}$ project into opposite open half-planes.*

*Proof.* If $\eta = f_G(\ell)$ is real, complex conjugation fixes the four labelled punctures $0, 1, \eta, \infty$ and their prescribed positive perimeters. The conjugate differential is Strebel with the same data. Uniqueness therefore makes the critical graph invariant under conjugation, with all metric lengths preserved. Passing to its dual gives the asserted orientation-reversing automorphism of $G$.

Conversely, an orientation-reversing metric ribbon automorphism of the dual critical graph extends through the flat-cylinder construction to an antiholomorphic automorphism of the marked sphere. Fixing the labelled points $0, 1, \infty$ forces this automorphism to be complex conjugation. Since it also fixes the fourth labelled point, $\eta$ is real. This proves the equivalence.

For $P$ and $C$, every vertex has degree at most two, so the identity on the graph reverses its cyclic orders and preserves every metric. For the simple types $S, T, H, K$, fixing all vertex labels fixes every edge and every dart. Each has a vertex of degree at least three, whose cyclic order cannot be reversed by the identity. They have no orientation-reversing labelled automorphism.

For the four remaining types, the two parallel edges are the only edges that a labelled graph automorphism can interchange. Swapping their endpoint darts reverses the cyclic orders in the prescribed reduced embedding and fixes all remaining darts. It is the unique orientation-reversing labelled automorphism: the composition of two such automorphisms would be an orientation-preserving one, which is trivial by Proposition 3.3. It preserves the metric precisely when $\ell_{12_a} = \ell_{12_b}$. This proves (36).

The positive length cones and each of the two stated half-cones are connected. Continuity and the absence of real image points make the sign of $\Im f_G$ constant on each. The parallel-edge swap changes $f_G$ to its complex conjugate, so the two half-cones have opposite signs. □

*Remark* 6.8 (Mirror sectors and total mass). Let $G^{\mathrm{op}}$ be obtained by reversing every cyclic order, with corresponding edge lengths, shapes, and rates identified, and write $\iota(\eta) = \overline{\eta}$. Naturality of the Strebel construction gives

$$f_{G^{\mathrm{op}}} = \iota \circ f_G, \qquad \nu_{G^{\mathrm{op}},m,\lambda} = \iota_* \nu_{G,m,\lambda}.$$

Thus the two oriented $K_4$ sectors have equal total masses and lie in opposite open half-planes. For a bounded measurable test function $\varphi$, their combined contribution is

$$\int \left[\varphi(\eta) + \varphi(\overline{\eta})\right] d\nu_{K,m,\lambda}(\eta).$$

The coefficient of a single oriented sector can be doubled when computing its total mass or integrating a conjugation-invariant test function. A general test-function integral retains both terms.

## 6.4 A complete criterion in terms of the operator degrees

Let $\mathbf{J} = (J_1, J_2, J_3, J_4) \in \mathbb{Z}^4_{>0}$, and put $M = \frac{1}{2}\sum_i J_i$. Let $\nu_{\mathbf{J}}$ denote the sum of the undecorated pushforwards over all compatible planar skeletons and positive integer multiplicities. All sector coefficients are positive. This positivity is essential below: distinct singular sectors cannot cancel.

**Lemma 6.9** (Nonvanishing of the connected planar sector)**.** *The measure $\nu_{\mathbf{J}}$ is nonzero if and only if*

$$M \in \mathbb{Z}, \qquad M \geq 3, \qquad \max_i J_i \leq M. \tag{37}$$

*Proof.* An admissible Wick graph has $M$ edges, is loop-free, and is connected on four vertices. These facts imply all three conditions.

Conversely, arrange $2M$ half-edge slots on a circle as four consecutive blocks of sizes $J_i$. Pair each slot with the slot $M$ positions away. No pair lies in one block, because every block has at most $M$ consecutive slots. This realizes a loop-free multigraph with the specified degrees. If it is disconnected, positivity of all degrees forces exactly two components, each on two vertices. Since $M \geq 3$, one component has at least two parallel edges. Remove one edge from each component and reconnect their four endpoints by two cross edges. The degrees are unchanged and the new multigraph is connected: the component with at least two edges retains an edge joining its vertices, and the cross edges attach the other two vertices to it.

Every simple graph on four vertices is a subgraph of planar $K_4$. Embed the underlying simple graph in the sphere and replace its edges by narrow parallel ribbons according to their multiplicities. This produces a planar ribbon graph of the required degrees and hence an admissible connected planar Wick contraction. □

**Proposition 6.10** (Allowed multiplicities in the singular sectors)**.** *For a path with vertex order $a$–$b$–$c$–$d$, positive integer multiplicities exist exactly when*

$$J_a + J_c = J_b + J_d, \qquad J_b > J_a. \tag{38}$$

*They are then unique:*

$$m_{ab} = J_a, \qquad m_{bc} = J_b - J_a = J_c - J_d, \qquad m_{cd} = J_d.$$

*For a cycle with cyclic order $a, b, c, d$, positive integer multiplicities exist exactly when*

$$J_a + J_c = J_b + J_d, \qquad \min_i J_i \geq 2. \tag{39}$$

*When they exist, there are exactly $\min_i J_i - 1$ choices. They are parametrized by*

$$\begin{aligned} m_{ab} &= t, & m_{bc} &= J_b - t, \\ m_{cd} &= J_c - J_b + t, & m_{da} &= J_a - t, \end{aligned}$$

*where*

$$\max(0, J_b - J_c) < t < \min(J_a, J_b), \qquad t \in \mathbb{Z}.$$

*Proof.* The displayed formulas follow successively from the four vertex-degree equations. In the cycle case, the length of the open interval for $t$ is

$$\min(J_a, J_b) - \max(0, J_b - J_c) = \min_i J_i$$

under the balance condition. Its integral endpoints give precisely $\min_i J_i - 1$ interior integers. □

**Theorem 6.11** (Charge classification of the Lebesgue decomposition)**.** *Assume* (37)*. The finite positive measure $\nu_{\mathbf{J}}$ has an atomless Lebesgue decomposition*

$$\nu_{\mathbf{J}} = \nu_{\mathbf{J}}^{\mathrm{ac}} + \nu_{\mathbf{J}}^{\mathrm{s}}$$

*with respect to area measure on $\mathcal{M}_{0,4}$. Its singular part is a finite sum of measures with densities along real cross-ratio intervals. The following statements are equivalent:*

1. *$\nu_{\mathbf{J}}^{\mathrm{s}} \neq 0$;*
2. *the four degrees admit a two-against-two balance, $J_a + J_c = J_b + J_d$ for a permutation $(a, b, c, d)$;*
3. *after sorting them as $a \leq b \leq c \leq d$, one has $a + d = b + c$.*

*Moreover,*

$$\nu_{\mathbf{J}}^{\mathrm{ac}} = 0 \quad \Longleftrightarrow \quad \{J_1, J_2, J_3, J_4\} = \{1, 1, 2, 2\} \text{ or } \{2, 2, 2, 2\}. \tag{40}$$

*Thus every nonzero four-point function with $M \geq 5$ has a nonzero absolutely continuous worldsheet component. For $M \geq 5$ it is purely absolutely continuous precisely when there is no two-against-two balance; otherwise it has both components.*

*Proof.* Theorem 6.5 identifies the singular part exactly with the allowed path and cycle sectors. These sectors are atomless. The remaining sectors are absolutely continuous, and positivity prevents cancellation. Their degree conditions in Proposition 6.10 show that a balance is necessary.

Suppose that a balance holds. If the four degrees are not all equal, choose an unequal pair from opposite parts of the balanced partition and orient the path so that the larger is an internal degree and the smaller is an endpoint degree. The other internal-minus-endpoint difference is the same positive integer, so an admissible path exists. If all four degrees equal $k$, nonvanishing excludes $k = 1$ and a cycle exists for every $k \geq 2$. This proves the first equivalence. The equivalence with the sorted condition is elementary: the other two possible pairings of four sorted numbers either force all four to be equal or force $a = b$ and $c = d$, and in both cases $a + d = b + c$ follows.

It remains to determine when an absolutely continuous sector exists in the balanced case. Sort the degrees as $a \leq b \leq c \leq d$. If $a < b$, attach a leaf of degree $a$ to the vertex of degree $d$. Make a triangle on the remaining three vertices, of degrees $b, c, d$, with edge multiplicities

$$m_{bc} = a, \qquad m_{bd} = b - a, \qquad m_{cd} = c - a.$$

All are positive, and the degree of the last vertex is $a + (b - a) + (c - a) = b + c - a = d$. This is a type-$T$ sector, with an absolutely continuous pushforward.

If $a = b$, the balance gives $c = d$. When $a = 1$ and $c \geq 3$, take the type $D_o$: the two degree-one leaves attach to different degree-$c$ vertices, and its two nonhomotopic middle edges have multiplicities 1 and $c - 2$. When $a \geq 2$ and $c > a$, use a diamond with the two degree-$a$

vertices nonadjacent. Its four spokes have multiplicities $(1, a-1)$ and $(a-1, 1)$, and its central edge has multiplicity $c-a$. All these sectors have rank two. Finally, if $a=b=c=d\geq 3$, choose a $K_4$ with multiplicities $a-2$ on one opposite pair of edges and 1 on the remaining four edges.

The only balanced nonzero cases not covered are $(1,1,2,2)$ and $(2,2,2,2)$. In the first case $M=3$, so every connected graph is a tree and its degrees force a path. In the second case every vertex has degree two. A connected loop-free multigraph on these four vertices must therefore be a four-cycle. Both measures are purely singular. If the degrees are unbalanced, the singular part is zero and nonvanishing already forces a nonzero absolutely continuous part. This completes the classification. □

**Corollary 6.12** (The exact mass of the singular component)**.** *Write $\Delta_{ij}=(4\pi^2 x_{ij}^2)^{-1}$. Let $\mathcal{P}$ be the twelve Hamiltonian paths on the four labels, with a path and its reversal identified, and let $\mathcal{C}$ be the three four-cycles, with cyclic shifts and reversal identified. Restrict these sets to the degree-compatible elements from Proposition 6.10. For $P=(a,b,c,d)$ put*

$$W_P=\Delta_{ab}^{J_a}\Delta_{bc}^{J_b-J_a}\Delta_{cd}^{J_d}.$$

*For $C=(a,b,c,d)$ put*

$$t_C=\max(0,J_b-J_c)+1, \qquad k=\min_i J_i-1,$$
$$R_C=\frac{\Delta_{ab}\Delta_{cd}}{\Delta_{bc}\Delta_{da}}, \qquad W_C=\Delta_{ab}^{t_C}\Delta_{bc}^{J_b-t_C}\Delta_{cd}^{J_c-J_b+t_C}\Delta_{da}^{J_a-t_C}.$$

*Then*

$$\nu_{\mathbf{J}}^{\mathrm{s}}(\mathcal{M}_{0,4})=\sqrt{\prod_i J_i}\left(\sum_{P\in\mathcal{P}_{\mathrm{adm}}} W_P+\sum_{C\in\mathcal{C}_{\mathrm{adm}}} W_C\mathcal{G}_k(R_C)\right), \tag{41}$$

*where*

$$\mathcal{G}_k(R)=\sum_{j=0}^{k-1}R^j=\begin{cases}(1-R^k)/(1-R), & R\neq 1,\\ k, & R=1.\end{cases}$$

*In particular, the exact fraction of singular worldsheet mass is the right-hand side of (41) divided by the connected planar correlator. Its computation requires no inversion of a Strebel period map.*

*Proof.* Each oriented, vertex-labelled four-point skeleton and positive multiplicity assignment has field-theory weight $\sqrt{\prod_i J_i}\prod_e \Delta_e^{m_e}$ by (9). There is one oriented ribbon cell for each labelled path or cycle. A path has its unique allowed multiplicity assignment. For a cycle, increasing $t$ by one multiplies its weight by $R_C$, and there are $k$ allowed values. Summing the resulting geometric progression proves the formula. □

# 7 Equal degrees: exact masses, stability, and large-degree limits

The degree classification can be strengthened to a quantitative theorem when $\mathbf{J}=(k,k,k,k)$. Throughout this section $k\geq 2$ and

$$A=\Delta_{12}\Delta_{34}, \qquad B=\Delta_{13}\Delta_{24}, \qquad C=\Delta_{14}\Delta_{23}. \tag{42}$$

All three variables are strictly positive. Let $h_j$ denote the complete homogeneous polynomial of degree $j$ in its displayed variables, with $h_0 = 1$ and $h_j = 0$ for $j < 0$. Define

$$I_k(A,B,C) = \sum_{\substack{a,b,c\geq 1\\ a+b+c=k}} A^a B^b C^c = ABC\, h_{k-3}(A,B,C), \tag{43}$$

$$S_k(A,B,C) = \sum_{\{U,V\}\subset\{A,B,C\}} \sum_{r=1}^{k-1} U^r V^{k-r}. \tag{44}$$

The outer sum has the three unordered pairs of the three named variables, even when some numerical values coincide.

## 7.1 Exact masses and the sharp lower bound

**Theorem 7.1** (Exact equal-degree mass decomposition)**.** *The only compatible reduced types are the four-cycle $C$ and the tetrahedral type $K$. The latter is absent at $k = 2$. Their masses are*

$$\nu^{\mathrm{ac}}_{kkkk}(\mathcal{M}_{0,4}) = 2k^2 I_k(A,B,C), \tag{45}$$

$$\nu^{\mathrm{s}}_{kkkk}(\mathcal{M}_{0,4}) = k^2 S_k(A,B,C). \tag{46}$$

*Consequently*

$$G^{(0)}_{kkkk} = k^2\big[2h_k(A,B,C) - h_k(A,B) - h_k(A,C) - h_k(B,C)\big]. \tag{47}$$

*Each pair sum in $S_k$ is explicitly*

$$\sum_{r=1}^{k-1} U^r V^{k-r} = \begin{cases} UV\dfrac{U^{k-1}-V^{k-1}}{U-V}, & U \neq V,\\ (k-1)U^k, & U = V. \end{cases} \tag{48}$$

*Proof.* A leaf would have multiplicity $k$ on its incident edge, exhausting the entire degree of its neighbour. That neighbour could have no other incident edge, contradicting connectedness on four vertices. Thus all six types with a leaf are excluded. Among the remaining types, the diamond $H$ and the double-hub triangulation $M$ have two nonadjacent outer vertices whose combined degree equals the combined degree of the two hubs. Subtracting the two sums gives twice the total multiplicity of edges joining the hubs. This would have to be zero, excluding both types.

For $K_4$, the four equal degree equations imply equality of opposite edge multiplicities. Write these values as $a, b, c \geq 1$, with $a + b + c = k$. Their propagator monomial is $A^a B^b C^c$. There are two oriented labelled $K_4$ cells, each of coefficient $\sqrt{k^4} = k^2$, and each has an absolutely continuous pushforward. This proves (45).

For a four-cycle, the multiplicities alternate between $r$ and $k - r$, where $1 \leq r \leq k - 1$. The three labelled cycles give the three unordered pairs of $A, B, C$. Each is singular and has coefficient $k^2$, proving (46). In $2h_k(A,B,C) - \sum_{\{U,V\}} h_k(U,V)$, pure powers cancel, monomials with exactly two positive exponents have coefficient one, and those with three positive exponents have coefficient two. This proves (47). The remaining formula is the finite geometric sum. □

**Theorem 7.2** (Sharp lower bound for the singular fraction)**.** *For every pairwise distinct Euclidean four-point configuration and every integer $k \geq 2$,*

$$\frac{\nu^{\mathrm{s}}_{kkkk}(\mathcal{M}_{0,4})}{G^{(0)}_{kkkk}} = \frac{S_k}{2I_k + S_k} \geq \frac{3}{k+1}. \tag{49}$$

*At $k = 2$ the fraction is identically one. For $k \geq 3$, equality holds if and only if $A = B = C$. The bound is attained by equidistant Euclidean insertion points and is therefore sharp on the physical configuration space.*

*Proof.* The case $k = 2$ follows from $I_2 = 0$. Suppose $k \geq 3$. For $b, c \geq 1$, weighted arithmetic–geometric mean gives

$$B^b C^c \leq \frac{b}{b+c} B^{b+c} + \frac{c}{b+c} C^{b+c}.$$

Multiply by $A^a$ and sum over positive triples with $a + b + c = k$. At fixed $a$, the sum of $b/(b+c)$ over $b = 1, \ldots, k-a-1$ equals $(k-a-1)/2$. Hence

$$I_k \leq \frac{1}{2} \sum_{a=1}^{k-2} (k-a-1)\big(A^a B^{k-a} + A^a C^{k-a}\big).$$

Add the three inequalities obtained by retaining $A$, $B$, and $C$. For each unordered variable pair, the coefficient of $U^r V^{k-r}$ on the right is

$$\frac{(k-r-1) + (r-1)}{2} = \frac{k-2}{2},$$

where a missing endpoint term contributes zero. Thus $6I_k \leq (k-2)S_k$. Rearranging gives (49). All weights in the underlying arithmetic–geometric mean inequalities are positive. Equality in the three summed inequalities forces $A = B = C$, and this condition also suffices. A regular tetrahedron embeds in $\mathbb{R}^4$ and has all $\Delta_{ij}$ equal, establishing attainment within the stated configuration space. □

**Corollary 7.3** (The symmetric configuration)**.** *If $A = B = C = T > 0$, then*

$$\nu^{\mathrm{ac}}_{kkkk}(\mathcal{M}_{0,4}) = k^2(k-1)(k-2)T^k, \tag{50}$$

$$\nu^{\mathrm{s}}_{kkkk}(\mathcal{M}_{0,4}) = 3k^2(k-1)T^k, \tag{51}$$

$$G^{(0)}_{kkkk} = k^2(k^2-1)T^k. \tag{52}$$

*In particular, the 3333 measure has absolutely continuous and singular fractions $1/4$ and $3/4$ at equal distances.*

*Proof.* There are $\binom{k-1}{2}$ positive triples with sum $k$ and $3(k-1)$ pair monomials in $S_k$. Substitute these counts into Theorem 7.1. □

At every finite equal degree the singular component has positive mass. Retaining the deficits in the arithmetic–geometric mean proof yields a quantitative estimate of the channel asymmetry, while the exact mass polynomials determine its large-degree behaviour.

## 7.2 Stability of the sharp bound

**Theorem 7.4** (Quantitative stability of the singular-mass bound)**.** *Fix an integer $k \geq 3$ and positive $A, B, C$. Set*

$$F_k(A,B,C) = \frac{S_k(A,B,C)}{2I_k(A,B,C) + S_k(A,B,C)}, \qquad c_k = \frac{k(k-2)}{24(k+1)},$$

*and define the scale-invariant quantities*

$$L = \max\{\log A, \log B, \log C\} - \min\{\log A, \log B, \log C\},$$

$$V = (\log A - \log B)^2 + (\log A - \log C)^2 + (\log B - \log C)^2.$$

*Then*

$$F_k(A,B,C) - \frac{3}{k+1} \geq c_k e^{-kL} V. \tag{53}$$

*Consequently, on every region* $\max(A,B,C)/\min(A,B,C) \leq R$*,*

$$V \leq \frac{24(k+1)}{k(k-2)} R^k \left( F_k(A,B,C) - \frac{3}{k+1} \right). \tag{54}$$

*Moreover, for fixed $k$ and $u_1 + u_2 + u_3 = 0$,*

$$F_k(e^{u_1}, e^{u_2}, e^{u_3}) = \frac{3}{k+1} + c_k \sum_{i<j} (u_i - u_j)^2 + O_k(\|u\|^3), \qquad u \to 0. \tag{55}$$

*Thus the coefficient $c_k$ in the quadratic behaviour at the symmetric configuration is exact.*

*Proof.* For positive integers $a, b, c$ with $a + b + c = k$, write

$$R_A(a,b,c) = A^a \left[ \frac{b}{b+c} B^{b+c} + \frac{c}{b+c} C^{b+c} - B^b C^c \right],$$

and define $R_B, R_C$ by retaining $B^b, C^c$, respectively. The coefficient count in the proof of Theorem 7.2 is the exact identity

$$D_k := (k-2)S_k - 6I_k = 2 \sum_{\substack{a,b,c \geq 1 \\ a+b+c=k}} \big( R_A(a,b,c) + R_B(a,b,c) + R_C(a,b,c) \big). \tag{56}$$

Let $x, y \in \mathbb{R}$ and $0 < \theta < 1$. Since the second derivative of $e^s$ is at least $e^{\min(x,y)}$ on the interval joining $x$ and $y$, the elementary strong-convexity form of Jensen's inequality gives

$$\theta e^x + (1-\theta) e^y - e^{\theta x + (1-\theta) y} \geq \frac{1}{2} \theta (1-\theta) e^{\min(x,y)} (x-y)^2.$$

Take $x = (b+c)\log B$, $y = (b+c)\log C$, and $\theta = b/(b+c)$. If $m = \min(A,B,C)$, then

$$R_A(a,b,c) \geq \frac{bc}{2} A^a \min(B,C)^{b+c} (\log B - \log C)^2 \geq \frac{bc}{2} m^k (\log B - \log C)^2.$$

The elementary sum

$$\sum_{\substack{a,b,c \geq 1 \\ a+b+c=k}} bc = \binom{k+1}{4}$$

follows by taking the coefficient of $z^k$ in $z^3(1-z)^{-5}$. Applying the same estimate to $R_B$ and $R_C$ in (56) yields

$$D_k \geq m^k \binom{k+1}{4} V.$$

If $M_* = \max(A, B, C)$, the polynomial $2I_k + S_k$ has $k^2 - 1$ monomials counted with multiplicity, all of degree $k$. Therefore

$$2I_k + S_k \leq (k^2 - 1) M_*^k.$$

Now use

$$F_k - \frac{3}{k+1} = \frac{D_k}{(k+1)(2I_k + S_k)}, \qquad (m/M_*)^k = e^{-kL},$$

and simplify the binomial coefficient to obtain (53). The next assertion is its immediate rearrangement.

For the local expansion, each retained-variable deficit satisfies

$$R_A(a, b, c) = \frac{bc}{2}(u_2 - u_3)^2 + O_k(\|u\|^3)$$

when $(A, B, C) = (e^{u_1}, e^{u_2}, e^{u_3})$. Thus $D_k = \binom{k+1}{4} \sum_{i<j} (u_i - u_j)^2 + O_k(\|u\|^3)$, while $2I_k + S_k = k^2 - 1 + O_k(\|u\|)$. Division gives (55). □

## 7.3 Large degree and fixed propagator ratios

The sharp lower bound tends to zero as the common degree grows. The actual singular fraction has a different limit at every fixed configuration with unequal channel monomials.

**Theorem 7.5** (The large-degree singular fraction)**.** *Fix $A, B, C > 0$ and relabel the channel variables so that $A = \max(A, B, C)$. Then*

$$\lim_{k \to \infty} F_k(A, B, C) = 1 - \frac{2BC}{A(B + C)}. \tag{57}$$

*The limit is positive unless $A = B = C$. If $A > \max(B, C)$ and $\rho = \max(B, C)/A < 1$, the error is $O_{A,B,C}(k\rho^k)$. If $A = B > C$, the error is $O_{A,B,C}(k^{-1})$. If all three are equal, $F_k = 3/(k+1)$ exactly.*

*Proof.* Homogeneity allows division of all three variables by $A$. First suppose that $b = B/A < 1$ and $c = C/A < 1$. Then

$$\frac{I_k}{A^k} = \sum_{\substack{r,s \geq 1 \\ r+s \leq k-1}} b^r c^s \longrightarrow \frac{bc}{(1-b)(1-c)},$$

while

$$\frac{S_k}{A^k} = \sum_{j=1}^{k-1} b^j + \sum_{j=1}^{k-1} c^j + \sum_{j=1}^{k-1} b^j c^{k-j} \longrightarrow \frac{b + c - 2bc}{(1-b)(1-c)}.$$

The omitted part of the first sum is bounded by $\sum_{q \geq k} (q-1)\rho^q = O_\rho(k\rho^k)$. The last sum in the second display is at most $(k-1)\rho^k$, and its two geometric tails satisfy the same error bound. The limiting denominator $(2I_k + S_k)/A^k$ is $(b + c)/[(1-b)(1-c)] > 0$. Taking the quotient gives both (57) and the stated error.

If $A = B > C$, put $c = C/A < 1$. Directly,

$$\frac{I_k}{A^k} = \sum_{j=1}^{k-2}(k-j-1)c^j = \frac{kc}{1-c} + O_c(1), \qquad \frac{S_k}{A^k} = k - 1 + 2\sum_{j=1}^{k-1} c^j = k + O_c(1).$$

The quotient tends to $(1-c)/(1+c)$ with error $O_c(k^{-1})$, which is the same expression as (57). The remaining tied case is the symmetric formula in Corollary 7.3. Finally,

$$A(B+C) - 2BC = B(A-C) + C(A-B)$$

is positive unless all three variables are equal. □

The constants in these error estimates depend on the fixed propagator ratios. The following simultaneous limit describes a neighbourhood of the symmetric configuration where those estimates are not uniform.

**Proposition 7.6** (A simultaneous limit near the symmetric configuration)**.** *For fixed $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, set $A_k = e^{u_1/k}$, $B_k = e^{u_2/k}$, and $C_k = e^{u_3/k}$. Define*

$$\mathcal{I}(u) = \int_{\substack{a,b>0 \\ a+b<1}} e^{u_1 a + u_2 b + u_3(1-a-b)}\, da\, db,$$
$$\mathcal{S}(u) = \sum_{1\le i<j\le 3} \int_0^1 e^{s u_i + (1-s)u_j}\, ds. \tag{58}$$

*Then*

$$kF_k(A_k, B_k, C_k) = \mathcal{R}(u) + O_K(k^{-1}), \qquad \mathcal{R}(u) = \frac{\mathcal{S}(u)}{2\mathcal{I}(u)}, \tag{59}$$

*uniformly for $u$ in any compact set $K \subset \mathbb{R}^3$. If $L_u = \max_i u_i - \min_i u_i$, then*

$$\mathcal{R}(u) - 3 \ge \frac{e^{-L_u}}{24} \sum_{i<j} (u_i - u_j)^2. \tag{60}$$

*In particular, $\mathcal{R}(u) = 3$ exactly when $u_1 = u_2 = u_3$.*

*Proof.* The positive composition sum defining $I_k$ is a Riemann sum on the triangle, and the three sums defining $S_k$ are Riemann sums on its three edges. Since the exponential integrands and their first derivatives are uniformly bounded for $u \in K$,

$$k^{-2} I_k(A_k, B_k, C_k) = \mathcal{I}(u) + O_K(k^{-1}), \qquad k^{-1} S_k(A_k, B_k, C_k) = \mathcal{S}(u) + O_K(k^{-1}).$$

The triangle integral has a uniform positive lower bound on $K$, so division gives (59). For these variables, $L = L_u/k$ and $V = k^{-2}\sum_{i<j}(u_i - u_j)^2$. Multiply (53) by $k$ and let $k \to \infty$, using $c_k/k \to 1/24$, to obtain (60). For equal $u_i$, the two integrals are $e^{u_1}/2$ and $3e^{u_1}$, proving the equality assertion. □

# 8 Exact four-point measures

We use labelled punctures

$$(z_1, z_2, z_3, z_4) = (0, 1, \eta, \infty).$$

The field-theory skeleton is denoted by $\Gamma$; its ribbon dual $\Gamma^\vee$ is the critical graph of the Strebel differential. Thus the vertices of $\Gamma$ label the faces of $\Gamma^\vee$. On a fixed labelled ribbon cell, the source measure is

$$d\mu_{\Gamma,\mathbf{m}} = C_\Gamma \prod_{e\in E(\Gamma)} \frac{\ell_e^{m_e-1}}{\Gamma(m_e)} e^{-\lambda_e \ell_e}\, d\ell_e, \qquad \lambda_e > 0, \quad m_e > 0. \tag{61}$$

In the free scalar application $m_e$ are positive integers and $\lambda_e = |x_{s(e)} - x_{t(e)}|^2/4$. The coefficient $C_\Gamma$ is the coefficient of this single labelled ribbon cell; it includes no further sum over its mirror image. The statements below also hold for arbitrary positive real shapes. Put $M = \sum_e m_e$.

Let $f_\Gamma$ be the map from positive edge lengths to the cross ratio, obtained by the Strebel correspondence followed by forgetting the perimeters. We compute

$$\nu_{\Gamma,\mathbf{m}} = (f_\Gamma)_* \mu_{\Gamma,\mathbf{m}}.$$

All these are finite positive measures, with total mass

$$\nu_{\Gamma,\mathbf{m}}(\mathcal{M}_{0,4}) = C_\Gamma \prod_e \lambda_e^{-m_e}. \tag{62}$$

In particular, all subsequent integrations and changes of order can be justified first for nonnegative test functions and then by linearity.

## 8.1 The square cell and its exact parameter domain

Let $\Gamma$ be the cycle $1-2-3-4-1$, with positive edge lengths

$$a = \ell_{12}, \qquad b = \ell_{23}, \qquad c = \ell_{34}, \qquad d = \ell_{14}.$$

Its perimeters satisfy

$$p_1 = a + d, \quad p_2 = a + b, \quad p_3 = b + c, \quad p_4 = c + d, \qquad p_1 + p_3 = p_2 + p_4. \tag{63}$$

The square-root form and real support were derived in [Aha+07, Appendix B, equations (B.1)–(B.3)]. The cross ratio in that reference is $\eta_{\mathrm{ref}} = 1/t \in (0, 1)$: the change $z_{\mathrm{ref}} = w/t$ puts the punctures at $(0, 1, 1/t, \infty)$ with perimeters $(p_1, p_3, p_2, p_4)$ in that order. We give the length coordinates and measure factors in the convention $t > 1$.

**Proposition 8.1** (Square coordinates)**.** *The cross ratio of the square cell is real and belongs to* $(1, \infty)$*. For positive perimeters satisfying* $p_1 + p_3 = p_2 + p_4$*, write* $t = \eta$ *and*

$$\mathcal{D}(t, p) = [p_1 + p_3 + (p_1 - p_2)t]^2 - 4p_1p_4t. \tag{64}$$

*The admissible domain is precisely*

$$t > 1, \qquad \mathcal{D}(t, p) < 0. \tag{65}$$

*On this domain the Strebel differential is*

$$q = -\frac{1}{4\pi^2}\left(\frac{p_1}{w} - \frac{p_2}{w-1} + \frac{p_3}{w-t}\right)^2 dw^2. \tag{66}$$

*Let*

$$\zeta_+ = \frac{p_1 + p_3 + (p_1 - p_2)t + i\sqrt{-\mathcal{D}(t,p)}}{2p_4},$$

*and set*

$$\theta_0 = \arg\zeta_+, \qquad \theta_1 = \arg(\zeta_+ - 1), \qquad \theta_t = \arg(\zeta_+ - t),$$

*where each argument is taken in* $(0,\pi)$*. Then*

$$a = \frac{p_1\theta_0 - p_2\theta_1 + p_3\theta_t}{\pi} + p_2 - p_3, \tag{67}$$

$$b = p_2 - a, \qquad c = p_3 - p_2 + a, \qquad d = p_1 - a. \tag{68}$$

*At fixed* $p_1, p_2, p_3$*, with* $p_4 = p_1 + p_3 - p_2$*, one has*

$$\frac{\partial a}{\partial t} = \frac{\sqrt{-\mathcal{D}(t,p)}}{2\pi t(t-1)} > 0. \tag{69}$$

*These formulas give a real-analytic bijection between the positive length cone and the domain in* (65) *together with the perimeter relation.*

*Proof.* The cycle has a reflection fixing its four labelled vertices and exchanging its two faces. Under the naturality of the Strebel construction, this reflection becomes an antiholomorphic involution fixing the punctures and exchanging the two zeros. After fixing the three labelled punctures at $0, 1, \infty$, the involution is complex conjugation. The cyclic order $1, 2, 3, 4$ of the punctures on its fixed circle gives $t > 1$. The two critical points are nonreal conjugates.

The dual critical graph has two vertices of valence four. Hence $q$ has two zeros of order two and is the negative of the square of a rational one-form with real residues. The bipartition $\{1,3\} \sqcup \{2,4\}$ fixes the alternating residue signs. The residue theorem and the perimeters give (66). The numerator of its square root is

$$p_4 w^2 - [p_1 + p_3 + (p_1 - p_2)t]w + p_1 t,$$

whose roots are $\zeta_+$ and $\zeta_- := \overline{\zeta_+}$. Consequently its discriminant is negative.

Integrate the square root along the edge crossing the interval $(0,1)$. Using the primitive

$$p_1 \log w - p_2 \log(w-1) + p_3 \log(w-t)$$

and continuing its logarithms along this edge gives (67). The other lengths then follow from the perimeters. The integral endpoints contribute nothing when differentiating in $t$, since the one-form vanishes there. Thus, with the orientation making $a$ positive,

$$\frac{\partial a}{\partial t} = \frac{p_3 \Im\zeta_+}{\pi|\zeta_+ - t|^2}.$$

Evaluation of the quadratic numerator at $w = t$ gives

$$|\zeta_+ - t|^2 = \frac{p_3 t(t-1)}{p_4}, \qquad \Im\zeta_+ = \frac{\sqrt{-\mathcal{D}}}{2p_4},$$

which proves (69).

For completeness, the domain and branches can also be checked without solving any trajectory equation. Put $r = p_1 + p_3$. At fixed perimeters, the positive square lengths are parametrized by

$$\max(0, p_2 - p_3) < a < \min(p_1, p_2).$$

The discriminant condition is the interval $t_- < t < t_+$, where

$$t_- = \frac{r^2}{(\sqrt{p_1 p_4} + \sqrt{p_2 p_3})^2}, \qquad t_+ = \frac{r^2}{(\sqrt{p_1 p_4} - \sqrt{p_2 p_3})^2}; \tag{70}$$

the second endpoint is interpreted as $+\infty$ when its denominator vanishes. Cauchy–Schwarz gives $t_- \geq 1$. The argument formula has the endpoint limits

$$a(t_-) = \max(0, p_2 - p_3), \qquad a(t_+) = \min(p_1, p_2).$$

They follow by letting the conjugate critical points merge on the real axis and taking the arguments from the upper half-plane. At the lower endpoint the vanishing edge is $a$ if $p_2 \leq p_3$, and $c$ otherwise; at the upper endpoint it is $d$ if $p_1 \leq p_2$, and $b$ otherwise. Equalities give the simultaneous collapses, including $t_- = 1$ and $t_+ = \infty$. Together with (69), these limits show that (67) parametrizes the entire positive length interval. Conversely, start from any admissible $(t, p)$ and compute these positive lengths. Strebel existence gives a marked surface with some $(t', p)$ and these lengths. The necessity already proved implies $a(t', p) = a(t, p)$; strict monotonicity forces $t' = t$. The residues and the square form then identify its differential with (66). This proves the converse and the asserted bijection without assuming that an arbitrary square differential is already Strebel. □

**Corollary 8.2** (A normalized delta formula)**.** *Let*

$$\rho(a, b, c, d) = C_\Gamma \prod_{e \in \{a,b,c,d\}} \frac{\ell_e^{m_e - 1}}{\Gamma(m_e)} e^{-\lambda_e \ell_e}.$$

*On the ambient coordinates $(\eta, p_1, p_2, p_3, p_4)$, the decorated pushforward measure is*

$$\begin{aligned} d\mathcal{W}_\square = \mathbf{1}_{\{\Re\eta > 1,\ \mathcal{D}(\Re\eta, p) < 0\}} \rho(a, b, c, d) \frac{\sqrt{-\mathcal{D}(\Re\eta, p)}}{2\pi \Re\eta(\Re\eta - 1)} \\ \times \delta(\Im\eta)\delta(p_1 + p_3 - p_2 - p_4)\, d^2\eta\, dp_1 dp_2 dp_3 dp_4. \end{aligned} \tag{71}$$

*In particular, the forgetful map of this cell has rank exactly one at every point.*

*Proof.* The absolute Jacobian from $(p_1, p_2, p_3, a)$ to $(a, b, c, d)$ is one. Now use (69). The two delta functions in (71) eliminate $\Im\eta$ and $p_4$ with unit derivative, so their elimination contributes no additional Jacobian factor. The strict positivity of $\partial a/\partial t$ proves the rank statement. □

## 8.2 The square marginal and a uniform cross-ratio law

We integrate the common scale and express the remaining perimeter integral over a fixed disk. For $t > 1$ and $X^2 + Y^2 < 1$, put

$$u = \frac{1 + \sqrt{(t-1)/t}\, X + Y/\sqrt{t}}{2}, \qquad v = \frac{1 + \sqrt{(t-1)/t}\, X - Y/\sqrt{t}}{2}. \tag{72}$$

Use Proposition 8.1 with normalized perimeters

$$(p_1, p_2, p_3, p_4) = (u, v, 1-u, 1-v)$$

to define normalized lengths

$$(L_a, L_b, L_c, L_d), \qquad \sum_e L_e = 1.$$

They are strictly positive on the open disk. Define

$$\Lambda(t, X, Y) = \sum_e \lambda_e L_e(t, X, Y).$$

**Theorem 8.3** (Square marginal)**.** *The square contribution on $\mathcal{M}_{0,4}$ is*

$$d\nu_\square(\eta) = \mathbf{1}_{\{\Re\eta > 1\}} H_\square(\Re\eta)\delta(\Im\eta)\, d^2\eta, \tag{73}$$

*where*

$$H_\square(t) = \frac{C_\Gamma \Gamma(M)}{4\pi t^2 \prod_e \Gamma(m_e)} \int_{X^2+Y^2<1} \frac{\sqrt{1-X^2-Y^2}\, \prod_e L_e(t, X, Y)^{m_e - 1}}{\Lambda(t, X, Y)^M}\, dX\, dY. \tag{74}$$

*For all positive real shapes this integral is finite and strictly positive at every $t > 1$. It is jointly smooth in $t$, the shapes, and the rates, as proved in Proposition 8.4 below.*

*If every $m_e = 1$ and all rates equal $\lambda$, then*

$$H_\square(t) = C_\Gamma \lambda^{-4} t^{-2}, \qquad t > 1. \tag{75}$$

*Consequently, under the normalized source measure, $\eta^{-1}$ is uniform on $(0, 1)$. It is independent of $(X, Y)$, which has density*

$$\frac{3}{2\pi}\sqrt{1-X^2-Y^2}\mathbf{1}_{\{X^2+Y^2<1\}}. \tag{76}$$

*Proof.* Put $r = a + b + c + d$, and write

$$p_1 = ru, \quad p_2 = rv, \quad p_3 = r(1-u), \quad p_4 = r(1-v).$$

The discriminant divided by $r^2$ is

$$\begin{aligned}\Delta(t, u, v) &= [1 + (u-v)t]^2 - 4u(1-v)t \\ &= 1 - t + t(u+v-1)^2 + t(t-1)(u-v)^2.\end{aligned}$$

Thus $\Delta < 0$ is the ellipse

$$\frac{t}{t-1}(u+v-1)^2 + t(u-v)^2 < 1.$$

This ellipse lies in $0 < u, v < 1$, with its boundary tangent to the four sides. Formula (72) maps the open unit disk onto it, and

$$\Delta = -(t-1)(1-X^2-Y^2), \qquad du\, dv = \frac{\sqrt{t-1}}{2t}\, dX\, dY.$$

The source volume element is

$$da\,db\,dc\,dd = r^3\frac{\sqrt{-\Delta(t,u,v)}}{2\pi t(t-1)}\,dr\,dt\,du\,dv$$
$$= \frac{r^3}{4\pi t^2}\sqrt{1-X^2-Y^2}\,dr\,dt\,dX\,dY.$$

Substitution of $\ell_e = rL_e$ into (61) leaves the radial integral

$$\int_0^\infty r^{M-1}e^{-r\Lambda}\,dr = \frac{\Gamma(M)}{\Lambda^M},$$

proving (74) as an identity of finite measures by nonnegative integration. For shapes at least one, the integrand is bounded by a constant times $\sqrt{1-X^2-Y^2}$ because $0 < L_e < 1$ and $\Lambda \geq \min_e \lambda_e > 0$. Pointwise finiteness for every positive shape and joint regularity are established in Proposition 8.4.

For unit shapes and a common rate, $M = 4$, $\Lambda = \lambda$, and

$$\int_{X^2+Y^2<1}\sqrt{1-X^2-Y^2}\,dX\,dY = \frac{2\pi}{3}.$$

Equation (75) follows. Since $dt/t^2 = -d(1/t)$, normalization gives the uniform law and the product factorization with (76). □

**Proposition 8.4** (Interior regularity for every positive shape)**.** *For arbitrary $m_e > 0$ and $\lambda_e > 0$, the square marginal $H_\square(t)$ is finite and strictly positive for every $t > 1$. It is jointly $C^\infty$ in $t$, the shapes, and the rates on this open parameter domain. If $m_e \geq 1$ for all four edges and $\lambda_{\min} = \min_e \lambda_e$, then*

$$0 < H_\square(t) \leq \frac{C_\Gamma\Gamma(M)}{6\prod_e \Gamma(m_e)\,\lambda_{\min}^M}\,t^{-2}, \qquad t > 1. \tag{77}$$

*Proof.* Use the upper critical point $\zeta = x + iy$, $y > 0$, as a coordinate on the perimeter fibre at fixed $t$. Put

$$F = (x-1)^2 + y^2 + t - 1.$$

The residue formulas for a square root with numerator $(w-\zeta)(w-\overline{\zeta})$ give the normalized perimeters

$$u = \frac{(t-1)(x^2+y^2)}{tF}, \qquad v = \frac{(x-1)^2+y^2}{F}, \qquad p_3 = 1-u, \qquad p_4 = \frac{t-1}{F} = 1-v. \tag{78}$$

These formulas identify the upper half-plane with the interior of the ellipse in the square-marginal proof. Direct differentiation gives

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{4y(t-1)^2}{tF^3}, \qquad \Delta(t,u,v) = -\frac{4y^2(t-1)^2}{F^2}.$$

Consequently the same marginal has the representation

$$H_\square(t) = \frac{4C_\Gamma\Gamma(M)(t-1)^2}{\pi t^2\prod_e\Gamma(m_e)}\int_{y>0}\frac{y^2}{F^4}\frac{\prod_e L_e(t,x,y)^{m_e-1}}{(\sum_e \lambda_e L_e(t,x,y))^M}\,dx\,dy. \tag{79}$$

Here the $L_e$ are the normalized lengths obtained from the argument formulas using (78).

We check every boundary of this fibre. Work uniformly with $t$ in a compact subinterval of $(1,\infty)$. At a real boundary point away from $0,1,t,\infty$, exactly one length vanishes. It has the form

$$L_e = y^3 A_e(t,x,y^2), \qquad A_e(t,x,0) > 0,$$

with $A_e$ real analytic, while the other three lengths stay positive. Indeed, a short path between $x-iy$ and $x+iy$ crosses no pole and its period equals the vanishing edge length. The numerator on the vertical path $w = x+is$ is $y^2 - s^2$. Integrating it from $-y$ to $y$ gives the nonzero leading coefficient

$$A_e(t,x,0) = \frac{2(t-1)}{3\pi F(t,x,0)\,|x(x-1)(x-t)|}.$$

Thus the local integrable power in (79) is $y^{3m_e-1}$.

At the corner $\zeta = 0$, write $\zeta = \varrho e^{i\theta}$ with $0 \le \theta \le \pi$ and define

$$h(\theta) = \frac{\theta - \sin\theta\cos\theta}{\pi}.$$

The adjacent lengths factor as

$$L_a = \varrho^2 h(\theta) A(t,\varrho,\theta), \qquad L_d = \varrho^2[1-h(\theta)]D(t,\varrho,\theta), \tag{80}$$

where $A, D$ extend smoothly to the closed corner and are strictly positive there after the neighbourhood is reduced. More precisely, $A(t,0,\theta) = D(t,0,\theta) = (t-1)/t^2$. To see the factorization, expand the logarithms at the other two finite poles in the argument formula. After the constant and linear terms cancel, the normalized length satisfies

$$L_a = \frac{t-1}{t^2}\varrho^2 h(\theta) + O(\varrho^3)\,.$$

The quotient $L_a/\varrho^2$ is analytic in $(t,\varrho,\theta)$. Its $\theta$ derivatives of orders zero, one, and two vanish at $\theta = 0$ for every positive $\varrho$, by the short-period calculation, and hence also at $\varrho = 0$ by continuity. The same statement holds for the quotient for $d$ at $\theta = \pi$. Hadamard's division lemma first factors $\theta^3$ and $(\pi-\theta)^3$ jointly in all parameters. Since $h/\theta^3$ and $(1-h)/(\pi-\theta)^3$ are smooth and positive at their respective endpoints, division by $h$ and $1-h$ extends smoothly. The displayed positive limits and compactness give the asserted positive factors. The other two lengths tend to $1/t$ and $(t-1)/t$.

Since $y^2\,dx\,dy = \varrho^3 \sin^2\theta\,d\varrho\,d\theta$, the corner integrand is bounded by a constant times

$$\varrho^{2(m_a+m_d)-1}\sin^2\theta\, h(\theta)^{m_a-1}[1-h(\theta)]^{m_d-1}.$$

The radial exponent is larger than $-1$. The angular exponents are $3m_a - 1$ at 0 and $3m_d - 1$ at $\pi$, since $h(\theta) \sim 2\theta^3/(3\pi)$ and $1 - h(\theta) \sim 2(\pi-\theta)^3/(3\pi)$. They too are larger than $-1$. Cyclic relabelling of the square transports these estimates to the corners $1, t, \infty$; the associated cross ratio is either $t$ or $t/(t-1)$, so compact parameter sets remain compact. This covers the compactified upper half-plane.

All the positive factors just used, and all their $t$ derivatives, are bounded on the corresponding compact coordinate patches. The rate denominator is bounded below by $\lambda_{\min} > 0$. On compact subsets of the positive shape domain, derivatives in the shapes only insert powers of logarithms of the boundary variables, which remain integrable against these powers. A finite smooth partition

subordinate to the boundary patches therefore permits differentiation of (79) to every order under the integral. This proves finiteness and the joint $C^\infty$ assertion. Positivity follows from the positive interior integrand.

Finally, when $m_e \geq 1$, the fixed-disk formula has $\prod_e L_e^{m_e-1} \leq 1$ and $\sum_e \lambda_e L_e \geq \lambda_{\min}$. Using $\int_{X^2+Y^2<1} \sqrt{1-X^2-Y^2}\,dX\,dY = 2\pi/3$ gives (77). □

For shapes at least one, let $K_\square$ denote the constant multiplying $t^{-2}$ in (77). The same estimate gives the explicit tail and moment bounds

$$\int_T^\infty H_\square(t)\,dt \leq \frac{K_\square}{T} \quad (T \geq 1), \qquad \int_1^\infty t^s H_\square(t)\,dt \leq \frac{K_\square}{1-s} \quad (s < 1). \tag{81}$$

Both density and tail bounds are attained for unit shapes and a common rate.

**Corollary 8.5** (The complete equal-distance 2222 measure)**.** *For the normal-ordered operators $\mathcal{O}_2 = (2N^2)^{-1/2} :\mathrm{tr}\,\phi^2:$, assume that all six spacetime squared distances equal $r_0^2 > 0$, and put $\Delta_0 = (4\pi^2 r_0^2)^{-1}$. The complete connected planar four-point measure is*

$$d\nu_{2222}(\eta) = 4\Delta_0^4\,\delta(\Im\eta)\,w(\Re\eta)\,d^2\eta, \tag{82}$$

*where*

$$w(t) = \begin{cases} (1-t)^{-2}, & t < 0, \\ 1, & 0 < t < 1, \\ t^{-2}, & t > 1. \end{cases} \qquad w(t) = \max\{1, |t|, |1-t|\}^{-2}. \tag{83}$$

*The measure is invariant under the anharmonic group generated by $\eta \mapsto 1-\eta$ and $\eta \mapsto 1/\eta$. Its total mass is $12\Delta_0^4$.*

*Proof.* A connected graph on four vertices of degree two is one of the three labelled four-cycles. Each has $2^4 = 16$ labelled pairings. The product of the four operator factors contributes $1/\sqrt{2^4} = 1/4$ after extracting the planar power of $N$. Thus each cycle has coefficient 4 multiplying its four propagators.

For the cycle $1-2-3-4-1$, Theorem 8.3 gives the density $4\Delta_0^4 t^{-2}$ on $(1,\infty)$. The other two cycles are its labelled images under $t \mapsto 1/t$ and $t \mapsto 1-t$. The corresponding pushforwards have densities $4\Delta_0^4$ on $(0,1)$ and $4\Delta_0^4(1-t)^{-2}$ on $(-\infty,0)$. This proves the formula. Each interval has mass $4\Delta_0^4$. Finally,

$$w(1-t) = w(t), \qquad \frac{w(1/t)}{t^2} = w(t)$$

away from $0, 1$. These are exactly the change-of-variables identities for invariance of the real-line measure. Equivalently they prove invariance of the distributional two-dimensional measure in (82); they do not assert scalar invariance of its coefficient separately from the delta function and the area element. □

## 8.3 The two tree types

For four labelled vertices a three-edge connected skeleton is either a star or a path. Their decorated cells have the same dimension, but their projections have different ranks.

**Proposition 8.6** (The star is absolutely continuous)**.** *Consider the star with centre* 4 *and edges* $14, 24, 34$*. Write its normalized lengths as* $a, b, c > 0$*,* $a + b + c = 1$*. Its two ribbon orientations give*

$$\eta_{\pm} = \frac{a - bc \;\pm\; 2i\sqrt{abc}}{(a+b)^2}. \tag{84}$$

*Each orientation maps the open length simplex real-analytically and bijectively onto the corresponding open half-plane. In particular, the forgetful map has rank two everywhere. If*

$$R = |\eta|, \quad S = |1 - \eta|, \quad T = 1 + R + S,$$

*the inverse coordinates are*

$$a = \frac{1 + R - S}{T}, \qquad b = \frac{1 - R + S}{T}, \qquad c = \frac{R + S - 1}{T}, \tag{85}$$

*and their Jacobian is*

$$\left| \det \frac{\partial(a, b)}{\partial(\Re\eta, \Im\eta)} \right| = \frac{4|\Im\eta|}{RST^3}. \tag{86}$$

*For one ribbon orientation the marginal density is*

$$\frac{d\nu_Y}{d^2\eta} = \frac{C_\Gamma \Gamma(M)}{\prod_{j=1}^3 \Gamma(m_j)} \frac{4|\Im\eta|}{RST^3} \frac{a^{m_1-1} b^{m_2-1} c^{m_3-1}}{(\lambda_1 a + \lambda_2 b + \lambda_3 c)^M}, \tag{87}$$

*on its half-plane, and zero on the other half-plane.*

*Proof.* The dual graph has one vertex of valence six, so its differential has one zero of order four. For the star the three finite residues have the same sign. Therefore

$$q = -\frac{r^2}{4\pi^2} \left( \frac{a}{w} + \frac{b}{w-1} + \frac{c}{w-\eta} \right)^2 dw^2, \qquad r > 0.$$

Its quadratic numerator has a double root exactly when

$$[(a + c) + (a + b)\eta]^2 = 4a\eta.$$

Solving gives (84). Directly, $|\eta| = (a + c)/(a + b)$ and $|1 - \eta| = (b + c)/(a + b)$, which yield (85). Strict triangle inequalities make its three entries positive for $\Im\eta \neq 0$. These formulas prove the bijection. Differentiating first with respect to $R, S$ gives the Jacobian $-4/T^3$, while

$$\det \frac{\partial(R, S)}{\partial(\Re\eta, \Im\eta)} = \frac{\Im\eta}{RS}.$$

This proves (86). Finally, write the three unnormalized edge lengths as $r(a, b, c)$, whose volume element is $r^2\, dr\, da\, db$, and integrate $r$ as in the square case. □

The star change of variables and its ordinary worldsheet density are already central to [DG07]. The proposition is included to distinguish the rank-two tree from the rank-one tree and to fix all measure normalizations in a common convention.

**Proposition 8.7** (The path is supported on a real interval)**.** *Let* $\Gamma$ *be the path* $1-2-3-4$ *with lengths* $a=\ell_{12}$, $b=\ell_{23}$, $c=\ell_{34}$. *This is the three-edge* $\Pi$ *diagram. Its cross ratio is*

$$\eta=\frac{\left(\sqrt{ac}+\sqrt{(a+b)(b+c)}\right)^2}{b^2}>1. \tag{88}$$

*Writing* $a/b=\sinh^2 u$, $c/b=\sinh^2 v$, *one obtains*

$$\eta=\cosh^2(u+v), \qquad u,v>0. \tag{89}$$

*Thus the forgetful map has rank exactly one everywhere and its image is* $(1,\infty)$.

*Proof.* The path has perimeters

$$(p_1,p_2,p_3,p_4)=(a,a+b,b+c,c).$$

It is the $d=0$ boundary of the square cell. All four perimeters remain positive; in the dual graph this is the contraction of an edge between the two distinct vertices. The two zeros therefore merge at the upper endpoint in (70). Substitution of these perimeters gives (88). Equivalently, one can impose a quadruple zero on (66) and select the branch with the missing 14 edge. The other quadratic root is a different path cell and is not included. Formula (89) follows from the addition law for cosh. Its derivative in either $u$ or $v$ is $\sinh(2(u+v))>0$, proving the rank and range statements. □

**Theorem 8.8** (Path marginal and endpoint exponents)**.** *Write the shapes and rates on* $12,23,34$ *as* $m_1,m_2,m_3$ *and* $\lambda_1,\lambda_2,\lambda_3$. *Put* $M=m_1+m_2+m_3$ *and* $w=\operatorname{arcosh}\sqrt{t}$. *Then*

$$d\nu_\Pi(\eta)=\mathbf{1}_{\{\Re\eta>1\}}H_\Pi(\Re\eta)\delta(\Im\eta)\,d^2\eta, \tag{90}$$

*where the following is a one-dimensional integral:*

$$H_\Pi(t)=\frac{C_\Gamma\Gamma(M)}{2\sqrt{t(t-1)}\prod_{j=1}^3\Gamma(m_j)} \times\int_0^w\frac{\sinh(2u)\sinh(2(w-u))\sinh^{2m_1-2}u\,\sinh^{2m_3-2}(w-u)}{[\lambda_1\sinh^2u+\lambda_2+\lambda_3\sinh^2(w-u)]^M}\,du. \tag{91}$$

*For every positive real shape this is positive and finite for* $t>1$. *Its two endpoint asymptotics are*

$$H_\Pi(t)\sim\frac{2C_\Gamma\Gamma(M)\,\mathrm{B}(2m_1,2m_3)}{\Gamma(m_1)\Gamma(m_2)\Gamma(m_3)\lambda_2^M}(t-1)^{m_1+m_3-1}, \qquad t\downarrow1, \tag{92}$$

$$H_\Pi(t)\sim\frac{C_\Gamma 2^{m_2-1}\Gamma(m_1+m_2/2)\Gamma(m_3+m_2/2)}{\Gamma(m_1)\Gamma(m_2)\Gamma(m_3)\lambda_1^{m_1+m_2/2}\lambda_3^{m_3+m_2/2}}t^{-1-m_2/2}, \qquad t\to\infty. \tag{93}$$

*If* $m_1=m_2=m_3=1$ *and all rates equal* $\lambda$, *then*

$$H_\Pi(t)=\frac{C_\Gamma}{2\lambda^3t^2}\left[1+\frac{t-2}{\sqrt{t-1}}\arctan\sqrt{t-1}\right], \qquad t>1. \tag{94}$$

*Equivalently, under the normalized measure,* $Q=\eta^{-1/2}$ *has density*

$$f_Q(q)=q+(1-2q^2)\frac{\arccos q}{\sqrt{1-q^2}}, \qquad 0<q<1. \tag{95}$$

*Proof.* Use $b = s$, $a = s\sinh^2 u$, $c = s\sinh^2 v$. The volume Jacobian is $s^2 \sinh(2u)\sinh(2v)$. The scale integral in $s$ gives $\Gamma(M)[\lambda_1 \sinh^2 u + \lambda_2 + \lambda_3 \sinh^2 v]^{-M}$. Then put $w = u + v$ and $t = \cosh^2 w$, so $dw = dt/(2\sqrt{t(t-1)})$. This proves (91); endpoint integrability follows from the powers $u^{2m_1-1}$ and $(w-u)^{2m_3-1}$.

For $w \downarrow 0$, put $u = w\theta$ in that integral. Uniform Taylor expansions on $0 \le \theta \le 1$, after factoring the integrable endpoint powers, give

$$\int_0^w \cdots du \sim \frac{4w^{2m_1+2m_3-1}}{\lambda_2^M} \mathrm{B}(2m_1, 2m_3).$$

Since $w^2 \sim t - 1$, this proves (92).

For the other endpoint write $q = t^{-1/2}$. With $a, c$ held fixed, (88) is equivalent to

$$b(q; a, c) = \frac{q^2(a+c) + 2q\sqrt{ac}}{1-q^2}, \qquad 0 < q < 1. \tag{96}$$

As $q \downarrow 0$, $b/q \to 2\sqrt{ac}$ and $\partial_q b \to 2\sqrt{ac}$. For $q$ in a fixed small interval, $q\partial_q b \le Kb$ and $b/q \le K(a+c)$ for a constant $K$. After division by $q^{m_2-1}$, the transformed integrand is therefore dominated by a constant multiple of

$$a^{m_1-1} c^{m_3-1} (a+c)^{m_2} e^{-\lambda_1 a - \lambda_3 c},$$

which is integrable. Dominated convergence gives the density in $q$ asymptotic to

$$\frac{C_\Gamma 2^{m_2} \Gamma(m_1 + m_2/2)\Gamma(m_3 + m_2/2)}{\prod_j \Gamma(m_j)\, \lambda_1^{m_1+m_2/2} \lambda_3^{m_3+m_2/2}} q^{m_2-1}.$$

Multiplication by $|dq/dt| = \frac{1}{2}t^{-3/2}$ proves (93).

Finally take unit shapes and a common rate in (91). Set $s = u - w/2$. The identities

$$1 + \sinh^2 u + \sinh^2(w-u) = \cosh w \cosh(2s),$$

$$\sinh(2u)\sinh(2(w-u)) = t - \cosh^2(2s)$$

reduce the integral to

$$\lambda^{-3} t^{-3/2} \left( t \int_0^w \operatorname{sech}^3 x \, dx - \int_0^w \operatorname{sech} x \, dx \right).$$

Using

$$\int_0^w \operatorname{sech} x \, dx = \arctan(\sinh w),$$

$$\int_0^w \operatorname{sech}^3 x \, dx = \tfrac{1}{2}\big(\operatorname{sech} w \tanh w + \arctan(\sinh w)\big)$$

gives (94). The change $q = t^{-1/2}$ gives (95). The normalization follows either by integration or directly from (62). □

# 9 Period coordinates on maximal cells

The four-simple-zero period system extends to every maximal planar Strebel cell. This extension also determines the rank of the forgetful map at every interior point of such a cell.

Fix $n \geq 4$, put $g = n-3$, and use labelled punctures

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3, \dots, z_{n-1} \in \mathbb{C}, \qquad z_n = \infty,$$

with all finite punctures distinct. Write, with $P$ a polynomial of degree at most $2n-4$,

$$D(w) = \prod_{i=1}^{n-1}(w - z_i), \qquad q = -\frac{1}{4\pi^2}\frac{P(w)}{D(w)^2}\,dw^2. \tag{97}$$

The prescribed positive perimeters impose the conditions

$$P(z_i) = p_i^2 D'(z_i)^2 \quad (1 \leq i \leq n-1), \qquad [w^{2n-4}]P = p_n^2. \tag{98}$$

There is a convenient explicit choice of reference polynomial:

$$P_0(w) = p_n^2 w^{2n-4} + \sum_{i=1}^{n-1}\bigl[p_i^2 D'(z_i)^2 - p_n^2 z_i^{2n-4}\bigr]\frac{D(w)}{(w-z_i)D'(z_i)}. \tag{99}$$

All other polynomials satisfying (98) are uniquely parametrized by $g$ complex accessory parameters:

$$P_{\boldsymbol{\kappa}}(w) = P_0(w) + D(w)\sum_{j=0}^{g-1}\kappa_j w^j, \qquad \boldsymbol{\kappa} \in \mathbb{C}^g. \tag{100}$$

Indeed, the difference of any two solutions has degree at most $2n-5$ and vanishes at all $n-1$ finite punctures, so division by $D$ leaves a polynomial of degree at most $n-4 = g-1$.

**Theorem 9.1** (Nondegeneracy of the real period system)**.** *Fix any accessory candidate in* (97) *satisfying* (98) *whose $2n-4$ zeros are all simple. No period reality assumption is imposed here. Its canonical cover*

$$\widehat{\Sigma}_{\boldsymbol{\kappa}} : y^2 = P_{\boldsymbol{\kappa}}(w)$$

*has genus $g = n-3$. Put*

$$\omega_{\boldsymbol{\kappa}} = \frac{i}{2\pi}\frac{y}{D(w)}\,dw.$$

*Choose cycles disjoint from the poles that represent a positively oriented symplectic basis $(\mathfrak{a}_1, \dots, \mathfrak{a}_g, \mathfrak{b}_1, \dots, \mathfrak{b}_g)$ of the compact cover. Continue these cycles in the punctured cover by Gauss–Manin transport in a simply connected parameter neighbourhood with no zero collisions. Define column vectors of periods $\boldsymbol{\Pi}_{\mathfrak{a}}$ and $\boldsymbol{\Pi}_{\mathfrak{b}}$ of $\omega_{\boldsymbol{\kappa}}$, and define the $g \times g$ matrices*

$$A_{ij} = \oint_{\mathfrak{a}_i}\frac{w^{j-1}\,dw}{y}, \qquad B_{ij} = \oint_{\mathfrak{b}_i}\frac{w^{j-1}\,dw}{y}, \qquad \tau = BA^{-1}. \tag{101}$$

*Writing $\boldsymbol{\kappa} = \boldsymbol{s} + i\boldsymbol{r}$ and ordering the real variables as $(\boldsymbol{s}, \boldsymbol{r})$, one has*

$$\det\frac{\partial(\Im\boldsymbol{\Pi}_{\mathfrak{a}}, \Im\boldsymbol{\Pi}_{\mathfrak{b}})}{\partial(\boldsymbol{s}, \boldsymbol{r})} = \frac{(-1)^g|\det A|^2\det(\Im\tau)}{(4\pi)^{2g}} \neq 0. \tag{102}$$

*Thus the imaginary-period map is a local real-analytic diffeomorphism in the accessory variables at every smooth candidate cover.*

*Proof.* The cover is branched exactly at the $2n-4=2g+2$ simple zeros, so Riemann–Hurwitz gives genus $g$. The forms $w^j\,dw/y$, $0\le j\le g-1$, form a basis of holomorphic one-forms on this hyperelliptic curve. Differentiation at fixed $w$ gives

$$\partial_{\kappa_j}\omega_{\boldsymbol{\kappa}}=\frac{i}{4\pi}\frac{w^j\,dw}{y}. \tag{103}$$

Thus the real period Jacobian is

$$\frac{1}{4\pi}\begin{pmatrix}\Re A & -\Im A\\ \Re B & -\Im B\end{pmatrix}. \tag{104}$$

The Riemann bilinear relations imply that $A$ is invertible and $\tau=BA^{-1}$ is symmetric with positive definite imaginary part. For example, a holomorphic form with all $\mathfrak{a}$-periods zero has zero $L^2$ norm by the bilinear relation, and hence vanishes; this proves invertibility. Applying the same relation after normalizing the $\mathfrak{a}$-periods proves positivity of $\Im\tau$.

To evaluate the determinant, first apply the real linear change $\boldsymbol{c}=A\boldsymbol{\kappa}$, whose determinant is $|\det A|^2$. The remaining real map has matrix

$$\begin{pmatrix}I & 0\\ \Re\tau & -\Im\tau\end{pmatrix},$$

with determinant $(-1)^g\det(\Im\tau)$. The factor $(4\pi)^{-2g}$ proves (102).

□

**Proposition 9.2** (Sufficiency of handle-period reality)**.** *Fix the distinct marked punctures and positive perimeters. Among candidates* (97)–(100) *with* $2n-4$ *distinct zeros, the conditions*

$$\Im\boldsymbol{\Pi}_{\mathfrak{a}}=\Im\boldsymbol{\Pi}_{\mathfrak{b}}=0 \tag{105}$$

*are equivalent to being the Strebel differential for these decorated data. This equivalence holds for any symplectic basis represented away from the poles. In particular, there is at most one accessory vector in the entire distinct-zero locus satisfying these conditions; such a vector exists precisely when the Strebel differential has only simple zeros.*

*Proof.* The residue constraints give quadratic residues $-p_i^2/(4\pi^2)$ for $q$ and residues $\pm ip_i/(2\pi)$ for $\omega_{\boldsymbol{\kappa}}$ at the two lifts of each pole. Consequently every small pole-loop period is real. The handle cycles together with these pole loops generate the first homology of the punctured canonical cover. Under (105), every closed period of $\omega_{\boldsymbol{\kappa}}$ is therefore real. Changing the representatives of the handle cycles by pole loops changes their periods by real numbers and hence leaves the reality condition unchanged.

There are no additional independent relative reality conditions in this stratum. Indeed, let $\mu$ be the deck involution and let $\gamma$ be any path between lifted zeros, avoiding the poles. Both endpoints are branch points fixed by $\mu$, so $\gamma-\mu\gamma$ is a closed cycle. Since $\mu^*\omega_{\boldsymbol{\kappa}}=-\omega_{\boldsymbol{\kappa}}$,

$$2\int_\gamma\omega_{\boldsymbol{\kappa}}=\oint_{\gamma-\mu\gamma}\omega_{\boldsymbol{\kappa}}\in\mathbb{R}.$$

In particular all anti-invariant homological coordinates are real. The real-period characterization of Jenkins–Strebel differentials in [BK20, Definition 2.1, with the homology convention (1.12)–(1.13)] now identifies $q$ as the Strebel differential. Conversely, for any closed cycle $c$ one has $\int_c\omega=\frac{1}{2}\int_{c-\mu c}\omega$, and $c-\mu c$ is anti-invariant. The characterization therefore implies (105) for every Strebel differential with simple zeros. Strebel uniqueness for the fixed marked surface and positive perimeters proves the global uniqueness assertion. Existence in the distinct-zero locus is exactly the stated condition on the unique Strebel differential. □

The simple-zero restriction in Proposition 9.2 ensures that all lifted zeros are fixed by the deck involution. At even-order zeros the two distinct lifts contribute relative periods, so this proof does not eliminate their additional reality conditions. The proposition identifies the global differential without selecting its ribbon type. To work in a specified maximal cell, choose its transported critical-edge chart and impose positivity of all signed edge periods $\ell_e = \int_{\gamma_e} \omega$; these inequalities specify the positive chamber of that chart.

**Corollary 9.3** (Local analytic Strebel section)**.** *At every decorated sphere whose Strebel differential has only simple zeros, its accessory parameters depend real-analytically on the marked punctures and positive perimeters in a neighbourhood.*

*Proof.* Transport the cycles in a simply connected parameter neighbourhood and apply the real-analytic implicit-function theorem to (105), using the nonzero determinant (102). This gives a unique local analytic accessory vector as the decorated data vary. Shrinking the neighbourhood keeps all zeros distinct. Proposition 9.2 then identifies every member of this local family with the Strebel differential for its data. No continuation through the zero-collision locus is used. □

For $n = 4$, $g = 1$, $z_3 = \eta$, and $D(w) = w(w-1)(w-\eta)$. Formula (99) is the unique reference quartic with the required values and leading coefficient and with zero cubic coefficient. The general determinant reduces to

$$\det \frac{\partial(\Im\Pi_{\mathfrak{a}}, \Im\Pi_{\mathfrak{b}})}{\partial(\Re\kappa, \Im\kappa)} = -\frac{|A|^2 \Im\tau}{16\pi^2}.$$

**Corollary 9.4** (Everywhere full rank on maximal cells)**.** *Let $G$ be a reduced planar skeleton on $n \geq 4$ individually labelled vertices with $E = 3n - 6$ edges. Its inverse Strebel map is a real-analytic diffeomorphism from the positive edge-length cone onto an open domain*

$$\mathcal{D}_G \subset \mathcal{M}_{0,n} \times (0, \infty)^n.$$

*The forgetful map $f_G$ has rank $2n - 6$ at every point. In particular, both four-point six-edge types $K$ and $M$ have rank two everywhere.*

*Proof.* Maximality implies that every face of $G$ is triangular, so every zero of the Strebel differential is simple. Its critical graph is therefore trivalent. The metric ribbon-graph correspondence identifies the positive length cone with its decorated cell, and label-preserving automorphisms are trivial by Proposition 3.3. Since both spaces have real dimension $E = 3n - 6$, invariance of domain shows that $\mathcal{D}_G$ is open.

Corollary 9.3 gives a real-analytic differential over this decorated domain. Label its simple zeros and transport the selected critical-edge paths locally. The length of each edge is its signed period, which is real analytic on these data. This gives a real-analytic inverse to $\mathfrak{S}_G$. The latter is real analytic by Lemma 6.1; hence it is a real-analytic diffeomorphism. The projection from the open domain $\mathcal{D}_G$ to $\mathcal{M}_{0,n}$ is a submersion, proving the rank assertion. The case $n = 3$ has a zero-dimensional undecorated target and satisfies the analogous assertion trivially. □

The accessory derivatives are computable from (104). For a real base coordinate $v$, differentiate the period reality conditions and solve the resulting $2g \times 2g$ linear system for $(\partial_v \Re\boldsymbol{\kappa}, \partial_v \Im\boldsymbol{\kappa})$. Differentiate the edge periods on the same transported branches; their endpoint terms vanish at the zeros. This determines the everywhere nonsingular length Jacobian within the selected maximal cell.

**Corollary 9.5** (A perimeter-simplex formula for maximal-cell densities)**.** *Use real coordinates* $z = (z^{(1)}, \dots, z^{(d)})$ *on* $\mathcal{M}_{0,n}$*, where* $d = 2n - 6$*, and let*

$$\Delta^{\circ}_{n-1} = \{u \in (0,\infty)^n : \textstyle\sum_i u_i = 1\}.$$

*For a maximal cell* $G$ *define its selected perimeter domain*

$$\mathcal{U}_G(z) = \{u \in \Delta^{\circ}_{n-1} : (z, 2u) \in \mathcal{D}_G\}, \qquad L(z,u) = \mathfrak{S}_G^{-1}(z, 2u).$$

*Then* $L_e > 0$ *and* $\sum_e L_e = 1$*. Eliminate* $u_n = 1 - \sum_{i=1}^{n-1} u_i$ *and set*

$$\mathcal{J}_G(z,u) = \left|\det\left[L, \partial_{z^{(1)}} L, \dots, \partial_{z^{(d)}} L, \partial_{u_1} L, \dots, \partial_{u_{n-1}} L\right]\right| > 0. \tag{106}$$

*For the general source* $C \prod_e \ell_e^{m_e - 1} e^{-\lambda_e \ell_e} d\ell_e / \Gamma(m_e)$*, with all shapes and rates positive and* $M = \sum_e m_e$*, an almost-everywhere version of its undecorated density is*

$$h_G(z) = \frac{C\Gamma(M)}{\prod_e \Gamma(m_e)} \int_{\mathcal{U}_G(z)} \frac{\mathcal{J}_G(z,u) \prod_e L_e(z,u)^{m_e - 1}}{\left(\sum_e \lambda_e L_e(z,u)\right)^M} \, du_1 \cdots du_{n-1}. \tag{107}$$

*Thus the four-point maximal-cell marginal is a three-dimensional integral with a Jacobian obtained by the period differentiation above.*

*Proof.* Every positive length vector has unique coordinates

$$r = \sum_e \ell_e = \frac{1}{2} \sum_i p_i, \qquad u_i = \frac{p_i}{2r}, \qquad z = f_G(\ell).$$

Homogeneity of the Strebel construction gives $\ell = rL(z,u)$. Corollary 9.4 shows that this is a real-analytic change of coordinates. Its Jacobian is

$$d^E \ell = r^{E-1} \mathcal{J}_G(z,u) \, dr \, d^d z \, d^{n-1} u,$$

because all columns except the scale column carry one factor of $r$. It is nonzero everywhere, which proves positivity in (106). Substituting the source density leaves the radial integral

$$\int_0^\infty r^{M-1} \exp\left(-r \sum_e \lambda_e L_e(z,u)\right) dr = \frac{\Gamma(M)}{\left(\sum_e \lambda_e L_e(z,u)\right)^M}.$$

Tonelli's theorem gives (107); finite total source mass makes this density finite almost everywhere. □

Maximal-cell projections have no interior critical points. Possible singular behaviour of their marginal densities must therefore be studied through the boundary or noncompact ends of the perimeter fibres. The formula above asserts an almost-everywhere density and does not impose pointwise boundedness or smoothness at such limits.

# 10 Admissible contraction of Schwinger measures

This section concerns an analytic extension of the edge weights to positive real shapes. Varying these shapes is not an assertion about a fixed integer-charge field-theory correlator. It provides a precise way to approximate a lower-dimensional Strebel measure by measures on incident higher-dimensional cells.

Let $A \subset E(\Gamma)$ be an admissible contraction set as in Lemma 6.3, and put $B = E \setminus A$ and $\Gamma_0 = \Gamma \setminus A$. For $e \in A$, fix $a_e > 0$ and $\lambda_e > 0$, and let

$$X_e^{(\varepsilon)} \sim \mathrm{Gamma}(\varepsilon a_e, \lambda_e)$$

be independent, with the second parameter denoting the rate. Let $Y = (Y_e)_{e\in B}$ be independent of these variables and have any fixed probability law on $(0,\infty)^B$. Denote the laws of $\mathfrak{S}_\Gamma(X^{(\varepsilon)}, Y)$ and $\mathfrak{S}_{\Gamma_0}(Y)$ by $P_\varepsilon$ and $P_0$.

**Theorem 10.1** (Contraction and a modulus-of-continuity estimate)**.** *The probability measures $P_\varepsilon$ converge weakly to $P_0$. The same holds after projection to undecorated moduli space. The limit is unchanged if the shapes of the edges in $A$ tend to zero at different positive rates, or if admissible subsets are contracted successively in any order.*

*Let $d$ be any bounded compatible metric on the target. For a compact $K \subset (0,\infty)^B$ and $0 < \delta < 1$, define*

$$\omega_K(\delta) = \sup_{y\in K,\ s\geq 0,\ \|s\|_1\leq\delta} d\big(\mathfrak{S}_\Gamma(s,y), \mathfrak{S}_{\Gamma_0}(y)\big).$$

*Then $\omega_K(\delta) \to 0$ as $\delta \downarrow 0$, and the bounded-Lipschitz distance satisfies*

$$d_{\mathrm{BL}}(P_\varepsilon, P_0) \leq 2\mathbb{P}\{Y \notin K\} + \omega_K(\delta) + \frac{2\varepsilon}{\delta}\sum_{e\in A}\frac{a_e}{\lambda_e}. \tag{108}$$

*No global Lipschitz or Hölder regularity of the inverse Strebel map is required for this conclusion.*

*Proof.* The means satisfy $\mathbb{E}\|X^{(\varepsilon)}\|_1 = \varepsilon\sum_{e\in A} a_e/\lambda_e$. The forest-contraction extension of the Strebel map is continuous on $[0,1]^A \times K$, hence uniformly continuous there. Couple the two target measures by using the same $Y$. On the event $Y \in K$ and $\|X^{(\varepsilon)}\|_1 \leq \delta$, the difference of any bounded 1-Lipschitz test function is at most $\omega_K(\delta)$. On its complement it is at most 2. Markov's inequality gives (108). First choose $K$ with arbitrarily large $Y$-probability, then choose $\delta$ small, and finally let $\varepsilon \downarrow 0$. This proves weak convergence. Projection is continuous, so the undecorated statement follows as well.

For unequal vanishing shapes, replace the mean above by $\sum_{e\in A}\alpha_e/\lambda_e$, which still tends to zero. Contracting a forest is independent of the order of its edges, and the product law has the unique limiting law $\delta_0^{\otimes A} \otimes \mathrm{Law}(Y)$. This proves order independence for both simultaneous and iterated limits. □

**Proposition 10.2** (A conditional quantitative refinement)**.** *On a compact set $K$ of surviving lengths, suppose that a bounded test function $\varphi$ has pullback $F(s,y) = \varphi(\mathfrak{S}_\Gamma(s,y))$ satisfying*

$$|F(s,y) - F(0,y)| \leq L_K\|s\|_1^\theta, \qquad \|s\|_1 \leq 1, \quad y \in K, \qquad 0 < \theta \leq 1.$$

*Then the contribution with $Y \in K$ to $\int \varphi\, d(P_\varepsilon - P_0)$ is $O_K(\varepsilon)$. An explicit bound is*

$$\begin{aligned}&\mathbb{E}\big[\mathbf{1}_{\{Y\in K\}}|F(X^{(\varepsilon)}, Y) - F(0,Y)|\big]\\ &\quad\leq L_K\sum_{e\in A}\lambda_e^{-\theta}\frac{\Gamma(\varepsilon a_e + \theta)}{\Gamma(\varepsilon a_e)} + 2\|\varphi\|_\infty\varepsilon\sum_{e\in A}\frac{a_e}{\lambda_e}.\end{aligned} \tag{109}$$

*The Hölder assumption is stated explicitly; it is not inferred from cellwise analyticity or from the topological Strebel theorem.*

*Proof.* On $\|X\|_1 \le 1$, use $(\sum_e X_e)^\theta \le \sum_e X_e^\theta$ and the exact gamma moment. On the complementary event, use Markov's inequality and boundedness. Finally,

$$\frac{\Gamma(\varepsilon a+\theta)}{\Gamma(\varepsilon a)} = \varepsilon a\Gamma(\theta) + O(\varepsilon^2),$$

which gives the claimed rate. In particular, a fractional-power modulus at a wall need not reduce the linear rate of this gamma-shape approximation. □

*Remark* 10.3. For the unnormalized Schwinger factors $\ell^{\alpha-1}e^{-\lambda\ell}d\ell/\Gamma(\alpha)$, the mass equals $\lambda^{-\alpha}$ and tends to 1 as $\alpha \downarrow 0$. Multiplying the probability statements by these explicitly convergent masses gives the corresponding finite measure limits. The theorem concerns admissible contractions inside decorated moduli space. It does not identify such a contraction with a Deligne–Mumford nodal degeneration or with physical OPE sewing.

## 10.1 First variation and total variation distance

**Proposition 10.4** (First variation at an admissible contraction)**.** *Retain the independent variables $X_e^{(\varepsilon)} \sim \mathrm{Gamma}(\varepsilon a_e, \lambda_e)$ for $e \in A$. Suppose that the law of $Y$ is supported on a compact set $K \subset (0,\infty)^B$. Let $F : [0,\infty)^A \times K \to \mathbb{R}$ be bounded and continuous, and assume that for some $L < \infty$ and $0 < \theta \le 1$,*

$$|F(s,y) - F(0,y)| \le L\|s\|_1^\theta, \qquad \|s\|_1 \le 1, \quad y \in K. \tag{110}$$

*Writing $\mathbf{e}_e$ for the $e$th coordinate vector, one has the expansion*

$$\begin{aligned}\mathbb{E}F(X^{(\varepsilon)}, Y) &= \mathbb{E}F(0,Y) \\ &\quad + \varepsilon \sum_{e\in A} a_e \mathbb{E}\left[\int_0^\infty \big(F(t\mathbf{e}_e, Y) - F(0,Y)\big)\frac{e^{-\lambda_e t}}{t}\,dt\right] + o(\varepsilon).\end{aligned} \tag{111}$$

*Every integral on the right is absolutely convergent. For a law of $Y$ with noncompact support, the same expansion holds after inserting $\mathbf{1}_{\{Y\in K\}}$ in each expectation, provided the stated hypotheses hold on $K$.*

*Proof.* The assertion is immediate when $A$ is empty. Otherwise, let $q = |A|$ and fix $0 < \delta < 1/q$. Set $F_0(y) = F(0,y)$. Divide the source space into the events that zero, exactly one, or at least two coordinates of $X^{(\varepsilon)}$ exceed $\delta$.

On the event that all coordinates are at most $\delta$, the norm is at most one. By (110) and $(\sum_e s_e)^\theta \le \sum_e s_e^\theta$, its contribution in absolute value is at most

$$L\sum_{e\in A}\frac{\lambda_e^{\varepsilon a_e}}{\Gamma(\varepsilon a_e)}\int_0^\delta t^{\varepsilon a_e+\theta-1}e^{-\lambda_e t}\,dt.$$

Since $\lambda_e^{\varepsilon a_e}/\Gamma(\varepsilon a_e) = \varepsilon a_e + O(\varepsilon^2)$, this is bounded by $C\varepsilon\delta^\theta$ for all sufficiently small $\varepsilon$, with $C$ independent of $\delta$.

For fixed $\delta$, each tail probability $\mathbb{P}\{X_e^{(\varepsilon)} > \delta\}$ is $O_\delta(\varepsilon)$. Independence and a union bound therefore show that the contribution from at least two coordinates above $\delta$ is $O_\delta(\varepsilon^2)$.

On the event where exactly coordinate $e$ exceeds $\delta$, integrate first in that coordinate. On $[\delta, \infty)$ its density divided by $\varepsilon$ converges to

$$a_e \frac{e^{-\lambda_e t}}{t}$$

and is dominated by an integrable function independent of small $\varepsilon$. The remaining coordinates converge to zero in probability. Boundedness and continuity of $F$, uniformly on $[0,\delta]^{A\setminus\{e\}} \times K$ for each fixed $t$, imply that their integral restricted to this event converges to $F(t\mathbf{e}_e, y) - F_0(y)$. Dominated convergence in $t$ and $y$ therefore gives the limit

$$a_e \mathbb{E}\left[\int_\delta^\infty \big(F(t\mathbf{e}_e, Y) - F_0(Y)\big) \frac{e^{-\lambda_e t}}{t}\, dt\right]$$

after division by $\varepsilon$.

The omitted interval $(0,\delta)$ in this integral has absolute value at most $a_e L\delta^\theta/\theta$ by the Hölder condition. The integral from one to infinity converges by boundedness and exponential decay. Letting first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ proves (111). The same proof applies to the finite restricted law $\mathbf{1}_K \operatorname{Law}(Y)$, proving the last assertion. □

For the geometric application, take $F(s,y) = \varphi(\mathfrak{S}_\Gamma(s,y))$ with bounded continuous $\varphi$. The admissible forest extension gives continuity on the partial contraction faces; the Hölder estimate (110) is an additional condition on this pullback. The first-order coefficient is a sum of integrals over faces where exactly one of the contracted edges has positive length. Excursions with two positive macroscopic contracted lengths contribute only at higher order in this limit.

For the unnormalized Schwinger factors $s_e^{\varepsilon a_e - 1} e^{-\lambda_e s_e}\, ds_e/\Gamma(\varepsilon a_e)$, the mass factor is

$$\prod_{e\in A} \lambda_e^{-\varepsilon a_e} = 1 - \varepsilon \sum_{e\in A} a_e \log \lambda_e + O(\varepsilon^2).$$

Consequently their first variation is the coefficient in (111) minus $\big(\sum_e a_e \log \lambda_e\big)\mathbb{E}F(0,Y)$.

**Corollary 10.5** (Total variation at a change of Lebesgue type)**.** *In the admissible-contraction setting of Theorem 10.1, suppose $n = 4$, the parent skeleton has rank two almost everywhere, and the contracted skeleton is a path or a four-cycle. Give the surviving lengths a fixed product of positive-shape gamma laws, independent of the vanishing-shape variables. Let $Q_\varepsilon$ and $Q_0$ be the projected probability measures. Then $Q_\varepsilon \Rightarrow Q_0$, but for every $\varepsilon > 0$,*

$$d_{\mathrm{TV}}(Q_\varepsilon, Q_0) := \sup_B |Q_\varepsilon(B) - Q_0(B)| = 1. \tag{112}$$

*Proof.* Weak convergence follows from Theorem 10.1. For every positive $\varepsilon$, all source shapes are positive, so Theorem 6.5 makes $Q_\varepsilon$ absolutely continuous with respect to area measure. The probability measure $Q_0$ is supported on the real cross-ratio locus. That locus consequently has $Q_\varepsilon$-mass zero and $Q_0$-mass one, proving the stated total variation distance. □

For an explicit admissible example, take a tetrahedral $K_4$ cell, retain the path edges $12, 23, 34$, and delete $13, 14, 24$. If the four dual vertices are labelled by the triangular faces $f_{123}, f_{124}, f_{134}, f_{234}$, the contracted dual edges form the tree

$$f_{123} \overset{13}{\longleftrightarrow} f_{134} \overset{14}{\longleftrightarrow} f_{124} \overset{24}{\longleftrightarrow} f_{234}.$$

The surviving field skeleton is connected and reduced, as required. The first variation in Proposition 10.4 is a compensated functional on the stated class of test functions. The subtraction at the contracted point is essential because $e^{-\lambda t}dt/t$ has infinite mass at zero; a finite signed Radon derivative is not asserted.

For an explicit normalization check, take $F(s, y) = \exp(-\sum_{e\in A} b_e s_e)$ with $b_e \geq 0$. The exact gamma Laplace transform is

$$\mathbb{E}F(X^{(\varepsilon)}, Y) = \prod_{e\in A} \left(\frac{\lambda_e}{\lambda_e + b_e}\right)^{\varepsilon a_e}.$$

Its derivative at zero is $-\sum_e a_e \log(1 + b_e/\lambda_e)$, in agreement with (111), since

$$\int_0^\infty (e^{-bt} - 1)\frac{e^{-\lambda t}}{t}\, dt = -\log(1 + b/\lambda).$$

# 11 Scope and further questions

The four-point classification determines the generic projection rank and Lebesgue type of every reduced normal-ordered skeleton. Positive integer degree equations and a finite list of path and cycle weights determine the entire singular mass. Its densities are explicit after perimeter integration, and the square density remains finite and smooth for every positive real shape. At equal degrees the sharp mass bound, its stability remainder, and the two large-degree regimes quantify distinct aspects of the decomposition. In particular, the decreasing lower bound $3/(k+1)$ does not imply vanishing singular fraction at fixed unequal channel ratios.

For arbitrary $n$, every maximal cell has projection rank $2n-6$ at every interior point. For the lower-dimensional cells with $n \geq 5$, the bound rank $df_G \leq E - 1$ and the inheritance principle remain available, while a complete graph-theoretic rank criterion requires further work. One approach is to identify spanning subgraphs of rank $2n - 6$ and apply the inheritance principle. The four-point comparison between the star and path shows why zero multiplicities and incidence ranks alone cannot supply such a criterion.

For the four-point maximal sectors, the period construction reduces the marginal to a three-dimensional integral over normalized perimeters. The forgetful projection has no interior critical points. Uniform estimates for the tetrahedral density must instead control the boundaries of this perimeter domain, including vanishing critical lengths and vanishing perimeters. The square and path formulas supply explicit models for some of these limits; a complete pointwise analysis of the maximal-cell marginal remains to be established.

The contraction first variation gives a precise coefficient on bounded continuous test functions whose pullbacks satisfy a stated local Hölder estimate. Establishing such estimates uniformly for specified families of degenerating metric graphs would extend its geometric applications. The total variation obstruction shows why weak convergence and the compensated test-function expansion are the appropriate conclusions when the support dimension changes.

Admissible forest contractions preserve a smooth marked sphere and merge critical vertices. Deligne–Mumford degenerations instead involve nodal curves and additional scaling data. Establishing factorization at those boundaries would require a separate analysis of plumbing coordinates, the induced measure, and the relation to spacetime OPE limits. Likewise, permutation covariance of the measure and its positive finite mass do not supply a local worldsheet operator algebra. Determining such an algebra remains a distinct part of the string-reconstruction programme.

The computational supplement contains the enumeration and numerical checks described in Appendix B. Direct unreduced Wick enumeration checks the complete propagator monomials independently of the skeleton reduction, and a separate ribbon-permutation check verifies the real-locus symmetry classification. The geometric, analytic, and asymptotic results are proved in the text.

# A Charge conditions and multiplicity polytopes

For each labelled skeleton, the complete charge condition is the positive integer system $B_G m = J$. We give all its solutions for the representatives in (16); relabelling the vertices relabels the conditions. Every parameter and every displayed edge multiplicity in the following formulas is required to be an integer.

**Proposition A.1** (Incidence rank)**.** *For a connected loopless graph with $n$ vertices,*

$$\operatorname{rank}_{\mathbb{R}} B_G = \begin{cases} n-1, & G \text{ is bipartite,} \\ n, & G \text{ is non-bipartite.} \end{cases} \tag{113}$$

*For a bipartition $V = A \sqcup B$, its only real linear compatibility condition is*

$$\sum_{i \in A} J_i = \sum_{i \in B} J_i. \tag{114}$$

*Whenever the fibre $B_G m = J$, $m > 0$, is nonempty, its real dimension is $E - \operatorname{rank} B_G$.*

*Proof.* A vector $a$ lies in the left kernel exactly when $a_i + a_j = 0$ on every edge. Connectedness determines all its entries from one of them. An odd cycle forces that entry to vanish; otherwise the two bipartition classes carry opposite constant values. This proves the rank and compatibility statements. A nonempty positive fibre is relatively open in the corresponding affine solution space, so has its full dimension. □

For later use, define the integer interval

$$I(v; J) := [\max\{1, J_1 - v - J_4 + 1\}, \min\{J_3 - 1, J_1 - v - 1\}] \cap \mathbb{Z}. \tag{115}$$

An interval with lower endpoint larger than its upper endpoint is empty.

**Trees.** For $S$, the solution is unique:

$$(m_{12}, m_{13}, m_{14}) = (J_2, J_3, J_4), \qquad J_1 = J_2 + J_3 + J_4. \tag{116}$$

For $P$, the solution is likewise unique when it exists:

$$(m_{12}, m_{23}, m_{34}) = (J_1, J_2 - J_1, J_4), \quad J_1 + J_3 = J_2 + J_4, \quad J_2 > J_1. \tag{117}$$

**The four-cycle.** For $C$, solutions exist only if $J_1 + J_3 = J_2 + J_4$, and all of them are

$$(m_{12}, m_{23}, m_{34}, m_{14}) = (t, J_2 - t, J_3 - J_2 + t, J_1 - t), \tag{118}$$

with

$$\max\{1, J_2 - J_3 + 1\} \le t \le \min\{J_1 - 1, J_2 - 1\}. \tag{119}$$

**A triangle with a pendant edge.** For $T$, put $r = J_1 - J_4$. The unique possible solution is

$$m_{12} = \frac{r + J_2 - J_3}{2}, \quad m_{13} = \frac{r + J_3 - J_2}{2}, \quad m_{23} = \frac{J_2 + J_3 - r}{2}, \quad m_{14} = J_4. \tag{120}$$

It exists precisely when the first three displayed quantities are positive integers.

**The two double-edge types with pendant edges.** For $D_s$, the necessary and sufficient conditions are $J_1 = J_2 + J_3 + J_4$ and $J_2 \geq 2$. The solutions are

$$(m_{12_a}, m_{12_b}, m_{13}, m_{14}) = (t, J_2 - t, J_3, J_4), \qquad 1 \leq t \leq J_2 - 1. \tag{121}$$

For $D_o$, put $v = J_1 - J_3$. The conditions are $v = J_2 - J_4 \geq 2$, and the solutions are

$$(m_{12_a}, m_{12_b}, m_{13}, m_{24}) = (t, v - t, J_3, J_4), \qquad 1 \leq t \leq v - 1. \tag{122}$$

The two parallel edges are distinct edges of the ribbon skeleton, so different values of $t$ are not identified by an additional edge swap.

**Five edges.** For $H$, put

$$v = \frac{J_1 + J_2 - J_3 - J_4}{2}. \tag{123}$$

The conditions are $v \in \mathbb{Z}_{>0}$ and $I(v; J) \neq \varnothing$. All solutions, with $t \in I(v; J)$, are

$$\begin{aligned} &m_{12} = v, && m_{13} = t, && m_{23} = J_3 - t, \\ &m_{14} = J_1 - v - t, && m_{24} = J_4 - J_1 + v + t. \end{aligned} \tag{124}$$

For $L$, put $r = J_1 - J_4$ and

$$v = \frac{r + J_2 - J_3}{2}, \qquad b = \frac{r + J_3 - J_2}{2}, \qquad c = \frac{J_2 + J_3 - r}{2}. \tag{125}$$

Solutions exist precisely when $v \geq 2$, $b \geq 1$, and $c \geq 1$ are integers. They are

$$(m_{12_a}, m_{12_b}, m_{13}, m_{23}, m_{14}) = (t, v - t, b, c, J_4), \qquad 1 \leq t \leq v - 1. \tag{126}$$

**Six edges.** For $K$, write $J_{\mathrm{tot}} = J_1 + J_2 + J_3 + J_4$. A solution exists if and only if

$$J_{\mathrm{tot}} \text{ is even}, \qquad J_i \geq 3 \ (1 \leq i \leq 4), \qquad 2 \max_i J_i \leq J_{\mathrm{tot}} - 6. \tag{127}$$

A complete two-parameter description is obtained by setting

$$d_{12} = \frac{J_1 + J_2 - J_3 - J_4}{2}, \quad d_{13} = \frac{J_1 + J_3 - J_2 - J_4}{2}, \quad d_{14} = \frac{J_1 + J_4 - J_2 - J_3}{2} \tag{128}$$

and taking

$$\begin{aligned} &m_{12} = a, && m_{13} = b, && m_{14} = J_1 - a - b, \\ &m_{34} = a - d_{12}, && m_{24} = b - d_{13}, && m_{23} = J_1 - a - b - d_{14}, \end{aligned} \tag{129}$$

for all integer pairs $(a, b)$ making the six entries positive.

For $M$, use $v$ from (123). The conditions are $v \in \mathbb{Z}$, $v \geq 2$, and $I(v;J) \neq \varnothing$. The solutions are

$$\begin{aligned} m_{12_a} &= s, & m_{12_b} &= v - s, & m_{13} &= t, \\ m_{23} &= J_3 - t, & m_{14} &= J_1 - v - t, & m_{24} &= J_4 - J_1 + v + t, \end{aligned} \tag{130}$$

where $1 \leq s \leq v-1$ and $t \in I(v;J)$ are independent. Thus a fixed oriented cell of type $M$ has exactly $(v-1)|I(v;J)|$ admissible multiplicity assignments.

All these parameterizations follow by solving the incidence equations. To verify the extra existence criterion (127), subtract one from every edge multiplicity of $K_4$. The remaining degrees are $r_i = J_i - 3$. A nonnegative loopless multigraph with prescribed degrees $r_i$ exists exactly when $\sum_i r_i$ is even and $2\max_i r_i \leq \sum_i r_i$. Sufficiency has a short induction: if the total degree is $2q > 0$, join two vertices of largest positive degrees, reduce both degrees by one, and observe that the new largest degree is at most $q-1$. Repeat until the degree sum is zero. Applying this criterion to the $r_i$ gives (127).

# B Computational verification

## B.1 Independent finite enumeration

The standard-library Python program in the computational supplement `enumerate_four_point.py` independently enumerates the complete classification. For each $3 \leq E \leq 6$ and each positive composition $d_1 + \cdots + d_4 = 2E$, it fixes a trace permutation with these cycle lengths and enumerates all fixed-point-free involutions pairing different vertex cycles. It retains precisely the pairings for which the generated permutation group is transitive, $c(\alpha\beta) = E - 2$, and every cycle of $\alpha\beta$ has length at least three. Finally it takes the exact quotient by the product of the four cyclic relabelling groups.

The file `four_point_skeletons.json` records all 115 orbit representatives, including their vertex cycles, pairing transpositions, face cycles, degree partitions, bipartitions, and stabilizer orders. It reproduces (17) and finds stabilizer order one in every case. This enumeration verifies the finite classification; the proof of Theorem 4.1 does not depend on the computation.

## B.2 Unreduced Wick contractions and ribbon reflections

The standard-library script `check_raw_wick.py` enumerates normal-ordered Wick pairings without first removing empty digons. It tests transitivity for connectedness and the permutation Euler characteristic for genus zero, then records the full monomial in the six propagators. Independently, it expands the supplied reduced-cell list over all positive edge multiplicities compatible with each degree vector and assigns the weight $\prod_i J_i$ from Proposition 3.3. For every ordered positive degree vector of even total at most 12, these two unreduced integer counts agree monomial by monomial: 295 degree vectors, 582 nonzero propagator monomials, and 51,792 connected planar labelled pairings.

The same script enumerates all label-preserving edge permutations of each of the 115 reduced cells. In dart notation, a reflection $\tau$ satisfies $\tau\beta = \beta\tau$ and $\tau\alpha = \alpha^{-1}\tau$. It finds exactly one such reflection for each cell of types $P, C, D_s, D_o, L, M$, and none for $S, T, H, K$. In the four types with parallel edges, that reflection exchanges precisely the two parallel edges. The program also rechecks that every label-preserving orientation-preserving automorphism is the identity. The counts and explicit dart permutations are recorded in `raw_wick_checks.json`.

### B.3 Degree classification and mass polynomials

The standard-library script `check_charge_classification.py` reads the enumerated cells and assigns positive multiplicities directly to each edge. It does not call the closed multiplicity formulas in Appendix A. With total operator degree at most 28, it checks 154,427 cell–multiplicity assignments and 20,475 ordered positive degree vectors, including odd totals. The observed classes are 4,816 purely absolutely continuous, 2,183 mixed, 7 purely singular, and 13,469 zero. The seven purely singular vectors are the six permutations of $(1, 1, 2, 2)$ and $(2, 2, 2, 2)$. There are no discrepancies with Theorem 6.11.

For $k = 2, \dots, 7$, the same direct enumeration recovers the complete equal-degree mass polynomials, separately for the rank-one and rank-two types. A further 190 checks use exact rational arithmetic to verify the polynomial identities and sharp inequality for ten positive triples $(A, B, C)$ and $k = 2, \dots, 20$. The geometric rank classification is an input to these checks; its analytic proof is not tested by this finite enumeration. Results are recorded in `charge_classification_checks.json`.

### B.4 Changes of variables and marginal measures

The script `check_strebel_measures.py`, using NumPy and SciPy, checks the square derivative and inverse, the star inverse, and the path marginal against independent numerical quadrature. With the recorded random seed 38271 and 4,000 square samples, the maximum square length-reconstruction error is less than $1.2 \times 10^{-13}$ and the derivative relative error is less than $1.3 \times 10^{-8}$. The star inverse error is less than $3.4 \times 10^{-16}$. Path quadrature agrees with the unit-shape closed density to within $1.4 \times 10^{-16}$ at the tested points, and its numerical total mass differs from one by less than $10^{-13}$. General-shape endpoint coefficients are also evaluated numerically. The uniform-law sampling diagnostic gives mean 0.49838 and variance 0.08227, compared with 1/2 and 1/12; these sample statistics are checks, not proofs of the law. The exact run output is `strebel_measure_checks.json`.

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