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The 2-adic valuation of the symmetric Prouhet–Tarry–Escott constant at sizes 13 and 15

David Niedbala Giraudin
ORCID 0009-0009-1526-1178

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Abstract

For symmetric ideal solutions of the Prouhet–Tarry–Escott problem of odd size n over the 2-adic integers, the associated constant C was known to satisfy $\mathrm{v}_2(C) \geq 12$ at $n = 13$, with a best known solution at $\mathrm{v}_2(C) = 17$, and $\mathrm{v}_2(C) \geq 14$ at $n = 15$, with no solution recorded. We show that the minimum is exactly 13 at $n = 13$ and exactly 14 at $n = 15$. The lower bounds are computer-assisted: an exhaustive 2-adic lifting, driven by the row law taken as an exact equality (units included) together with the vanishing of the odd power sums, eliminates every valuation profile below these values. The upper bounds are explicit 2-adic solutions, certified by the multivariate Hensel lemma. The same procedure reproduces exactly the known values $4, 6, 9, 11$ at $n = 5, 7, 9, 11$. A self-contained verifier accompanies the note.

1 Setting and result

Let $n \geq 3$ be odd. A *symmetric ideal solution of size n over $\mathbb{Z}_2$* is a multiset $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}_2$ such that

$$p_{2k+1}(A) = \sum_{i=1}^{n} a_i^{2k+1} = 0 \qquad (0 \leq k \leq (n-3)/2), \tag{1}$$

and $a_i + a_j \neq 0$ for all i, j. Then A and $-A$ are disjoint and have equal power sums of every degree $1, \ldots, n-1$, so they form an ideal solution of the Prouhet–Tarry–Escott (PTE) problem. Writing $F(X) = \prod_i (X - a_i)$, condition (1) says that $F(X) = XH(X^2) - c$ with H monic, and $F(0) = -c$ gives $c = \prod_i a_i$. The associated constant is $C = \prod_i (X - a_i) - \prod_i (X + a_i) = -2c$, hence

$$\mathrm{v}_2(C) = 1 + \sum_i \mathrm{v}_2(a_i) = 1 + t, \qquad t := \sum_{a_i \text{ even}} \mathrm{v}_2(a_i). \tag{2}$$

The *profile* of A is the multiset of the valuations of its even entries; u denotes their number and $w = n - u$ the number of odd entries. Since $p_1(A) = 0$, w is even and u is odd.

We use the conventions of [5], where it is shown that $\mathrm{v}_2(C) \geq 12$ at $n = 13$ and an explicit 2-adic solution with $\mathrm{v}_2(C) = 17$ is given (Kleiman's theorem [7] gives 10); the bound $\mathrm{v}_2(C) \geq 14$ at $n = 15$ is proved in [6], which also isolates the two profiles $(7, 1, 1, 1, 1)$ and $(5, 2, 2, 1, 1)$, $w = 8$, as the only open cases at $n = 13$, $t = 11$.

Theorem 1. *For symmetric ideal solutions of size 13 over $\mathbb{Z}_2$, the minimum of $\mathrm{v}_2(C)$ is 13. More precisely:*

(i) every such solution has $\mathrm{v}_2(C) \geq 13$;

(ii) there is a solution with $\mathrm{v}_2(C) = 13$*, of profile* $(4, 2, 2, 1, 1, 1, 1)$*,* $w = 6$*.*

In particular 2^{13} *divides the constant of every symmetric ideal solution of size* 13 *over* $\mathbb{Z}$*.*

Theorem 2. *For symmetric ideal solutions of size* 15 *over* $\mathbb{Z}_2$*, the minimum of* $\mathrm{v}_2(C)$ *is* 14*: the lower bound is that of [6], recovered independently below, and it is attained by a solution of profile* $(5, 2, 2, 1, 1, 1, 1)$*,* $w = 8$*.*

The lower bounds are computer-assisted (Section 3); the upper bounds are explicit certificates (Section 4).

2 The row law as an exact equality

The following is Proposition 2.1 of [5] (the evaluation goes back to Rees–Smyth [9] and to Coppersmith, Mossinghoff, Scheinerman and VanderKam [2]); in [5] it is used at the level of valuations. Here it is used as an equality, units included.

Lemma 3 (row law)**.** *Let* $A = \{a_1, \dots, a_n\}$ *be a symmetric ideal solution of odd size* n *over* $\mathbb{Z}_2$*. Then for every* j

$$R_j := \prod_{i=1}^{n} (a_j + a_i) = 2 \prod_{i=1}^{n} a_i. \tag{3}$$

Proof. $\prod_i (X + a_i) = -F(-X) = XH(X^2) + c$ and $\prod_i (X - a_i) = XH(X^2) - c$, so their difference is the constant $2c$; at $X = a_j$ the second product vanishes, and $c = \prod_i a_i$. □

3 The lower bound

Fix n, a profile $(f_1, \dots, f_u)$ with $t = \sum f_k$, and suppose that a solution A with this profile exists. Multiplying A by a 2-adic unit preserves (1), the profile and $\mathrm{v}_2(C)$; we may therefore assume that one odd entry equals 1. For $L \geq 1$ let $\tilde{a}_i \in [0, 2^L)$ be the residue of a_i modulo 2^L.

Lemma 4 (necessary conditions at level L)**.** *The residues* $\tilde{a}_i$ *satisfy:*

(N1) for an even entry of valuation f*:* $\tilde{a}_i \equiv 2^f \pmod{2^{f+1}}$ *if* $L > f$*, and* $\tilde{a}_i = 0$ *if* $L \leq f$*; the odd entries are odd, and the normalised entry equals* 1*;*

(N2) $p_{2k+1}(\tilde{a}) \equiv 0 \pmod{2^L}$ *for* $0 \leq k \leq (n-3)/2$*;*

(N3) for every j*, with* $s_{ji} = (\tilde{a}_j + \tilde{a}_i) \bmod 2^L$ *and* $v'(s) = \mathrm{v}_2(s)$ *if* $s \neq 0$*,* $v'(0) = L$*:* $\sum_i v'(s_{ji}) \leq 1 + t$*; if moreover every* $s_{ji} \neq 0$*, then* $\sum_i \mathrm{v}_2(s_{ji}) = 1 + t$*, and the units* $U_j = \prod_i s_{ji} 2^{-\mathrm{v}_2(s_{ji})}$ *satisfy* $U_j \equiv U_k \pmod{2^{\min(\pi_j, \pi_k)}}$*, where* $\pi_j = L - \max_i \mathrm{v}_2(s_{ji})$*.*

Proof. (N1) is the definition of the profile. (N2): p_{2k+1} has integer coefficients and $p_{2k+1}(A) = 0$. (N3): $a_j + a_i \equiv s_{ji} \pmod{2^L}$, so $\mathrm{v}_2(a_j + a_i) = \mathrm{v}_2(s_{ji})$ when $s_{ji} \neq 0$ and $\geq L$ otherwise, and the unit part of $a_j + a_i$ is congruent to that of s_{ji} modulo $2^{L - \mathrm{v}_2(s_{ji})}$. By Lemma 3 and (2), every R_j equals $2 \prod a_i$, of valuation $1 + t$; the units of the R_j are therefore all equal. □

All conditions are invariant under permutations of entries of equal valuation (other than the normalised one). Let S_L be the set of tuples of residues modulo 2^L satisfying (N1)–(N3), taken up to these permutations. Every element of S_{L+1} reduces modulo 2^L to an element of S_L, and S_{L+1} is computed exactly by testing all 2^n bit extensions of every element of S_L compatible with (N1). *If* $S_L = \emptyset$ *for some* L*, no solution with that profile exists.*

Proposition 5. *For $n = 13$ and each of the 98 profiles with u odd and $t \le 11$, and for $n = 15$ and each of the 135 profiles with u odd and $t \le 12$, the set S_L is empty for some $L \le 4$.*

This is a finite computation (a few seconds); it is replayed by the verifier. By (2), $\mathrm{v}_2(C) \le 12$ at $n = 13$ means $t \le 11$, and then $u \le t < n$ forces $w \ge 2$, so the normalisation is available; Proposition 5 therefore proves part (i) of Theorem 1, and the lower bound of Theorem 2.

Calibration. The same procedure, applied without change to the sizes where the answer is known, gives the following; "killed" means $S_L = \emptyset$, "alive" means $S_6 \neq \emptyset$.

n	profiles, $t \le$ known-2	killed	bound	known	witness profiles alive
5	2	yes	4	4	(3)
7	6	yes	6	6	$(3,1,1)$
9	23	yes	9	9	$(4,1,1,1,1)$, Letac [8]
11	49	yes	11	11	$(4,2,2,1,1)$, [2]
13	98	yes	13	—	$(4,2,2,1,1,1,1)$ (this note); $(6,3,3,1,1,1,1)$ [5]
15	135	yes	14	—	$(5,2,2,1,1,1,1)$ (this note)

The known values are due to Rees–Smyth [9, Prop. 2.4] at $n = 5$, Caley [4] at $n = 7$, Filaseta–Markovich [3] at $n = 9$, and [5] at $n = 11$ (with the solutions of [2]).

4 The upper bound

Proposition 6. *There is a symmetric ideal solution of size 13 over $\mathbb{Z}_2$ with entries congruent modulo 2^{48} to*

222431608172976, 261066633817732, 68195931130444, 34160080136710,
226052145870210, 40449915965842, 121758620517478, 1,
253186536047243, 36685992419751, 148962866138169, 95119401340895,
180780128706485,

whose valuations are $4,2,2,1,1,1,1$ and six times 0; hence $\mathrm{v}_2(C) = 1 + 12 = 13$.

Proof. The verifier contains 13 integers b_i (the residues above are their reductions) such that $p_{2k+1}(b) \equiv 0 \pmod{2^{380}}$ for $0 \le k \le 5$. Keep the seven entries other than the six odd ones fixed and regard $f = (p_1, p_3, \dots, p_{11})$ as a function of the six odd entries. Its Jacobian determinant is

$$\det\left((2k+1)\, b_i^{2k}\right) = \prod_{k=0}^{5} (2k+1) \prod_{x<y} (b_y^2 - b_x^2),$$

a Vandermonde determinant in the squares, of valuation $d = 59$. Since $380 > 2d$, the multivariate Hensel lemma [1, Theorem 4.1] gives an exact zero a of f with $a_i \equiv b_i \pmod{2^{380-59}}$. As 321 exceeds every valuation involved, a has the valuations of b, and $a_i + a_j \neq 0$ because $\prod_i (b_j + b_i) \not\equiv 0 \pmod{2^{321}}$ for every j. By (2), $\mathrm{v}_2(C) = 13$. □

Proposition 7. *There is a symmetric ideal solution of size 15 over $\mathbb{Z}_2$ with entries congruent modulo 2^{48} to*

47066783113760, 247202769948004, 207640644201852, 240838204572682,
53708447253522, 173853332231430, 118966605088222, 1,
210029373016823, 54748232418229, 10125580704469, 274115814525415,
129269563341747, 152037868484161, 50721618074275,

whose valuations are $5, 2, 2, 1, 1, 1, 1$ *and eight times* 0*; hence* $\mathrm{v}_2(C) = 1 + 13 = 14$.

Proof. As for Proposition 6: the verifier contains 15 integers with $p_{2k+1} \equiv 0 \pmod{2^{480}}$ for $0 \le k \le 6$; for a suitable choice of seven of the eight odd entries the Jacobian determinant of $(p_1, \ldots, p_{13})$ has valuation $d = 76$, and $480 > 2d$. □

5 Status

- PROVED (computer-assisted): $\mathrm{v}_2(C) \ge 13$ for every symmetric ideal solution of size 13 over $\mathbb{Z}_2$, hence over $\mathbb{Z}$ (Proposition 5).
- PROVED (explicit certificate): the value 13 is attained over $\mathbb{Z}_2$ (Proposition 6). Together: the minimum is exactly 13 at $n = 13$.
- PROVED: at $n = 15$ the minimum is exactly 14 (Propositions 5 and 7, with the lower bound of [6]).
- VERIFIED: the procedure reproduces the known values at $n = 5, 7, 9, 11$, the bound of [6] at $n = 15$, and keeps alive the profiles of all known witnesses.
- OPEN: no ideal solution of size 13 or 15 over $\mathbb{Z}$ is known; whether these values are attained over $\mathbb{Z}$ is open.

Ancillary file. `anc/VERIF_PTE2ADIC_N13_N15.py` (Python 3, standard library only) replays the arithmetic self-test, the row law on integer witnesses, the calibration in both directions, the 98 eliminations at $n = 13$ and the 135 at $n = 15$, and the two Hensel certificates.

Credits. The row law in valuation form, the profile framework and the bound 12 at $n = 13$ are from [5], the bound 14 at $n = 15$ from [6]; the evaluation underlying the row law is that of [9, 2]. The values at $n = 5, 7, 9$ are due to Rees–Smyth [9], Caley [4] and Filaseta–Markovich [3]; the witness at $n = 9$ is Letac's [8], those at $n = 11$ are from [2]. The research and the writing were carried out with Claude (Anthropic); the computations are replayed by the ancillary verifier.

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