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REPRESENTATION OF DEGREES OF CONVERGENCE BY NONADDITIVE MEASURES

PAOLO LEONETTI

Abstract. We characterize the functions that arise as degrees of convergence of sequences in metric spaces with respect to normalized capacities on $\mathbb{N}$. More precisely, answering a question of Mišík and Tóth in [Fuzzy Sets and Systems **375** (2019), 179–190], we show that these functions are precisely the $[0,1]$-valued upper semicontinuous functions whose positive supports are separable. We obtain further characterizations when the representing capacities are required to be maxitive, subadditive, submodular, superadditive, supermodular, or countably subadditive, as well as for finitely and countably additive probability measures. Remarkably, representation by a normalized submeasure is equivalent to representation by the upper asymptotic density $\mathsf{d}^\star$. As a byproduct, if (X,d) is an uncountable compact metric space, then a map $f : X \to [0,1]$ is upper semicontinuous if and only if there exists a sequence (x_n) in X such that

$$f(x) = \inf_{\varepsilon>0} \mathsf{d}^\star(\{n \in \mathbb{N} : d(x_n,x) < \varepsilon\}) \quad \text{for all } x \in X.$$

We also characterize the pairs of upper and lower degrees that can be represented simultaneously by a single sequence, with a sharp distinction between countable compact metric spaces and the remaining metric spaces. Lastly, we characterize the existence of universal representing sequences, and investigate the comeagerness of the family of maldistributed sequences.

1. Introduction and Main Results

The *degree of convergence* of a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ quantifies its concentration near a point of the underlying space. It was introduced by Mišík and Tóth in [11]; see Definition 1.1 below.

Throughout, X will be a given metric space with metric d containing at least two points. Setting $\mathbb{N} := \{0,1,2,\ldots\}$, a map $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is said to be:

(i) *normalized* if $\mu(\mathbb{N}) = 1$;
(ii) *monotone* if $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq \mathbb{N}$;
(iii) *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;
(iv) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;
(v) *submodular* if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ for all $A, B \subseteq \mathbb{N}$;
(vi) *maxitive* if $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \subseteq \mathbb{N}$;
(vii) *finite-invariant* if $\mu(A) = \mu(B)$ for all $A, B \subseteq \mathbb{N}$ with $|A \triangle B| < \infty$.

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The notions of superadditivity and supermodularity are analogous, while we use σ-additivity and σ-subadditivity with their usual meanings. Accordingly, a map $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is a *capacity* if it is monotone and $\mu(\emptyset) = 0$, and it is a *submeasure* if it is a subadditive capacity. Observe also that a maxitive capacity is submodular, and that a submodular capacity is a submeasure.

Following [11, Definition 2.2], we recall the definition of degree of convergence.

Definition 1.1. Let $\boldsymbol{x} = (x_n)_{n\geq 0}$ be a sequence with values in X, and let $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ be a normalized capacity. Then the *degree of convergence* is the map $\mathrm{DC}_{\mu,\boldsymbol{x}} : X \to \mathbb{R}$ defined by

$$\forall \eta \in X, \qquad \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) := \inf_{\varepsilon>0} \mu\left(\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}\right).$$

1.1. **Representations of degrees of convergence.** Given a nonempty family $\mathscr{M}$ of capacities, we say that "*a map $f : X \to \mathbb{R}$ is represented by some $\mu \in \mathscr{M}$*" if there exist $\mu \in \mathscr{M}$ and a sequence $\boldsymbol{x}$ with values in X such that

$$f = \mathrm{DC}_{\mu,\boldsymbol{x}}.$$

In [11, Example 3.6], Mišík and Tóth proved that, if μ is the upper or lower asymptotic density on $\mathbb{N}$, then for each $\alpha \in [0,1]$ there exists a sequence $\boldsymbol{x}$ with values in X and a point $\eta \in X$ such that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = \alpha$. Thus, they concluded with the following open question [11, Open problem 3.1]:

Question 1. Characterize the family of functions $f : X \to \mathbb{R}$ which are represented by some normalized capacity.

First, we answer Question 1 in Theorem 1.2. Then, we characterize the family of functions which are represented by capacities satisfying additional structural properties. To this end, for each map $f : X \to \mathbb{R}$ we define

$$P_f := \{\eta \in X : f(\eta) > 0\}.$$

Also, the image of f is denoted by $\mathrm{im}(f)$.

Theorem 1.2. *Let X be a metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(A1) *f is represented by some normalized capacity*;
(A2) *f is represented by some finite-invariant normalized capacity*;
(A3) *f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, and P_f is separable.*

If, in addition, X is not compact, then they are also equivalent to:

(A4) *f is represented by some normalized submeasure*;
(A5) *f is represented by some finite-invariant normalized submeasure*;
(A6) *f is represented by some normalized submodular capacity*;
(A7) *f is represented by some finite-invariant normalized submodular capacity*;
(A8) *f is represented by some normalized maxitive capacity*;
(A9) *f is represented by some finite-invariant normalized maxitive capacity.*

In the following, we study the remaining cases of compact metrizable spaces X for the conditions (A4)–(A9). In such case, observe that a compact metric space X is separable, hence all its subspaces (including P_f) are automatically separable. We start with the characterization of functions f represented by normalized maxitive capacities.

Theorem 1.3. *Let X be a compact metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(B1) *f is represented by some normalized maxitive capacity*;
(B2) *f is represented by some finite-invariant normalized maxitive capacity*;
(B3) *f is upper semicontinuous,* $\operatorname{im}(f) \subseteq [0,1]$, *and* $\max f = 1$.

Before we proceed to the characterization of maps f represented by normalized submeasures or normalized submodular capacities, we recall that, in an attempt to formalize the notion of "upper density on $\mathbb{N}$," a normalized finite-invariant submeasure $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ (as in condition (A5)) has been termed *abstract upper density* by Di Nasso and Jin [5], cf. also [6]. This class is really large and contains all the upper densities considered in [9], which we will call *arithmetic upper densities* (that is, normalized monotone subadditive maps $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ such that $\mu(\{kx + h : x \in S\}) = \mu(S)/k$ for all $S \subseteq \mathbb{N}$ and $k, h \in \mathbb{N}$ with $k \neq 0$). Examples include the upper Buck, upper Pólya, and upper analytic densities, as well as the family of upper α-densities (subsuming the upper logarithmic and upper asymptotic densities), where α is a real parameter ≥ -1. More explicitly, recall that the upper asymptotic density $\mathsf{d}^\star : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is defined by

$$\forall S \subseteq \mathbb{N}, \qquad \mathsf{d}^\star(S) := \limsup_{N\to\infty} \frac{|S \cap [0,N)|}{N}.$$

Quite unexpectedly, a map $f : X \to \mathbb{R}$ can be represented by any of the above classes of capacities if and only if it can be represented by $\mathsf{d}^\star$. Throughout, if f is nonnegative, we write $\sum f := \sup\left\{\sum_{x\in F} f(x) : F \subseteq X, |F| < \infty\right\}$.

Theorem 1.4. *Let X be a metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(C1) *f is represented by some normalized submeasure*;
(C2) *f is represented by some finite-invariant normalized submeasure (i.e., by some abstract upper density)*;
(C3) *f is represented by some normalized submodular capacity*;
(C4) *f is represented by some finite-invariant normalized submodular capacity*;
(C5) *f is represented by some arithmetic upper density*;
(C6) *f is represented by $\mathsf{d}^\star$.*

If X is compact and countable, conditions (C1)–(C6) are also equivalent to:

(C7′) *f is upper semicontinuous,* $\operatorname{im}(f) \subseteq [0,1]$, *and* $\sum f \geq 1$.

If X is noncompact or uncountable, conditions (C1)–(C6) are also equivalent to:

(C7″) *f is upper semicontinuous,* $\operatorname{im}(f) \subseteq [0,1]$, *and P_f is separable.*

In particular, Theorem 1.4 shows that the equivalences (A4) $\iff$ (A5) $\iff$ (A6) $\iff$ (A7) in Theorem 1.2 hold also if X is an uncountable compact metric space. In addition, we state the following consequence.

Corollary 1.5. *Let X be an uncountable compact metric space. Then a function $f : X \to [0,1]$ is upper semicontinuous if and only if there exists a sequence $(x_n)_{n\geq 0}$ with values in X such that*

$$\forall y \in X, \qquad f(y) = \inf_{\varepsilon>0} \limsup_{N\to\infty} \frac{|\{n \in \mathbb{N} \cap [0,N) : d(x_n,y) < \varepsilon\}|}{N}.$$

It is remarkable that the analogue of item (C6) does *not* hold for arbitrary arithmetic upper densities, even if $X = [0,1]$; see Remark 2.4 below.

We next characterize the functions represented by normalized σ-subadditive capacities.

Theorem 1.6. *Let X be a metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(D1) *f is represented by some normalized σ-subadditive capacity*;
(D2) *f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, P_f is separable, and $\sum f \geq 1$.*

Observe that we cannot expect a representation of f by some finite-invariant normalized σ-subadditive capacity: indeed, such capacities do not exist since their existence would imply $1 = \mu(\mathbb{N}) \leq \sum_n \mu(\{n\}) = 0$. The same observation applies to the first part of Theorem 1.11 below.

We next characterize the functions represented by superadditive or supermodular capacities. If X is noncompact or uncountable, these functions are precisely those represented by *probability charges* (that is, normalized additive capacities). Here, we will need also the definition of the lower asymptotic density, that is,

$$\forall S \subseteq \mathbb{N}, \qquad \mathsf{d}_\star(S) := \liminf_{N\to\infty} \frac{|S \cap [0,N)|}{N}.$$

Observe that $\mathsf{d}_\star$ is a finite-invariant normalized superadditive capacity.

Theorem 1.7. *Let X be a metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(E1) *f is represented by some normalized superadditive capacity*;
(E2) *f is represented by some finite-invariant normalized superadditive capacity*;
(E3) *f is represented by $\mathsf{d}_\star$*;
(E4) *f is represented by some normalized supermodular capacity*;
(E5) *f is represented by some finite-invariant normalized supermodular capacity*;
(E6) *$\mathrm{im}(f) \subseteq [0,1]$ and $\sum f \leq 1$.*

If, in addition, X is noncompact or uncountable, then they are also equivalent to:

(E7) *f is represented by some probability charge*;
(E8) *f is represented by some finite-invariant probability charge.*

In the case of uncountable compact metric spaces, we are able to provide a joint representation by $\mathsf{d}^\star$ and $\mathsf{d}_\star$ using a single sequence $\boldsymbol{x}$.

Theorem 1.8. *Let X be an uncountable compact metric space and let $f, g : X \to [0,1]$. Then there is a sequence $\boldsymbol{x}$ with values in X such that*

$$DC_{\mathsf{d}^\star,\boldsymbol{x}} = f \qquad \text{and} \qquad DC_{\mathsf{d}_\star,\boldsymbol{x}} = g \tag{1}$$

if and only if the following three conditions hold:

(i) $f - g$ *is upper semicontinuous*;
(ii) $\sum g \leq 1$;
(iii) $0 \leq f(\eta) - g(\eta) \leq 1 - \sum g$ *for every* $\eta \in X$.

The following example shows that separate representability of f and g by $\mathsf{d}^\star$ and $\mathsf{d}_\star$, respectively, is not sufficient for the existence of a sequence $\boldsymbol{x}$ satisfying (1).

Example 1.9. Let $X = [0,1]$, and define $f, g : X \to [0,1]$ by

$$f(\eta) := \begin{cases} 1/2 & \text{if } \eta = 0, \\ 1/4 & \text{if } \eta = 1/n \text{ for some } n \geq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(\eta) := \begin{cases} 1/2 & \text{if } \eta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \geq g$, and both functions are upper semicontinuous. By Theorem 1.4, f is represented by $\mathsf{d}^\star$. Moreover, $\sum g = 1/2 \leq 1$, so Theorem 1.7 shows that g is represented by $\mathsf{d}_\star$. On the other hand,

$$(f - g)(0) = 0, \quad \text{and} \quad (f - g)(1/n) = 1/4 \ \text{ for all } n \geq 1.$$

Observe that $f - g$ is not upper semicontinuous. Therefore, by Theorem 1.8, there is no sequence $\boldsymbol{x}$ in X satisfying (1).

Remark 1.10. If X is an uncountable compact metric space without isolated points, then every continuous function $g : X \to [0,1]$ represented by $\mathsf{d}_\star$ is identically zero. Indeed, otherwise continuity would imply $\sum g = \infty$, contradicting Theorem 1.7.

Hence, if $f, g : X \to [0,1]$ are continuous, $f \geq g$, and they are separately represented by $\mathsf{d}^\star$ and $\mathsf{d}_\star$, respectively, then they are also jointly representable by the same sequence, thanks to Theorem 1.8 and the above observation.

Finally, we characterize the functions represented by *probability measures*, that is, σ-additive probability charges. When X is countable and compact, these are precisely the functions represented by probability charges.

Theorem 1.11. *Let X be a metric space with $|X| \geq 2$, and pick a function $f : X \to \mathbb{R}$. Then the following are equivalent*:

(F1) *f is represented by some probability measure*;
(F2) $\mathrm{im}(f) \subseteq [0,1]$ *and* $\sum f = 1$.

If, in addition, X is compact and countable, then they are also equivalent to:

(F3) *f is represented by some probability charge*;
(F4) *f is represented by some finite-invariant probability charge.*

Example 1.12. The following examples distinguish some of the representation classes considered above.

(i) Let X be a finite metric space with at least two points. Then the zero function is represented by a normalized capacity, by Theorem 1.2, and by $\mathsf{d}_\star$, by Theorem 1.7. However, it is represented neither by $\mathsf{d}^\star$ nor by a normalized maxitive capacity, thanks to Theorems 1.4 and 1.3, respectively.

(ii) Let $X = [0,1]$ and $f := \mathbf{1}_{\{0,1\}}$. Then f is upper semicontinuous, hence represented by $\mathsf{d}^\star$ by Theorem 1.4. However, since $\sum f = 2 > 1$, it cannot be represented by any normalized superadditive capacity, by Theorem 1.7. In particular, it cannot be represented by $\mathsf{d}_\star$.

(iii) Let X be an uncountable compact metric space. Then Theorem 1.8, applied with $f = g \equiv 0$, yields a sequence $\boldsymbol{x}$ such that

$$\mathrm{DC}_{\mathsf{d}^\star,\boldsymbol{x}} = \mathrm{DC}_{\mathsf{d}_\star,\boldsymbol{x}} \equiv 0.$$

On the other hand, if X is a countable compact metric space, the zero function cannot be represented by $\mathsf{d}^\star$, thanks to Theorem 1.4.

Table 1 summarizes our main characterizations for functions $f : X \to [0,1]$, where X is a metric space containing at least two points. The first column specifies the representing normalized capacity, and the remaining three columns specify the type of space X. Here Usc means that f is upper semicontinuous, and Sep means that P_f is separable. Observe that every nonnegative function f with $\sum f < \infty$ is upper semicontinuous: for every $t > 0$, the superlevel set $\{\eta \in X : f(\eta) \geq t\}$ is finite and hence closed. Moreover, P_f is at most countable.

Normalized map μ	Countable compact	Uncountable compact	Noncompact
Capacity	Usc	Usc	$\mathsf{Usc}, \mathsf{Sep}$
Maxitive capacity	$\mathsf{Usc}, \max f = 1$	$\mathsf{Usc}, \max f = 1$	$\mathsf{Usc}, \mathsf{Sep}$
Submeasure	$\mathsf{Usc}, \sum f \geq 1$	Usc	$\mathsf{Usc}, \mathsf{Sep}$
Submodular capacity	$\mathsf{Usc}, \sum f \geq 1$	Usc	$\mathsf{Usc}, \mathsf{Sep}$
Abstract upper density	$\mathsf{Usc}, \sum f \geq 1$	Usc	$\mathsf{Usc}, \mathsf{Sep}$
Arithmetic upper density	$\mathsf{Usc}, \sum f \geq 1$	Usc	$\mathsf{Usc}, \mathsf{Sep}$
$\mathsf{d}^\star$	$\mathsf{Usc}, \sum f \geq 1$	Usc	$\mathsf{Usc}, \mathsf{Sep}$
σ-subadditive capacity	$\mathsf{Usc}, \sum f \geq 1$	$\mathsf{Usc}, \sum f \geq 1$	$\mathsf{Usc}, \mathsf{Sep}, \sum f \geq 1$
Superadditive capacity	$\sum f \leq 1$	$\sum f \leq 1$	$\sum f \leq 1$
Supermodular capacity	$\sum f \leq 1$	$\sum f \leq 1$	$\sum f \leq 1$
$\mathsf{d}_\star$	$\sum f \leq 1$	$\sum f \leq 1$	$\sum f \leq 1$
Probability charge	$\sum f = 1$	$\sum f \leq 1$	$\sum f \leq 1$
Probability measure	$\sum f = 1$	$\sum f = 1$	$\sum f = 1$

TABLE 1. Summary of the characterizations.

1.2. **Additional universal results.** Differently from the case of the "universal submeasure" $\mathsf{d}^\star$ in Theorem 1.4 and the case of the "universal superadditive capacity" $\mathsf{d}_\star$ in Theorem 1.7, there is no "universal probability measure" (or σ-subadditive capacity) for infinite metric spaces:

Proposition 1.13. *Let X be a metric space with $|X| \geq 2$. Then the following are equivalent*:

(G1) *X is finite*;
(G2) *there exists a normalized σ-subadditive capacity μ on $\mathbb{N}$ representing every function that can be represented by some normalized σ-subadditive capacity*;
(G3) *there exists a probability measure μ on $\mathbb{N}$ representing every function that can be represented by some probability measure.*

However, the situation is different in the case of probability charges. Recall that a *density measure* is a probability charge $\lambda : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ which extends the asymptotic density (that is, $\lambda(S) = \mathsf{d}^\star(S)$ for all $S \subseteq \mathbb{N}$ such that $\mathsf{d}^\star(S) = \mathsf{d}_\star(S)$). Examples of density measures include the maps $\lambda_{\mathcal{U}} : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ defined by

$$\forall S \subseteq \mathbb{N}, \qquad \lambda_{\mathcal{U}}(S) := \mathcal{U}\text{-}\lim_{N\to\infty} \frac{|S \cap [0, N)|}{N},$$

where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, see e.g. [4, 10]. As the next result shows, every density measure is universal for the representation of functions by probability charges.

Proposition 1.14. *Let X be a metric space with $|X| \geq 2$. Pick a function $f : X \to \mathbb{R}$ and a density measure $\lambda : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$. Then the following are equivalent*:
(H1) *f is represented by λ*;
(H2) *f is represented by some probability charge.*

Next, we study the symmetric results for "universal sequences $\boldsymbol{x}$." We start with the cases of normalized capacities, submeasures, and probability charges.

Proposition 1.15. *Let X be a metric space with $|X| \geq 2$.*
Then the following are equivalent:
(I1) *X is separable*;
(I2) *there exists a sequence $\boldsymbol{x}$ with values in X such that every function $f : X \to [0, 1]$ represented by some normalized capacity is represented by a normalized capacity using the same sequence $\boldsymbol{x}$*;
(I3) *there exists a sequence $\boldsymbol{x}$ with values in X such that every function $f : X \to [0, 1]$ represented by some normalized submeasure is represented by a normalized submeasure using the same sequence $\boldsymbol{x}$*;
(I4) *there exists a sequence $\boldsymbol{x}$ with values in X such that every function represented by some probability charge is represented by a probability charge using the same sequence $\boldsymbol{x}$;*
In such case, one can choose a sequence $\boldsymbol{x}$ if and only if it has dense range.

Thus, we continue with the analogous result for probability measures and normalized σ-subadditive capacities.

Proposition 1.16. *Let X be a metric space with $|X| \geq 2$.*
Then the following are equivalent:
(J1) *X is countable*;
(J2) *there exists a sequence $\boldsymbol{x}$ with values in X such that every function $f : X \to [0, 1]$ represented by some normalized σ-subadditive capacity is represented by a normalized σ-subadditive capacity using the same sequence $\boldsymbol{x}$*;

(J3) *there exists a sequence* $\boldsymbol{x}$ *with values in* X *such that every function* $f : X \to [0,1]$ *represented by some probability measure is represented by a probability measure using the same sequence* $\boldsymbol{x}$.

In such case, one can choose a sequence $\boldsymbol{x}$ *if and only if it is surjective.*

1.3. **Maldistribution.** A direction related to degrees of convergence concerns the study of maldistributed sequences, see [8, 12]: given a normalized capacity $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$, a sequence $\boldsymbol{x}$ taking values in a given topological space X is said to be *ν-maldistributed* if

$$\nu(\{n \in \mathbb{N} : x_n \in U\}) = 1 \qquad \text{for every nonempty open set } U \subseteq X.$$

In fact, if X is a metric space, this is equivalent to $\mathrm{DC}_{\nu,\boldsymbol{x}} \equiv 1$.

In [12], Mišík and Tóth introduced the family of ν-maldistributed sequences

$$\Sigma_\nu(X) := \{\boldsymbol{x} \in X^{\mathbb{N}} : \boldsymbol{x} \text{ is } \nu\text{-maldistributed}\},$$

Here, $X^{\mathbb{N}}$ is the topological space endowed with its product topology. Mišík and Tóth provided in [12, Theorem 3.1] a sufficient condition on ν to ensure that, from a topological viewpoint, almost every sequence is ν-maldistributed. More precisely, they proved that, if X is a separable metric space and ν is a normalized capacity such that $\nu(F) = 0$ and $\nu(\mathbb{N} \setminus F) = 1$ for all finite $F \subseteq \mathbb{N}$, then $\Sigma_\nu(X)$ is comeager (that is, it has meager complement in $X^{\mathbb{N}}$) whenever the following technical condition holds:

(W) For each $\alpha \in (0,1)$ there exists a map $g_\alpha : \mathbb{N} \to \mathbb{N}$ such that, for all $A \subseteq \mathbb{N}$ with $\nu(\mathbb{N} \setminus A) \le 1 - \alpha$, we have

$$A \cap [n, n + g_\alpha(n)] \neq \emptyset$$

for all sufficiently large n.

Then, they asked in [12, Open Problem 5.1] whether the converse holds, namely:

Question 2. Do there exist a separable metric space X and a normalized capacity ν with $\nu(F) = 0$ and $\nu(\mathbb{N} \setminus F) = 1$ for all finite $F \subseteq \mathbb{N}$ such that $\Sigma_\nu(X)$ is comeager and condition (W) fails?

First, we show that condition (W) can be slightly simplified (we do not require that $\nu(F) = 0$ and $\nu(\mathbb{N} \setminus F) = 1$ for all finite $F \subseteq \mathbb{N}$; cf. also Remark 2.6 below). Hereafter, $\mathcal{P}(\mathbb{N})$ is regarded as the Cantor space $\{0,1\}^{\mathbb{N}}$.

Proposition 1.17. *Let* $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ *be a normalized capacity. Then the following are equivalent*:

(W1) *condition (W) holds*;

(W2) *There exists a map* $g : \mathbb{N} \to \mathbb{N}$ *such that, for all* $A \subseteq \mathbb{N}$ *with* $\nu(\mathbb{N} \setminus A) < 1$, *we have* $A \cap [n, n + g(n)] \neq \emptyset$ *for all sufficiently large* n;

(W3) *There exists a partition* $\{F_k : k \in \mathbb{N}\}$ *of* $\mathbb{N}$ *into nonempty finite intervals such that* $\nu\left(\bigcup_{k \in S} F_k\right) = 1$ *for all infinite* $S \subseteq \mathbb{N}$;

(W4) *The family* $\mathscr{F}_\nu := \{S \subseteq \mathbb{N} : \nu(S) = 1\}$ *is comeager on* $\mathcal{P}(\mathbb{N})$.

Proposition 1.17 allows us to check easily whether suitable normalized capacities satisfy condition (W). As an application, we recover the equivalence (i) $\iff$ (ii) in [8,

Theorem 1.2]. To this end, recall that a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an ideal if it is stable under subsets and finite unions. In addition, we always assume that $\mathbb{N} \notin \mathcal{I}$ and that $F \in \mathcal{I}$ for all finite $F \subseteq \mathbb{N}$. Also, we set $\mathcal{I}^+ := \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$.

Corollary 1.18. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then $\mathcal{I}$ is meager if and only if $\mathbf{1}_{\mathcal{I}^+}$ satisfies condition (W).*

Second, we improve [12, Theorem 3.1] by showing that the analogous statement holds for all second countable spaces and all normalized capacities ν.

Proposition 1.19. *Let X be a second countable topological space, and let $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ be a normalized capacity. If condition (W) holds, then $\Sigma_\nu(X)$ is comeager.*

Together with Corollary 1.18, this immediately implies that, if $\mathcal{I}$ is a meager ideal on $\mathbb{N}$ and X a second countable topological space, then $\Sigma_{\mathbf{1}_{\mathcal{I}^+}}(X)$ is comeager in $X^{\mathbb{N}}$.

Now, recall that a topological space X is a *Baire space* if every countable intersection of open dense subsets is dense. Examples of Baire spaces include the complete metrizable spaces and the locally compact Hausdorff spaces, see e.g. [7, Theorem 8.4]. In addition, X is *meager in itself* (or simply meager) if X can be written as a countable union of nowhere dense subsets. Lastly, X is *Polish* it is separable and completely metrizable. It is known that, if X is second countable, then

$$X \text{ Polish} \implies X \text{ Baire} \iff X^{\mathbb{N}} \text{ not meager} \implies X \text{ not meager}$$

As anticipated, the first implication $\implies$ follows by the Baire category theorem [7, Theorem 8.4]; this cannot be reversed (choose e.g. a non-Borel comeager subset of $\mathbb{R}$ with its relative topology, which is a Baire separable metric space that is not completely metrizable, since a Polish subspace of $\mathbb{R}$ is necessarily G_δ). The equivalence $\iff$ follows by [7, Proposition 8.1] and [14, Theorem 3], while the last implication $\implies$ is obtained by [7, Lemma 8.43] and the homeomorphism $X^{\mathbb{N}} \cong X \times X^{\mathbb{N}}$. This last implication cannot be reversed (pick for instance $X = (\mathbb{Q} \cap (0,1)) \cup \{2\}$ with the relative Euclidean topology, which is a separable metric space, not Baire and not meager in itself).

In particular, if X is not a Baire space then every subset of $X^{\mathbb{N}}$ would be meager and comeager. In particular, if $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then $X := \mathbb{Q}$ is a separable metric space which is not Baire, $\nu := \mathbf{1}_{\mathcal{U}}$ does not satisfy condition (W) (by Corollary 1.18 since $\mathcal{U}$ is not meager), and $\Sigma_\nu(X)$ is comeager. This provides a (rather trivial) affirmative answer to Question 2.

However, we show that this is essentially the unique counterexample for equivalence between the comeagerness of $\Sigma_\nu(X)$ and the condition (W).

Theorem 1.20. *Let X be a second countable Baire space which admits two disjoint nonempty open sets, and let $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ be a normalized capacity.*

Then $\Sigma_\nu(X)$ is comeager in $X^{\mathbb{N}}$ if and only if condition (W) holds.

In particular, Theorem 1.20 holds for Baire separable metric spaces (and we do not require that $\nu(F) = 0$ and $\nu(\mathbb{N} \setminus F) = 1$ for all finite $F \subseteq \mathbb{N}$). This provides a negative answer to Question 2 in the remaining case of Baire spaces.

1.4. **Concluding remarks.** It is worth noting that the degree of convergence appeared in disguise in the literature on ideal convergence. More precisely, suppose that $\varphi : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a lower semicontinuous submeasure (that is, a submeasure such that $\varphi(A) = \lim_n \varphi(A \cap [0,n])$ for all $A \subseteq \mathbb{N}$ and $\varphi(\{n\}) < \infty$ for all $n \in \mathbb{N}$). Assume that $M := \lim_n \varphi(\mathbb{N} \setminus [0,n]) \in (0,\infty)$, and define the finite-invariant normalized submeasure $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ by

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \frac{1}{M} \lim_{n\to\infty} \varphi(A \setminus [0,n]).$$

Thus $\mathcal{I} := \mathrm{Exh}(\varphi) = \{A \subseteq \mathbb{N} : \mu(A) = 0\}$ is an ideal on $\mathbb{N}$ (that is, a proper family stable under finite unions and subsets, and containing the finite sets). Taking into account [2, Theorem 2.2], it follows that, if $\boldsymbol{x}$ is a sequence taking values in a metric space X, then

$$\{\eta \in X : \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) > 0\} = \{\eta \in X : \exists A \subseteq \mathbb{N}, A \notin \mathcal{I} \text{ and } \lim_{\substack{n\to\infty,\\ n\in A}} x_n = \eta\}.$$

In other words, the zero set of $\mathrm{DC}_{\mu,\boldsymbol{x}}$ coincides with the set of all $\eta \in X$ which are not $\mathcal{I}$-limit points of $\boldsymbol{x}$.

We leave as open question to establish a joint representation as in Theorem 1.8 for metric space which are not necessarily uncountable and compact.

2. Proofs of Main Results

Throughout, $B(\eta,\varepsilon)$ denotes the open ball with center $\eta \in X$ and radius $\varepsilon > 0$, and $\overline{B(\eta,\varepsilon)}$ denotes its closure in the metric space X. In addition, a probability measure is nonatomic if it assigns measure 0 to all singletons.

Proof of Theorem 1.2. (A1) $\implies$ (A3). Suppose that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$ for some normalized capacity μ and some sequence $\boldsymbol{x} \in X^{\mathbb{N}}$. By monotonicity, we have $0 = \mu(\emptyset) \le \mu(A) \le \mu(\mathbb{N}) = 1$ for all $A \subseteq \mathbb{N}$, hence $\mathrm{im}(f) \subseteq [0,1]$.

To prove upper semicontinuity, fix $t \in \mathbb{R}$ and $\eta \in X$ such that $f(\eta) < t$. Then there exists $\varepsilon > 0$ such that $\mu\big(\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}\big) < t$. For all $\xi \in X$ with $d(\xi,\eta) < \varepsilon$, setting $r := \varepsilon - d(\xi,\eta) > 0$, we obtain

$$f(\xi) \le \mu\big(\{n \in \mathbb{N} : d(x_n,\xi) < r\}\big) \le \mu\big(\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}\big) < t.$$

Thus $\{\xi \in X : f(\xi) < t\}$ is open, proving the upper semicontinuity of f.

Moreover,

$$P_f \subseteq \overline{\{x_n : n \in \mathbb{N}\}}.$$

Indeed, every point outside this closure has a neighborhood containing no terms of $\boldsymbol{x}$, and hence has degree of convergence zero. The closure on the right is a separable metric space, hence second countable. Therefore P_f is separable.

(A3) $\implies$ (A2). Suppose that f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, and P_f is separable. For every $q \in \mathbb{Q} \cap (0,1]$, choose a countable dense subset

$$D_q \subseteq \{\eta \in X : f(\eta) \ge q\}.$$

Such a choice is possible because every closed subset of a separable metric space is separable; when the set on the right is empty, choose $D_q = \emptyset$.

Fix distinct points $a, b \in X$ (which is possible since $|X| \geq 2$), and choose a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ whose range is

$$\{a, b\} \cup \bigcup_{q\in\mathbb{Q}\cap(0,1]} D_q,$$

so that every point of this range occurs infinitely often. Define

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \begin{cases} 0 & \text{if } A \text{ is finite,} \\ 1 & \text{if } A \text{ is cofinite,} \\ \limsup\limits_{\substack{n\to\infty \\ n\in A}} f(x_n) & \text{otherwise.} \end{cases}$$

Then it is routine to check that μ takes values in $[0, 1]$, satisfies $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = 1$, and is monotone and finite-invariant. To complete the proof, we need to show that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$.

To this end, fix $\eta \in X$. Suppose that $f(\eta) > 0$, and pick $q \in \mathbb{Q} \cap (0, f(\eta))$. For every $\varepsilon > 0$, the denseness of D_q in $\{\xi \in X : f(\xi) \geq q\}$ gives a point $\xi \in D_q$ such that $d(\xi, \eta) < \varepsilon$. Since ξ occurs infinitely often in $\boldsymbol{x}$ and $f(\xi) \geq q$, the definition of μ yields

$$\mu\big(\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}\big) \geq q.$$

Taking the infimum over $\varepsilon > 0$ and then letting $q \uparrow f(\eta)$, we obtain $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \geq f(\eta)$. The same inequality is trivially satisfied when $f(\eta) = 0$.

For the reverse inequality, fix $\delta > 0$ and choose $c \in \{a, b\} \setminus \{\eta\}$. By upper semicontinuity of f, there exists $\varepsilon \in (0, d(\eta, c))$ such that

$$f(\xi) < f(\eta) + \delta \qquad \text{whenever } d(\xi, \eta) < \varepsilon.$$

Since c occurs infinitely often in the sequence $\boldsymbol{x}$, the set $\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}$ is not cofinite. If it is finite, its μ-value is zero. Otherwise, its μ-value is the limsup of values $f(x_n)$ bounded above by $f(\eta) + \delta$. In either case,

$$\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \leq \mu\big(\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}\big) \leq f(\eta) + \delta.$$

Letting $\delta \downarrow 0$ proves that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \leq f(\eta)$. Therefore $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$.

(A2) $\implies$ (A1). This is immediate.

The above implications show that (A1) $\iff$ (A2) $\iff$ (A3). At this point, suppose that X is not compact. It is clear that (A9) $\implies$ (A8) $\implies$ (A6) $\implies$ (A4) $\implies$ (A1) and also (A9) $\implies$ (A7) $\implies$ (A5) $\implies$ (A2). Hence, to complete the proof, it is enough to show the following implication.

(A3) $\implies$ (A9). Suppose that X is not compact, and that f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0, 1]$, and P_f is separable. Since X is a noncompact metric space, there exists a sequence $\boldsymbol{y} = (y_n)_{n\geq 0}$ in X which has no convergent subsequences. In particular, for every $\eta \in X$ there exists $\delta_\eta > 0$ such that

$$|\{n \in \mathbb{N} : d(y_n, \eta) < \delta_\eta\}| < \infty. \tag{2}$$

Indeed, otherwise there would exist $\eta \in X$ such that, for every $k \geq 1$, infinitely many terms of $\boldsymbol{y}$ belong to the open ball $B(\eta, 1/k)$. It would then be possible to choose a

strictly increasing sequence (n_k) such that $d(y_{n_k}, \eta) < 1/k$ for all k, contradicting the choice of $\boldsymbol{y}$.

For every $q \in \mathbb{Q} \cap (0,1]$, consider the set

$$F_q := \{\eta \in X : f(\eta) \geq q\},$$

which is closed since f is upper semicontinuous. Moreover, $F_q \subseteq P_f$, hence F_q is separable. Thus, we can choose a countable dense subset G_q of F_q, with $G_q = \emptyset$ whenever $F_q = \emptyset$. Now, define the countable set

$$G := \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_q, \tag{3}$$

and choose a sequence $\boldsymbol{z} = (z_n)_{n \geq 0}$ in X such that every point of G occurs infinitely often in $\boldsymbol{z}$. If $G = \emptyset$, we take $\boldsymbol{z}$ to be any constant sequence.

Define the sequence $\boldsymbol{x} = (x_n)_{n \geq 0}$ by setting

$$\forall n \in \mathbb{N}, \qquad x_{2n} := z_n \quad \text{and} \quad x_{2n+1} := y_n.$$

At this point, set $E := \{2n+1 : n \in \mathbb{N}\}$. For every $A \subseteq \mathbb{N}$, define

$$\lambda(A) := \inf_{m \in \mathbb{N}} \sup\{f(x_n) : n \in A,\ n \geq m\},$$

where $\sup \emptyset := 0$. Note that $\lambda(A) = 0$ whenever A is finite, and that λ is finite-invariant. Finally, define $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ by

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \begin{cases} \lambda(A) & \text{if } |A \cap E| < \infty, \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

We claim that μ is a finite-invariant normalized maxitive capacity. Of course, $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = 1$. Moreover, μ is monotone by construction. To prove maxitivity, fix $A, B \subseteq \mathbb{N}$. Observe, by the definition of λ, that $\lambda(A \cup B) = \max\{\lambda(A), \lambda(B)\}$. Also, $|(A \cup B) \cap E| = \infty$ if and only if at least one of $A \cap E$ and $B \cap E$ is infinite. Hence $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$. Furthermore, if $A \triangle B$ is finite, then $\lambda(A) = \lambda(B)$ and $|A \cap E| = \infty$ if and only if $|B \cap E| = \infty$. Therefore $\mu(A) = \mu(B)$, proving that μ is finite-invariant.

It remains to show that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$. To this end, fix $\eta \in X$ and put

$$\forall \varepsilon > 0, \qquad A_\varepsilon := \{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}.$$

First, we prove that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \geq f(\eta)$. The inequality is trivial if $f(\eta) = 0$, so suppose that $f(\eta) > 0$ and choose $q \in \mathbb{Q} \cap (0, f(\eta))$. Since $\eta \in F_q = \overline{G_q}$, for every $\varepsilon > 0$ there exists $\xi \in G_q$ such that $d(\xi, \eta) < \varepsilon$. By the construction of $\boldsymbol{z}$, the point ξ occurs infinitely often among the even-indexed terms of $\boldsymbol{x}$. Hence A_ε contains infinitely many indices n such that $f(x_n) = f(\xi) \geq q$. It follows that $\lambda(A_\varepsilon) \geq q$, and hence $\mu(A_\varepsilon) \geq q$. Taking the infimum over $\varepsilon > 0$, we obtain $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \geq q$. By the arbitrariness of q, we conclude that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \geq f(\eta)$.

For the reverse inequality, proceeding as in the argument above, fix $\delta > 0$. By the upper semicontinuity of f, there exists $\rho > 0$ such that $f(\xi) < f(\eta) + \delta$ whenever $d(\xi, \eta) < \rho$. Moreover, it is possible to pick $\delta_\eta > 0$ satisfying (2). Fix also $\varepsilon \in (0, \min\{\rho, \delta_\eta\})$.

Recalling that $x_{2n+1} = y_n$, we obtain $|A_\varepsilon \cap E| < \infty$. Therefore, by (4), $\mu(A_\varepsilon) = \lambda(A_\varepsilon)$. On the other hand, by the choice of ε, every $n \in A_\varepsilon$ satisfies $f(x_n) < f(\eta) + \delta$. Hence $\lambda(A_\varepsilon) \leq f(\eta) + \delta$, so that

$$\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \leq \mu(A_\varepsilon) = \lambda(A_\varepsilon) \leq f(\eta) + \delta.$$

Letting $\delta \downarrow 0$, we conclude that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \leq f(\eta)$. Therefore $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = f(\eta)$. □

Proof of Theorem 1.3. (B1) $\implies$ (B3). Suppose that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$ for some normalized maxitive capacity μ and some sequence $\boldsymbol{x}$ with values in X. By Theorem 1.2, the function f is upper semicontinuous and $\mathrm{im}(f) \subseteq [0,1]$. Since X is compact and f is upper semicontinuous, it follows by Weierstrass' theorem that there exists $\eta_0 \in X$ such that

$$f(\eta_0) = \max f.$$

We claim that $\max f = 1$. Indeed, suppose for the sake of contradiction that $f(\eta) < 1$ for every $\eta \in X$. By the definition of degree of convergence, for every $\eta \in X$ there exists $\varepsilon_\eta > 0$ such that $\mu(\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon_\eta\}) < 1$. Since X is compact, there exist finitely many points $\eta_1, \ldots, \eta_k \in X$ such that

$$X = \bigcup_{i=1}^{k} B(\eta_i, \varepsilon_{\eta_i}).$$

For every $i \in \{1, \ldots, k\}$, set $A_i := \{n \in \mathbb{N} : d(x_n, \eta_i) < \varepsilon_{\eta_i}\}$. Then $\bigcup_{i=1}^{k} A_i = \mathbb{N}$. By the normalization and maxitivity of μ, we obtain

$$1 = \mu(\mathbb{N}) = \mu\left(\bigcup_{i=1}^{k} A_i\right) = \max_{1 \leq i \leq k} \mu(A_i) < 1,$$

which is the claimed contradiction. Therefore $\max f = 1$.

(B3) $\implies$ (B2). Suppose that f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, and $\max f = 1$. Fix $\eta_0 \in X$ such that $f(\eta_0) = 1$. We proceed exactly as in the proof of the implication (A3) $\implies$ (A9) of Theorem 1.2: with the same definitions of F_q and G_q, replace the definition of G in (3) with

$$G := \{\eta_0\} \cup \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_q. \tag{5}$$

Choose a sequence $\boldsymbol{x} = (x_n)_{n \geq 0}$ in X whose range is G, so that every point of G occurs infinitely often in $\boldsymbol{x}$, and consider the finite-invariant normalized maxitive capacity $\mu := \lambda$. Using the same argument, we conclude that $f = \mathrm{DC}_{\lambda,\boldsymbol{x}}$.

(B2) $\implies$ (B1). This is immediate. □

Next, we isolate some intermediate results which will be useful in the proof of Theorem 1.4.

Lemma 2.1. *Let $(\nu_j)_{j\geq 1}$ be a sequence of finitely supported probability measures on a given nonempty set X. Then there exists a sequence $\boldsymbol{x}$ with values in X such that*

$$\forall A \subseteq X, \qquad \mathsf{d}^\star(\{n : x_n \in A\}) = \limsup_{j\to\infty} \nu_j(A). \tag{6}$$

Proof. For each $j \geq 1$, construct a nonempty finite X-valued string

$$w_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,\ell_j})$$

such that its empirical probability measure q_j (defined by $q_j(A) := |\{t \in [1,\ell_j] : x_{j,t} \in A\}|/\ell_j$ for each $A \subseteq X$) satisfies

$$\sup_{A\subseteq X} |q_j(A) - \nu_j(A)| \leq \frac{1}{j}.$$

Set $s_0 := 0$, and choose recursively positive multiples b_j of ℓ_j such that, setting $s_j := s_{j-1} + b_j$, we have

$$\forall j \geq 1, \qquad \max\left\{\frac{s_{j-1}}{b_j}, \frac{\ell_{j+1}}{s_j}\right\} \leq \frac{1}{j}.$$

At this point, construct the sequence $\boldsymbol{x} = (x_n : n \in \mathbb{N})$ by concatenating b_j/ℓ_j copies of the string w_j, for each $j = 1, 2, \ldots$, and write

$$\forall N \geq 1, \qquad e_N := \frac{1}{N}\sum_{n=0}^{N-1} \delta_{x_n}.$$

Observe that, at the end of the jth block, we have

$$\forall j \geq 2, \qquad e_{s_j} = \frac{s_{j-1}}{s_j} e_{s_{j-1}} + \frac{b_j}{s_j} q_j,$$

Since probability measures take values in $[0,1]$, for every $j \geq 2$ and $A \subseteq X$ we obtain

$$\begin{aligned} |e_{s_j}(A) - \nu_j(A)| &\leq \frac{s_{j-1}}{s_j}|e_{s_{j-1}}(A) - \nu_j(A)| + \frac{b_j}{s_j}|q_j(A) - \nu_j(A)| \\ &\leq \frac{s_{j-1}}{s_j} + \frac{1}{j} \leq \frac{2}{j}. \end{aligned}$$

For $j = 1$, we have $e_{s_1} = q_1$, so the same bound holds. Thus

$$\forall j \geq 1, \qquad \sup_{A\subseteq X} |e_{s_j}(A) - \nu_j(A)| \leq \frac{2}{j} \tag{7}$$

Now, let $N = s_{j-1} + r$, where $j \geq 2$ and $0 \leq r \leq b_j$. The first r terms of the jth block consist of copies of w_j and a remaining word of length less than ℓ_j. Hence

$$\left|e_N(A) - \frac{s_{j-1}}{N} e_{s_{j-1}}(A) - \frac{r}{N} q_j(A)\right| \leq \frac{\ell_j}{N} \leq \frac{\ell_j}{s_{j-1}} \leq \frac{1}{j-1}$$

for all $A \subseteq X$. This implies, for all $j \geq 2$ and $A \subseteq X$, that

$$\left|e_N(A) - \frac{s_{j-1}}{N} \nu_{j-1}(A) - \frac{r}{N} \nu_j(A)\right| \leq \frac{2}{j-1} + \frac{1}{j} + \frac{\ell_j}{s_{j-1}} \leq \frac{4}{j-1}. \tag{8}$$

Fix $A \subseteq X$. Since $s_j \to \infty$, the estimate (7) implies

$$\limsup_{N\to\infty} e_N(A) \geq \limsup_{j\to\infty} e_{s_j}(A) = \limsup_{j\to\infty} \nu_j(A).$$

Conversely, if $N = s_{j-1} + r$ with $0 \leq r \leq b_j$, the estimate (8) implies

$$e_N(A) \leq \max\{\nu_{j-1}(A), \nu_j(A)\} + \frac{4}{j-1},$$

so that

$$\limsup_{N\to\infty} e_N(A) \leq \limsup_{j\to\infty} \max\{\nu_{j-1}(A), \nu_j(A)\} = \limsup_{j\to\infty} \nu_j(A).$$

The conclusion follows by the fact that

$$e_N(A) = \frac{|\{n \in \{0, 1, \ldots, N-1\} : x_n \in A\}|}{N},$$

hence $\mathsf{d}^\star(\{n \in \mathbb{N} : x_n \in A\}) = \limsup_N e_N(A)$. This proves (6). □

Lemma 2.2. *Let X be a compact metric space, and let $\mathscr{C}$ be a nonempty compact convex subset of the space Δ of Borel probability measures on X, endowed with the weak topology. Then there exists a sequence $\boldsymbol{x}$ with values in X such that $\mathscr{C}$ is exactly the set of weak accumulation points of $(e_N : N \geq 1)$, where*

$$\forall N \geq 1, \qquad e_N := \frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n}.$$

Proof. Choose a sequence $(g_k)_{k\geq 1}$ dense in the closed unit ball of $C(X; \mathbb{R})$ for the uniform norm. The metric

$$\forall \alpha, \beta \in \Delta, \qquad D(\alpha, \beta) := \sum_{k\geq 1} \frac{1}{2^k} \left| \int_X g_k \, \mathrm{d}\alpha - \int_X g_k \, \mathrm{d}\beta \right|$$

metrizes weak convergence on Δ. It is upper bounded by 2 and satisfies

$$D(t\alpha + (1-t)\beta, t\alpha' + (1-t)\beta') \leq tD(\alpha, \alpha') + (1-t)D(\beta, \beta') \tag{9}$$

for all $t \in [0, 1]$ and $\alpha, \alpha', \beta, \beta' \in \Delta$.

For every $m \geq 1$, compactness of $(\mathscr{C}, D)$ implies that $\mathscr{C}$ is D-totally bounded, hence there exist finitely many measures $\nu_{m,1}, \ldots, \nu_{m,k_m} \in \mathscr{C}$ such that

$$\forall \nu \in \mathscr{C}, \qquad \min_{1\leq i\leq k_m} D(\nu, \nu_{m,i}) < \frac{1}{m}.$$

Concatenate these finite lists $(\nu_{m,1}, \ldots, \nu_{m,k_m})$, for $m = 1, 2, \ldots$, to obtain a sequence $(\nu_j)_{j\geq 1}$. Then every tail of this sequence is D-dense in $\mathscr{C}$, that is,

$$\forall \nu \in \mathscr{C}, \forall \varepsilon > 0, \forall J \geq 1, \exists j \geq J, \qquad D(\nu_j, \nu) < \varepsilon.$$

For each $j \geq 1$, construct a nonempty finite X-valued string $w_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,\ell_j})$ such that its empirical probability measure q_j (as in the previous proof) satisfies $D(q_j, \nu_j) \leq 1/j$. This is possible because finitely supported probability measures with rational values are weakly dense in Δ.

Define $(s_j)_{j\geq 0}$, together with $(b_j)_{j\geq 1}$, $(e_N)_{N\geq 1}$, and the sequence $\boldsymbol{x}$ as in the previous proof. Using the convexity estimate (9) and the upper bound $D \leq 2$, for every $j \geq 2$ we obtain

$$D(e_{s_j}, \nu_j) \leq \frac{s_{j-1}}{s_j} D(e_{s_{j-1}}, \nu_j) + \frac{b_j}{s_j} D(q_j, \nu_j) \leq 2\frac{s_{j-1}}{s_j} + \frac{1}{j} \leq \frac{3}{j}.$$

For $j = 1$, we have $e_{s_1} = q_1$, so the same bound holds. Thus

$$\forall j \geq 1, \qquad D(e_{s_j}, \nu_j) \leq \frac{3}{j}. \tag{10}$$

Now, let $N = s_{j-1} + r$, where $j \geq 2$ and $0 \leq r \leq b_j$. Write $r = k\ell_j + t$, where $k \geq 0$ and $0 \leq t < \ell_j$. As before, the first r terms of the jth block consist of k copies of w_j followed by its first t terms. Since $\|g_u\|_\infty \leq 1$ for every $u \geq 1$, we have

$$\left| \int_X g_u \, \mathrm{d}e_N - \frac{s_{j-1}}{N} \int_X g_u \, \mathrm{d}e_{s_{j-1}} - \frac{r}{N} \int_X g_u \, \mathrm{d}q_j \right| = \frac{1}{N} \left| \sum_{v=1}^{t} g_u(x_{j,v}) - t \int_X g_u \, \mathrm{d}q_j \right| \leq \frac{2t}{N}.$$

Dividing by 2^u and summing over all $u \geq 1$, it follows that

$$D\left(e_N, \frac{s_{j-1}}{N} e_{s_{j-1}} + \frac{r}{N} q_j\right) \leq \frac{2\ell_j}{N} \leq \frac{2\ell_j}{s_{j-1}} \leq \frac{2}{j-1}.$$

Together with (10), the triangle inequality and the convexity estimate (9) therefore yield

$$D\left(e_N, \frac{s_{j-1}}{N} \nu_{j-1} + \frac{r}{N} \nu_j\right) \leq \frac{3}{j-1} + \frac{1}{j} + \frac{2\ell_j}{s_{j-1}} \leq \frac{6}{j-1}. \tag{11}$$

Since $\mathscr{C}$ is convex, we have

$$\frac{s_{j-1}}{N} \nu_{j-1} + \frac{r}{N} \nu_j \in \mathscr{C}.$$

Hence (11) implies

$$\lim_{N\to\infty} \inf_{\lambda\in\mathscr{C}} D(e_N, \lambda) = 0.$$

Since $\mathscr{C}$ is compact (and hence closed in Δ), every weak accumulation point of $(e_N)_{N\geq 1}$ belongs to $\mathscr{C}$.

Conversely, fix $\nu \in \mathscr{C}$. Since every tail of $(\nu_j)_{j\geq 1}$ is D-dense in $\mathscr{C}$, choose strictly increasing indices $(j_m)_{m\geq 1}$ such that $D(\nu_{j_m}, \nu) \leq 1/m$ for all $m \geq 1$. By (10), we get

$$\forall m \geq 1, \qquad D(e_{s_{j_m}}, \nu) \leq D(e_{s_{j_m}}, \nu_{j_m}) + D(\nu_{j_m}, \nu) \leq \frac{3}{j_m} + \frac{1}{m} \leq \frac{4}{m}.$$

Thus $(e_{s_{j_m}})$ converges weakly to ν. This completes the proof. □

The following proposition is closely related to the theory of random closed sets. Indeed, the function defined in (12) is the coverage function associated with a random closed set, a notion studied extensively in stochastic geometry and random-set theory; see [1, 13]. Our interest here is in representing such functions as degrees of convergence of deterministic sequences.

Proposition 2.3. *Let X be a compact metric space. Pick another compact metric space Ω, a Borel probability measure ρ on Ω, and a closed set $R \subseteq \Omega \times X$ such that the section $R_\omega := \{\eta \in X : (\omega, \eta) \in R\}$ is nonempty for each $\omega \in \Omega$.*

Then the function $f : X \to \mathbb{R}$ defined by

$$\forall \eta \in X, \qquad f(\eta) := \rho(\{\omega \in \Omega : \eta \in R_\omega\}) \tag{12}$$

satisfies both items (C4) and (C6).

Proof. We start proving that f satisfies item (C4). Since X is a compact metric space, then it is a Polish space. Choose a countable dense subset of X and a sequence $\boldsymbol{x}$ in which every point of this subset occurs infinitely often. For $A \subseteq \mathbb{N}$, denote by

$$C(A) := \bigcap_{m \in \mathbb{N}} \overline{\{x_n : n \in A,\ n \geq m\}}$$

the set of accumulation points of $\boldsymbol{x}$ along A. For each $A \subseteq \mathbb{N}$ and $\eta \in X$, define

$$E_A := \{\omega \in \Omega : R_\omega \cap C(A) \neq \emptyset\} \qquad \text{and} \qquad E_\eta := \{\omega \in \Omega : \eta \in R_\omega\}.$$

Note that E_A is compact, since it is the projection onto Ω of the compact set $R \cap (\Omega \times C(A))$. In particular, each E_A is Borel, hence it is possible to define

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \rho(E_A).$$

Observe that $E_{A \cup B} = E_A \cup E_B$ and $E_{A \cap B} \subseteq E_A \cap E_B$ for all $A, B \subseteq \mathbb{N}$, hence

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &\leq \rho(E_A \cup E_B) + \rho(E_A \cap E_B) \\ &= \rho(E_A) + \rho(E_B) = \mu(A) + \mu(B). \end{aligned}$$

Observe also that $C(\emptyset) = \emptyset$, $C(\mathbb{N}) = X$, and $C(A \cup B) = C(A) \cup C(B)$, while $C(A \cap B) \subseteq C(A) \cap C(B)$. Thus, μ is a normalized monotone submodular capacity. It is also finite-invariant, since $C(A) = C(B)$ whenever $A \triangle B$ is finite.

Moreover, for every nonempty open set $U \subseteq X$, the choice of $\boldsymbol{x}$ implies

$$C(\{n \in \mathbb{N} : x_n \in U\}) = \overline{U}.$$

Fix $\eta \in X$. For every $k \geq 1$, define

$$F_k := \overline{B(\eta, 1/k)} \qquad \text{and} \qquad E_{\eta,k} := \{\omega \in \Omega : R_\omega \cap F_k \neq \emptyset\}.$$

Each $E_{\eta,k}$ is compact, being the projection onto Ω of $R \cap (\Omega \times F_k)$. The sets $E_{\eta,k}$ are decreasing and satisfy $\bigcap_k E_{\eta,k} = E_\eta$. Indeed, if ω belongs to every $E_{\eta,k}$, choose $\eta_k \in R_\omega \cap F_k$. Then $\eta_k \to \eta$, and closedness of R_ω implies $\eta \in R_\omega$. The converse inclusion is immediate.

Hence, continuity from above of ρ gives

$$\begin{aligned} \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) &= \inf_{k \geq 1} \mu(\{n \in \mathbb{N} : d(x_n, \eta) < 1/k\}) \\ &= \inf_{k \geq 1} \rho(E_{\eta,k}) = \rho(E_\eta) = f(\eta), \end{aligned}$$

where the last equality follows from (12). Thus item (C4) holds.

Next, we prove item (C6). Let $\mathscr{J}$ be the family of Borel probability measures π on $\Omega \times X$ whose first marginal is ρ and which are concentrated on R (namely, $\pi(R) = 1$). Let $\mathscr{C}$ be the family of their second marginals, that is,

$$\mathscr{C} := \left\{ \nu : \begin{array}{c} \text{there exists } \pi \in \mathscr{J} \text{ such that} \\ \nu(B) = \pi(\Omega \times B) \text{ for every Borel set } B \subseteq X \end{array} \right\}.$$

Since $\Omega \times X$ is compact, the family of all Borel probability measures on $\Omega \times X$ is uniformly tight. By Prokhorov's theorem [3, Theorem 5.1], every sequence in this family has a weakly convergent subsequence. Since the weak topology is metrizable [3, Theorem 6.8], this space of probability measures is weakly compact. The requirement that the first marginal be ρ is preserved under weak convergence. The requirement $\pi(R) = 1$ is also preserved under weak convergence, by closedness of R and the Portmanteau theorem [3, Theorem 2.1]. Thus $\mathscr{J}$ is compact. It is also convex. Since the map taking a probability measure to its second marginal is continuous and affine, $\mathscr{C}$ is compact and convex.

CLAIM 1. *$\mathscr{J}$ and $\mathscr{C}$ are nonempty.*

Proof. Equip $\Omega \times X$ with the maximum product metric. For every $j \geq 1$, choose a finite Borel partition $\{H_{j,i} : i = 1, \dots, m_j\}$ of Ω into nonempty sets of diameter at most $1/j$. For each $i \leq m_j$, pick some $\omega_{j,i} \in H_{j,i}$ and $\xi_{j,i} \in R_{\omega_{j,i}}$, and define $s_j : \Omega \to X$ by

$$\forall \omega \in H_{j,i}, \qquad s_j(\omega) := \xi_{j,i}.$$

Then s_j is Borel measurable. Moreover, for each $\omega \in H_{j,i}$, the point $(\omega_{j,i}, \xi_{j,i})$ belongs to R and is at distance at most $1/j$ from $(\omega, s_j(\omega))$, so that $\operatorname{dist}((\omega, s_j(\omega)), R) \leq 1/j$ for all $\omega \in \Omega$. Define a Borel probability measure π_j on $\Omega \times X$ by

$$\pi_j(H) := \rho\big(\{\omega \in \Omega : (\omega, s_j(\omega)) \in H\}\big)$$

for every Borel set $H \subseteq \Omega \times X$. Observe that its first marginal is ρ since $\pi_j(A \times X) = \rho(A)$ for every Borel set $A \subseteq \Omega$. By compactness, there exists a subsequence of (π_j) which converges weakly to a Borel probability measure π. Its first marginal is again ρ. In addition, the function $z \mapsto \operatorname{dist}(z, R)$ is continuous on $\Omega \times X$, and

$$0 \leq \int_{\Omega \times X} \operatorname{dist}(z, R) \, \mathrm{d}\pi_j(z) \leq \frac{1}{j}.$$

Hence, passing to the weak limit, it implies

$$\int_{\Omega \times X} \operatorname{dist}(z, R) \, \mathrm{d}\pi(z) = 0.$$

Since R is closed, we obtain $\pi(R) = 1$. Thus $\pi \in \mathscr{J}$, proving that both $\mathscr{J}$ and $\mathscr{C}$ are nonempty. □

CLAIM 2. *For each $\eta \in X$, we have $\sup_{\nu \in \mathscr{C}} \nu(\{\eta\}) = \rho(E_\eta) = f(\eta)$.*

Proof. Fix $\eta \in X$. The set E_η is closed, since R is closed. For every $\pi \in \mathscr{J}$, we have

$$\pi(\Omega \times \{\eta\}) = \pi(R \cap (\Omega \times \{\eta\})) \leq \pi(E_\eta \times X) = \rho(E_\eta).$$

Taking second marginals, we obtain

$$\sup_{\nu\in\mathscr{C}} \nu(\{\eta\}) \le \rho(E_\eta).$$

For the reverse inequality, modify the maps s_j constructed in the proof of Claim 1 by setting

$$\forall j \ge 1, \forall \omega \in \Omega, \qquad s_j^\eta(\omega) := \begin{cases} \eta & \text{if } \omega \in E_\eta, \\ s_j(\omega) & \text{if } \omega \notin E_\eta. \end{cases}$$

These maps s_j^η are Borel measurable, and $\operatorname{dist}((\omega, s_j^\eta(\omega)), R) \le 1/j$ for all $\omega \in \Omega$. Indeed, on E_η we have $(\omega, s_j^\eta(\omega)) = (\omega, \eta) \in R$.

For each $j \ge 1$, define a Borel probability measure π_j^η on $\Omega \times X$ by

$$\pi_j^\eta(H) := \rho\big(\{\omega \in \Omega : (\omega, s_j^\eta(\omega)) \in H\}\big)$$

for every Borel set $H \subseteq \Omega \times X$. Choose a weakly convergent subsequence $(\pi_{j_k}^\eta)$, and denote its limit by π^η. The same argument as in the proof of Claim 1 shows that $\pi^\eta \in \mathscr{J}$. Moreover, $\pi_j^\eta(\Omega \times \{\eta\}) \ge \rho(E_\eta)$ for all $j \ge 1$. Since $\Omega \times \{\eta\}$ is closed, by the Portmanteau theorem [3, Theorem 2.1] we obtain

$$\rho(E_\eta) \le \limsup_{k\to\infty} \pi_{j_k}^\eta(\Omega \times \{\eta\}) \le \pi^\eta(\Omega \times \{\eta\}).$$

The second marginal of π^η belongs to $\mathscr{C}$. This proves the reverse inequality. □

Thanks to Lemma 2.2, it is possible to choose a sequence $\boldsymbol{y} = (y_n)_{n\ge0}$ with values in X whose empirical measures $(e_N : N \ge 1)$ defined by $e_N := \frac{1}{N}\sum_{n=0}^{N-1} \delta_{y_n}$ have precisely $\mathscr{C}$ as their set of weak accumulation points. Fix $\eta \in X$ and $\varepsilon > 0$, and define

$$A_\varepsilon := \{n \in \mathbb{N} : d(y_n, \eta) < \varepsilon\}.$$

CLAIM 3. $\sup_{\nu\in\mathscr{C}} \nu(\{\eta\}) \le \mathsf{d}^\star(A_\varepsilon) \le \sup_{\nu\in\mathscr{C}} \nu(\overline{B(\eta,\varepsilon)})$.

Proof. For every $\nu \in \mathscr{C}$, choose a subsequence (e_{N_k}) converging weakly to ν. Since $B(\eta,\varepsilon)$ is open, the Portmanteau theorem [3, Theorem 2.1] gives

$$\nu(\{\eta\}) \le \nu(B(\eta,\varepsilon)) \le \liminf_{k\to\infty} e_{N_k}(B(\eta,\varepsilon)) \le \mathsf{d}^\star(A_\varepsilon).$$

Taking the supremum over $\nu \in \mathscr{C}$ proves the first inequality.

For the second inequality, choose a subsequence (N_k) such that $\lim_k e_{N_k}(B(\eta,\varepsilon)) = \mathsf{d}^\star(A_\varepsilon)$. By weak compactness of the space of Borel probability measures on X, it is possible to choose further subsequence along which e_{N_k} converges weakly to some $\nu \in \mathscr{C}$. The Portmanteau theorem [3, Theorem 2.1] then gives

$$\mathsf{d}^\star(A_\varepsilon) \le \limsup_{k\to\infty} e_{N_k}(\overline{B(\eta,\varepsilon)}) \le \nu(\overline{B(\eta,\varepsilon)}) \le \sup_{\lambda\in\mathscr{C}} \lambda(\overline{B(\eta,\varepsilon)}),$$

which completes the proof. □

CLAIM 4. $\inf_{\varepsilon>0} \sup_{\nu\in\mathscr{C}} \nu(\overline{B(\eta,\varepsilon)}) = \sup_{\nu\in\mathscr{C}} \nu(\{\eta\})$.

Proof. For every $k \geq 1$, define

$$F_k := \overline{B(\eta, 1/k)} \qquad \text{and} \qquad a_k := \max_{\nu \in \mathscr{C}} \nu(F_k).$$

The maximum exists because $\mathscr{C}$ is compact (and nonempty by Claim 1), and $\nu \mapsto \nu(F_k)$ is upper semicontinuous by the Portmanteau theorem [3, Theorem 2.1]. Choose $\nu_k \in \mathscr{C}$ with $\nu_k(F_k) = a_k$. The sequence (a_k) is nonincreasing, hence it is possible to define $a := \lim_k a_k$. By compactness of $\mathscr{C}$, it is possible to choose a subsequence (ν_{k_j}) converging weakly to some $\nu \in \mathscr{C}$.

Fix $m \geq 1$. Since $F_{k_j} \subseteq F_m$ whenever $k_j \geq m$, the Portmanteau theorem [3, Theorem 2.1] implies

$$a \leq \limsup_{j \to \infty} \nu_{k_j}(F_m) \leq \nu(F_m).$$

Since $F_m \downarrow \{\eta\}$, continuity from above yields

$$a \leq \nu(\{\eta\}) \leq \sup_{\lambda \in \mathscr{C}} \lambda(\{\eta\}).$$

Conversely, $\{\eta\} \subseteq F_k$ for every k, so

$$\forall k \geq 1, \quad \sup_{\lambda \in \mathscr{C}} \lambda(\{\eta\}) \leq a_k.$$

The reverse inequality follows by letting $k \to \infty$. Therefore we conclude observing that $\inf_{\varepsilon > 0} \sup_{\lambda \in \mathscr{C}} \lambda(\overline{B(\eta, \varepsilon)}) = \inf_{k \geq 1} a_k = a$. □

Taking the infimum over $\varepsilon > 0$ in Claim 3, and using Claim 2 and Claim 4, we conclude that

$$\mathrm{DC}_{\mathsf{d}^\star, \boldsymbol{y}}(\eta) = \inf_{\varepsilon > 0} \mathsf{d}^\star(A_\varepsilon) = \sup_{\nu \in \mathscr{C}} \nu(\{\eta\}) = f(\eta).$$

Since $\eta \in X$ was arbitrary, item (C6) holds. This completes the proof. □

We finally proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. Since $\mathsf{d}^\star$ is an arithmetic upper density, and every arithmetic upper density is a finite-invariant normalized submeasure, cf. [9, Proposition 1(ii) and Proposition 6], we have (C6) $\implies$ (C5) $\implies$ (C2) $\implies$ (C1). Moreover, since every submodular capacity is subadditive, we have (C4) $\implies$ (C3) $\implies$ (C1), and also (C4) $\implies$ (C2) $\implies$ (C1). Thus, to complete the proof we will show that item (C1) implies (C7′) or (C7″) (depending on the underlying set X), and that the latter ones imply both (C4) and (C6).

(C1) $\implies$ (C7″) *when X is noncompact or uncountable, and* (C1) $\implies$ (C7′) *when X is countable and compact.* Suppose that (C1) holds. Choose a normalized submeasure μ and a sequence $\boldsymbol{x}$ with values in X such that $f = \mathrm{DC}_{\mu, \boldsymbol{x}}$. By Theorem 1.2, the function f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0, 1]$, and P_f is separable. This proves (C7″) whenever X is noncompact or uncountable.

Assume now that X is countable and compact. Enumerate X as $\{\eta_j : j \in J\}$, for some finite or countably infinite J with $|J| \geq 2$. Fix also $\delta > 0$. For every $j \in J$, choose $\varepsilon_j > 0$ such that

$$\mu(\{n \in \mathbb{N} : d(x_n, \eta_j) < \varepsilon_j\}) < f(\eta_j) + \frac{\delta}{2^j}.$$

By compactness, there exists a finite family of open balls $\{B(\eta_j, \varepsilon_j) : j \in F\}$ whose union covers X. Thus, normalization and subadditivity yield

$$\begin{aligned} 1 = \mu(\mathbb{N}) &\leq \sum_{j \in F} \mu(\{n \in \mathbb{N} : d(x_n, \eta_j) < \varepsilon_j\}) \\ &\leq \sum_{j \in F} f(\eta_j) + \delta \sum_{j \in F} \frac{1}{2^j} < \sum f + 2\delta. \end{aligned}$$

Letting $\delta \downarrow 0$, we obtain that $\sum f \geq 1$. Thus (C7′) holds.

(C7″) $\implies$ (C4) *and* (C6) *when* X *is noncompact.* Suppose that (C7″) holds and X is noncompact. By Theorem 1.2, f is represented by a finite-invariant normalized maxitive capacity. Since every maxitive capacity is submodular, item (C4) holds.

To prove (C6), choose a sequence $(y_j)_{j \geq 1}$ with values in X having no convergent subsequence. For every $q \in \mathbb{Q} \cap (0, 1]$, define F_q and G_q as in the proof of the implication (A3) $\implies$ (A9), and $G := \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_q$. Similarly, pick a sequence $\boldsymbol{z} = (z_n)_{n \geq 0}$ with values in X and $\mathrm{im}(\boldsymbol{z}) = G$ such that every point of G occurs infinitely often in $\boldsymbol{z}$; if $G = \emptyset$, we take $\boldsymbol{z}$ to be any constant sequence. For every $j \geq 1$, define the finitely supported probability measure

$$\nu_j := f(z_j)\delta_{z_j} + (1 - f(z_j))\delta_{y_j}.$$

By Lemma 2.1, there exists a sequence $\boldsymbol{x}$ with values in X such that

$$\forall A \subseteq X, \qquad \mathsf{d}^\star(\{n \in \mathbb{N} : x_n \in A\}) = \limsup_{j \to \infty} \nu_j(A).$$

Fix $\eta \in X$. Since (y_j) has no convergent subsequence, there exists $r > 0$ such that $y_j \notin B(\eta, r)$ for all sufficiently large j. Indeed, otherwise one could choose strictly increasing indices (j_k) such that $d(y_{j_k}, \eta) < 1/k$. Hence, for every $\varepsilon \in (0, r)$,

$$\begin{aligned} \mathsf{d}^\star(\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}) &= \limsup_{j \to \infty} f(z_j)\mathbf{1}_{B(\eta,\varepsilon)}(z_j) \\ &= \sup\{f(\xi) : \xi \in G \cap B(\eta, \varepsilon)\}, \end{aligned}$$

where $\sup \emptyset := 0$. The last equality follows because every point of G occurs infinitely often in (z_j); when $G = \emptyset$, both sides are zero.

By upper semicontinuity of f,

$$\inf_{0 < \varepsilon < r} \sup\{f(\xi) : \xi \in G \cap B(\eta, \varepsilon)\} \leq f(\eta).$$

For the converse inequality, if $f(\eta) > 0$, fix $q \in \mathbb{Q} \cap (0, f(\eta))$. Then $\eta \in F_q$, so denseness of G_q in F_q implies $G_q \cap B(\eta, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Hence

$$\sup\{f(\xi) : \xi \in G \cap B(\eta, \varepsilon)\} \geq q.$$

Letting $q \uparrow f(\eta)$ gives the converse inequality. When $f(\eta) = 0$, it follows from nonnegativity. Therefore $\mathrm{DC}_{\mathtt{d}^\star,\boldsymbol{x}}(\eta) = f(\eta)$. Since $\eta \in X$ was arbitrary, item (C6) holds.

(C7″) $\implies$ (C4) *and* (C6) *when* X *is uncountable and compact.* Suppose that (C7″) holds and X is uncountable and compact. Then X is a Polish space, hence it contains a closed subset homeomorphic to the Cantor space, see e.g. [7, Corollary 6.5]. Transporting the standard nonatomic probability measure on the Cantor space to this subset and extending it by zero yields a nonatomic Borel probability measure P on X. Define also $M := \max f$: the maximum exists by compactness and upper semicontinuity, and $0 \le M \le 1$.

Let Ω be the topological disjoint union of $[0, M]$ and a copy of X. Observe that Ω is a compact metric space. Define a Borel probability measure ρ by taking Lebesgue measure on the first component and $(1 - M)P$ on the second component. Define also a relation $R \subseteq \Omega \times X$ through its sections by

$$\forall t \in [0, M], \forall z \in X, \qquad R_t := \{\eta \in X : f(\eta) \ge t\} \quad \text{and} \quad R_z := \{z\}.$$

Every section is nonempty, since f attains its maximum. On the first component, the relation is closed by upper semicontinuity of f. On the second component, it is the diagonal of $X \times X$, so it is closed there as well. Thus R is closed in $\Omega \times X$.

For every $\eta \in X$, we have

$$\rho(\{\omega \in \Omega : \eta \in R_\omega\}) = f(\eta) + (1 - M)P(\{\eta\}) = f(\eta)$$

because P is nonatomic. All the hypotheses of Proposition 2.3 are therefore satisfied. It follows that both (C4) and (C6) hold.

(C7′) $\implies$ (C4) *and* (C6) *when* X *is countable and compact.* Suppose that (C7′) holds and X is countable and compact. We again construct the objects required by Proposition 2.3.

If $\sum f = 1$, define a Borel probability measure P on X by

$$P(B) := \sum_{\eta \in B} f(\eta)$$

for all Borel sets $B \subseteq X$. Take $\Omega := X$, $\rho := P$, and let R be the diagonal of $X \times X$. Then R is closed, every section is a singleton, and

$$\forall \eta \in X, \qquad \rho(\{\omega \in \Omega : \eta \in R_\omega\}) = P(\{\eta\}) = f(\eta).$$

Thus Proposition 2.3 implies both (C4) and (C6).

Lastly, assume that $\sum f > 1$, and choose distinct points $\eta_1, \dots, \eta_k \in X$ such that $\sum_{i=1}^{k} f(\eta_i) > 1$. Set $t_1 := 0$ and

$$\forall i = 2, \dots, k, \qquad t_i := \sum_{j=1}^{i-1} f(\eta_j).$$

Since $\{\eta_1, \dots, \eta_k\}$ is closed in X, Tietze's extension theorem [7, Theorem 1.3] implies the existence of a continuous function $\theta : X \to \mathbb{R}$ such that $\theta(\eta_i) = t_i$ for every $i = 1, \dots, k$.

Let $\Omega := \mathbb{T} = \mathbb{R}/\mathbb{Z}$, endowed with normalized Lebesgue measure ρ, and define

$$R := \{(u, \eta) \in \mathbb{T} \times X : u \in \theta(\eta) + [0, f(\eta)] \pmod 1\}.$$

To see that R is closed, observe that

$$\{(\eta, t) \in X \times [0,1] : 0 \le t \le f(\eta)\}$$

is compact by upper semicontinuity of f. Its image under the continuous map $(\eta, t) \longmapsto (\theta(\eta) + t \pmod 1, \eta)$ is exactly R, so R is compact and therefore closed in $\mathbb{T} \times X$.

Every section R_u is nonempty. Indeed, the family of intervals $\{[t_i, t_i + f(\eta_i)] : i = 1, \dots, k\}$ is a cover of $[0, \sum_{i=1}^k f(\eta_i)]$. However, since the latter interval has length greater than 1, their images modulo 1 cover $\mathbb{T}$. Thus every $u \in \mathbb{T}$ belongs to the arc corresponding to at least one η_i.

Finally, for every $\eta \in X$, the arc $\theta(\eta) + [0, f(\eta)] \pmod 1$ has normalized Lebesgue measure $f(\eta)$, since $0 \le f(\eta) \le 1$. Hence

$$\rho(\{u \in \mathbb{T} : \eta \in R_u\}) = f(\eta).$$

Proposition 2.3 again yields both (C4) and (C6). □

Proof of Corollary 1.5. This follows from the equivalence (C6) $\iff$ (C7″) in Theorem 1.4, once we recall that P_f is separable if X is compact metrizable. □

Remark 2.4. The conclusion of Corollary 1.5 does not hold for every arithmetic upper density. Indeed, fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define

$$\mu^\star(A) := \mathcal{U}\text{-}\lim_{N\to\infty} \frac{1}{H_N} \sum_{n \in A \cap [1,N]} \frac{1}{n}, \qquad \text{where } H_N := \sum_{n\in[1,N]} \frac{1}{n}.$$

Then $\mu^\star$ is a well-defined finite-invariant normalized additive capacity. Moreover, for all $A \subseteq \mathbb{N}$ and $k, h \in \mathbb{N}$ with $k \neq 0$, we have $\mu^\star(\{kx + h : x \in A\}) = \mu^\star(A)/k$: indeed, the corresponding harmonic sums differ, up to the factor $1/k$, by a quantity bounded independently of N. Thus $\mu^\star$ is an additive arithmetic upper density.

Now let $X = [0,1]$ and $f := \mathbf{1}_{\{0,1\}}$. Then f is upper semicontinuous and hence, by Corollary 1.5, it is represented by $\mathsf{d}^\star$. However, we claim that it cannot be represented by $\mu^\star$. Indeed, suppose that $f = \mathrm{DC}_{\mu^\star, \boldsymbol{x}}$ for some sequence $\boldsymbol{x} = (x_n)_{n\ge0}$ in X. Define $A := \{n \in \mathbb{N} : x_n < 1/4\}$ and $B := \{n \in \mathbb{N} : x_n > 3/4\}$. Since $f(0) = f(1) = 1$, the definition of degree of convergence yields $\mu^\star(A) = \mu^\star(B) = 1$. On the other hand, $A \cap B = \emptyset$, so that by finite additivity and monotonicity,

$$2 = \mu^\star(A) + \mu^\star(B) = \mu^\star(A \cup B) \le \mu^\star(\mathbb{N}) = 1.$$

This provides the required contradiction.

Proof of Theorem 1.6. (D1) $\implies$ (D2). Suppose that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$ for some normalized σ-subadditive capacity μ. By Theorem 1.2, the function f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, and P_f is separable. For every η in the range of $\boldsymbol{x}$, put $A_\eta := \{n \in \mathbb{N} : x_n = \eta\}$. Since the sets A_η form a countable partition of $\mathbb{N}$ and $\mu(A_\eta) \le f(\eta)$, we obtain

$$1 = \mu(\mathbb{N}) \le \sum_{\eta \in \{x_n : n \in \mathbb{N}\}} \mu(A_\eta) \le \sum f.$$

(D2) $\implies$ (D1). Suppose that f is upper semicontinuous, $\mathrm{im}(f) \subseteq [0,1]$, P_f is separable, and $\sum f \ge 1$. By the definition of $\sum f$, there exist pairwise distinct points $(\eta_j)_{j\in J}$ and numbers $t_j \in [0, f(\eta_j)]$ such that $\sum_{j\in J} t_j = 1$, where J is finite or countably infinite. Let G be the countable set defined in (3), and choose a sequence $\boldsymbol{x}$ whose range contains $G \cup \{\eta_j : j \in J\}$, with every point of this set occurring infinitely often. For each $j \in J$, choose an index n_j such that $x_{n_j} = \eta_j$, and define

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \max\left\{ \sup_{n\in A} f(x_n), \sum_{\substack{j\in J \\ n_j \in A}} t_j \right\},$$

with the convention $\sup \emptyset := 0$. Then μ is a normalized capacity. It is also σ-subadditive, since both maps appearing in the maximum are σ-subadditive.

Fix $\eta \in X$ and put $A_\varepsilon := \{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}$. The upper semicontinuity of f and the denseness of G_q in F_q for every rational $q \in (0, f(\eta))$ imply that $\inf_{\varepsilon>0} \sup_{n\in A_\varepsilon} f(x_n) = f(\eta)$, as in the proof of Theorem 1.2. Let $P := \sum_{j\in J} t_j \delta_{\eta_j}$, which is a probability measure. By continuity from above,

$$\inf_{\varepsilon>0} P(B(\eta,\varepsilon)) = P(\{\eta\}) \le f(\eta).$$

Since the second term defining $\mu(A_\varepsilon)$ equals $P(B(\eta,\varepsilon))$ and both terms decrease as $\varepsilon \downarrow 0$, it follows that

$$\begin{aligned} \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) &= \inf_{\varepsilon>0} \max\left\{ \sup_{n\in A_\varepsilon} f(x_n), P(B(\eta,\varepsilon)) \right\} \\ &= \max\{f(\eta), P(\{\eta\})\} = f(\eta). \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.7. Since every normalized supermodular capacity is superadditive, we have (E5) $\implies$ (E4) $\implies$ (E1) and (E5) $\implies$ (E2) $\implies$ (E1). Moreover, $\mathsf{d}_\star$ is a finite-invariant normalized superadditive capacity, so that (E3) $\implies$ (E2). It is therefore enough to establish (E1) $\implies$ (E6), (E6) $\implies$ (E3), and (E6) $\implies$ (E5).

(E1) $\implies$ (E6). This implication was proved in [11, Theorem 2.1]. For completeness, suppose that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$ for some normalized superadditive capacity μ. By Theorem 1.2,

$\mathrm{im}(f) \subseteq [0,1]$. Fix pairwise distinct $\eta_1, \dots, \eta_k \in X$ and choose $\varepsilon > 0$ so that the balls $B(\eta_i, \varepsilon)$ are pairwise disjoint. Putting $A_i := \{n : d(x_n, \eta_i) < \varepsilon\}$, superadditivity gives

$$\sum_{i=1}^{k} f(\eta_i) \leq \sum_{i=1}^{k} \mu(A_i) \leq \mu\left(\bigcup_{i=1}^{k} A_i\right) \leq 1.$$

(E6) $\implies$ (E3). Suppose that $\mathrm{im}(f) \subseteq [0,1]$ and define $s := \sum f \leq 1$. Then $P_f = \{\eta \in X : f(\eta) > 0\}$ is at most countable, and

$$\forall A \subseteq X, \qquad P(A) := \sum_{\eta \in A} f(\eta)$$

defines a σ-additive measure on X, with $P(X) = s$. Since $|X| \geq 2$, we can choose distinct points $a, b \in X$. Then, we define the probability measures

$$Q_a := P + (1-s)\delta_a \qquad \text{and} \qquad Q_b := P + (1-s)\delta_b.$$

Choose increasing finite subsets $F_j \subseteq P_f$ such that $\bigcup_{j \geq 1} F_j = P_f$, and set

$$P_j := \sum_{\eta \in F_j} f(\eta)\delta_\eta \qquad \text{and} \qquad s_j := P_j(X).$$

(If $P_f = \emptyset$, take $F_j = \emptyset$ for every $j \geq 1$.) For each $j \geq 1$, define also the finitely supported probability measures

$$\nu_{2j-1} := P_j + (1-s_j)\delta_a, \qquad \text{and} \qquad \nu_{2j} := P_j + (1-s_j)\delta_b.$$

Since $s_j \uparrow s$, for every $A \subseteq X$ we have

$$\lim_{j \to \infty} \nu_{2j-1}(A) = Q_a(A) \quad \text{and} \quad \lim_{j \to \infty} \nu_{2j}(A) = Q_b(A).$$

At this point, thanks to Lemma 2.1, there exists a sequence $\boldsymbol{x} = (x_n)_{n \geq 0}$ with values in X such that

$$\forall A \subseteq X, \qquad \mathsf{d}^\star(S_A) = \limsup_{j \to \infty} \nu_j(A) = \max\{Q_a(A), Q_b(A)\},$$

where $S_A := \{n \in \mathbb{N} : x_n \in A\}$. Using that $S_{X \setminus A} = \mathbb{N} \setminus S_A$, we obtain that

$$\begin{aligned} \forall A \subseteq X, \qquad \mathsf{d}_\star(S_A) &= 1 - \mathsf{d}^\star(S_{X \setminus A}) \\ &= 1 - \max\{Q_a(X \setminus A), Q_b(X \setminus A)\} \\ &= \min\{Q_a(A), Q_b(A)\}. \end{aligned}$$

For each $\eta \in X$, continuity from above of the probability measures Q_a, Q_b now implies

$$\begin{aligned} \mathrm{DC}_{\mathsf{d}_\star, \boldsymbol{x}}(\eta) &= \inf_{\varepsilon > 0} \min\{Q_a(B(\eta, \varepsilon)), Q_b(B(\eta, \varepsilon))\} \\ &= \min\{Q_a(\{\eta\}), Q_b(\{\eta\})\} = f(\eta), \end{aligned}$$

where the last equality uses $a \neq b$. Thus f is represented by $\mathsf{d}_\star$.

(E6) $\implies$ (E5). Suppose that (E6) holds, and set $s := \sum f$. Then P_f is at most countable; write $P_f = \{\eta_j : j \in J\}$, with distinct points, where $J \subseteq \mathbb{N} \setminus \{0\}$ is finite or countably infinite, and $J = \emptyset$ if $P_f = \emptyset$. Fix distinct $a, b \in X$ and a partition $\{E_a, E_b\} \cup \{D_j : j \in J\}$ of $\mathbb{N}$ into infinite sets. Choose $\boldsymbol{x}$ with $x_n = a$ on E_a, $x_n = b$ on

E_b, and $x_n = \eta_j$ on D_j. For every infinite $D \subseteq \mathbb{N}$, put $\nu_D(A) := 1$ if $D \setminus A$ is finite, and $\nu_D(A) := 0$ otherwise. Since $\{A : |D \setminus A| < \infty\}$ is a filter, each ν_D is a finite-invariant normalized supermodular capacity. Thus the map

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := (1-s)\nu_{\mathbb{N}}(A) + \sum_{j\in J} f(\eta_j)\nu_{D_j}(A)$$

is a finite-invariant normalized supermodular capacity. To compute its degree of convergence, fix $\eta \in X$ and choose $c \in \{a,b\} \setminus \{\eta\}$. For every $\varepsilon \in (0, d(c,\eta))$, the set $A_\varepsilon := \{n : d(x_n,\eta) < \varepsilon\}$ is not cofinite, while $\nu_{D_j}(A_\varepsilon) = 1$ precisely when $d(\eta_j,\eta) < \varepsilon$. Hence,

$$\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = \inf_{\varepsilon>0} \sum_{\substack{j\in J \\ d(\eta_j,\eta)<\varepsilon}} f(\eta_j) = f(\eta),$$

where the last equality follows by continuity from above of the finite atomic measure $\sum_{j\in J} f(\eta_j)\delta_{\eta_j}$. This proves (E5).

Finally, suppose that X is noncompact or uncountable. It is clear that (E8) $\implies$ (E7) $\implies$ (E1). Since conditions (E1)–(E6) are equivalent, the next implication will conclude the proof.

(E6) $\implies$ (E8). If X is noncompact, choose a sequence $\boldsymbol{y} = (y_n)$ with no convergent subsequence and a partition $\{D_0\} \cup \{D_j : j \in J\}$ of $\mathbb{N}$ into infinite sets. Put $x_n = \eta_j$ on D_j, and enumerate $\boldsymbol{y}$ on D_0. For each $j \in J \cup \{0\}$, choose a free ultrafilter $\mathcal{U}_j$ on $\mathbb{N}$ containing D_j, and define

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := (1-s)\mathbf{1}_{\mathcal{U}_0}(A) + \sum_{j\in J} f(\eta_j)\mathbf{1}_{\mathcal{U}_j}(A).$$

Then μ is a finite-invariant probability charge. For each $\eta \in X$, the indices in D_0 with $x_n \in B(\eta,\varepsilon)$ form a finite set for all sufficiently small ε. The other ultrafilters contribute precisely when $\eta_j \in B(\eta,\varepsilon)$. It follows, as above, that $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = f(\eta)$.

If X is uncountable and compact, then it is possible to pick a nonatomic Borel probability measure P on X (as in the proof of (C7″) $\implies$ (C4) and (C6) when X is uncountable and compact). Accordingly, define the Borel probability

$$Q := (1-s)P + \sum_{j\in J} f(\eta_j)\delta_{\eta_j}.$$

Applying Lemma 2.2 to $\mathscr{C} = \{Q\}$, there exists a sequence $\boldsymbol{x}$ whose empirical measures converge weakly to Q. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and put

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \mathcal{U}\text{-}\lim_{N\to\infty} \frac{|A \cap [0,N)|}{N}.$$

Then μ is a finite-invariant probability charge. By the Portmanteau theorem [3, Theorem 2.1], for every $\eta \in X$ and $\varepsilon > 0$ we have

$$Q(B(\eta,\varepsilon)) \le \mu(\{n : d(x_n,\eta) < \varepsilon\}) \le Q(\overline{B(\eta,\varepsilon)}).$$

Taking the infimum over $\varepsilon > 0$, we obtain $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = Q(\{\eta\}) = f(\eta)$. □

Proof of Theorem 1.8. First, let $\boldsymbol{x} = (x_n)_{n\geq 0}$ be a sequence in X and let $\mathscr{C}$ be the set of weak accumulation points of its empirical measures $(e_N : N \geq 1)$ defined by $e_N := \frac{1}{N}\sum_{n=0}^{N-1} \delta_{x_n}$. Observe that $\mathscr{C}$ is a nonempty compact family of Borel probability measures on X.

We claim that

$$\forall \eta \in X, \qquad \mathrm{DC}_{\mathsf{d}^\star,\boldsymbol{x}}(\eta) = \sup_{\nu\in\mathscr{C}} \nu(\{\eta\}), \quad \text{and} \quad \mathrm{DC}_{\mathsf{d}_\star,\boldsymbol{x}}(\eta) = \inf_{\nu\in\mathscr{C}} \nu(\{\eta\}). \tag{13}$$

To this end, fix $\eta \in X$. The first identity follows exactly as in the proof of Proposition 2.3. For the second identity, fix $\eta \in X$, and set $A_\varepsilon := \{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\}$ for each $\varepsilon > 0$. By the Portmanteau theorem [3, Theorem 2.1], we obtain

$$\forall \varepsilon > 0, \qquad \inf_{\nu\in\mathscr{C}} \nu(B(\eta,\varepsilon)) \leq \mathsf{d}_\star(A_\varepsilon) \leq \inf_{\nu\in\mathscr{C}} \nu(\overline{B(\eta,\varepsilon)}).$$

Indeed, the left-hand inequality follows by extracting a weakly convergent subsequence along which $e_N(B(\eta,\varepsilon))$ tends to its liminf; for the right-hand inequality, use any subsequence converging weakly to a prescribed $\nu \in \mathscr{C}$. Thus, continuity from above implies

$$\inf_{\varepsilon>0}\inf_{\nu\in\mathscr{C}} \nu(B(\eta,\varepsilon)) = \inf_{\nu\in\mathscr{C}} \nu(\{\eta\}) = \inf_{\varepsilon>0}\inf_{\nu\in\mathscr{C}} \nu(\overline{B(\eta,\varepsilon)}),$$

which proves the second identity in (13).

Necessity. Suppose that $\boldsymbol{x}$ represents f and g simultaneously, and let $\mathscr{C}$ be as above. Thanks to (13), we have

$$\forall \eta \in X, \qquad f(\eta) = \sup_{\nu\in\mathscr{C}} \nu(\{\eta\}) \quad \text{and} \quad g(\eta) = \inf_{\nu\in\mathscr{C}} \nu(\{\eta\}).$$

Fixing $\nu \in \mathscr{C}$ implies that $\sum g = \sup \sum_{\eta\in F} g(\eta) \leq \sup \sum_{\eta\in F} \nu(\{\eta\}) \leq \nu(X) \leq 1$, where each sup is over all finite $F \subseteq X$. This proves (ii). Define $s := \sum g$ and

$$Q_g := \sum_{\eta\in X} g(\eta)\delta_\eta.$$

For every $\nu \in \mathscr{C}$ we have $\nu \geq Q_g$ as Borel measures, since $\nu(\{\eta\}) \geq g(\eta)$ for every $\eta \in X$. Now, set $r := 1 - s$. If $r = 0$, then $\mathscr{C} = \{Q_g\}$ and $f = g$; hence all three conditions hold.

Suppose that $r > 0$, and consider the nonempty compact family

$$\mathscr{D} := \left\{ \frac{\nu - Q_g}{r} : \nu \in \mathscr{C} \right\}$$

of Borel probability measures on X. Then

$$\forall \eta \in X, \qquad (f - g)(\eta) = r \sup_{\lambda\in\mathscr{D}} \lambda(\{\eta\}). \tag{14}$$

In particular, $0 \leq f - g \leq r$, proving (iii). To prove (i), define $h(\eta) := \sup_{\lambda\in\mathscr{D}} \lambda(\{\eta\})$. The argument of Claim 4 in the proof of Proposition 2.3 implies

$$h(\eta) = \inf_{\varepsilon>0} \sup_{\lambda\in\mathscr{D}} \lambda(\overline{B(\eta,\varepsilon)}).$$

If $h(\eta) < t$, choose $\varepsilon > 0$ such that the supremum on the right for this ε is less than t. For every $\xi \in B(\eta, \varepsilon/2)$, we have $\overline{B(\xi,\varepsilon/2)} \subseteq \overline{B(\eta,\varepsilon)}$, so $h(\xi) < t$. Thus h, and hence $f - g$ by (14), is upper semicontinuous.

Sufficiency. Suppose that (i)–(iii) hold, and retain the notation $s := \sum g$, $r := 1 - s$, and Q_g as before. If $r = 0$, then $f = g$ and Q_g is a probability measure. Applying Lemma 2.2 to the singleton $\mathscr{C} := \{Q_g\}$, the conclusion follows from (13).

Assume that $r > 0$ and put $h := (f - g)/r$. Then $h : X \to [0, 1]$ is upper semicontinuous. Apply the construction in the proof of (C7″) $\implies$ (C4) and (C6) when X is uncountable and compact, with h in place of f. As in the proof of Proposition 2.3, let $\mathscr{C}_h$ be the family of second marginals of the corresponding measures on $\Omega \times X$. It is nonempty, compact, and convex, and Claim 2 implies

$$\forall \eta \in X, \qquad \sup_{\nu \in \mathscr{C}_h} \nu(\{\eta\}) = h(\eta). \tag{15}$$

Let P_0 be the nonatomic Borel probability measure used in the same construction, and define the family

$$\mathscr{K} := \{(1 - t)\nu + tP_0 : \nu \in \mathscr{C}_h,\, t \in [0, 1]\}.$$

Then $\mathscr{K}$ is nonempty, compact, and convex. Since $P_0(\{\eta\}) = 0$ for every $\eta \in X$, (15) implies

$$\forall \eta \in X, \qquad \sup_{\nu \in \mathscr{K}} \nu(\{\eta\}) = h(\eta) \quad \text{and} \quad \inf_{\nu \in \mathscr{K}} \nu(\{\eta\}) = 0.$$

Hence, the nonempty compact convex family

$$\mathscr{C} := \{Q_g + r\nu : \nu \in \mathscr{K}\}$$

is made of Borel probability measures on X, and satisfies

$$\forall \eta \in X, \qquad \sup_{\lambda \in \mathscr{C}} \lambda(\{\eta\}) = f(\eta) \quad \text{and} \quad \inf_{\lambda \in \mathscr{C}} \lambda(\{\eta\}) = g(\eta).$$

By Lemma 2.2, there exists a sequence $\boldsymbol{x}$ whose empirical measures have precisely $\mathscr{C}$ as their set of weak accumulation points. The conclusion follows from (13). □

Proof of Theorem 1.11. (F1) $\implies$ (F2). Suppose that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$ for some probability measure μ on $\mathbb{N}$. By continuity from above, for every $\eta \in X$,

$$f(\eta) = \mu(\{n \in \mathbb{N} : x_n = \eta\}).$$

The nonempty sets on the right form a countable partition of $\mathbb{N}$, so $\mathrm{im}(f) \subseteq [0, 1]$ and $\sum f = \mu(\mathbb{N}) = 1$.

(F2) $\implies$ (F1). Suppose that $\mathrm{im}(f) \subseteq [0, 1]$ and $\sum f = 1$. Enumerate P_f as $\{\eta_j : j \in J\}$, and choose distinct indices $(n_j)_{j \in J}$ in $\mathbb{N}$. Choose a sequence $\boldsymbol{x}$ such that $x_{n_j} = \eta_j$ for every $j \in J$, and define

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \sum_{\substack{j \in J \\ n_j \in A}} f(\eta_j).$$

Then μ is a probability measure, and continuity from above gives $\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = \mu(\{n : x_n = \eta\}) = f(\eta)$ for every $\eta \in X$.

Finally, suppose that X is compact and countable. Clearly, (F4) $\implies$ (F3).

(F3) $\implies$ (F2). If (F3) holds, then Theorem 1.4 implies $\sum f \geq 1$. At the same time, by Theorem 1.7 we get $\sum f \leq 1$. Hence $\sum f = 1$.

(F2) $\implies$ (F4). Enumerate P_f as $\{\eta_j : j \in J\}$ and choose a partition $\{D_j : j \in J\}$ of $\mathbb{N}$ into infinite sets. Define the sequence $\boldsymbol{x}$ by $x_n = \eta_j$ for each $n \in D_j$ and $j \in J$. For each $j \in J$, choose a free ultrafilter $\mathcal{U}_j$ containing D_j, and put

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \sum_{j \in J} f(\eta_j) \mathbf{1}_{\mathcal{U}_j}(A).$$

Then μ is a finite-invariant probability charge. For every $\eta \in X$ and $\varepsilon > 0$ we have $\mu(\{n : d(x_n, \eta) < \varepsilon\}) = \sum_{d(\eta_j, \eta) < \varepsilon} f(\eta_j)$, whose infimum over $\varepsilon > 0$ is $f(\eta)$. Hence (F4) holds, completing the proof. □

Proof of Proposition 1.13. (G1) $\implies$ (G3). Suppose that X is finite, let us say $X = \{\eta_1, \dots, \eta_m\}$ with $m \geq 2$. Define the probability measure $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ by

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \frac{1}{m} \sum_{n \in A} q^n, \quad \text{where } \; q := 1 - \frac{1}{m}. \tag{16}$$

Note that $\mu(\mathbb{N}) = 1$, since $\sum_n q^n = m$.

Fix a function $f : X \to [0, 1]$ which is represented by some probability measure. By Theorem 1.11, we have $\sum_i f(\eta_i) = 1$. Thus, we construct a sequence $\boldsymbol{x} = (x_n)_{n \geq 0}$ with values in X such that $f = \mathrm{DC}_{\mu, \boldsymbol{x}}$.

To this end, define $r_{i,0} := f(\eta_i)$ for each $i \in \{1, \dots, m\}$. Suppose that, for some $n \in \mathbb{N}$, the nonnegative numbers $r_{1,n}, \dots, r_{m,n}$ have been defined and satisfy

$$\sum_{i=1}^{m} r_{i,n} = q^n.$$

Choose $i_n \in \{1, \dots, m\}$ such that $r_{i_n,n} = \max_{1 \leq i \leq m} r_{i,n}$, and note that

$$r_{i_n,n} \geq \frac{q^n}{m} = \mu(\{n\}).$$

Set $x_n := \eta_{i_n}$ and define

$$\forall i \in \{1, \dots, m\}, \qquad r_{i,n+1} := \begin{cases} r_{i,n} - \frac{q^n}{m} & \text{if } i = i_n, \\ r_{i,n} & \text{otherwise.} \end{cases}$$

By the choice of i_n, all these numbers are nonnegative. Moreover,

$$\sum_{i=1}^{m} r_{i,n+1} = q^n - \frac{q^n}{m} = q^{n+1}.$$

Hence the construction can be continued recursively for every $n \in \mathbb{N}$.

By induction, for every $i \in \{1, \dots, m\}$ and every $N \geq 1$, we obtain

$$r_{i,N} = f(\eta_i) - \frac{1}{m} \sum_{\substack{n < N \\ x_n = \eta_i}} q^n.$$

Since $0 \le r_{i,N} \le q^N$ and $\lim_N q^N = 0$, the continuity from above of μ implies

$$f(\eta_i) = \frac{1}{m} \sum_{\substack{n\in\mathbb{N}\\ x_n=\eta_i}} q^n = \mu(\{n \in \mathbb{N} : x_n = \eta_i\}) = \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta_i)$$

for every $i \in \{1,\dots,m\}$. Therefore $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$, proving that μ represents every function represented by some probability measure.

(G1) $\implies$ (G2). Write $X = \{\eta_1,\dots,\eta_m\}$, where $m \ge 2$. Identify $\mathbb{N}$ with $\{1,\dots,m\}\times\mathbb{N}$ through a fixed bijection. For every $i \in \{1,\dots,m\}$, define

$$P_i(A) := \frac{1}{m} \sum_{n\in\mathbb{N}:\,(i,n)\in A} q^n \quad \text{and} \quad \mu(A) := \max_{i\in\{1,\dots,m\}} P_i(A)$$

for all $A \subseteq \{1,\dots,m\}\times\mathbb{N}$, where q is defined as above. Then each P_i is a probability measure, and μ is a normalized σ-subadditive capacity.

Let $f : X \to [0,1]$ be represented by a normalized σ-subadditive capacity. Thanks to Theorem 1.6, we have $\sum_j f(\eta_j) \ge 1$. For every $i \in \{1,\dots,m\}$, choose a probability vector $(p_{i,1},\dots,p_{i,m})$ such that $p_{i,j} \in [0, f(\eta_j)]$ for each $j \in \{1,\dots,m\}$, and $p_{i,i} = f(\eta_i)$. As in the proof of (G1) $\implies$ (G3), pick a partition $\{A_{i,1},\dots,A_{i,m}\}$ of $\{i\}\times\mathbb{N}$ such that $P_i(A_{i,j}) = p_{i,j}$. Define $x_{(i,n)} := \eta_j$ whenever $(i,n) \in A_{ij}$. Since every point of the finite metric space X is isolated,

$$\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta_j) = \mu\left(\bigcup_{i=1}^{m} A_{i,j}\right) = \max_{i\in\{1,\dots,m\}} p_{i,j} = f(\eta_j).$$

This proves (G2).

(G2) $\implies$ (G1). Fix a normalized σ-subadditive capacity μ on $\mathbb{N}$. Since $1 = \mu(\mathbb{N}) \le \sum_n \mu(\{n\})$, there exists $n_0 \in \mathbb{N}$ such that $\alpha := \mu(\{n_0\}) > 0$.

Now, fix a sequence $\boldsymbol{x} = (x_n)_{n\ge0}$ with values in X. For every $\varepsilon > 0$, the set $A_\varepsilon := \{n \in \mathbb{N} : d(x_n, x_{n_0}) < \varepsilon\}$ contains n_0. Hence, by monotonicity, $\mu(A_\varepsilon) \ge \alpha$, and therefore

$$\sup_{\eta\in X} \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) \ge \mathrm{DC}_{\mu,\boldsymbol{x}}(x_{n_0}) = \inf_{\varepsilon>0} \mu(A_\varepsilon) \ge \alpha.$$

At this point, choose an integer $m > 1/\alpha$ and pairwise distinct points $\eta_1,\dots,\eta_m \in X$, which is possible since X is infinite. Define the function $f : X \to [0,1]$ by setting

$$\forall \eta \in X, \qquad f(\eta) := \begin{cases} 1/m & \text{if } \eta \in \{\eta_1,\dots,\eta_m\},\\ 0 & \text{otherwise.}\end{cases}$$

Then f is upper semicontinuous, since $\{\eta_1,\dots,\eta_m\}$ is closed. Moreover, P_f is finite, hence separable, and $\sum f = \sum_{i=1}^m f(\eta_i) = 1$. By Theorem 1.6, the function f is represented by some normalized σ-subadditive capacity.

On the other hand, f cannot be represented by μ. Indeed, otherwise there would exist a sequence $\boldsymbol{x}$ with values in X such that $f = \mathrm{DC}_{\mu,\boldsymbol{x}}$, which would imply

$$\alpha \le \sup_{\eta\in X} \mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = \sup_{\eta\in X} f(\eta) = \frac{1}{m} < \alpha,$$

which is the claimed contradiction.

(G3) $\implies$ (G1). If X is infinite, then there is no "universal probability measure," proceeding verbatim as in the proof of the implication (G2) $\implies$ (G1) above (we omit details). □

Remark 2.5. Note that the definition of μ in (16) depends only on $|X|$.

Proof of Proposition 1.14. (H1) $\implies$ (H2). This is immediate, since every density measure is a probability charge.

(H2) $\implies$ (H1). Suppose that f is represented by some probability charge. By Theorem 1.7, we have $\mathrm{im}(f) \subseteq [0,1]$ and $s := \sum f \leq 1$. In particular, P_f is at most countable, and hence

$$P := \sum_{\eta \in P_f} f(\eta)\delta_\eta$$

is a finite Borel measure on X with $P(X) = s$.

We distinguish two cases.

Case 1: X is compact. If X is countable, then Theorem 1.11 implies that $s = 1$. Otherwise, X is uncountable, so it admits a nonatomic Borel probability measure P_0, as recalled in the proof of Theorem 1.4. Thus, in either case, there exists a Borel probability measure Q on X such that

$$\forall \eta \in X, \qquad Q(\{\eta\}) = f(\eta).$$

Indeed, take $Q := P$ if $s = 1$, and $Q := P + (1-s)P_0$ otherwise.

Applying Lemma 2.2 to $\mathscr{C} = \{Q\}$, there exists a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ with values in X whose empirical measures converge weakly to Q. Fix $\eta \in X$, and choose a sequence of positive numbers $\varepsilon_k \downarrow 0$ such that $Q(\partial B(\eta, \varepsilon_k)) = 0$ for all $k \geq 1$. By the Portmanteau theorem [3, Theorem 2.1], for each $k \geq 1$ the set

$$A_k := \{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon_k\}$$

has asymptotic density $Q(B(\eta, \varepsilon_k))$. Since λ is a density measure, we obtain $\lambda(A_k) = Q(B(\eta, \varepsilon_k))$. Hence, by continuity from above,

$$\mathrm{DC}_{\lambda,\boldsymbol{x}}(\eta) = \inf_{k\geq 1} \lambda(A_k) = \inf_{k\geq 1} Q(B(\eta, \varepsilon_k)) = Q(\{\eta\}) = f(\eta).$$

Since $\eta \in X$ was arbitrary, we conclude that $f = \mathrm{DC}_{\lambda,\boldsymbol{x}}$.

Case 2: X is noncompact. Choose an increasing sequence $(F_j)_{j\geq 1}$ of finite subsets of P_f such that $\bigcup_j F_j = P_f$, and set

$$P_j := \sum_{\eta \in F_j} f(\eta)\delta_\eta \quad \text{and} \quad s_j := P_j(X).$$

Since X is noncompact, there exists a sequence $(y_j)_{j\geq 1}$ with values in X having no convergent subsequences. For each $j \geq 1$, define the finitely supported probability measure

$$\nu_j := P_j + (1 - s_j)\delta_{y_j}.$$

By Lemma 2.1, there exists a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ with values in X such that

$$\forall A \subseteq X, \qquad \mathsf{d}^\star(\{n : x_n \in A\}) = \limsup_{j\to\infty} \nu_j(A).$$

Fix $\eta \in X$. By the choice of (y_j), there exists $r > 0$ such that $y_j \notin B(\eta, r)$ for all sufficiently large j. Hence, for every $\varepsilon \in (0, r)$, we have

$$\lim_{j\to\infty} \nu_j(B(\eta,\varepsilon)) = P(B(\eta,\varepsilon)),$$

since P_j increases to P. Applying Lemma 2.1 to $B(\eta,\varepsilon)$ and its complement, we obtain that the set $\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}$ has asymptotic density $P(B(\eta,\varepsilon))$. Since λ agrees with the asymptotic density whenever the latter exists, it follows that

$$\forall \varepsilon \in (0, r), \qquad \lambda(\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}) = P(B(\eta,\varepsilon)).$$

Hence, by continuity from above,

$$\mathrm{DC}_{\lambda,\boldsymbol{x}}(\eta) = \inf_{0<\varepsilon<r} P(B(\eta,\varepsilon)) = P(\{\eta\}) = f(\eta).$$

Since $\eta \in X$ was arbitrary, we conclude again that $f = \mathrm{DC}_{\lambda,\boldsymbol{x}}$, completing the proof. □

Proof of Proposition 1.15. (I1) $\implies$ (I2), (I3), and (I4). Suppose that X is separable, and fix a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ with dense range in X. Let $f : X \to [0,1]$ be represented by some normalized capacity, normalized submeasure, or probability charge. By Theorem 1.2, Theorem 1.4, and Proposition 1.14, respectively, there exist a sequence $\boldsymbol{y} = (y_n)_{n\geq 0}$ with values in X and a finite-invariant normalized capacity μ of the same type such that $f = \mathrm{DC}_{\mu,\boldsymbol{y}}$. Since the range of $\boldsymbol{x}$ is dense, we can choose $h : \mathbb{N} \to \mathbb{N}$ such that $d(z_n, y_n) < 2^{-n}$ for every $n \in \mathbb{N}$, where $z_n := x_{h(n)}$. Define $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by

$$\forall A \subseteq \mathbb{N}, \qquad \nu(A) := \mu(h^{-1}[A]).$$

Then ν is a normalized capacity. Moreover, it is a submeasure if μ is a submeasure, a probability charge if μ is a probability charge, and $\mathrm{DC}_{\nu,\boldsymbol{x}} = \mathrm{DC}_{\mu,\boldsymbol{z}}$.

Now, fix $\eta \in X$ and $\varepsilon > 0$. Since $\lim_n d(y_n, z_n) = 0$, we have that, if $n \in \mathbb{N}$ is sufficiently large and such that $d(y_n,\eta) < \varepsilon/2$, then $d(z_n,\eta) < \varepsilon$. By monotonicity and finite-invariance of μ, we get

$$\mu(\{n : d(y_n,\eta) < \varepsilon/2\}) \leq \mu(\{n : d(z_n,\eta) < \varepsilon\}).$$

Taking the infimum over $\varepsilon > 0$, it follows that $\mathrm{DC}_{\mu,\boldsymbol{y}}(\eta) \leq \mathrm{DC}_{\mu,\boldsymbol{z}}(\eta)$. Interchanging $\boldsymbol{y}$ and $\boldsymbol{z}$ proves the converse inequality. Therefore

$$f = \mathrm{DC}_{\mu,\boldsymbol{y}} = \mathrm{DC}_{\mu,\boldsymbol{z}} = \mathrm{DC}_{\nu,\boldsymbol{x}}.$$

This proves (I2), (I3), and (I4).

(I2), (I3), or (I4) $\implies$ (I1). Suppose that $\boldsymbol{x} = (x_n)_{n\geq 0}$ satisfies any one of these conditions. Fix $\eta \in X$, and define $f : X \to \mathbb{R}$ by

$$\forall \xi \in X, \qquad f_\eta(\xi) := \begin{cases} 1 & \text{if } \xi = \eta, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Then f_η is represented by a probability measure: if $\boldsymbol{y}$ is the constant sequence η then δ_0 is probability measure and $f_\eta = \mathrm{DC}_{\delta_0,\boldsymbol{y}}$. In particular, f_η is represented by a probability charge (hence also by a normalized capacity and a normalized submeasure). Thus, there exists a normalized capacity ν_η such that $f_\eta = \mathrm{DC}_{\nu_\eta,\boldsymbol{x}}$.

If η did not belong to the closure of the range of $\boldsymbol{x}$, then $\{n \in \mathbb{N} : d(x_n, \eta) < \varepsilon\} = \emptyset$ for some $\varepsilon > 0$. This would imply

$$1 = f_\eta(\eta) = \mathrm{DC}_{\nu_\eta,\boldsymbol{x}}(\eta) \leq \nu(\emptyset) = 0,$$

which is a contradiction. Thus the range of $\boldsymbol{x}$ is dense in X, so X is separable. The same argument also shows that every sequence satisfying one of conditions (I2)–(I4) must have dense range, completing the proof. □

Proof of Proposition 1.16. (J1) $\implies$ (J2) and (J3). Suppose that X is countable, and fix a sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ whose range coincides with X. For each $\eta \in X$, choose $n_\eta \in \mathbb{N}$ such that $x_{n_\eta} = \eta$.

Pick a function $f : X \to [0,1]$ which is represented by some normalized σ-subadditive capacity. By Theorem 1.6, the function f is upper semicontinuous and $\sum f \geq 1$. Since X is countable, we can choose $t_\eta \in [0, f(\eta)]$ for every $\eta \in X$ such that $\sum_{\eta\in X} t_\eta = 1$. Define a probability measure $P : \mathcal{P}(\mathbb{N}) \to [0,1]$ by

$$\forall A \subseteq \mathbb{N}, \qquad P(A) := \sum_{\eta\in X\colon n_\eta\in A} t_\eta,$$

and set

$$\forall A \subseteq \mathbb{N}, \qquad \nu(A) := \max\left\{P(A), \sup_{n\in A} f(x_n)\right\},$$

where $\sup\emptyset := 0$. Observe that ν is a normalized σ-subadditive capacity. Indeed, the map $A \mapsto \sup_{n\in A} f(x_n)$ is monotone and satisfies

$$\sup_{n\in\bigcup_{j\geq 0} A_j} f(x_n) = \sup_{j\geq 0}\sup_{n\in A_j} f(x_n) \leq \sum_{j\geq 0}\sup_{n\in A_j} f(x_n),$$

and hence is σ-subadditive. Since P is a probability measure, their pointwise maximum is also σ-subadditive. Moreover, $\nu(\emptyset) = 0$ and $\nu(\mathbb{N}) = 1$.

We claim that $f = \mathrm{DC}_{\nu,\boldsymbol{x}}$. To this end, fix $\eta \in X$. Since f is upper semicontinuous and $x_{n_\eta} = \eta$, we have

$$\inf_{\varepsilon>0}\sup_{\substack{n\in\mathbb{N}\\ d(x_n,\eta)<\varepsilon}} f(x_n) = f(\eta).$$

On the other hand, we have

$$\forall \varepsilon > 0, \qquad P(\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}) = \sum_{\substack{\xi\in X\\ d(\xi,\eta)<\varepsilon}} t_\xi.$$

Since $\sum_{\xi\in X} t_\xi = 1$, continuity from above gives

$$\inf_{\varepsilon>0} P(\{n \in \mathbb{N} : d(x_n,\eta) < \varepsilon\}) = t_\eta.$$

it follows that

$$\mathrm{DC}_{\nu,\boldsymbol{x}}(\eta) = \max\{f(\eta), t_\eta\} = f(\eta),$$

where the first equality follows from the fact that both terms in the definition of ν are nondecreasing with respect to ε. This proves (J2).

Next, let $f : X \to [0,1]$ be represented by some probability measure. By Theorem 1.11, we have $\sum_{\eta\in X} f(\eta) = 1$. Define the probability measure $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ by

$$\forall A \subseteq \mathbb{N}, \qquad \mu(A) := \sum_{\substack{\eta\in X\\ n_\eta\in A}} f(\eta).$$

Then, for every $\eta \in X$,

$$\mathrm{DC}_{\mu,\boldsymbol{x}}(\eta) = \inf_{\varepsilon>0} \mu(\{n\in\mathbb{N} : d(x_n,\eta) < \varepsilon\}) = \inf_{\varepsilon>0} \sum_{\substack{\xi\in X\\ d(\xi,\eta)<\varepsilon}} f(\xi) = f(\eta),$$

where the last equality follows from continuity from above. This proves (J3).

(J2) $\implies$ (J1) and (J3) $\implies$ (J1). Suppose that either (J2) or (J3) holds, and let $\boldsymbol{x} = (x_n)_{n\geq 0}$ be the corresponding universal sequence.

Fix $\eta \in X$, and define $f_\eta : X \to [0,1]$ as in (17). Then f_η is represented by a probability measure. By the assumed universal property, there exists a normalized σ-subadditive capacity ν_η such that $f_\eta = \mathrm{DC}_{\nu_\eta,\boldsymbol{x}}$.

Suppose, towards a contradiction, that η does not belong to the range of $\boldsymbol{x}$. Then $f_\eta(x_n) = 0$ for every $n \in \mathbb{N}$. Hence, by monotonicity of ν,

$$0 = f_\eta(x_n) = \inf_{\varepsilon>0} \nu_\eta(\{k\in\mathbb{N} : d(x_k,x_n) < \varepsilon\}) \geq \nu_\eta(\{n\})$$

for every $n \in \mathbb{N}$. Hence $\nu_\eta(\{n\}) = 0$ for all $n \in \mathbb{N}$. This is impossible because, by the σ-subadditivity, we have $1 = \nu_\eta(\mathbb{N}) \leq \sum_n \nu_\eta(\{n\}) = 0$. By the arbitrariness of η, the sequence $\boldsymbol{x}$ is surjective. In particular, X is countable.

We have also shown that every sequence satisfying (J2) or (J3) must be surjective. Conversely, the first part above shows that every surjective sequence satisfies both conditions. □

Proof of Proposition 1.17. (W1) $\implies$ (W4). For each integer $j \geq 2$, define the family

$$\mathscr{G}_j := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{S \subseteq \mathbb{N} : [n, n + g_{1/j}(n)] \subseteq S\right\}.$$

For every $n \in \mathbb{N}$, the set $\left\{S \subseteq \mathbb{N} : [n, n + g_{1/j}(n)] \subseteq S\right\}$ is clopen in $\mathcal{P}(\mathbb{N})$. Moreover, for each $m \in \mathbb{N}$, the union over $n \geq m$ is dense. Hence $\mathscr{G}_j$ is comeager.

Now, we claim that

$$\mathscr{G}_j \subseteq \{S \subseteq \mathbb{N} : \nu(S) > 1 - 1/j\}.$$

Indeed, otherwise there would exist $S \in \mathscr{G}_j$ such that $\nu(S) \leq 1 - 1/j$. Setting $A := \mathbb{N} \setminus S$, condition (W1) would imply that $A \cap [n, n + g_{1/j}(n)] \neq \emptyset$ for all sufficiently large n, contradicting the definition of $\mathscr{G}_j$.

It follows that

$$\bigcap_{j\geq 2} \mathscr{G}_j \subseteq \mathscr{F}_\nu.$$

Since ν is normalized and monotone, it takes values in $[0,1]$. Thus the intersection on the left is comeager, and so is $\mathscr{F}_\nu$.

(W4) $\implies$ (W3). Suppose that $\mathscr{F}_\nu$ is comeager. Then there exists a sequence $(\mathscr{U}_j)_{j\in\mathbb{N}}$ of dense open subsets of $\mathcal{P}(\mathbb{N})$ such that

$$\bigcap_{j\in\mathbb{N}} \mathscr{U}_j \subseteq \mathscr{F}_\nu.$$

We construct recursively a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of nonnegative integers, with $n_0 := 0$, having the following property: for every $k \in \mathbb{N}$ and every $H \subseteq [0, n_k)$, there exists $T \subseteq [n_k, n_{k+1})$ such that

$$\{S \subseteq \mathbb{N} : S \cap [0, n_{k+1}) = H \cup T\} \subseteq \bigcap_{j=0}^{k} \mathscr{U}_j.$$

Indeed, suppose that n_k has been chosen. Since $\bigcap_{j=0}^{k} \mathscr{U}_j$ is dense and open, for every $H \subseteq [0, n_k)$ there exists a finite extension of H whose corresponding basic clopen set is contained in $\bigcap_{j=0}^{k} \mathscr{U}_j$. Since there are only finitely many choices of H, we can choose $n_{k+1} > n_k$ sufficiently large so that the required property holds simultaneously for all of them. At this point, define

$$\forall k \in \mathbb{N}, \qquad F_k := [n_k, n_{k+1}).$$

Of course, $\{F_k : k \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ into nonempty finite intervals.

Fix an infinite set $S \subseteq \mathbb{N}$, and put

$$B := \bigcup_{k\in S} F_k.$$

We claim that $B \in \mathscr{F}_\nu$. To this end, choose a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ with values in S. By the property above, we construct recursively a set $A \subseteq B$ such that

$$\forall j \in \mathbb{N}, \qquad \big\{T \subseteq \mathbb{N} : T \cap [0, n_{k_j+1}) = A \cap [0, n_{k_j+1})\big\} \subseteq \bigcap_{i=0}^{k_j} \mathscr{U}_i.$$

By the choice of (k_j), it follows that $A \in \bigcap_{j\in\mathbb{N}} \mathscr{U}_j \subseteq \mathscr{F}_\nu$. Hence $\nu(A) = 1$. Since $A \subseteq B$, monotonicity and normalization imply $1 = \nu(A) \leq \nu(B) \leq 1$. Therefore $\nu(B) = 1$. Since S was arbitrary, condition (W3) holds.

(W3) $\implies$ (W2). Suppose that (W3) holds and define $n_k := \min F_k$ for all $k \in \mathbb{N}$. Without loss of generality, we can assume that (n_k) is strictly increasing, and hence $n_0 = 0$ and $F_k = [n_k, n_{k+1})$ for all $k \in \mathbb{N}$. Define $g : \mathbb{N} \to \mathbb{N}$ by setting

$$\forall k \in \mathbb{N},\ \forall n \in F_k, \qquad g(n) := n_{k+2} - n - 1.$$

Fix $A \subseteq \mathbb{N}$ such that $\nu(\mathbb{N} \setminus A) < 1$.

We claim that A intersects all but finitely many sets F_k. Indeed, in the opposite case, the set $S := \{k \in \mathbb{N} : A \cap F_k = \emptyset\}$ would be infinite. By (W3) and monotonicity, we would obtain

$$1 = \nu\left(\bigcup_{k\in S} F_k\right) \leq \nu(\mathbb{N} \setminus A) < 1,$$

which is the claimed contradiction. Hence, for every sufficiently large k and every $n \in F_k$, we have $\emptyset \neq A \cap F_{k+1} \subseteq A \cap [n, n + g(n)]$. This proves (W2).

(W2) $\implies$ (W1). This is immediate. □

Remark 2.6. An inspection of the proof of Proposition 1.17 reveals that the analogous statement holds for all normalized maps $\nu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ such that $\mathrm{im}(\nu) \subseteq [0, 1]$ and $\nu(B) = 1$ whenever there exists $A \subseteq B$ with $\nu(A) = 1$.

Proof of Corollary 1.18. Thanks to Talagrand's characterization of meager ideals [15, Theorem 21], an ideal $\mathcal{I}$ is meager if and only if there exists a partition $\{F_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ into nonempty finite intervals such that $\bigcup_{k\in S} F_k \in \mathcal{I}^+$ for all infinite $S \subseteq \mathbb{N}$. This is equivalent to item (W3). The conclusion follows by Proposition 1.17. □

Proof of Proposition 1.19. By Proposition 1.17, there exists a partition $\{F_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ into nonempty finite intervals such that $\nu\left(\bigcup_{k\in S} F_k\right) = 1$ for every infinite set $S \subseteq \mathbb{N}$. Since X is second countable, we can fix a countable base $(U_j)_{j\in\mathbb{N}}$ consisting of nonempty open sets. For every $j, m \in \mathbb{N}$, define

$$\mathscr{G}_{j,m} := \bigcup_{k\geq m} \left\{\boldsymbol{x} \in X^{\mathbb{N}} : x_n \in U_j \text{ for all } n \in F_k\right\}.$$

Observe that each $\mathscr{G}_{j,m}$ is open, since every F_k is finite. Moreover, it is dense in $X^{\mathbb{N}}$. Hence

$$\mathscr{G} := \bigcap_{j,m\in\mathbb{N}} \mathscr{G}_{j,m}$$

is comeager in $X^{\mathbb{N}}$.

To complete the proof, it is enough to show that $\mathscr{G} \subseteq \Sigma_\nu(X)$. To this end, fix $\boldsymbol{x} \in \mathscr{G}$ and a nonempty open set $U \subseteq X$. Choose $j \in \mathbb{N}$ such that $U_j \subseteq U$, and define

$$S := \{k \in \mathbb{N} : x_n \in U_j \text{ for all } n \in F_k\}.$$

Since $\boldsymbol{x} \in \mathscr{G}_{j,m}$ for every $m \in \mathbb{N}$, the set S is infinite. Moreover, $\bigcup_{k\in S} F_k \subseteq \{n \in \mathbb{N} : x_n \in U\}$. Hence, by monotonicity and normalization,

$$1 = \nu\left(\bigcup_{k\in S} F_k\right) \leq \nu(\{n \in \mathbb{N} : x_n \in U\}) \leq 1.$$

Since U was arbitrary, we conclude that $\boldsymbol{x} \in \Sigma_\nu(X)$. Therefore $\Sigma_\nu(X)$ is comeager. □

Proof of Theorem 1.20. The sufficiency follows from Proposition 1.19.

Conversely, suppose that $\Sigma_\nu(X)$ is comeager. Choose two disjoint nonempty open subsets of X, and denote one of them by U. Define

$$V := X \setminus \overline{U} \qquad \text{and} \qquad Y := U \cup V.$$

Then V is nonempty and open, while Y is open and dense in X. Moreover, U and V are disjoint nonempty clopen subsets of Y.

Since $\Sigma_\nu(X)$ is comeager, there exist open dense subsets $G_m \subseteq X^{\mathbb{N}}$, $m \in \mathbb{N}$, such that

$$\bigcap_{m\in\mathbb{N}} G_m \subseteq \Sigma_\nu(X).$$

Since $Y^{\mathbb{N}}$ is dense in $X^{\mathbb{N}}$, the sets $G_m \cap Y^{\mathbb{N}}$ are open and dense in $Y^{\mathbb{N}}$.

Consider the space

$$Z := U^{\mathbb{N}} \times V^{\mathbb{N}}.$$

Since U and V are nonempty open subspaces of the Baire space X, they are Baire spaces, see e.g. [7, Proposition 8.3]. In addition, both are second countable. Hence, it follows by [14, Theorem 3] that Z is a nonempty second countable Baire space.

At this point, define the map $\Phi : 2^{\mathbb{N}} \times Z \longrightarrow Y^{\mathbb{N}}$ by

$$\forall (A, \boldsymbol{u}, \boldsymbol{v}) \in 2^{\mathbb{N}} \times Z, \qquad \Phi(A, \boldsymbol{u}, \boldsymbol{v})_n := \begin{cases} u_n & \text{if } n \in A, \\ v_n & \text{if } n \notin A. \end{cases}$$

Since U and V are clopen in Y, the map Φ is continuous, open, and surjective.

For every $m \in \mathbb{N}$, set

$$H_m := \Phi^{-1}(G_m \cap Y^{\mathbb{N}}).$$

Then each H_m is open and dense in $2^{\mathbb{N}} \times Z$. Fix a countable base $(B_j)_{j\in\mathbb{N}}$ of Z. For every $j, m \in \mathbb{N}$, define

$$D_{j,m} := \{A \subseteq \mathbb{N} : H_m \cap (\{A\} \times B_j) \neq \emptyset\}.$$

Since H_m is open, the set $D_{j,m}$ is open in $2^{\mathbb{N}}$. Moreover, it is dense: indeed, for every nonempty open subset $O \subseteq 2^{\mathbb{N}}$, the set $O \times B_j$ is nonempty and open, and hence intersects H_m. It follows that

$$D_\infty := \bigcap_{j,m\in\mathbb{N}} D_{j,m}$$

is comeager in $2^{\mathbb{N}}$.

Fix $A \in D_\infty$. For every $m \in \mathbb{N}$, the set $H_m(A) := \{z \in Z : (A, z) \in H_m\}$ is open and intersects every member of the base (B_j). Thus $H_m(A)$ is open and dense in Z. Since Z is a nonempty Baire space, it is possible to pick

$$(\boldsymbol{u}, \boldsymbol{v}) \in \bigcap_{m\in\mathbb{N}} H_m(A).$$

It follows that

$$\boldsymbol{x} := \Phi(A, \boldsymbol{u}, \boldsymbol{v}) \in \bigcap_{m\in\mathbb{N}} G_m \subseteq \Sigma_\nu(X).$$

Note that, by construction, $\{n \in \mathbb{N} : x_n \in U\} = A$. Since U is a nonempty open subset of X and $\boldsymbol{x} \in \Sigma_\nu(X)$, we obtain $\nu(A) = 1$. Therefore $D_\infty \subseteq \mathscr{F}_\nu := \{A \subseteq \mathbb{N} : \nu(A) = 1\}$.

Since D_∞ is comeager, so is $\mathscr{F}_\nu$. Thus condition (W4) holds, and by Proposition 1.17 condition (W) holds as well. □

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UNIVERSITÁ DEGLI STUDI DELL'INSUBRIA, VIA MONTE GENEROSO 71, VARESE 21100, ITALY