hexagon:2611.00569v1

UNIQUE CONTINUATION FOR THE PAULI–FIERZ HAMILTONIAN

MIHÁLY A. CSIRIK

Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Göteborg, Sweden

Abstract. We prove a unique continuation property for eigenstates of the Pauli–Fierz Hamiltonian, viewed as functions of the electron configuration with values in the spin–photon space. The vector potential may lie in L^6_{loc}, with its divergence, curl, and the scalar potential in L^p_{loc} for some $p > 2$. As an application, we prove a Hohenberg–Kohn theorem at positive coupling strength: the spin-summed ground-state density and combined current density determine the external potentials up to a gauge transformation and an additive constant.

1. Introduction

A differential operator is said to have the unique continuation property if every solution that vanishes on an open set, or to infinite order at a point, vanishes identically. Carleman [1] introduced the weighted inequalities now bearing his name to prove such results. For Schrödinger operators $-\Delta + V$ on $\mathbb{R}^n$, Jerison and Kenig [2] proved unique continuation under the scale-invariant condition $V \in L^{n/2}_{\mathrm{loc}}$; see [3] for a review of such sharp results. Singular magnetic fields and nonsmooth coefficients were treated by Kurata [4] and by Koch and Tataru [5]. In density-functional theory, the property enters through the Hohenberg–Kohn theorem [6]; see [8] for a review. As Lieb emphasized [7], for the usual L^p potentials its proof requires that ground states do not vanish on sets of positive measure. For Laplace form-bounded distributional potentials, Corso recently showed that the relevant condition is instead strict positivity of the density quasi-everywhere, and verified it for non-interacting systems [9]; see also [10, 11] in one dimension.

For N electrons, the unique continuation theorems of [2, 4, 5] must be applied on $\mathbb{R}^{3N}$, where their conditions deteriorate with N, although the physical potentials are sums of one- and two-body terms. For instance, the condition of Jerison and Kenig becomes $V \in L^{3N/2}_{\mathrm{loc}}$. Garrigue [12] removed this dependence. He proved strong unique continuation for many-body Schrödinger operators whose one- and two-body potentials lie in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ with $p > 2$, for every N, and deduced the Hohenberg–Kohn theorem under the same assumptions. In a sequel [13] he treated the many-body Pauli Hamiltonian with external magnetic fields. Garrigue's main tool is a Carleman estimate with the singular weight $-\log|\mathbf{x}| + (-\log|\mathbf{x}|)^{-\alpha}$ that also controls fractional derivatives, so that each potential is estimated in its own

Email: csirik@gmail.com.

three-dimensional variables. For the Hohenberg–Kohn theorem, unique continuation from sets of positive measure suffices. He obtains it from an argument of de Figueiredo and Gossez [14], in which a local energy inequality shows that an eigenfunction vanishes to infinite order at every Lebesgue density point of its zero set.

Quantum-electrodynamical density-functional theory (QEDFT) extends density-functional theory to electrons coupled to quantized photons. Rajagopal formulated a time-dependent version [15]. Ruggenthaler and coworkers developed a time-dependent Kohn–Sham approach to relativistic quantum electrodynamics [16] and a hierarchy of such theories [17]. In the non-relativistic limit, described by the Pauli–Fierz Hamiltonian, the basic variables are the electronic current density and the expectation value of the quantized vector potential [17]. The ground-state theory [18] derives a Hohenberg–Kohn result: the external scalar potential and a transversal external current acting on the photon field are in one-to-one correspondence with the electronic density and the expectation value of the vector potential. As in the original argument [6], the step determining the scalar potential tacitly assumes that the ground state does not vanish on a set of positive measure. In the dipole approximation, Tokatly [19] used the electronic density, or current density, together with the expectation values of the photon mode coordinates. The time-dependent works [16, 17, 19] derive the time-dependent analogue of the Hohenberg–Kohn theorem, the Runge–Gross theorem, for their variables.

In this paper, we carry out Garrigue's program to derive a unique continuation property for the Pauli–Fierz Hamiltonian. Two new challenges arise: the eigenstates take values in the tensor product of the electron spin space and the photon Fock space, and the quantized vector potential is, at each point, an unbounded operator on the photon Fock space rather than a function. Our Carleman estimate uses the same weight as in [12, 13] but keeps the covariant kinetic energy and the radiation Hamiltonian together in the principal part. Under an additional ultraviolet moment condition on the cutoff, Theorem 3.1 gives unique continuation for external vector potentials in L^6_{loc}, with the divergence, curl, and scalar potential in L^p_{loc} for some $p > 2$. As an application, Theorem 3.2 gives a Hohenberg–Kohn theorem at positive coupling. For this theorem we follow [20]: unlike in [18], the photon field carries no external current, and the basic variables are the spin-summed density and the combined current density, including the curl of the magnetization density. The proofs are deferred to Section 4, and the Carleman estimates they rely on are proved in Appendix A. Counterexamples to the Hohenberg–Kohn theorem at zero coupling, both without and with electron spin, are constructed in Appendix B.

Acknowledgements. The author thanks T. C. Corso for his useful comments on the manuscript. The author acknowledges funding from the European Research Council through ERC-2021-STG Grant Agreement No. 101041487 (REGAL).

Lean formalization. The unique continuation and Hohenberg–Kohn theorems, the Carleman estimates in Appendix A, and both zero-coupling counterexamples in Appendix B are formalized in Lean 4, building on the formalization of [20]. The formalization is in the directory `lean` of the repository of this paper[1]. Standard results from analysis and operator theory enter as common assumptions.

AI disclosure. The large language models GPT-6 Astra (OpenAI), and Claude Opus 5.5 and Claude Fable 5.2 (Anthropic), were used throughout the development of this paper and its Lean formalization. The author has reviewed all content and takes full responsibility for the contents of this paper.

[1] https://github.com/xyzw/pf_ucp

2. Setting

We consider $N \geq 1$ spin-1/2 electrons. We follow the conventions and notations of [20], although that paper only considers spinless electrons. The Hilbert space and spin–photon fiber are

$$\mathcal{H}_N = \bigwedge^N L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{\mathrm{ph}}, \qquad \mathcal{K}_N = (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{F}_{\mathrm{ph}},$$

$$\mathcal{F}_{\mathrm{ph}} = \mathcal{F}_s(\mathfrak{H}_{\mathrm{ph}}), \qquad \mathfrak{H}_{\mathrm{ph}} = L^2(\mathbb{R}^3 \times \{1,2\}).$$

We write $\Omega \in \mathcal{F}_{\mathrm{ph}}$ for the vacuum. The dispersion relation is $\omega(\mathbf{k}) = |\mathbf{k}|$, and the radiation Hamiltonian is

$$H_{\mathrm{rad}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \omega(\mathbf{k})\, a^*(\mathbf{k},\lambda) a(\mathbf{k},\lambda) \mathrm{d}\mathbf{k},$$

where $a^*(\mathbf{k},\lambda)$ and $a(\mathbf{k},\lambda)$ create and annihilate a photon with momentum $\mathbf{k}$ and polarization λ. To define the Hamiltonian, the photon field must be smeared out by an ultraviolet cutoff $\widehat{\chi}$; without it, the quantized vector potential $\mathfrak{A}(\mathbf{x})$ is only an operator-valued distribution in $\mathbf{x}$, not an operator-valued function [21]. Throughout, the cutoff satisfies

$$\widehat{\chi}(\mathbf{k}) = \overline{\widehat{\chi}(-\mathbf{k})}, \qquad \widehat{\chi} \in L^2(\mathbb{R}^3), \qquad \widehat{\chi}/\sqrt{\omega} \in L^2(\mathbb{R}^3). \tag{1}$$

With real transverse polarization vectors $\boldsymbol{\varepsilon}_\lambda$, set

$$G^\mu_{\mathbf{x}}(\mathbf{k},\lambda) = \frac{\varepsilon^\mu_\lambda(\mathbf{k}) \widehat{\chi}(-\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2}}, \qquad \mathfrak{A}^\mu(\mathbf{x}) = a^*(G^\mu_{\mathbf{x}}) + a(G^\mu_{\mathbf{x}}). \tag{2}$$

Here, for $f \in \mathfrak{H}_{\mathrm{ph}}$, $a^*(f)$ and $a(f)$ are the photon creation and annihilation operators on $\mathcal{F}_{\mathrm{ph}}$, as in [20]. The operator $a^*(f)$ is linear and $a(f)$ is antilinear in f, they are mutually adjoint, and they satisfy the canonical commutation relations

$$[a(g), a^*(f)] = \langle g, f \rangle_{\mathfrak{H}_{\mathrm{ph}}}, \qquad [a(g), a(f)] = [a^*(g), a^*(f)] = 0, \qquad f, g \in \mathfrak{H}_{\mathrm{ph}}.$$

By (1), $G^\mu_{\mathbf{x}} \in \mathfrak{H}_{\mathrm{ph}}$, so $\mathfrak{A}^\mu(\mathbf{x})$ is well-defined.

For fixed $e \geq 0$, the covariant derivative and external Hamiltonian are

$$\begin{aligned} \mathbf{D}_n &= -\mathrm{i}\boldsymbol{\nabla}_{\mathbf{x}_n} - e\mathfrak{A}(\mathbf{x}_n), \\ H^{v,\mathbf{A}} &= \sum_{n=1}^N \left[\frac{1}{2} \big(\boldsymbol{\sigma}_n \cdot (\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n)) \big)^2 + v(\mathbf{x}_n) \right] + \sum_{1 \leq n < m \leq N} \frac{1}{|\mathbf{x}_n - \mathbf{x}_m|} + H_{\mathrm{rad}}. \end{aligned} \tag{3}$$

Here $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is the vector of Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\boldsymbol{\sigma}_n$ denotes $\boldsymbol{\sigma}$ acting on the spin factor of the nth electron in $(\mathbb{C}^2)^{\otimes N}$, so that $\boldsymbol{\sigma}_n \cdot \mathbf{b} = \sum_{\mu=1}^3 \sigma^\mu_n b^\mu$ for $\mathbf{b} \in \mathbb{C}^3$. Moreover, we assume that the real external potentials belong to

$$v \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3), \qquad \mathbf{A} \in L^3(\mathbb{R}^3, \mathbb{R}^3) + L^\infty(\mathbb{R}^3, \mathbb{R}^3). \tag{4}$$

The common closed form domain $\mathcal{Q}_{\mathrm{PF}}$ is the antisymmetric subspace of

$$\mathcal{Q} = H^1(\mathbb{R}^{3N}, \mathcal{K}_N) \cap D(H_{\mathrm{rad}}^{1/2}),$$

where the radiation operator acts fiberwise. Closedness and semiboundedness under these assumptions are established in Proposition 4.1. We denote by $H^{v,\mathbf{A}}$ the self-adjoint operator associated with this closed form, and write $\langle\Phi, H^{v,\mathbf{A}}\Psi\rangle$ for the form evaluated at $\Phi, \Psi \in \mathcal{Q}_{\mathrm{PF}}$. By the first representation theorem [22, Theorem VI.2.1], $\Psi \in \mathcal{Q}_{\mathrm{PF}}$ is an eigenstate of $H^{v,\mathbf{A}}$ with eigenvalue E if and only if it satisfies the weak eigenvalue equation

$$\langle\Phi, H^{v,\mathbf{A}}\Psi\rangle = E\langle\Phi, \Psi\rangle \qquad \text{for all } \Phi \in \mathcal{Q}_{\mathrm{PF}}. \tag{5}$$

We also use the spinless operator

$$H^{v,\mathbf{A}}_{\mathrm{sl}} = \sum_{n=1}^{N}\left[\frac{1}{2}\big(\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n)\big)^2 + v(\mathbf{x}_n)\right] + \sum_{1\le n<m\le N}\frac{1}{|\mathbf{x}_n - \mathbf{x}_m|} + H_{\mathrm{rad}}.$$

It acts on the space $\mathcal{H}^{\mathrm{sl}}_N = \bigwedge^N L^2(\mathbb{R}^3)\otimes\mathcal{F}_{\mathrm{ph}}$, with form domain $\mathcal{Q}^{\mathrm{sl}}_{\mathrm{PF}}$, or componentwise on spin when explicitly stated. As for $H^{v,\mathbf{A}}$, it is realized as the self-adjoint operator associated with its closed form, and its eigenstates are the solutions $\Psi \in \mathcal{Q}^{\mathrm{sl}}_{\mathrm{PF}}$ of the corresponding weak eigenvalue equation. With

$$\boldsymbol{\pi}_n = \mathbf{D}_n + \mathbf{A}(\mathbf{x}_n), \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \mathfrak{B} = \boldsymbol{\nabla}\times\mathfrak{A},$$

the following identity of quadratic forms holds whenever $\omega\widehat{\chi} \in L^2(\mathbb{R}^3)$ and the distributional curl satisfies $\mathbf{B} \in L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$: for every compactly supported $\Psi \in \mathcal{Q}$,

$$\|(\boldsymbol{\sigma}_n\cdot\boldsymbol{\pi}_n)\Psi\|^2 = \|\boldsymbol{\pi}_n\Psi\|^2 + \langle\Psi, \boldsymbol{\sigma}_n\cdot\big(\mathbf{B}(\mathbf{x}_n) - e\mathfrak{B}(\mathbf{x}_n)\big)\Psi\rangle. \tag{6}$$

For $\Phi, \Psi \in L^2(\mathbb{R}^{3N}, \mathcal{K}_N)$, the partial inner product

$$\langle\Phi, \Psi\rangle_{\mathbf{x}} = \int_{\mathbb{R}^{3(N-1)}}\langle\Phi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N), \Psi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N)\rangle_{\mathcal{K}_N}\,\mathrm{d}\mathbf{x}_2\cdots\mathrm{d}\mathbf{x}_N$$

integrates out all electronic positions except $\mathbf{x}_1 = \mathbf{x}$; the inner product of $\mathcal{K}_N$ sums over all spin indices and integrates out the photons. It is defined for a.e. $\mathbf{x} \in \mathbb{R}^3$ and is integrable in $\mathbf{x}$. For a normalized $\Psi \in \mathcal{Q}_{\mathrm{PF}}$, the spin-summed density, magnetization density, and current densities are

$$\begin{aligned}
\rho_\Psi(\mathbf{x}) &= N\langle\Psi, \Psi\rangle_{\mathbf{x}},\\
\mathbf{m}_\Psi(\mathbf{x}) &= N\langle\Psi, \boldsymbol{\sigma}_1\Psi\rangle_{\mathbf{x}},\\
\mathbf{j}^{\mathrm{p}}_\Psi(\mathbf{x}) &= N\,\mathrm{Im}\langle\Psi, \boldsymbol{\nabla}_{\mathbf{x}}\Psi\rangle_{\mathbf{x}},\\
\mathbf{j}^{\mathrm{ph}}_\Psi(\mathbf{x}) &= -eN\,\mathrm{Re}\langle\Psi, \mathfrak{A}(\mathbf{x})\Psi\rangle_{\mathbf{x}},\\
\mathbf{j}^{\mathrm{c}}_\Psi &= \mathbf{j}^{\mathrm{p}}_\Psi + \mathbf{j}^{\mathrm{ph}}_\Psi + \tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{m}_\Psi,\\
\mathbf{j}_\Psi &= \mathbf{j}^{\mathrm{c}}_\Psi + \rho_\Psi\mathbf{A}.
\end{aligned} \tag{7}$$

Here $\mathbf{j}^{\mathrm{p}}_\Psi$ is the paramagnetic current density, $\mathbf{j}^{\mathrm{ph}}_\Psi$ the field-induced current density, and $\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{m}_\Psi$ the magnetization current density; their sum $\mathbf{j}^{\mathrm{c}}_\Psi$ is the combined current density. Finally, $\mathbf{j}_\Psi$ is the physical current density for the specified external vector potential. In contrast, $\mathbf{j}^{\mathrm{c}}_\Psi$ is a function of the state alone at fixed e, χ. The bounds in [20, Proposition 4.1] are stated for spinless states. Their proof regards $\mathbf{x}_1 \mapsto \Psi(\mathbf{x}_1, \cdot)$ as an H^1 function with values in the Hilbert space of the remaining variables, so it applies verbatim when this space contains the spin factor. They give $\rho_\Psi \in L^1\cap L^3$ and $\mathbf{j}^{\mathrm{p}}_\Psi, \mathbf{j}^{\mathrm{ph}}_\Psi \in L^1\cap L^{3/2}$. Similarly, $\boldsymbol{\nabla}\mathbf{m}_\Psi \in L^1\cap L^{3/2}$, and hence $\mathbf{j}^{\mathrm{c}}_\Psi \in L^1\cap L^{3/2}$.

Let $\mathcal{C}_{\mathrm{PF}} \subset \mathcal{Q}_{\mathrm{PF}}$ be the antisymmetric part of $C_c^\infty(\mathbb{R}^{3N}, (\mathbb{C}^2)^{\otimes N})$ tensored algebraically with finite-photon vectors whose one-photon factors have bounded momentum support. Since the form norm of $H^{v,\mathbf{A}}$ is equivalent to that of $\mathcal{Q}$ (Proposition 4.1), $\mathcal{C}_{\mathrm{PF}}$ is a form core for $H^{v,\mathbf{A}}$. For smooth potentials, expanding the square in (3) gives, as operators on $\mathcal{C}_{\mathrm{PF}}$,

$$\big(\boldsymbol{\sigma}_n \cdot (\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n))\big)^2 = (\boldsymbol{\sigma}_n \cdot \mathbf{D}_n)^2 + \mathbf{D}_n \cdot \mathbf{A}(\mathbf{x}_n) + \mathbf{A}(\mathbf{x}_n) \cdot \mathbf{D}_n + |\mathbf{A}(\mathbf{x}_n)|^2 + \boldsymbol{\sigma}_n \cdot \mathbf{B}(\mathbf{x}_n).$$

Next, we take the expectation in a normalized $\Psi \in \mathcal{C}_{\mathrm{PF}}$ and sum over n. By the antisymmetry of Ψ, the terms linear in $\mathbf{A}$ contribute $\int_{\mathbb{R}^3} \mathbf{A} \cdot (\mathbf{j}_\Psi^{\mathrm{p}} + \mathbf{j}_\Psi^{\mathrm{ph}})$, the quadratic term contributes $\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{A}|^2 \rho_\Psi$, and the external Zeeman term contributes $\frac{1}{2}\int_{\mathbb{R}^3} \mathbf{B} \cdot \mathbf{m}_\Psi = \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{m}_\Psi)$ after an integration by parts. Collecting terms gives the energy decomposition (cf. [13, Section 2.3.2])

$$\langle \Psi, H^{v,\mathbf{A}} \Psi \rangle = \langle \Psi, H^{0,0} \Psi \rangle + \langle v + \tfrac{1}{2}|\mathbf{A}|^2, \rho_\Psi \rangle + \langle \mathbf{A}, \mathbf{j}_\Psi^{\mathrm{c}} \rangle. \tag{8}$$

We derived (8) for smooth potentials and normalized $\Psi \in \mathcal{C}_{\mathrm{PF}}$. It extends to all potentials satisfying (4) and all normalized $\Psi \in \mathcal{Q}_{\mathrm{PF}}$ by smooth approximation of the potentials, density of $\mathcal{C}_{\mathrm{PF}}$ in the form norm, and continuity of both sides. We write $E(v, \mathbf{A}) = \inf \operatorname{spec}(H^{v,\mathbf{A}})$, which is finite since $H^{v,\mathbf{A}}$ is bounded from below.

3. Main results

We start by establishing a unique continuation property for the Pauli–Fierz Hamiltonian $H^{v,\mathbf{A}}$. A state $\Psi \in \mathcal{H}_N$ is said to *vanish to infinite order* at $\mathbf{X}_0 \in \mathbb{R}^{3N}$ if, for every $m > 0$,

$$\int_{B_r(\mathbf{X}_0)} \|\Psi(\mathbf{X})\|_{\mathcal{K}_N}^2 \, \mathrm{d}\mathbf{X} = \mathcal{O}(r^m) \qquad (r \downarrow 0).$$

This notion connects unique continuation with the positivity of densities. The *de Figueiredo–Gossez argument* [14] (Proposition 4.6) uses a local energy estimate to show that an eigenstate vanishes to infinite order at every Lebesgue density point of its zero set. Hence an eigenstate without zeros of infinite order cannot vanish on a set of positive measure.

In order to state our result, define the collision-free configuration space via

$$\mathcal{U} = \{\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N} : \mathbf{x}_n \neq \mathbf{x}_m \text{ if } n \neq m\}. \tag{9}$$

Our first main result shows that a nonzero eigenstate cannot vanish to infinite order at any point of $\mathcal{U}$.

Theorem 3.1 (Pauli–Fierz unique continuation)**.** *Assume that*

$$\omega \widehat{\chi} \in L^2(\mathbb{R}^3) \tag{10}$$

and that, for some $p > 2$,

$$v \in L^p_{\mathrm{loc}}(\mathbb{R}^3), \qquad \mathbf{A} \in L^6_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3),$$
$$\operatorname{div} \mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^3), \qquad \mathbf{B} \in L^p_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3).$$

An eigenstate Ψ of $H^{v,\mathbf{A}}$ that vanishes to infinite order at any point of $\mathcal{U}$ is identically zero. Moreover, every nonzero eigenstate satisfies

$$\|\Psi(\mathbf{X})\|_{\mathcal{K}_N} > 0 \quad \text{for a.a. } \mathbf{X} \in \mathbb{R}^{3N}, \qquad \rho_\Psi(\mathbf{x}) > 0 \quad \text{for a.a. } \mathbf{x} \in \mathbb{R}^3.$$

The same conclusions hold without the condition on $\mathbf{B}$ for the spinless operator $H_{\mathrm{sl}}^{v,\mathbf{A}}$ on $\mathcal{H}_N^{\mathrm{sl}}$, with the fiber $\mathcal{K}_N$ replaced by $\mathcal{F}_{\mathrm{ph}}$.

The proof in Section 4.2 follows the method of Garrigue [12, 13] and relies on the covariant Carleman estimates of Appendix A.

Theorem 3.2 (Hohenberg–Kohn)**.** *Let $e > 0$, and let the cutoff satisfy $\widehat{\chi} \not\equiv 0$ and $\omega\widehat{\chi} \in L^2(\mathbb{R}^3)$. Let $(v_k, \mathbf{A}_k)$, $k = 1, 2$, be two potential pairs with $\mathbf{B}_k = \boldsymbol{\nabla} \times \mathbf{A}_k$, and suppose that, for at least one k and some $p > 2$,*

$$\begin{aligned} v_k &\in L^p_{\mathrm{loc}}(\mathbb{R}^3), \qquad & \mathbf{A}_k &\in L^6_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3), \\ \operatorname{div} \mathbf{A}_k &\in L^p_{\mathrm{loc}}(\mathbb{R}^3), \qquad & \mathbf{B}_k &\in L^p_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3). \end{aligned} \tag{11}$$

Let Ψ_k be normalized ground states of $H^{v_k, \mathbf{A}_k}$, with energies $E_k = E(v_k, \mathbf{A}_k)$, $k = 1, 2$. Assume that there is a real function $\theta \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$ with $\boldsymbol{\nabla}\theta \in L^3(\mathbb{R}^3, \mathbb{R}^3) + L^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\rho_{\Psi_1} = \rho_{\Psi_2} = \rho, \qquad \mathbf{j}^{\mathrm{c}}_{\Psi_2} = \mathbf{j}^{\mathrm{c}}_{\Psi_1} + \rho \boldsymbol{\nabla}\theta \quad \text{a.e.}$$

Assume moreover that at least one of the following holds: $\widehat{\chi} \neq 0$ almost everywhere, or ρ has an exponential moment, that is, for some $\alpha > 0$,

$$\int_{\mathbb{R}^3} \mathrm{e}^{\alpha|\mathbf{x}|} \rho(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty. \tag{12}$$

Then

$$\boxed{\mathbf{A}_2 = \mathbf{A}_1 - \boldsymbol{\nabla}\theta, \qquad v_1 = v_2 + c \quad \text{a.e.}, \qquad c = \frac{E_1 - E_2}{N}.} \tag{13}$$

The same conclusion holds without the condition on $\mathbf{B}_k$ for ground states of the spinless operators $H^{v_k, \mathbf{A}_k}_{\mathrm{sl}}$, with combined current density $\mathbf{j}^{\mathrm{c}}_\Psi = \mathbf{j}^{\mathrm{p}}_\Psi + \mathbf{j}^{\mathrm{ph}}_\Psi$.

Condition (11) includes molecular Coulomb scalar potentials with locally L^p perturbations for $p > 2$. The singularities at the nuclei are allowed, since $|\mathbf{x}|^{-1} \in L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for every $q < 3$. In particular, the global class $L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $p > 2$, satisfies the scalar hypothesis.

The hard ultraviolet cutoff $\widehat{\chi}_\Lambda(\mathbf{k}) = \mathbb{1}_{\{|\mathbf{k}| \leq \Lambda\}} / \sqrt{|\mathbf{k}|}$, $\Lambda > 0$, satisfies $\omega\widehat{\chi}_\Lambda \in L^2(\mathbb{R}^3)$ but vanishes for $|\mathbf{k}| > \Lambda$, so it is covered only by the exponential-moment alternative (12). Exponential decay below an ionization threshold is known for Pauli–Fierz models under appropriate binding hypotheses [24].

The vector-potential part of the conclusion, restricted to the support of the density, does not need the local regularity (11). In fact, for identical combined current densities, Proposition 4.4 below gives $\rho(\mathbf{A}_1 - \mathbf{A}_2) = 0$ under the remaining hypotheses alone. This is the only step that uses $e > 0$, through the photon variations in Proposition 4.3 below. Note that the unique continuation theorem holds for every $e \geq 0$.

Remark 1 (Regularity away from an exceptional set). The conclusion of Theorem 3.2 remains valid if the conditions in (11) are required only on a connected open set $\mathcal{O} \subset \mathbb{R}^3$ with $|\mathbb{R}^3 \setminus \mathcal{O}| = 0$. The divergence and curl conditions are then understood in the sense of distributions on $\mathcal{O}$. This extension follows from Remark 3 by the same proof. It allows singularities that fail the stated local integrability conditions near the excluded set. For example, $v(\mathbf{x}) = \eta(\mathbf{x})|\mathbf{x}|^{-3/2}$, with a smooth compactly supported cutoff η equal to one near the origin, belongs to $L^{3/2}(\mathbb{R}^3)$ and is locally bounded on $\mathcal{O} = \mathbb{R}^3 \setminus \{0\}$, but is not in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for any $p > 2$.

The positive coupling assumption $e > 0$ cannot be dropped from Theorem 3.2, either without or with electron spin. In Appendix B we construct, for $N = 1$ and $e = 0$, families of bounded azimuthal vector potentials $\mathbf{A}_t$, linear in t and with different magnetic fields for different t, together with scalar potentials v_t, such that every $H^{v_t,\mathbf{A}_t}$ has the same nondegenerate ground state, the hydrogen $1s$ orbital times the photon vacuum (Propositions B.1 and B.2). Hence the density and combined current density do not depend on t, and for a nonzero compact ultraviolet cutoff all hypotheses of Theorem 3.2 other than $e > 0$ hold. The same examples work at $e > 0$ if $\widehat{\chi} = 0$.

4. Proofs

The rest of the paper is devoted to proofs.

4.1. **Preliminaries.** For the convenience of the reader, we recall that all Hamiltonians $H^{v,\mathbf{A}}$ with potentials satisfying (4) are well defined, with a common form domain.

Proposition 4.1 (Form domain)**.** *Let v and $\mathbf{A}$ satisfy (4). The form*

$$q^{v,\mathbf{A}}[\Phi] = \frac{1}{2}\sum_{n=1}^{N} \|\boldsymbol{\sigma}_n \cdot \boldsymbol{\pi}_n \Phi\|^2 + \|H_{\mathrm{rad}}^{1/2}\Phi\|^2 + \int_{\mathbb{R}^{3N}} \Big(\sum_{n=1}^{N} v(\mathbf{x}_n) + \sum_{1\le n<m\le N} \frac{1}{|\mathbf{x}_n - \mathbf{x}_m|} \Big) \|\Phi(\mathbf{X})\|_{\mathcal{K}_N}^2 \, \mathrm{d}\mathbf{X}$$

on $L^2(\mathbb{R}^{3N}, \mathcal{K}_N)$ is closed and bounded below on $\mathcal{Q}$, and its shifted form norm is equivalent to the norm of $H^1(\mathbb{R}^{3N}, \mathcal{K}_N) \cap D(H_{\mathrm{rad}}^{1/2})$. In particular, $H^{v,\mathbf{A}}$ is the self-adjoint operator associated with the restriction of this form to $\mathcal{Q}_{\mathrm{PF}}$, and there are constants $c > 0$ and $C < \infty$ such that

$$q^{v,\mathbf{A}}[\Phi] + C\|\Phi\|^2 \ge c\Big(\sum_{n=1}^{N} \|\boldsymbol{\pi}_n \Phi\|^2 + \|\boldsymbol{\nabla}_{\mathbf{X}}\Phi\|^2 + \|H_{\mathrm{rad}}^{1/2}\Phi\|^2 \Big), \qquad \Phi \in \mathcal{Q}. \tag{14}$$

Proof. For the spinless Hamiltonian, this is the KLMN argument of [20, Section 3.4]. The same argument applies with spin. The only additional facts needed are the free Pauli identity $\|\boldsymbol{\nabla}_{\mathbf{X}}\Phi\|^2 = \sum_n \|\boldsymbol{\sigma}_n \cdot (-\mathrm{i}\boldsymbol{\nabla}_{\mathbf{x}_n})\Phi\|^2$ and $(\boldsymbol{\sigma}_n \cdot \mathbf{a})^2 = |\mathbf{a}|^2$ for real $\mathbf{a} \in \mathbb{R}^3$. □

Next, we derive a consequence of the variational principle: if the ground states of two Hamiltonians share their density and combined current density, then each is also a ground state of the other Hamiltonian.

Proposition 4.2 (Common ground states)**.** *Let $e \ge 0$, let $(v_k, \mathbf{A}_k)$, $k = 1, 2$, be two potential pairs, and let Ψ_k be normalized ground states of $H^{v_k,\mathbf{A}_k}$ with energies $E_k = E(v_k, \mathbf{A}_k)$. Suppose that*

$$\rho_{\Psi_1} = \rho_{\Psi_2} = \rho, \qquad \mathbf{j}^{\mathrm{c}}_{\Psi_1} = \mathbf{j}^{\mathrm{c}}_{\Psi_2} = \mathbf{j}^{\mathrm{c}}.$$

Then Ψ_1 and Ψ_2 are ground states of both $H^{v_1,\mathbf{A}_1}$ and $H^{v_2,\mathbf{A}_2}$, and

$$E_1 - E_2 = \langle v_1 - v_2 + \tfrac{1}{2}(|\mathbf{A}_1|^2 - |\mathbf{A}_2|^2), \rho \rangle + \langle \mathbf{A}_1 - \mathbf{A}_2, \mathbf{j}^{\mathrm{c}} \rangle.$$

This is the analogue of the "HK1 theorem" for paramagnetic CDFT [25, Theorem 1], with the combined current density in place of the paramagnetic one; (8) supplies the required linearity of the energy in $(v+\frac{1}{2}|\mathbf{A}|^2, \mathbf{A})$. Note that the ground states may be degenerate.

Proof. Let $d = \langle v_1 - v_2 + \frac{1}{2}(|\mathbf{A}_1|^2 - |\mathbf{A}_2|^2), \rho\rangle + \langle \mathbf{A}_1 - \mathbf{A}_2, \mathbf{j}^{\mathrm{c}}\rangle$. The variational principle and (8) give

$$E_1 \le \langle \Psi_2, H^{v_1,\mathbf{A}_1}\Psi_2\rangle = E_2 + d, \qquad E_2 \le \langle \Psi_1, H^{v_2,\mathbf{A}_2}\Psi_1\rangle = E_1 - d.$$

The differences $(E_2+d)-E_1$ and $(E_1-d)-E_2$ are nonnegative and sum to zero, so both vanish. Hence $E_1 - E_2 = d$, $\langle \Psi_2, H^{v_1,\mathbf{A}_1}\Psi_2\rangle = E_1$ and $\langle \Psi_1, H^{v_2,\mathbf{A}_2}\Psi_1\rangle = E_2$. Since $E_k = \inf \mathrm{spec}(H^{v_k,\mathbf{A}_k})$, the spectral theorem shows that every normalized $\Phi \in \mathcal{Q}_{\mathrm{PF}}$ with $\langle \Phi, H^{v_k,\mathbf{A}_k}\Phi\rangle = E_k$ satisfies $H^{v_k,\mathbf{A}_k}\Phi = E_k\Phi$. This proves both assertions. □

At positive coupling, we obtain the following restriction on $\rho(\mathbf{A}_1 - \mathbf{A}_2)$.

Proposition 4.3 (The ambiguity left by the cutoff)**.** *Let $e > 0$, let $(v_k, \mathbf{A}_k)$, $k = 1, 2$, be two potential pairs, and let Ψ_k be normalized ground states of $H^{v_k,\mathbf{A}_k}$ with energies $E_k = E(v_k, \mathbf{A}_k)$. Suppose that*

$$\rho_{\Psi_1} = \rho_{\Psi_2} = \rho, \qquad \mathbf{j}^{\mathrm{c}}_{\Psi_1} = \mathbf{j}^{\mathrm{c}}_{\Psi_2} = \mathbf{j}^{\mathrm{c}}.$$

Then

$$\boxed{\begin{aligned} &\mathrm{div}_{\mathbf{x}}\big[\rho(\mathbf{x})\big(\mathbf{A}_1(\mathbf{x}) - \mathbf{A}_2(\mathbf{x})\big)\big] = 0, \\ &\widehat{\chi(-\mathbf{k})\,\Big(\rho(\cdot)\big(\mathbf{A}_1(\cdot) - \mathbf{A}_2(\cdot)\big)\Big)}(\mathbf{k}) = 0 \quad \textit{for a.a. } \mathbf{k}. \end{aligned}} \tag{15}$$

The first identity is distributional in $\mathbf{x}$.

Proof. By Proposition 4.2, we may use the same state $\Psi = \Psi_1$ as a ground state of both Hamiltonians. Its density and combined current density are ρ and $\mathbf{j}^{\mathrm{c}}$. Let $\mathbf{j}_k$ be its physical current density (7) for the vector potential $\mathbf{A}_k$. Then

$$\mathbf{j}_1 - \mathbf{j}_2 = \rho(\mathbf{A}_1 - \mathbf{A}_2).$$

Both $\mathbf{j}_k$ belong to L^1, since $\rho \in L^1 \cap L^{3/2}$ and $\mathbf{A}_k \in L^3 + L^\infty$.

For a real function θ on $\mathbb{R}^3$, let U_θ denote multiplication by $\mathrm{e}^{\mathrm{i}\sum_n \theta(\mathbf{x}_n)}$. For $\theta \in C_c^\infty(\mathbb{R}^3)$ and $t \in \mathbb{R}$, the state $U_{t\theta}\Psi$ belongs to $\mathcal{Q}_{\mathrm{PF}}$, and its energy expectation in $H^{v_k,\mathbf{A}_k}$ is

$$\langle U_{t\theta}\Psi, H^{v_k,\mathbf{A}_k}U_{t\theta}\Psi\rangle = E_k + t\int_{\mathbb{R}^3} \boldsymbol{\nabla}\theta \cdot \mathbf{j}_k + \frac{t^2}{2}\int_{\mathbb{R}^3} \rho|\boldsymbol{\nabla}\theta|^2.$$

Stationarity at $t = 0$ gives $\mathrm{div}\,\mathbf{j}_k = 0$. Subtraction proves $\mathrm{div}\,\big(\rho(\mathbf{A}_1 - \mathbf{A}_2)\big) = 0$. Here the magnetization density is unchanged by the phase, and $\int_{\mathbb{R}^3} \boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\times\mathbf{m}_\Psi = 0$.

For the photon variation, let $f \in D(\omega) \subset \mathfrak{H}_{\mathrm{ph}}$, and set

$$W(tf) = \exp\big(ta^*(f) - ta(f)\big), \qquad c_f^\mu(\mathbf{x}) = 2\,\mathrm{Re}\langle G^\mu_{\mathbf{x}}, f\rangle.$$

The Weyl field shift of [20, Proposition A.6(iii)] gives $W(tf)^*\boldsymbol{\pi}_n W(tf) = \boldsymbol{\pi}_n - et\mathbf{c}_f(\mathbf{x}_n)$. By [20, Corollary A.8], $W(tf)$ also preserves the common form domain. Apply (8) to this shift of the external vector potential and add the radiation-energy

variation. The Pauli cross term includes $-(et/2)\int_{\mathbb{R}^3}\mathbf{c}_f\cdot(\boldsymbol{\nabla}\times\mathbf{m}_\Psi)$, which is already part of the physical current density $\mathbf{j}_k$. Thus

$$\begin{aligned}\langle W(tf)\Psi, H^{v_k,\mathbf{A}_k}W(tf)\Psi\rangle = E_k + t\left(2\,\mathrm{Re}\langle\Psi, a(\omega f)\Psi\rangle - e\int_{\mathbb{R}^3}\mathbf{c}_f\cdot\mathbf{j}_k\right)\\ + t^2\left(\|\sqrt{\omega}f\|^2 + \frac{e^2}{2}\int_{\mathbb{R}^3}\rho|\mathbf{c}_f|^2\right).\end{aligned} \tag{16}$$

All terms are finite. In particular, the annihilation expectation is controlled by $\|\sqrt{\omega}f\|\|H_{\mathrm{rad}}^{1/2}\Psi\|$, and $\mathbf{c}_f$ is bounded.

The linear coefficient in (16) vanishes for both k. Since the photon expectation refers to the same Ψ, subtraction and $e>0$ give

$$\int_{\mathbb{R}^3}\rho\,\mathbf{c}_f\cdot(\mathbf{A}_1-\mathbf{A}_2)=0 \qquad (f\in D(\omega)). \tag{17}$$

The Bochner integral

$$g=\sum_{\mu=1}^{3}\int_{\mathbb{R}^3}\rho(\mathbf{x})\left(A_1^\mu(\mathbf{x})-A_2^\mu(\mathbf{x})\right)G_{\mathbf{x}}^\mu\mathrm{d}\mathbf{x}\in\mathfrak{H}_{\mathrm{ph}} \tag{18}$$

exists, since $\rho(\mathbf{A}_1-\mathbf{A}_2)\in L^1$ and $\|G_{\mathbf{x}}^\mu\|$ is bounded uniformly in $\mathbf{x}$. By (17), $2\,\mathrm{Re}\langle g,f\rangle=0$. Using also $\mathrm{i}f$ and density of $D(\omega)$ in $\mathfrak{H}_{\mathrm{ph}}$ proves $g=0$. Inserting (2) into (18) and evaluating the Bochner integral pointwise gives

$$g(\mathbf{k},\lambda)=\frac{(2\pi)^{3/2}}{\sqrt{2}}\widehat{\chi}(-\mathbf{k})\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\widehat{\left(\rho\,(\mathbf{A}_1-\mathbf{A}_2)\right)}(\mathbf{k}).$$

Thus $\widehat{\chi}(-\mathbf{k})P_{\mathrm{sol}}(\mathbf{k})\widehat{\left(\rho\,(\mathbf{A}_1-\mathbf{A}_2)\right)}(\mathbf{k})=0$ almost everywhere, where $P_{\mathrm{sol}}(\mathbf{k})=\mathbb{1}-\mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2$. On the other hand, $\mathrm{div}\left(\rho(\mathbf{A}_1-\mathbf{A}_2)\right)=0$ gives $\mathbf{k}\cdot\widehat{\left(\rho\,(\mathbf{A}_1-\mathbf{A}_2)\right)}(\mathbf{k})=0$. The transverse projection is therefore redundant for $\mathbf{k}\neq0$, proving (15). □

The continuity equation, which gives the first identity in (15), and the photon variation, which gives (17), have distinct roles. Write $\mathbf{F}=\rho(\mathbf{A}_1-\mathbf{A}_2)$ and decompose, for $\mathbf{k}\neq0$, $\widehat{\mathbf{F}}(\mathbf{k})=P_{\mathrm{sol}}(\mathbf{k})\widehat{\mathbf{F}}(\mathbf{k})+(\mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2)\widehat{\mathbf{F}}(\mathbf{k})$. The photon variation gives $\widehat{\chi}(-\mathbf{k})P_{\mathrm{sol}}(\mathbf{k})\widehat{\mathbf{F}}(\mathbf{k})=0$, so it detects the transverse part only at momenta with $\widehat{\chi}(-\mathbf{k})\neq0$. The continuity equation $\mathrm{div}\,\mathbf{F}=0$ gives $\mathbf{k}\cdot\widehat{\mathbf{F}}(\mathbf{k})=0$, which removes the longitudinal part at all momenta.

Under an additional condition on either the cutoff or the density, (15) determines the vector potential on the support of the density.

Proposition 4.4 (Vector potential on the density support)**.** *Let $e>0$ and $\widehat{\chi}\not\equiv0$, let $(v_k,\mathbf{A}_k)$, $k=1,2$, be two potential pairs, and let Ψ_k be normalized ground states of $H^{v_k,\mathbf{A}_k}$. Suppose that*

$$\rho_{\Psi_1}=\rho_{\Psi_2}=\rho, \qquad \mathbf{j}^{\mathrm{c}}_{\Psi_1}=\mathbf{j}^{\mathrm{c}}_{\Psi_2}=\mathbf{j}^{\mathrm{c}},$$

and that at least one of the following conditions holds.

(i) $\widehat{\chi}(\mathbf{k})\neq0$ for almost every $\mathbf{k}\in\mathbb{R}^3$.
(ii) The density has the exponential moment (12) for some $\alpha>0$.

Then $\rho(\mathbf{A}_1-\mathbf{A}_2)=0$ almost everywhere. In particular, if $\rho>0$ almost everywhere, then $\mathbf{A}_1=\mathbf{A}_2$ almost everywhere.

Proof. By Hölder's inequality, $\rho(\mathbf{A}_1 - \mathbf{A}_2) \in L^1(\mathbb{R}^3, \mathbb{R}^3)$. Under condition (i), (15) of Proposition 4.3 gives $\widehat{\big(\rho\,(\mathbf{A}_1 - \mathbf{A}_2)\big)} = 0$ almost everywhere. Fourier uniqueness on L^1 then gives $\rho(\mathbf{A}_1 - \mathbf{A}_2) = 0$.

Under condition (ii), write $\mathbf{A}_k = \mathbf{A}_{k,3} + \mathbf{A}_{k,\infty}$ with $\mathbf{A}_{k,3} \in L^3$ and $\mathbf{A}_{k,\infty} \in L^\infty$ for $k = 1, 2$. Interpolation gives

$$\|\mathrm{e}^{\alpha|\mathbf{x}|/2}\rho\|_{3/2} \leq \|\mathrm{e}^{\alpha|\mathbf{x}|}\rho\|_1^{1/2}\|\rho\|_3^{1/2}.$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathrm{e}^{\alpha|\mathbf{x}|/2}\rho(\mathbf{x})|\mathbf{A}_1(\mathbf{x}) - \mathbf{A}_2(\mathbf{x})|\mathrm{d}\mathbf{x} \\ &\quad\leq \sum_{k=1}^{2}\Big(\|\mathbf{A}_{k,3}\|_3\|\mathrm{e}^{\alpha|\mathbf{x}|/2}\rho\|_{3/2} + \|\mathbf{A}_{k,\infty}\|_\infty\|\mathrm{e}^{\alpha|\mathbf{x}|/2}\rho\|_1\Big) < \infty.\end{aligned} \tag{19}$$

Since $|\mathrm{e}^{-\mathrm{i}\boldsymbol{\zeta}\cdot\mathbf{x}}| \leq \mathrm{e}^{|\operatorname{Im}\boldsymbol{\zeta}||\mathbf{x}|}$, the bound (19) makes the Fourier integral of each component of $\rho(\mathbf{A}_1 - \mathbf{A}_2)$ converge absolutely and locally uniformly on $\{\boldsymbol{\zeta} \in \mathbb{C}^3 : |\operatorname{Im}\boldsymbol{\zeta}| < \alpha/2\}$. Each component of $\widehat{\big(\rho\,(\mathbf{A}_1 - \mathbf{A}_2)\big)}$ therefore extends holomorphically to this strip and is real analytic on $\mathbb{R}^3$.

Because $\widehat{\chi} \not\equiv 0$ in L^2, the set on which $\widehat{\chi}(-\mathbf{k}) \neq 0$ has positive Lebesgue measure. Equation (15) of Proposition 4.3 makes every component of $\widehat{\big(\rho\,(\mathbf{A}_1 - \mathbf{A}_2)\big)}$ zero almost everywhere on this set. A real-analytic function on a connected open set that vanishes on a set of positive measure is identically zero. Applying this fact to real and imaginary parts gives $\widehat{\big(\rho\,(\mathbf{A}_1 - \mathbf{A}_2)\big)} = 0$ on $\mathbb{R}^3$. Fourier uniqueness gives $\rho(\mathbf{A}_1 - \mathbf{A}_2) = 0$. Division by ρ on $\{\rho > 0\}$ proves the last assertion. □

Remark 2 (The role of conditions (i) and (ii)). Without condition (i) or (ii) of Proposition 4.4, the identities (15) alone do not force $\rho(\mathbf{A}_1 - \mathbf{A}_2) = 0$. For example, the hard cutoff $\widehat{\chi}_\Lambda$ vanishes for $|\mathbf{k}| > \Lambda$. Let $\eta \in C_c^\infty(\mathbb{R}^3)$ be nonzero, real and even, with support in $\{\Lambda + 1 < |\mathbf{k}| < \Lambda + 2\}$, and let $\mathbf{F}$ be the vector field with $\widehat{\mathbf{F}}(\mathbf{k}) = \mathrm{i}\eta(\mathbf{k})\,\mathbf{k} \times \mathbf{e}_3$. Then $\mathbf{F}$ is a nonzero, real, divergence-free Schwartz field with $\widehat{\chi}_\Lambda(-\mathbf{k})\widehat{\mathbf{F}}(\mathbf{k}) = 0$ for all $\mathbf{k}$, so it satisfies both identities in (15) in place of $\rho(\mathbf{A}_1 - \mathbf{A}_2)$. Condition (i) rules this out, since it forces the Fourier transform to vanish almost everywhere. Condition (ii) rules it out as well, since it makes the Fourier transform of $\rho(\mathbf{A}_1 - \mathbf{A}_2)$ real analytic, whereas $\widehat{\mathbf{F}}$ is nonzero with compact support. The example concerns only the identities (15) and is not built from two Pauli–Fierz ground states, so it is not a counterexample at positive coupling. Removing conditions (i) and (ii) would require information beyond (15).

4.2. **Proof of Theorem 3.1.** The total scalar potential

$$V(\mathbf{X}) = \sum_{n=1}^{N} v(\mathbf{x}_n) + \sum_{1\leq n<m\leq N} \frac{1}{|\mathbf{x}_n - \mathbf{x}_m|}$$

has one-body terms in L^p_{loc} and an interaction term that is locally bounded on the collision-free configuration space $\mathcal{U}$ from (9). The scalar perturbation argument below treats the one-body terms in their three-dimensional variables.

We first prove continuation for the spinless Hamiltonian. Its local estimates will then be combined with the magnetic terms to prove Theorem 3.1. The first step of Garrigue's argument [13] applies to the full spinless Pauli–Fierz form. In

particular, H_{rad} need not be estimated as a scalar multiplication operator in this step. All configuration-space balls below are balls in $\mathbb{R}^{3N}$. The form of $H_{\mathrm{sl}}^{v,\mathbf{A}}$ is also defined on the full space $L^2(\mathbb{R}^{3N},\mathcal{F}_{\mathrm{ph}})$, without antisymmetry, with form domain $H^1(\mathbb{R}^{3N},\mathcal{F}_{\mathrm{ph}})\cap D(H_{\mathrm{rad}}^{1/2})$, and we denote it by $\langle\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle$ as well. This is needed because localization by a function of $\mathbf{X}$ does not preserve antisymmetry. We also use the standard creation and annihilation operator bounds [20, Section 3.3], which with (1) give, uniformly in $\mathbf{x}$,

$$\sum_{\mu=1}^{3}\|\mathfrak{A}^{\mu}(\mathbf{x})\xi\|_{\mathcal{F}_{\mathrm{ph}}}^2\le C_\chi\left(\|H_{\mathrm{rad}}^{1/2}\xi\|_{\mathcal{F}_{\mathrm{ph}}}^2+\|\xi\|_{\mathcal{F}_{\mathrm{ph}}}^2\right),\qquad \xi\in D(H_{\mathrm{rad}}^{1/2}).\tag{20}$$

We begin with a Caccioppoli-type inequality, which bounds the local energy of an eigenstate by its L^2 mass on a larger ball. For $\Phi\in L^2(\mathbb{R}^{3N},\mathcal{F}_{\mathrm{ph}})$, write $u_\Phi(\mathbf{X})=\|\Phi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}$.

Lemma 4.5 (Local kinetic energy estimates)**.** *Assume (1) and (4), and let $\Psi\in\mathcal{Q}_{\mathrm{PF}}^{\mathrm{sl}}$ be an eigenstate of $H_{\mathrm{sl}}^{v,\mathbf{A}}$ with eigenvalue E. There is a constant C such that, for every $\mathbf{X}_0\in\mathbb{R}^{3N}$ and $0<r\le 1$, the following hold.*

(i) The local kinetic and radiation energy is bounded by the L^2 mass on the doubled ball,

$$\int_{B_r(\mathbf{X}_0)}\left(\sum_{n=1}^{N}\|\boldsymbol{\pi}_n\Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2+\|H_{\mathrm{rad}}^{1/2}\Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2\right)\mathrm{d}\mathbf{X}\le\frac{C}{r^2}\int_{B_{2r}(\mathbf{X}_0)}\|\Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2\mathrm{d}\mathbf{X}.\tag{21}$$

(ii) For every $\eta\in C_c^\infty(\mathbb{R}^{3N},\mathbb{R})$ with $0\le\eta\le 1$, $\eta=1$ on $B_r(\mathbf{X}_0)$, $\operatorname{supp}\eta\subset B_{2r}(\mathbf{X}_0)$, and $|\boldsymbol{\nabla}\eta|\le C_1/r$,

$$\|\boldsymbol{\nabla}_{\mathbf{X}}(\eta u_\Psi)\|^2\le\frac{C}{r^2}\int_{B_{2r}(\mathbf{X}_0)}u_\Psi(\mathbf{X})^2\mathrm{d}\mathbf{X}.\tag{22}$$

The constant C depends on the Hamiltonian, E, and C_1, but not on r, $\mathbf{X}_0$, or η.

Proof. Let $\mathcal{Q}^{\mathrm{sl}}=H^1(\mathbb{R}^{3N},\mathcal{F}_{\mathrm{ph}})\cap D(H_{\mathrm{rad}}^{1/2})$ be the form domain without antisymmetry, so that $\mathcal{Q}_{\mathrm{PF}}^{\mathrm{sl}}$ is its antisymmetric subspace. Permutations of the electronic coordinates act unitarily on $\mathcal{Q}^{\mathrm{sl}}$ and leave the form invariant. Averaging over permutations, the antisymmetrizer Π_- satisfies $\langle\Pi_-\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle=\langle\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Pi_-\Psi\rangle$ for all $\Phi,\Psi\in\mathcal{Q}^{\mathrm{sl}}$. Since the eigenstate satisfies $\Pi_-\Psi=\Psi$, this gives

$$\langle\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle=\langle\Pi_-\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle,\qquad\langle\Phi,\Psi\rangle=\langle\Pi_-\Phi,\Psi\rangle.$$

The weak eigenvalue equation on the antisymmetric subspace consequently gives

$$\langle\Phi,H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle=E\langle\Phi,\Psi\rangle\tag{23}$$

for every $\Phi\in\mathcal{Q}^{\mathrm{sl}}$.

Claim. *For every $\Phi\in\mathcal{Q}^{\mathrm{sl}}$, $u_\Phi\in H^1(\mathbb{R}^{3N})$ and*

$$|\boldsymbol{\nabla}_{\mathbf{X}}u_\Phi(\mathbf{X})|^2\le\sum_{n=1}^{N}\|(\boldsymbol{\pi}_n\Phi)(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2\qquad\text{for a.a. }\mathbf{X}\in\mathbb{R}^{3N}.\tag{24}$$

Proof of the claim. Fix $\varepsilon > 0$ and put $u_\varepsilon(\mathbf{X}) = (u_\Phi(\mathbf{X})^2 + \varepsilon^2)^{1/2} - \varepsilon$. The map $\xi \mapsto (\|\xi\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \varepsilon^2)^{1/2}$ is continuously differentiable and Lipschitz on $\mathcal{F}_{\mathrm{ph}}$, so the chain rule for $\Phi \in H^1(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$ gives $u_\varepsilon \in H^1(\mathbb{R}^{3N})$ with

$$\partial_{x_n^\mu} u_\varepsilon(\mathbf{X}) = \frac{\mathrm{Re}\langle \Phi(\mathbf{X}), \partial_{x_n^\mu}\Phi(\mathbf{X})\rangle_{\mathcal{F}_{\mathrm{ph}}}}{(u_\Phi(\mathbf{X})^2 + \varepsilon^2)^{1/2}}.$$

By the definition of $\boldsymbol{\pi}_n$,

$$\partial_{x_n^\mu}\Phi(\mathbf{X}) = \mathrm{i}(\pi_n^\mu\Phi)(\mathbf{X}) - \mathrm{i}\big(A^\mu(\mathbf{x}_n) - e\mathfrak{A}^\mu(\mathbf{x}_n)\big)\Phi(\mathbf{X}).$$

The operator $A^\mu(\mathbf{x}_n) - e\mathfrak{A}^\mu(\mathbf{x}_n)$ is symmetric, so its expectation in $\Phi(\mathbf{X})$ is real, and therefore

$$\mathrm{Re}\langle \Phi(\mathbf{X}), \partial_{x_n^\mu}\Phi(\mathbf{X})\rangle_{\mathcal{F}_{\mathrm{ph}}} = \mathrm{Re}\langle \Phi(\mathbf{X}), \mathrm{i}(\pi_n^\mu\Phi)(\mathbf{X})\rangle_{\mathcal{F}_{\mathrm{ph}}} \le u_\Phi(\mathbf{X})\|(\pi_n^\mu\Phi)(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}.$$

These expectations are defined by (1), since $\Phi(\mathbf{X}) \in D(H_{\mathrm{rad}}^{1/2})$ for almost every $\mathbf{X}$. Summing over n and μ gives

$$|\boldsymbol{\nabla}_{\mathbf{X}} u_\varepsilon(\mathbf{X})|^2 \le \sum_{n=1}^N \|(\boldsymbol{\pi}_n\Phi)(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \qquad \text{for almost every } \mathbf{X}.$$

Since $0 \le u_\Phi(\mathbf{X}) - u_\varepsilon(\mathbf{X}) \le \min(u_\Phi(\mathbf{X}), \varepsilon)$, $u_\varepsilon \to u_\Phi$ in L^2 as $\varepsilon \downarrow 0$, while $\boldsymbol{\nabla}_{\mathbf{X}} u_\varepsilon$ stays bounded in L^2. Hence $u_\Phi \in H^1(\mathbb{R}^{3N})$, and $\boldsymbol{\nabla}_{\mathbf{X}} u_\varepsilon$ converges weakly to $\boldsymbol{\nabla}_{\mathbf{X}} u_\Phi$ along a subsequence. The set of $g \in L^2(\mathbb{R}^{3N}, \mathbb{R}^{3N})$ with $|g(\mathbf{X})|^2 \le \sum_n \|(\boldsymbol{\pi}_n\Phi)(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2$ for almost every $\mathbf{X}$ is convex and closed, hence weakly closed, which proves (24). □

The assumption on v and the three-dimensional Sobolev inequality, applied separately to every electronic coordinate and then integrated in the other coordinates, give, with $v_- = \max(-v, 0)$ the negative part of v, for every $\varepsilon > 0$,

$$\sum_{n=1}^N \int_{\mathbb{R}^{3N}} v_-(\mathbf{x}_n) u_\Phi(\mathbf{X})^2 \mathrm{d}\mathbf{X} \le \varepsilon \|\boldsymbol{\nabla}_{\mathbf{X}} u_\Phi\|^2 + C_\varepsilon \|\Phi\|^2. \tag{25}$$

To obtain the arbitrarily small coefficient, split the $L^{3/2}$ part of v_- into a bounded function and an $L^{3/2}$ function of arbitrarily small norm. Since the electron–electron interaction and H_{rad} are nonnegative, (24) and (25) with $\varepsilon = 1/4$ imply

$$\langle \Phi, H_{\mathrm{sl}}^{v,\mathbf{A}}\Phi\rangle \ge \frac{1}{4}\sum_{n=1}^N \|\boldsymbol{\pi}_n\Phi\|^2 + \|H_{\mathrm{rad}}^{1/2}\Phi\|^2 - C_0\|\Phi\|^2. \tag{26}$$

Let η be as in (ii). Multiplication by η preserves $\mathcal{Q}^{\mathrm{sl}}$. Since η is a real scalar function, $\boldsymbol{\pi}_n(\eta\Psi) = \eta\boldsymbol{\pi}_n\Psi - \mathrm{i}(\boldsymbol{\nabla}_{\mathbf{x}_n}\eta)\Psi$. This and the eigenvalue equation (23), tested with $\eta^2\Psi$, give the IMS localization identity

$$\begin{aligned}\langle \eta\Psi, H_{\mathrm{sl}}^{v,\mathbf{A}}\eta\Psi\rangle &= \mathrm{Re}\langle \eta^2\Psi, H_{\mathrm{sl}}^{v,\mathbf{A}}\Psi\rangle + \frac{1}{2}\int_{\mathbb{R}^{3N}} |\boldsymbol{\nabla}_{\mathbf{X}}\eta(\mathbf{X})|^2 u_\Psi(\mathbf{X})^2 \mathrm{d}\mathbf{X} \\ &= E\|\eta\Psi\|^2 + \frac{1}{2}\int_{\mathbb{R}^{3N}} |\boldsymbol{\nabla}_{\mathbf{X}}\eta(\mathbf{X})|^2 u_\Psi(\mathbf{X})^2 \mathrm{d}\mathbf{X}.\end{aligned} \tag{27}$$

Combining (26) and (27) proves

$$\sum_{n=1}^N \|\boldsymbol{\pi}_n(\eta\Psi)\|^2 + \|H_{\mathrm{rad}}^{1/2}\eta\Psi\|^2 + \|\boldsymbol{\nabla}_{\mathbf{X}}(\eta u_\Psi)\|^2 \le Cr^{-2}\int_{B_{2r}(\mathbf{X}_0)} u_\Psi(\mathbf{X})^2 \mathrm{d}\mathbf{X}. \tag{28}$$

Since $\eta = 1$ and $\boldsymbol{\nabla}\eta = 0$ on $B_r(\mathbf{X}_0)$, the product rule above gives $\boldsymbol{\pi}_n(\eta\Psi) = \boldsymbol{\pi}_n\Psi$ and $H_{\mathrm{rad}}^{1/2}(\eta\Psi) = H_{\mathrm{rad}}^{1/2}\Psi$ on $B_r(\mathbf{X}_0)$. Restricting the norms on the left of (28) to $B_r(\mathbf{X}_0)$ therefore proves (21), and dropping the first two terms on the left proves (22). □

Next, the de Figueiredo–Gossez argument [14], in the form used by Garrigue [13], turns the preceding inequality into infinite-order vanishing: an eigenstate vanishes to infinite order at every Lebesgue density point of its zero set.

Proposition 4.6 (Zeros of infinite order, cf. [14])**.** *Assume* (1) *and* (4)*, let* $\Psi \in \mathcal{Q}_{\mathrm{PF}}^{\mathrm{sl}}$ *be an eigenstate of* $H_{\mathrm{sl}}^{v,\mathbf{A}}$ *with eigenvalue* E*, and put*

$$\mathcal{Z} = \{\mathbf{X} \in \mathbb{R}^{3N} : \Psi(\mathbf{X}) = 0\}.$$

At every Lebesgue density point $\mathbf{X}_0$ *of* $\mathcal{Z}$*,* Ψ *vanishes to infinite order, with the norm of* $\mathcal{F}_{\mathrm{ph}}$ *in place of that of* $\mathcal{K}_N$*. Moreover, for every* $m > 0$*,*

$$\int_{B_r(\mathbf{X}_0)} \Big(\sum_{n=1}^{N} \|\boldsymbol{\pi}_n \Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \|H_{\mathrm{rad}}^{1/2}\Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \Big) \mathrm{d}\mathbf{X} = \mathcal{O}(r^m) \qquad (r \downarrow 0). \tag{29}$$

In particular, if $|\mathcal{Z}| > 0$*, such a point exists.*

Proof. Without loss of generality, we can translate $\mathbf{X}_0$ to the origin and set

$$U(r) = \int_{B_r} u_\Psi(\mathbf{X})^2 \mathrm{d}\mathbf{X}.$$

Then $\theta(r) := |B_r \setminus \mathcal{Z}|/|B_r| \to 0$ as $r \downarrow 0$. Let $\eta \in C_c^\infty(\mathbb{R}^{3N}, \mathbb{R})$ satisfy $0 \le \eta \le 1$, $\eta = 1$ on B_r, $\operatorname{supp}\eta \subset B_{2r}$, and $|\boldsymbol{\nabla}\eta| \le C_1/r$, as in Lemma 4.5(ii). Hölder's inequality, the Sobolev inequality on $\mathbb{R}^{3N}$, and (22) of Lemma 4.5 give

$$\begin{aligned} U(r) &\le \|\eta u_\Psi\|_2^2 \\ &\le |B_{2r} \setminus \mathcal{Z}|^{2/(3N)} \|\eta u_\Psi\|_{6N/(3N-2)}^2 \\ &\le C|B_{2r} \setminus \mathcal{Z}|^{2/(3N)} \|\boldsymbol{\nabla}_{\mathbf{X}}(\eta u_\Psi)\|_2^2 \\ &\le C\theta(2r)^{2/(3N)} U(2r). \end{aligned} \tag{30}$$

Fix $m > 0$. Since $\theta(s) \to 0$ as $s \downarrow 0$, there is $r_m \in (0,1)$ such that $C\theta(s)^{2/(3N)} \le 2^{-m}$ whenever $0 < s \le r_m$. Let $k \ge 1$ and apply (30) with $r = 2^{-k} r_m \le 1$. Since $2r = 2^{-k+1} r_m \le r_m$, the choice of r_m gives

$$U(2^{-k} r_m) \le 2^{-m} U(2^{-k+1} r_m).$$

Induction on k yields

$$U(2^{-k} r_m) \le 2^{-km} U(r_m), \qquad k \ge 0.$$

Now let $0 < r \le r_m$, and choose $k \ge 0$ with $2^{-k-1} r_m < r \le 2^{-k} r_m$. Then $2^{-k} < 2r/r_m$. Since U is nondecreasing and $U(r_m) \le \|\Psi\|^2$,

$$U(r) \le U(2^{-k} r_m) \le 2^{-km} U(r_m) \le \Big(\frac{2r}{r_m}\Big)^m \|\Psi\|^2.$$

Thus

$$U(r) \le C_m r^m \quad (0 < r \le r_m), \qquad C_m = \Big(\frac{2}{r_m}\Big)^m \|\Psi\|^2, \tag{31}$$

so Ψ vanishes to infinite order at the origin.

For (29), let $0 < r \le r_{m+2}/2$. Then $r \le 1$ and $2r \le r_{m+2}$, so Lemma 4.5 and (31) with $m+2$ in place of m give

$$\int_{B_r} \Big(\sum_{n=1}^{N} \|\boldsymbol{\pi}_n \Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \|H_{\mathrm{rad}}^{1/2} \Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \Big) \mathrm{d}\mathbf{X} \\ \le \frac{C}{r^2} U(2r) \le \frac{C}{r^2} C_{m+2} (2r)^{m+2} = 2^{m+2} C C_{m+2}\, r^m.$$

The two extra powers of r compensate for the factor r^{-2} in Lemma 4.5.

Finally, by the Lebesgue density theorem, almost every point of $\mathcal{Z}$ is a Lebesgue density point of $\mathcal{Z}$, that is, a point $\mathbf{X}_0$ with $|B_r(\mathbf{X}_0) \setminus \mathcal{Z}|/|B_r(\mathbf{X}_0)| \to 0$ as $r \downarrow 0$. If $|\mathcal{Z}| > 0$, the set of such points has positive measure, and in particular it is nonempty. □

The Carleman estimates in Appendix A are proved for smooth functions. To apply them to a localized eigenstate, we show that the localized state lies in the operator domain of $-\Delta_{\mathbf{X}} + H_{\mathrm{rad}}$ and can be approximated there by smooth functions. This step uses the first ultraviolet moment (10), which includes the hard ultraviolet cutoff.

We repeatedly use the moment inequality: for a nonnegative self-adjoint operator A and $0 \le \theta \le 1$,

$$\|A^\theta u\| \le \|Au\|^\theta \|u\|^{1-\theta}, \qquad u \in D(A). \tag{32}$$

It follows from the spectral theorem and Hölder's inequality for the spectral measure of u. For $\theta < 1$, Young's inequality turns it into $\|A^\theta u\| \le \varepsilon \|Au\| + C_{\varepsilon,\theta} \|u\|$ for every $\varepsilon > 0$.

Lemma 4.7 (Local operator domain)**.** *Assume (1), (4) and (10). Let $\mathcal{O} \subset \mathcal{U}$ be open, with $\mathcal{U}$ as in (9), and let $\mathcal{O}_1 \subset \mathbb{R}^3$ be the union of its electronic coordinate projections. Suppose that, for some $p > 2$,*

$$v \in L^p_{\mathrm{loc}}(\mathcal{O}_1), \qquad \mathbf{A} \in L^6_{\mathrm{loc}}(\mathcal{O}_1, \mathbb{R}^3), \qquad \operatorname{div} \mathbf{A} \in L^p_{\mathrm{loc}}(\mathcal{O}_1),$$

where the divergence is distributional. If Ψ is an eigenstate of $H_{\mathrm{sl}}^{v,\mathbf{A}}$ with eigenvalue E, then for every $\eta \in C_c^\infty(\mathcal{O})$,

$$\eta \Psi \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}}), \tag{33}$$

and

$$(H_{\mathrm{sl}}^{v,\mathbf{A}} - E)(\eta \Psi) = -\mathrm{i} \sum_{n=1}^{N} (\boldsymbol{\nabla}_{\mathbf{x}_n} \eta) \cdot \boldsymbol{\pi}_n \Psi - \frac{1}{2} (\Delta_{\mathbf{X}} \eta) \Psi. \tag{34}$$

Here $H_{\mathrm{sl}}^{v,\mathbf{A}}$ acts on $u \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ with compact support in $\mathcal{O}$ by

$$H_{\mathrm{sl}}^{v,\mathbf{A}} u = T^0 u + \sum_{n=1}^{N} \Big(\mathbf{A}(\mathbf{x}_n) \cdot \mathbf{D}_n u - \frac{\mathrm{i}}{2} (\operatorname{div} \mathbf{A})(\mathbf{x}_n) u + \frac{1}{2} |\mathbf{A}(\mathbf{x}_n)|^2 u \Big) + V u, \tag{35}$$

where $T^0 = \frac{1}{2} \sum_{n=1}^{N} \mathbf{D}_n^2 + H_{\mathrm{rad}}$, V is the total scalar potential, and each term belongs to $L^2(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$.

Proof. Fix $\eta \in C_c^\infty(\mathcal{O})$. Since $\operatorname{supp} \eta$ is compact, there is an open set $\mathcal{V} \supset \operatorname{supp} \eta$ with compact closure in $\mathcal{O}$. Choose $\zeta \in C_c^\infty(\mathcal{V})$ equal to one near $\operatorname{supp} \eta$. We replace $H_{\mathrm{sl}}^{v,\mathbf{A}}$ by an operator $\widetilde{H}_{\mathrm{sl}}$ on $L^2(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$ whose potentials are cut off by ζ. It agrees with $H_{\mathrm{sl}}^{v,\mathbf{A}}$ near $\operatorname{supp} \eta$, and its perturbations are relatively bounded with

respect to $-\Delta_{\mathbf{X}} + H_{\mathrm{rad}}$. We show in Steps 1–3 that $D(\widetilde{H}_{\mathrm{sl}}) = D(-\Delta_{\mathbf{X}} + H_{\mathrm{rad}})$, with equivalent graph norms, and in Step 4 that $\eta\Psi \in D(\widetilde{H}_{\mathrm{sl}})$ and

$$\widetilde{H}_{\mathrm{sl}}(\eta\Psi) = E\eta\Psi - \mathrm{i}\sum_{n=1}^{N}(\boldsymbol{\nabla}_{\mathbf{x}_n}\eta)\cdot\boldsymbol{\pi}_n\Psi - \frac{1}{2}(\Delta_{\mathbf{X}}\eta)\Psi. \tag{36}$$

Since $D(-\Delta_{\mathbf{X}}+H_{\mathrm{rad}}) = D(-\Delta_{\mathbf{X}})\cap D(H_{\mathrm{rad}})$, these two facts prove (33). By Step 1, $\widetilde{H}_{\mathrm{sl}}u$ is given by (35) for $u \in D(\widetilde{H}_{\mathrm{sl}})$ supported where $\zeta = 1$, so (36) also proves (34). Since η is arbitrary, Steps 1–3, applied with a cutoff equal to one near the support of a given u, also show that each term of (35) belongs to L^2.

Step 1: The localized Hamiltonian. Let $K_n \subset \mathcal{O}_1$ be the compact projection of $\overline{\mathcal{V}}$ onto the coordinate $\mathbf{x}_n$, and put $v_n = v\mathbb{1}_{K_n} \in L^p(\mathbb{R}^3)$. Since $\overline{\mathcal{V}}$ is a compact subset of $\mathcal{U}$, the electron–electron interaction is bounded on $\mathcal{V}$. Hence the total scalar potential V from the beginning of this subsection satisfies

$$V(\mathbf{X}) = \sum_{n=1}^{N} v_n(\mathbf{x}_n) + W(\mathbf{X}), \qquad W \in L^\infty(\mathcal{V}),$$

for $\mathbf{X} \in \mathcal{V}$. Let $\widetilde{V}$ be the extension of ζV by zero. It is a real multiplication operator, equal to V near $\operatorname{supp}\eta$. For $n = 1,\dots,N$, let $P_n \subset \mathcal{O}_1$ be the compact projection of $\operatorname{supp}\zeta$ onto the coordinate $\mathbf{x}_n$. Since $\mathbf{A} \in L^6_{\mathrm{loc}}(\mathcal{O}_1,\mathbb{R}^3)$, we have $\mathbb{1}_{P_n}\mathbf{A} \in L^6(\mathbb{R}^3,\mathbb{R}^3)$. Put

$$\widetilde{\mathbf{A}}_n(\mathbf{X}) = \zeta(\mathbf{X})\mathbf{A}(\mathbf{x}_n), \qquad n = 1,\dots,N,$$

with $\widetilde{\mathbf{A}}_n(\mathbf{X}) = 0$ for $\mathbf{X} \notin \operatorname{supp}\zeta$. Then $\widetilde{\mathbf{A}}_n(\mathbf{X}) = \mathbf{A}(\mathbf{x}_n)$ for $\mathbf{X}$ near $\operatorname{supp}\eta$, and $\widetilde{\mathbf{A}}_n(\mathbf{X}) = \zeta(\mathbf{X})(\mathbb{1}_{P_n}\mathbf{A})(\mathbf{x}_n)$ for all $\mathbf{X}$. The distributional product rule gives

$$\boldsymbol{\nabla}_{\mathbf{x}_n}\cdot\widetilde{\mathbf{A}}_n = \zeta\,(\operatorname{div}\mathbf{A})(\mathbf{x}_n) + \boldsymbol{\nabla}_{\mathbf{x}_n}\zeta\cdot\mathbf{A}(\mathbf{x}_n).$$

The first term is bounded in modulus by a one-body L^p multiplier, and the second by a one-body L^6 multiplier. Also $|\widetilde{\mathbf{A}}_n|^2$ is bounded in modulus by a one-body L^3 multiplier. The localized Hamiltonian is

$$\widetilde{H}_{\mathrm{sl}} = \frac{1}{2}\sum_{n=1}^{N}\big(\mathbf{D}_n + \widetilde{\mathbf{A}}_n\big)^2 + H_{\mathrm{rad}} + \widetilde{V} = T^0 + \widetilde{R} + \widetilde{V}, \tag{37}$$

with T^0 as in (35), and

$$\widetilde{R} = \sum_{n=1}^{N}\Big(\widetilde{\mathbf{A}}_n\cdot\mathbf{D}_n - \frac{\mathrm{i}}{2}\boldsymbol{\nabla}_{\mathbf{x}_n}\cdot\widetilde{\mathbf{A}}_n + \frac{1}{2}|\widetilde{\mathbf{A}}_n|^2\Big).$$

Note that $\widetilde{H}_{\mathrm{sl}}$ is not of the form $H_{\mathrm{sl}}^{v',\mathbf{A}'}$, since $\widetilde{\mathbf{A}}_n$ and $\widetilde{V}$ depend on all electronic coordinates. Steps 2 and 3 realize (37) as a self-adjoint operator.

Step 2: The free operator T^0. Theorem 6 of [23] gives $D(T^0) = D(-\Delta_{\mathbf{X}}+H_{\mathrm{rad}})$ and states that any operator core for $-\Delta_{\mathbf{X}} + H_{\mathrm{rad}}$ is an operator core for T^0. Theorem 7 there shows that T^0, defined through its closed form, acts on this domain by the expanded expression $\frac{1}{2}\sum_n \mathbf{D}_n^2 + H_{\mathrm{rad}}$. To check the hypotheses of both theorems, write $\mathfrak{H}_\omega$ for the one-photon space equipped with the norm

$$\|g\|_{\mathfrak{H}_\omega}^2 = \|g\|^2 + \|g/\sqrt{\omega}\|^2.$$

Our coupling functions lie uniformly in $\mathfrak{H}_\omega$. Since $\sqrt{\omega}\widehat{\chi} \in L^2$ follows from $\widehat{\chi}, \omega\widehat{\chi} \in L^2$, the extra assumption (10) puts $\omega G^\mu_{\mathbf{x}}$ and all first electronic derivatives of $G^\mu_{\mathbf{x}}$

uniformly in this space. Moreover, $\sum_{\mu=1}^{3} \partial_{x^\mu} G^\mu_{\mathbf{x}} = 0$, since $\mathbf{k} \cdot \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0$; that is, $\operatorname{div} \mathfrak{A} = 0$. These are Hypothesis (G) of Theorem 6 and the hypotheses of Theorem 7 in that reference. The different positive coefficient of the electronic kinetic energy is accommodated by a linear rescaling of electronic coordinates. The coupling strength is arbitrary. In particular, T^0 and $-\Delta_{\mathbf{X}} + H_{\mathrm{rad}}$ have equivalent graph norms. Put

$$L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1. \tag{38}$$

Step 3: Relative bounds and the domain of $\widetilde{H}_{\mathrm{sl}}$. Let $b \in L^q(\mathbb{R}^3)$ with $q > 2$ and put $s = 3/q$. Then the three-dimensional Sobolev inequality in the nth electronic coordinate gives

$$\begin{aligned} \|b(\mathbf{x}_n)\Phi\|_2 &\le C_q \|b\|_q \|(-\Delta_{\mathbf{x}_n})^{s/2}\Phi\|_2 \\ &\le C_q \|b\|_q \|(-\Delta_{\mathbf{X}})^{s/2}\Phi\|_2. \end{aligned} \tag{39}$$

The Sobolev inequality holds for Hilbert-space-valued functions. We apply it with values in $\mathcal{F}_{\mathrm{ph}}$ at fixed values of the remaining electronic coordinates and then integrate over these coordinates. For $q = 6$, apply this estimate to electronic derivatives and to $(H_{\mathrm{rad}} + 1)^{1/2}\Phi$. The electronic and radiation spectral multipliers commute, and

$$\|(-\Delta_{\mathbf{x}_n})^{1/4}(H_{\mathrm{rad}} + 1)^{1/2}\Phi\|_2 \le \|L^{3/4}\Phi\|_2.$$

Together with the pointwise field bound (20), this yields, for $\mathbf{b} \in L^6(\mathbb{R}^3, \mathbb{R}^3)$,

$$\|\mathbf{b}(\mathbf{x}_n) \cdot \mathbf{D}_n \Phi\|_2 \le C \|\mathbf{b}\|_6 \|L^{3/4}\Phi\|_2. \tag{40}$$

Since $\widetilde{\mathbf{A}}_n = \zeta\, (\mathbb{1}_{P_n}\mathbf{A})(\mathbf{x}_n)$ and ζ is bounded, (40) with $\mathbf{b} = \mathbb{1}_{P_n}\mathbf{A}$ bounds $\widetilde{\mathbf{A}}_n \cdot \mathbf{D}_n$. The scalar terms are bounded by $\|L^{s/2}\Phi\|_2$ with $s/2 < 3/4$, and the first-order term by $\|L^{3/4}\Phi\|_2$. Since these powers of L are below one, the moment inequality (32) gives, for every $\varepsilon > 0$,

$$\|\widetilde{R}\Phi\|_2 + \|\widetilde{V}\Phi\|_2 \le \varepsilon \|L\Phi\|_2 + C_\varepsilon \|\Phi\|_2.$$

By Step 2, both perturbations are infinitesimally bounded with respect to T^0. The distributional product rule, justified first by smooth approximation and then by these bounds, shows that $\widetilde{R}$ is symmetric on $D(T^0)$ and agrees there with the difference of the kinetic forms. The Kato–Rellich theorem [22, Theorem V.4.3] identifies $T^0 + \widetilde{R} + \widetilde{V}$ with the form realization of (37) and gives

$$D(\widetilde{H}_{\mathrm{sl}}) = D(-\Delta_{\mathbf{X}} + H_{\mathrm{rad}}), \tag{41}$$

again with equivalent graph norms and the same operator cores.

Step 4: The localized eigenvalue equation. Since $\zeta = 1$ near $\operatorname{supp}\eta$, the form of $\widetilde{H}_{\mathrm{sl}}$ agrees with that of $H^{v,\mathbf{A}}_{\mathrm{sl}}$ on pairs $(\Phi, \eta\Psi)$ with $\Phi \in \mathcal{Q}^{\mathrm{sl}}$. The weak eigenvalue equation, extended to full-space tests as in (23), is tested with $\eta\Phi$. The product rule $\boldsymbol{\pi}_n(\eta\Psi) = \eta\boldsymbol{\pi}_n\Psi - \mathrm{i}(\boldsymbol{\nabla}_{\mathbf{x}_n}\eta)\Psi$ then gives (36) in the form sense. The right-hand side of (36) is in L^2, since Ψ is in the form domain. The first representation theorem for closed forms [22, Theorem VI.2.1] puts $\eta\Psi$ in $D(\widetilde{H}_{\mathrm{sl}})$. □

Let $\mathcal{F}_{\mathrm{fin}} \subset \mathcal{F}_{\mathrm{ph}}$ be the span of Ω and of the vectors $a^*(h_1)\cdots a^*(h_m)\Omega$ with $m \ge 1$ and $h_1, \dots, h_m \in \mathfrak{H}_{\mathrm{ph}}$ of bounded support, and let $\mathcal{C} \subset L^2(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$ be the span of the functions $\mathbf{X} \mapsto \varphi(\mathbf{X})\xi$ with $\varphi \in C^\infty_c(\mathbb{R}^{3N})$ and $\xi \in \mathcal{F}_{\mathrm{fin}}$.

Lemma 4.8 (Approximation with prescribed support)**.** *Let w be a compactly supported element of $D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$, and let $\mathcal{G} \subset \mathbb{R}^{3N}$ be an open neighborhood of* $\operatorname{supp} w$*. Then there are $w_j \in \mathcal{C}$ with* $\operatorname{supp} w_j \subset \mathcal{G}$ *and*

$$\|w_j - w\| + \|\Delta_{\mathbf{X}}(w_j - w)\| + \|H_{\mathrm{rad}}(w_j - w)\| \to 0 \qquad (j \to \infty). \tag{42}$$

The same holds for $w \in L^2(\mathbb{R}^{3N}, \mathcal{K}_N)$, with $\mathcal{C}$ replaced by $\mathcal{C} \otimes (\mathbb{C}^2)^{\otimes N}$.

Proof. Let $L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1$ as in (38). Since $-\Delta_{\mathbf{X}}$ and H_{rad} are commuting nonnegative self-adjoint operators, $D(L) = D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$, and the graph norm of L is equivalent to the norm in (42). The set $\mathcal{C}$ is the algebraic tensor product of the core $C_c^\infty(\mathbb{R}^{3N})$ of $-\Delta_{\mathbf{X}}$ and the core $\mathcal{F}_{\mathrm{fin}}$ of H_{rad}, and hence an operator core for L. Choose $\Phi_j \in \mathcal{C}$ with $\Phi_j \to w$ in the graph norm of L, and let $\zeta \in C_c^\infty(\mathcal{G})$ be equal to one near $\operatorname{supp} w$. Then $w_j = \zeta\Phi_j \in \mathcal{C}$, $\operatorname{supp} w_j \subset \mathcal{G}$, and

$$Lw_j = \zeta L\Phi_j - 2\boldsymbol{\nabla}_{\mathbf{X}}\zeta \cdot \boldsymbol{\nabla}_{\mathbf{X}}\Phi_j - (\Delta_{\mathbf{X}}\zeta)\Phi_j.$$

Since $\|\boldsymbol{\nabla}_{\mathbf{X}}\Phi\|^2 \le \|\Phi\|\|L\Phi\|$, also $\boldsymbol{\nabla}_{\mathbf{X}}\Phi_j \to \boldsymbol{\nabla}_{\mathbf{X}}w$, and therefore $w_j \to \zeta w = w$ in the graph norm of L. This proves (42). The version with spin follows by applying this argument to each spin component, since $-\Delta_{\mathbf{X}}$ and H_{rad} act componentwise on $(\mathbb{C}^2)^{\otimes N}$. □

Proof of Theorem 3.1 in the spinless case. Suppose first that Ψ vanishes to infinite order at $\mathbf{X}_0 \in \mathcal{U}$. Without loss of generality, we can translate this point to the origin. Fix $\alpha > 0$, and let

$$r = |\mathbf{X}|, \qquad t = -\log r, \qquad f(t) = t + t^{-\alpha}, \qquad \beta(t) = \ddot{f}(t) = \alpha(\alpha+1)t^{-\alpha-2}.$$

Choose $R > 0$ with $\overline{B_R} \subset \mathcal{U}$. On B_R the electron–electron interaction is bounded, so $V - E$ is a sum of one-body L^p terms and a bounded term. Since $H_{\mathrm{sl}}^{v,\mathbf{A}} - E = T^{\mathbf{A}} + (V - E)$ with $T^{\mathbf{A}}$ as in Appendix A, Lemma A.4 applies with $W = V - E$. After decreasing R, it gives $\tau_0 \ge 1$ and $C < \infty$ such that, for $\tau \ge \tau_0$ and every u in the local operator domain of $H_{\mathrm{sl}}^{v,\mathbf{A}}$ with compact support in $B_R \setminus \{0\}$,

$$\tau^3 \int_{B_R} \beta(t)\mathrm{e}^{2\tau f(t)} r^{-2} \|u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \le C \int_{B_R} \mathrm{e}^{2\tau f(t)} r^2 \|(H_{\mathrm{sl}}^{v,\mathbf{A}} - E)u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X}. \tag{43}$$

This is (73) of Lemma A.4, keeping only the first term on its left-hand side. By Lemma 4.5, the covariant derivatives of Ψ also vanish to infinite order at the origin, in the sense that $\int_{B_s} \sum_n \|\boldsymbol{\pi}_n\Psi(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} = \mathcal{O}(s^m)$ as $s \downarrow 0$ for every $m > 0$.

Choose $\zeta \in C_c^\infty(B_R)$ with $\zeta = 1$ on $B_{R/2}$, and choose a radial cutoff η_ε equal to zero on B_ε and one outside $B_{2\varepsilon}$, with $|\boldsymbol{\nabla}\eta_\varepsilon| \le C/\varepsilon$ and $|\Delta\eta_\varepsilon| \le C/\varepsilon^2$. For $0 < \varepsilon < R/4$, the localized vector $u_\varepsilon = \zeta\eta_\varepsilon\Psi$ belongs to the local operator domain by Lemma 4.7, and (34) of Lemma 4.7 with $\zeta\eta_\varepsilon$ in place of η gives

$$(H_{\mathrm{sl}}^{v,\mathbf{A}} - E)u_\varepsilon = -\mathrm{i}\sum_{n=1}^N \boldsymbol{\nabla}_{\mathbf{x}_n}(\zeta\eta_\varepsilon) \cdot \boldsymbol{\pi}_n\Psi - \frac{1}{2}\Delta_{\mathbf{X}}(\zeta\eta_\varepsilon)\Psi.$$

Inserting $u = u_\varepsilon$ in (43) therefore gives

$$\begin{aligned} &\tau^3 \int_{B_R} \beta(t)\mathrm{e}^{2\tau f(t)} r^{-2} \|u_\varepsilon(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \\ &\quad \le C \int_{B_R} \mathrm{e}^{2\tau f(t)} r^2 \Big\| \sum_{n=1}^N \boldsymbol{\nabla}_{\mathbf{x}_n}(\zeta\eta_\varepsilon) \cdot \boldsymbol{\pi}_n\Psi(\mathbf{X}) - \frac{\mathrm{i}}{2}\Delta_{\mathbf{X}}(\zeta\eta_\varepsilon)\Psi(\mathbf{X}) \Big\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X}. \end{aligned} \tag{44}$$

The terms with derivatives of η_ε are supported in $B_{2\varepsilon} \setminus B_\varepsilon$, where $f(t) \leq f(-\log \varepsilon)$. For fixed τ, their contribution to the right-hand side of (44) is at most

$$C\mathrm{e}^{2\tau f(-\log\varepsilon)} \left(\varepsilon^{-2} \int_{B_{2\varepsilon}} \|\Psi(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} + \int_{B_{2\varepsilon}} \sum_{n=1}^{N} \|\boldsymbol{\pi}_n \Psi(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} \right).$$

Since $\mathrm{e}^{2\tau f(-\log\varepsilon)} = \mathcal{O}_\tau(\varepsilon^{-2\tau})$, infinite-order vanishing makes this expression tend to zero. The terms with derivatives of ζ are supported where $R/2 \leq r < R$ and do not depend on ε. Taking $\varepsilon \downarrow 0$ and using Fatou's lemma on the left gives

$$\tau^3 \int_{B_{R/2}} \beta(t) \mathrm{e}^{2\tau f(t)} r^{-2} \|\Psi(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} \leq C_\Psi \mathrm{e}^{2\tau f(-\log(R/2))}, \tag{45}$$

where C_Ψ is independent of τ. For $r < R/4$, the weight $f(-\log r)$ is at least $f(-\log(R/4)) > f(-\log(R/2))$, and $\beta(-\log r) r^{-2}$ has a positive lower bound. Letting $\tau \to \infty$ in (45) proves $\Psi = 0$ on $B_{R/4}$.

This local vanishing propagates throughout $\mathcal{U}$. The set $\mathcal{U}$ is open, connected and of full measure. Indeed, paths in $\mathbb{R}^3$ can avoid any finite set of points. Moving the electron positions one at a time, through distinct auxiliary positions if necessary, therefore connects any two configurations in $\mathcal{U}$. Its complement in $\mathbb{R}^{3N}$ consists of diagonals of codimension three. To justify the propagation, let $\mathcal{Z}_{\mathrm{open}}$ be the set of points having a neighborhood on which $\Psi = 0$ almost everywhere. It is nonempty and open. On each sufficiently small ball compactly contained in $\mathcal{U}$, the preceding argument gives a vanishing radius that is uniform in the center. If a point belongs to the relative closure of $\mathcal{Z}_{\mathrm{open}}$, choose a point of $\mathcal{Z}_{\mathrm{open}}$ closer than half this uniform radius. Infinite-order vanishing at that interior point then supplies a zero neighborhood of the closure point. Thus $\mathcal{Z}_{\mathrm{open}}$ is also relatively closed. Connectedness of $\mathcal{U}$ implies $\mathcal{Z}_{\mathrm{open}} = \mathcal{U}$. Since its complement has measure zero, $\Psi = 0$ almost everywhere on the full configuration space $\mathbb{R}^{3N}$.

Now suppose that the configuration-space zero set of a nonzero Ψ has positive measure. It has a Lebesgue density point in $\mathcal{U}$, because $\mathcal{U}$ has full measure. Proposition 4.6 gives infinite-order vanishing there, contradicting the conclusion just proved. This establishes the nonvanishing assertion of Theorem 3.1 for the spinless Hamiltonian. If ρ_Ψ vanished on a set $\Sigma \subset \mathbb{R}^3$ of positive measure, then by the definition of ρ_Ψ and Tonelli's theorem, Ψ would vanish almost everywhere on $\Sigma \times \mathbb{R}^{3(N-1)}$, a set of positive measure, contradicting the nonvanishing just proved. This gives the density-positivity assertion. □

With spin, the local kinetic energy estimates take the following form.

Lemma 4.9 (Local kinetic energy estimates with spin)**.** *Assume* (1) *and* (4)*, and let* $\Psi \in \mathcal{Q}_{\mathrm{PF}}$ *be an eigenstate of* $H^{v,\mathbf{A}}$ *with eigenvalue* E*. There is a constant* C *such that, for every* $\mathbf{X}_0 \in \mathbb{R}^{3N}$ *and* $0 < r \leq 1$*, the following hold.*

(i) The local kinetic and radiation energy is bounded by the L^2 *mass on the doubled ball,*

$$\int_{B_r(\mathbf{X}_0)} \Big(\sum_{n=1}^{N} \|\boldsymbol{\pi}_n \Psi(\mathbf{X})\|^2_{\mathcal{K}_N} + \|H^{1/2}_{\mathrm{rad}} \Psi(\mathbf{X})\|^2_{\mathcal{K}_N} \Big) \mathrm{d}\mathbf{X} \leq \frac{C}{r^2} \int_{B_{2r}(\mathbf{X}_0)} \|\Psi(\mathbf{X})\|^2_{\mathcal{K}_N} \mathrm{d}\mathbf{X}. \tag{46}$$

(ii) For every η as in Lemma 4.5(ii),

$$\|\boldsymbol{\nabla}_{\mathbf{X}}(\eta\Psi)\|^2 \le \frac{C}{r^2}\int_{B_{2r}(\mathbf{X}_0)} \|\Psi(\mathbf{X})\|^2_{\mathcal{K}_N}\mathrm{d}\mathbf{X}. \tag{47}$$

The constant C depends on the Hamiltonian, E, and C_1, but not on r, $\mathbf{X}_0$, or η.

Proof. Simultaneous permutations of the position and spin variables act unitarily on $\mathcal{Q}$ and leave $q^{v,\mathbf{A}}$ invariant. As in the proof of Lemma 4.5, averaging over permutations and $\Pi_-\Psi = \Psi$ show that Ψ satisfies (5) for every $\Phi \in \mathcal{Q}$, not only for antisymmetric Φ. Let η be as in Lemma 4.5(ii). The product rule and

$$\|(\boldsymbol{\sigma}_n \cdot \boldsymbol{\nabla}_{\mathbf{x}_n}\eta(\mathbf{X}))\Psi(\mathbf{X})\|^2_{\mathcal{K}_N} = |\boldsymbol{\nabla}_{\mathbf{x}_n}\eta(\mathbf{X})|^2\|\Psi(\mathbf{X})\|^2_{\mathcal{K}_N}$$

give the IMS localization identity for the Pauli form

$$q^{v,\mathbf{A}}[\eta\Psi] = E\|\eta\Psi\|^2 + \frac{1}{2}\int_{\mathbb{R}^{3N}} |\boldsymbol{\nabla}_{\mathbf{X}}\eta(\mathbf{X})|^2\|\Psi(\mathbf{X})\|^2_{\mathcal{K}_N}\mathrm{d}\mathbf{X}.$$

Applying (14) of Proposition 4.1 to $\eta\Psi$ and using $r \le 1$ and $|\boldsymbol{\nabla}\eta| \le C_1/r$, we obtain

$$\sum_{n=1}^{N}\|\boldsymbol{\pi}_n(\eta\Psi)\|^2 + \|H_{\mathrm{rad}}^{1/2}\eta\Psi\|^2 + \|\boldsymbol{\nabla}_{\mathbf{X}}(\eta\Psi)\|^2 \le \frac{C}{r^2}\int_{B_{2r}(\mathbf{X}_0)}\|\Psi(\mathbf{X})\|^2_{\mathcal{K}_N}\mathrm{d}\mathbf{X}.$$

Restricting the norms on the left to $B_r(\mathbf{X}_0)$, as in the proof of Lemma 4.5, proves (46), and dropping the first two terms on the left proves (47). □

With spin, we split off the Zeeman terms. Set

$$S_{\mathrm{ext}}(\mathbf{X}) = \frac{1}{2}\sum_{n=1}^{N}\boldsymbol{\sigma}_n \cdot \mathbf{B}(\mathbf{x}_n), \qquad S_{\mathrm{ph}}(\mathbf{X}) = -\frac{e}{2}\sum_{n=1}^{N}\boldsymbol{\sigma}_n \cdot \boldsymbol{\mathfrak{B}}(\mathbf{x}_n).$$

Thus, by (6), $H^{v,\mathbf{A}} = H_{\mathrm{sl}}^{v,\mathbf{A}} + S_{\mathrm{ext}} + S_{\mathrm{ph}}$ in the sense of forms on compactly supported functions, where the spinless operator on the right acts componentwise on spin.

The quantized magnetic field has coupling functions

$$K^{\mu}_{\mathbf{x}}(\mathbf{k},\lambda) = -\frac{\mathrm{i}}{\sqrt{2}}(\mathbf{k}\times\boldsymbol{\varepsilon}_\lambda(\mathbf{k}))^\mu\widehat{\chi}(-\mathbf{k})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \mathfrak{B}^\mu(\mathbf{x}) = a^*(K^\mu_{\mathbf{x}}) + a(K^\mu_{\mathbf{x}}).$$

Put $M_{1/2} = \|\sqrt{\omega}\widehat{\chi}\|_2$ and $M_1 = \|\omega\widehat{\chi}\|_2$. Cauchy–Schwarz gives $M_{1/2}^2 \le \|\widehat{\chi}\|_2 M_1$. The orthonormal transverse polarizations give, uniformly in $\mathbf{x}$,

$$\sum_\mu \|K^\mu_{\mathbf{x}}/\sqrt{\omega}\|^2_{\mathfrak{H}_{\mathrm{ph}}} = M_{1/2}^2, \qquad \sum_\mu \|K^\mu_{\mathbf{x}}\|^2_{\mathfrak{H}_{\mathrm{ph}}} = M_1^2.$$

The creation and annihilation bounds [20, Section 3.3] therefore yield

$$\|\boldsymbol{\mathfrak{B}}(\mathbf{x})\xi\|^2 \le 8M_{1/2}^2\|H_{\mathrm{rad}}^{1/2}\xi\|^2 + 2M_1^2\|\xi\|^2, \qquad \xi \in D(H_{\mathrm{rad}}^{1/2}). \tag{48}$$

The same bound holds componentwise on the spin space. In particular, S_{ph} is bounded from the radiation form domain to the fiber, uniformly in $\mathbf{X}$. This uses only the first ultraviolet moment (10).

The local operator domain of Lemma 4.7 carries over to the Hamiltonian with spin.

Lemma 4.10 (Local operator domain with spin)**.** *Assume (1), (4) and (10), let $\mathcal{O}$, $\mathcal{O}_1$, p, v and $\mathbf{A}$ be as in Lemma 4.7, and suppose in addition that $\mathbf{B} \in L^p_{\mathrm{loc}}(\mathcal{O}_1, \mathbb{R}^3)$. If $\Psi \in \mathcal{Q}_{\mathrm{PF}}$ is an eigenstate of $H^{v,\mathbf{A}}$ with eigenvalue E, then for every $\eta \in C_c^\infty(\mathcal{O})$, $\eta\Psi \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ and*

$$(H^{v,\mathbf{A}} - E)(\eta\Psi) = -\mathrm{i}\sum_{n=1}^{N}(\boldsymbol{\nabla}_{\mathbf{x}_n}\eta)\cdot\boldsymbol{\pi}_n\Psi - \frac{1}{2}(\Delta_{\mathbf{X}}\eta)\Psi. \tag{49}$$

Here $H^{v,\mathbf{A}}u = H^{v,\mathbf{A}}_{\mathrm{sl}}u + S_{\mathrm{ext}}u + S_{\mathrm{ph}}u$ for $u \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ with compact support in $\mathcal{O}$, with $H^{v,\mathbf{A}}_{\mathrm{sl}}u$ given by (35) componentwise on the spin space, and each term belongs to $L^2(\mathbb{R}^{3N}, \mathcal{K}_N)$.

Proof. Fix $\eta \in C_c^\infty(\mathcal{O})$, and let $\mathcal{V} \supset \operatorname{supp}\eta$ be an open set with compact closure in $\mathcal{O}$, as in the proof of Lemma 4.7. On $\mathcal{V}$, take the local realization $\widetilde{H}_{\mathrm{sl}}$ from Lemma 4.7 and let it act componentwise on the spin space. Choose a real cutoff of S_{ext} that agrees with S_{ext} near $\operatorname{supp}\eta$, and denote it by $\widetilde{S}_{\mathrm{ext}}$. It can be bounded in operator norm by a finite sum of one-body L^p multipliers, since $\mathbf{B} \in L^p_{\mathrm{loc}}$. For $L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1$, the estimate (39) and the magnetic-field bound give

$$\|\widetilde{S}_{\mathrm{ext}}\Phi\|_2 \le C\|L^{3/(2p)}\Phi\|_2, \qquad \|S_{\mathrm{ph}}\Phi\|_2 \le C\|L^{1/2}\Phi\|_2.$$

Both operators are infinitesimally bounded relative to $\widetilde{H}_{\mathrm{sl}}$, whose graph norm is equivalent to that of L. The Kato–Rellich theorem [22, Theorem V.4.3] thus gives

$$D(\widetilde{H}) = D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}}), \qquad \widetilde{H} = \widetilde{H}_{\mathrm{sl}} + \widetilde{S}_{\mathrm{ext}} + S_{\mathrm{ph}}, \tag{50}$$

with equivalent graph norms and the operator core $\mathcal{C} \otimes (\mathbb{C}^2)^{\otimes N}$. Both added terms also have infinitesimal form bounds on $\mathcal{Q}$. For $\widetilde{S}_{\mathrm{ext}}$, use the compactly supported L^p coefficients in the $L^{3/2}$ Sobolev form bound. For S_{ph}, use (48). Thus this operator is associated with the corresponding closed form sum. The form identity (6) with the external field holds on this neighborhood. It follows first on $\mathcal{C} \otimes (\mathbb{C}^2)^{\otimes N}$ by integration by parts and then on localized form vectors by continuity, using (4), the Sobolev inequality, and $\mathbf{B} \in L^p_{\mathrm{loc}} \subset L^{3/2}_{\mathrm{loc}}$. Thus $\widetilde{H}$ agrees locally with the form realization of $H^{v,\mathbf{A}}$. The independently chosen truncation of S_{ext} need not be the curl of the truncated vector potential outside that neighborhood.

Both Zeeman terms commute with scalar electronic cutoffs. Therefore the localized weak equation is still

$$\widetilde{H}(\eta\Psi) = E\eta\Psi - \mathrm{i}\sum_{n=1}^{N}(\boldsymbol{\nabla}_{\mathbf{x}_n}\eta)\cdot\boldsymbol{\pi}_n\Psi - \frac{1}{2}(\Delta_{\mathbf{X}}\eta)\Psi.$$

Its right-hand side belongs to L^2, since $\boldsymbol{\pi}_n\Psi \in L^2$ by (14) of Proposition 4.1. The first representation theorem [22, Theorem VI.2.1] and (50) give $\eta\Psi \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$. For u in this domain supported where the cutoffs of $\widetilde{H}_{\mathrm{sl}}$ and $\widetilde{S}_{\mathrm{ext}}$ equal one, $\widetilde{H}u$ is the expression for $H^{v,\mathbf{A}}u$ in the statement, by Step 1 of the proof of Lemma 4.7 and the choice of $\widetilde{S}_{\mathrm{ext}}$. Taking $u = \eta\Psi$ proves (49). The terms $S_{\mathrm{ext}}u$ and $S_{\mathrm{ph}}u$ belong to L^2 by the bounds above. □

Proof of Theorem 3.1. All norms in this proof use the spin–photon fiber $\mathcal{K}_N = (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{F}_{\mathrm{ph}}$. The Sobolev chain rule gives, for almost every $\mathbf{X} \in \mathbb{R}^{3N}$, the pointwise bound $|\boldsymbol{\nabla}_{\mathbf{X}}(\eta(\mathbf{X})\|\Psi(\mathbf{X})\|_{\mathcal{K}_N})| \le \|\boldsymbol{\nabla}_{\mathbf{X}}(\eta(\mathbf{X})\Psi(\mathbf{X}))\|_{\mathcal{K}_N}$, so (47) of Lemma 4.9

replaces (22) of Lemma 4.5 in the proof of Proposition 4.6. Hence that proof, including the density recurrence (30), applies to the total spinor norm, and Ψ vanishes to infinite order at every density point of its zero set. Moreover, by Lemma 4.9(i), infinite-order vanishing of Ψ at a point implies that $\int_{B_s} \sum_n \|\boldsymbol{\pi}_n \Psi(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} = \mathcal{O}(s^m)$ as $s \downarrow 0$ for every $m > 0$.

Now let Ψ vanish to infinite order at a point of $\mathcal{U}$, translated to the origin. Choose $R > 0$ with $\overline{B_R} \subset \mathcal{U}$. On B_R, the total scalar potential V is the sum of the one-body terms $v(\mathbf{x}_n)$ and the bounded electron–electron interaction. Let $H^{V,\mathbf{A}} = T^{\mathbf{A}} + V + S_{\mathrm{ext}} + S_{\mathrm{ph}}$. We apply Lemma A.5 with $\mathbf{X}_0 = 0$, $v_n = v$, V_b the electron–electron interaction and E the eigenvalue, and decrease R so that $R \le r_0$. Keeping only the first term on the left of (75) of Lemma A.5, we obtain $\tau_0 \ge 1$ and $C < \infty$ such that, for $\tau \ge \tau_0$ and every $u \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ with compact support in $B_R \setminus \{0\}$,

$$\tau^3 \int_{B_R} \beta(t) \mathrm{e}^{2\tau f(t)} r^{-2} \|u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \le C \int_{B_R} \mathrm{e}^{2\tau f(t)} r^2 \|(H^{V,\mathbf{A}} - E) u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X}.$$

Let ζ, η_ε and $u_\varepsilon = \zeta \eta_\varepsilon \Psi$ be as in the spinless case. By (35) of Lemma 4.7, $H^{v,\mathbf{A}} u = H^{V,\mathbf{A}} u$ for $u \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ with compact support in B_R. Thus Lemma 4.10, applied with $\mathcal{O} = B_R$ and $\eta = \zeta\eta_\varepsilon$, gives $u_\varepsilon \in D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ and

$$(H^{V,\mathbf{A}} - E) u_\varepsilon = -\mathrm{i} \sum_{n=1}^{N} \boldsymbol{\nabla}_{\mathbf{x}_n}(\zeta\eta_\varepsilon) \cdot \boldsymbol{\pi}_n \Psi - \frac{1}{2} \Delta_{\mathbf{X}}(\zeta\eta_\varepsilon) \Psi.$$

Inserting $u = u_\varepsilon$ in the Carleman estimate above gives, for $0 < \varepsilon < R/4$,

$$\begin{aligned} &\tau^3 \int_{B_R} \beta(t) \mathrm{e}^{2\tau f(t)} r^{-2} \|u_\varepsilon(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \\ &\quad \le C \int_{B_R} \mathrm{e}^{2\tau f(t)} r^2 \Big\| \sum_{n=1}^{N} \boldsymbol{\nabla}_{\mathbf{x}_n}(\zeta\eta_\varepsilon) \cdot \boldsymbol{\pi}_n \Psi(\mathbf{X}) - \frac{\mathrm{i}}{2} \Delta_{\mathbf{X}}(\zeta\eta_\varepsilon) \Psi(\mathbf{X}) \Big\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X}, \end{aligned}$$

which is (44) with norms in $\mathcal{K}_N$. As in the spinless case, the infinite-order local energy bounds make the inner-cutoff error vanish as $\varepsilon \downarrow 0$, and Fatou's lemma on the left gives

$$\tau^3 \int_{B_{R/2}} \beta(t) \mathrm{e}^{2\tau f(t)} r^{-2} \|\Psi(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \le C_\Psi \mathrm{e}^{2\tau f(-\log(R/2))},$$

where C_Ψ is independent of τ. Letting $\tau \to \infty$ proves $\Psi = 0$ on $B_{R/4}$. This vanishing propagates through the connected collision-free configuration space $\mathcal{U}$. If the zero set of a nonzero spinor had positive measure, the density-point argument would produce infinite-order vanishing at a point of $\mathcal{U}$, a contradiction. Positivity almost everywhere of the spin-summed density follows as in the spinless case: if ρ_Ψ vanished on a set $\Sigma \subset \mathbb{R}^3$ of positive measure, then by Tonelli's theorem Ψ would vanish almost everywhere on $\Sigma \times \mathbb{R}^{3(N-1)}$. □

Remark 3 (Unique continuation off an exceptional set). The local integrability assumptions of Theorem 3.1 may instead be imposed on a connected open set $\mathcal{O} \subset \mathbb{R}^3$ with $|\mathbb{R}^3 \setminus \mathcal{O}| = 0$. The same conclusions hold with $\mathcal{U}$ replaced by $\mathcal{U} \cap \mathcal{O}^N$ and the divergence and curl understood in the sense of distributions on $\mathcal{O}$. For the spinless version the curl condition is omitted. This configuration space is still connected, since paths in a connected open subset of $\mathbb{R}^3$ can avoid finitely many

points, and it has full measure. The local regularity of v and $\mathbf{A}$ enters the proof only through Lemmas 4.7 and 4.10 and the Carleman estimates of Appendix A. Their constants depend only on local norms of the potentials and are uniform on compact subsets of $\mathcal{U} \cap \mathcal{O}^N$. Hence the proof applies unchanged with $\mathcal{U}$ replaced by $\mathcal{U} \cap \mathcal{O}^N$.

4.3. **Proof of Theorem 3.2.** First consider the case $\theta = 0$, in which $\rho_{\Psi_1} = \rho_{\Psi_2} = \rho$ and $\mathbf{j}^{\mathrm{c}}_{\Psi_1} = \mathbf{j}^{\mathrm{c}}_{\Psi_2} = \mathbf{j}^{\mathrm{c}}$. The general case is reduced to this one below by a gauge transformation. By Proposition 4.2, Ψ_1 is a common ground state. We apply Theorem 3.1 to the potential pair satisfying (11). It gives

$$\|\Psi_1(\mathbf{X})\|_{\mathcal{K}_N} > 0 \quad \text{for a.a. } \mathbf{X} \in \mathbb{R}^{3N}, \qquad \rho(\mathbf{x}) > 0 \quad \text{for a.a. } \mathbf{x} \in \mathbb{R}^3. \tag{51}$$

Proposition 4.4 now gives $\mathbf{A}_1 = \mathbf{A}_2$. Subtracting the weak eigenvalue equations cancels the kinetic, interaction, and radiation forms, leaving

$$M(\mathbf{X})\Psi_1(\mathbf{X}) = 0, \qquad M(\mathbf{X}) = \sum_{n=1}^{N} \big(v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)\big) - (E_1 - E_2). \tag{52}$$

The equation $M\Psi_1 = 0$ holds in the sense of distributions. The weak eigenvalue equations only provide antisymmetric test functions $\Phi \in \mathcal{Q}_{\mathrm{PF}}$. Since the multiplier M is symmetric under permutations, $M\Psi_1$ is antisymmetric, so $\langle \Phi, M\Psi_1 \rangle = \langle \Pi_- \Phi, M\Psi_1 \rangle$ for every test function Φ. Hence (52) holds against all test functions. The products $\big(v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)\big)\Psi_1(\mathbf{X})$ are locally Bochner integrable. Indeed, on each bounded configuration-space set K, Cauchy–Schwarz gives

$$\begin{aligned} &\int_K |v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)| \|\Psi_1(\mathbf{X})\|_{\mathcal{K}_N} \mathrm{d}\mathbf{X} \\ &\quad \le \left(\int_K |v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)| \|\Psi_1(\mathbf{X})\|^2_{\mathcal{K}_N} \mathrm{d}\mathbf{X} \right)^{1/2} \\ &\qquad \times \left(\int_K |v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)| \mathrm{d}\mathbf{X} \right)^{1/2} < \infty, \end{aligned}$$

by scalar form boundedness and local integrability of $v_1 - v_2$. Consequently the multiplier equation holds almost everywhere. Using (51), it reduces to

$$\sum_{n=1}^{N} \big(v_1(\mathbf{x}_n) - v_2(\mathbf{x}_n)\big) = E_1 - E_2 \qquad \text{for a.a. } \mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N}.$$

Write $f = v_1 - v_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and fix a ball $B \subset \mathbb{R}^3$. Averaging the identity over $(\mathbf{x}_2, \dots, \mathbf{x}_N) \in B^{N-1}$ and using Fubini's theorem gives

$$f(\mathbf{x}_1) = E_1 - E_2 - \frac{N-1}{|B|} \int_B f(\mathbf{y}) \mathrm{d}\mathbf{y} \qquad \text{for a.a. } \mathbf{x}_1 \in \mathbb{R}^3.$$

Thus $f = c$ almost everywhere for a constant c, and substituting this into the identity gives $Nc = E_1 - E_2$.

Now let θ be any function satisfying the hypotheses of the theorem, so that

$$\rho_{\Psi_1} = \rho_{\Psi_2} = \rho, \qquad \mathbf{j}^{\mathrm{c}}_{\Psi_2} = \mathbf{j}^{\mathrm{c}}_{\Psi_1} + \rho \boldsymbol{\nabla} \theta.$$

As in the proof of Proposition 4.3, U_θ denotes multiplication by $\mathrm{e}^{\mathrm{i} \sum_n \theta(\mathbf{x}_n)}$. The weak chain rule and the multiplier bounds for $L^3 + L^\infty$ show that U_θ maps the

fermionic form domain onto itself. The covariant derivatives satisfy

$$\big(\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n) - \boldsymbol{\nabla}\theta(\mathbf{x}_n)\big)U_\theta\Psi = U_\theta\big(\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n)\big)\Psi.$$

The phase also commutes with the Pauli matrices, so this identity holds after contraction with $\boldsymbol{\sigma}_n$. Since U_θ commutes with the scalar, interaction, and radiation terms, the closed forms give

$$H^{v,\mathbf{A}-\boldsymbol{\nabla}\theta} = U_\theta H^{v,\mathbf{A}} U_\theta^{-1}.$$

Thus gauge changes preserve ground-state energies and transform the data by $\rho_{U_\theta\Psi} = \rho_\Psi$ and $\mathbf{j}^{\mathrm{c}}_{U_\theta\Psi} = \mathbf{j}^{\mathrm{c}}_\Psi + \rho_\Psi\boldsymbol{\nabla}\theta$. The magnetization density is unchanged by the phase.

We now reduce to the case $\theta = 0$ by gauge transforming one of the two pairs. The pair satisfying (11) is left unchanged, since that condition is not gauge invariant: $\operatorname{div}(\mathbf{A}_k \pm \boldsymbol{\nabla}\theta) = \operatorname{div}\mathbf{A}_k \pm \Delta\theta$ need not lie in L^p_{loc}.

Suppose first that the first potential pair satisfies (11). We apply the case $\theta = 0$ to the pairs $(v_1, \mathbf{A}_1)$ and $(v_2, \mathbf{A}_2 + \boldsymbol{\nabla}\theta)$, with the states Ψ_1 and $U_{-\theta}\Psi_2$. Its hypotheses hold for these data. First, since $\boldsymbol{\nabla}\theta \in L^3 + L^\infty$, the pair $(v_2, \mathbf{A}_2 + \boldsymbol{\nabla}\theta)$ satisfies (4). Second, the conjugation identity above with $-\theta$ in place of θ gives $H^{v_2,\mathbf{A}_2+\boldsymbol{\nabla}\theta} = U_{-\theta}H^{v_2,\mathbf{A}_2}U_{-\theta}^{-1}$. Hence $U_{-\theta}\Psi_2$ is a normalized ground state of $H^{v_2,\mathbf{A}_2+\boldsymbol{\nabla}\theta}$, and its energy is $E(v_2, \mathbf{A}_2 + \boldsymbol{\nabla}\theta) = E_2$. Third, by the transformation rule above,

$$\rho_{U_{-\theta}\Psi_2} = \rho = \rho_{\Psi_1}, \qquad \mathbf{j}^{\mathrm{c}}_{U_{-\theta}\Psi_2} = \mathbf{j}^{\mathrm{c}}_{\Psi_2} - \rho\boldsymbol{\nabla}\theta = \mathbf{j}^{\mathrm{c}}_{\Psi_1},$$

so the two states have the same density and combined current density. Fourth, the first pair is unchanged and still satisfies (11). Finally, e, $\widehat{\chi}$, and the common density ρ are unchanged, so whichever cutoff alternative holds for the original data also holds for the new data. The case $\theta = 0$ therefore gives $\mathbf{A}_1 = \mathbf{A}_2 + \boldsymbol{\nabla}\theta$ and $v_1 - v_2 = (E_1 - E_2)/N$ almost everywhere.

If instead the second pair supplies the local regularity, we apply the case $\theta = 0$ to the pairs $(v_1, \mathbf{A}_1 - \boldsymbol{\nabla}\theta)$ and $(v_2, \mathbf{A}_2)$, with the states $U_\theta\Psi_1$ and Ψ_2. The same checks apply. The pair $(v_1, \mathbf{A}_1 - \boldsymbol{\nabla}\theta)$ satisfies (4), and $U_\theta\Psi_1$ is a normalized ground state of $H^{v_1,\mathbf{A}_1-\boldsymbol{\nabla}\theta} = U_\theta H^{v_1,\mathbf{A}_1}U_\theta^{-1}$ with energy E_1. Moreover,

$$\rho_{U_\theta\Psi_1} = \rho = \rho_{\Psi_2}, \qquad \mathbf{j}^{\mathrm{c}}_{U_\theta\Psi_1} = \mathbf{j}^{\mathrm{c}}_{\Psi_1} + \rho\boldsymbol{\nabla}\theta = \mathbf{j}^{\mathrm{c}}_{\Psi_2}.$$

The unchanged second pair satisfies (11), and the cutoff alternative is preserved. The case $\theta = 0$ gives $\mathbf{A}_1 - \boldsymbol{\nabla}\theta = \mathbf{A}_2$ and the same scalar constant. This proves (13).

Appendix A. Covariant Pauli–Fierz Carleman estimates

Define the auxiliary spinless operator on $L^2(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$ by

$$T^{\mathbf{A}} = \frac{1}{2}\sum_{n=1}^{N}\boldsymbol{\pi}_n^2 + H_{\mathrm{rad}} = \frac{1}{2}\sum_{n=1}^{N}\big(\mathbf{D}_n + \mathbf{A}(\mathbf{x}_n)\big)^2 + H_{\mathrm{rad}}.$$

We first prove a Carleman estimate for T^0, keeping the covariant derivatives and the radiation energy in the principal operator, so that the left-hand side controls $\mathbf{D}_n u$ rather than $\boldsymbol{\nabla}u$. This covariant form is essential for two reasons. First, $\mathfrak{A}(\mathbf{x})$ is an unbounded operator on $\mathcal{F}_{\mathrm{ph}}$, so the first-order term $e\mathfrak{A}(\mathbf{x}_n)\cdot\boldsymbol{\nabla}_{\mathbf{x}_n}$ cannot be treated as a perturbation of a Carleman estimate for the Laplacian. It is controlled only through $H_{\mathrm{rad}}^{1/2}$. Second, in a local radial gauge the commutators in the Carleman computation involve the quantized field only through its curvature, the quantized

magnetic field. This is controlled by the first ultraviolet moment (10) and is small near the center of the weight. Fractional Carleman estimates in the mixed electronic and radiation norms of $L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1$ then permit the external vector potential in L^6_{loc} as a first-order perturbation.

For an electronic configuration $\mathbf{X}_0$ and $\alpha > 0$, we use the weights

$$r = |\mathbf{X} - \mathbf{X}_0|, \qquad t = -\log r, \qquad f(t) = t + t^{-\alpha}, \qquad \beta(t) = \ddot{f}(t) = \alpha(\alpha+1)t^{-\alpha-2}.$$

For $0 < r_0 < 1$, $\tau > 0$ and $u \in D(T^0)$ with compact support in $B_{r_0}(\mathbf{X}_0) \setminus \{\mathbf{X}_0\}$, put

$$\mathcal{T}^0(u) = \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|T^0 u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X}.$$

We suppress the dependence of $\mathcal{T}^0(u)$ on $\mathbf{X}_0$, α, r_0 and τ.

Lemma A.1 (A Carleman estimate with $\mathbf{A} = 0$)**.** *Assume (1) and $\omega\widehat{\chi} \in L^2(\mathbb{R}^3)$. Fix an electronic configuration $\mathbf{X}_0$ and $\alpha > 0$, with r, t, f and β as above. There are $r_0 \in (0, \mathrm{e}^{-2})$, $\tau_0 \geq 1$ and $C < \infty$ such that, for $\tau \geq \tau_0$ and every $u \in D(T^0)$ with compact support in $B_{r_0}(\mathbf{X}_0) \setminus \{\mathbf{X}_0\}$,*

$$\begin{aligned} \tau^3 \int_{B_{r_0}(\mathbf{X}_0)} \beta(t) \mathrm{e}^{2\tau f(t)} r^{-2} \|u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} \\ + \tau \int_{B_{r_0}(\mathbf{X}_0)} \beta(t) \mathrm{e}^{2\tau f(t)} \sum_{n=1}^{N} \|\mathbf{D}_n u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} \\ + \tau \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} \|H_{\mathrm{rad}}^{1/2} u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} \leq C\mathcal{T}^0(u). \end{aligned} \tag{53}$$

The constants are independent of $\mathbf{X}_0$, and the same estimate holds on every smaller ball with the same C and τ_0.

Proof. The proof has three steps. We first prove (53) by a computation in a radial gauge, assuming that $\widehat{\chi}$ has compact support and that $u \in \mathcal{C}$, with $\mathcal{C}$ as defined before Lemma 4.8. The resulting constants depend only on the first ultraviolet moment. We then remove the compact support of $\widehat{\chi}$ by an ultraviolet limit, and finally extend the estimate to all of $D(T^0)$ by an approximation argument.

Step 1: Compactly supported cutoff. Without loss of generality, we translate $\mathbf{X}_0$ to the origin. Translating the nth electron by $(\mathbf{X}_0)_n$ multiplies $G^\mu_{\mathbf{x}_n}$ by the phase $\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{X}_0)_n}$, which leaves all properties used below unchanged, and we suppress it from the notation. We write $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ and $\Phi(h) = a^*(h) + a(h)$, so that $\mathfrak{A}^\mu(\mathbf{x}) = \Phi(G^\mu_{\mathbf{x}})$. The cutoff assumptions give uniform bounds for every spatial derivative of $G^\mu_{\mathbf{x}}$ in

$$\mathfrak{H}_\omega = \{h \in \mathfrak{H}_{\mathrm{ph}} : h/\sqrt{\omega} \in \mathfrak{H}_{\mathrm{ph}}\}, \qquad \|h\|^2_\omega = \|h\|^2 + \|h/\sqrt{\omega}\|^2,$$

and the same holds after multiplying these derivatives by ω. The reality assumption and the real, even transverse polarization projector imply

$$\mathrm{Im}\langle G^\mu_{\mathbf{x}}, G^\nu_{\mathbf{y}}\rangle = 0 \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^3).$$

Consequently all the fields $\mathfrak{A}^\mu(\mathbf{x})$ strongly commute [20, Proposition A.6(iii)]. Their spatial derivatives have the same commutation property.

Define the radial line integral and its unitary exponential by

$$\Theta(\mathbf{X}) = -e\sum_{n=1}^{N}\int_0^1 \mathbf{x}_n\cdot\mathfrak{A}(s\mathbf{x}_n)\mathrm{d}s = \Phi(h_{\mathbf{X}}), \qquad U_{\mathbf{X}} = \mathrm{e}^{\mathrm{i}\Theta(\mathbf{X})},$$
$$h_{\mathbf{X}} = -e\sum_{n=1}^{N}\sum_{\mu=1}^{3}\int_0^1 x_n^\mu G^\mu_{s\mathbf{x}_n}\mathrm{d}s.$$

These formulas may first be read on vectors with finitely many photons. Strong commutativity, the field bounds and the Weyl relations then give their closed-operator and form interpretations. Write U for the unitary $(Uu)(\mathbf{X}) = U_{\mathbf{X}}u(\mathbf{X})$ on $L^2(\mathbb{R}^{3N},\mathcal{F}_{\mathrm{ph}})$. Since $U_{\mathbf{X}}$ commutes with all the fields, conjugation gives

$$U\mathbf{D}_nU^* = -\mathrm{i}\boldsymbol{\nabla}_{\mathbf{x}_n} + \mathfrak{b}_n,$$

where

$$\mathfrak{b}_n(\mathbf{X}) = -e\mathfrak{A}(\mathbf{x}_n) - \boldsymbol{\nabla}_{\mathbf{x}_n}\Theta(\mathbf{X}).$$

The operator-valued connection $\mathfrak{b} = (\mathfrak{b}_1,\dots,\mathfrak{b}_N)$ is $-e\mathfrak{A}$ in radial gauge, that is, it has no radial component: $\mathbf{x}_n\cdot\mathfrak{b}_n(\mathbf{X}) = 0$. This follows from the classical radial-gauge identity: for a smooth vector field $\mathbf{A}$ on $\mathbb{R}^3$, differentiation under the integral and the fundamental theorem of calculus in s give

$$\mathbf{A}(\mathbf{x}) - \boldsymbol{\nabla}_{\mathbf{x}}\int_0^1 \mathbf{x}\cdot\mathbf{A}(s\mathbf{x})\mathrm{d}s = -\int_0^1 s\,\mathbf{x}\times(\boldsymbol{\nabla}\times\mathbf{A})(s\mathbf{x})\mathrm{d}s.$$

The identity is real-linear in $\mathbf{A}$, so it also holds for the $\mathfrak{H}_{\mathrm{ph}}$-valued field $\mathbf{x}\mapsto(G^1_{\mathbf{x}},G^2_{\mathbf{x}},G^3_{\mathbf{x}})$. Applying Φ, with $\mathfrak{B} = \boldsymbol{\nabla}\times\mathfrak{A}$, gives

$$\mathfrak{b}_n(\mathbf{X}) = e\int_0^1 s\,\mathbf{x}_n\times\mathfrak{B}(s\mathbf{x}_n)\mathrm{d}s.$$

Let $\mathbf{X} = r\boldsymbol{\Omega}$ with $\boldsymbol{\Omega} = (\boldsymbol{\Omega}_1,\dots,\boldsymbol{\Omega}_N)\in\mathbb{S}^{3N-1}$, and recall $t = -\log r$, so that $r = \mathrm{e}^{-t}$ and $\partial_t = -r\partial_r$. Put $t_0 = -\log r_0 > 2$. In $(t,\boldsymbol{\Omega})$-coordinates, $B_{r_0}\setminus\{0\}$ is the cylinder $(t_0,\infty)\times\mathbb{S}^{3N-1}$. For each $t > t_0$, define the angular connection, an operator-valued one-form on the unit sphere $\mathbb{S}^{3N-1}$ acting on $\mathcal{F}_{\mathrm{ph}}$-valued functions, by

$$\mathfrak{a}(t,\boldsymbol{\Omega})[\boldsymbol{\zeta}] = r\,\boldsymbol{\zeta}\cdot\mathfrak{b}(r\boldsymbol{\Omega}), \qquad \boldsymbol{\zeta} = (\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_N)\in T_{\boldsymbol{\Omega}}\mathbb{S}^{3N-1}.$$

Since $\mathbf{X}\cdot\mathfrak{b}(\mathbf{X}) = 0$, the vector $\mathfrak{b}(r\boldsymbol{\Omega})$ is tangent to the sphere, so $\mathfrak{a}$ determines $\mathfrak{b}$ completely. Differentiating the radial integral in t, with $\boldsymbol{\Omega}$ fixed, gives

$$\dot{\mathfrak{a}}(t,\boldsymbol{\Omega})[\boldsymbol{\zeta}] = er^2\sum_{n=1}^{N}(\boldsymbol{\Omega}_n\times\boldsymbol{\zeta}_n)\cdot\mathfrak{B}(r\boldsymbol{\Omega}_n). \tag{54}$$

Let $\nabla_{\mathbb{S}}$ denote the spherical gradient on $\mathbb{S}^{3N-1}$, acting on $\mathcal{F}_{\mathrm{ph}}$-valued functions. Set

$$\nabla_{\mathbb{S},\mathfrak{a}} = \nabla_{\mathbb{S}} + \mathrm{i}\mathfrak{a}, \qquad K(t) = \nabla^*_{\mathbb{S},\mathfrak{a}}\nabla_{\mathbb{S},\mathfrak{a}}, \qquad \Upsilon(t,\boldsymbol{\Omega}) = 2U_{r\boldsymbol{\Omega}}H_{\mathrm{rad}}U^*_{r\boldsymbol{\Omega}}.$$

Since $\mathbf{X}\cdot\mathfrak{b}(\mathbf{X}) = 0$, the gauge-transformed covariant Laplacian has no connection in the radial direction, so

$$(-\mathrm{i}\boldsymbol{\nabla}_{\mathbf{X}} + \mathfrak{b})^2 = -\partial_r^2 - \frac{3N-1}{r}\partial_r + \frac{1}{r^2}K(t).$$

Since $UT^0U^* = \frac{1}{2}(-\mathrm{i}\boldsymbol{\nabla}_{\mathbf{X}} + \mathfrak{b})^2 + UH_{\mathrm{rad}}U^*$, multiplying by $2r^2$ and using $\partial_t = -r\partial_r$ gives

$$2r^2\, UT^0U^* = -\partial_t^2 + 2\gamma\,\partial_t + K(t) + r^2\Upsilon(t), \qquad \gamma = \frac{3N-2}{2}. \tag{55}$$

Here $\Upsilon(t) = \Upsilon(t,\cdot)$ is a nonnegative multiplication operator on the angular Hilbert space $L^2(\mathbb{S}^{3N-1}, \mathcal{F}_{\mathrm{ph}})$, with the surface measure $\mathrm{d}\boldsymbol{\Omega}$. For Ξ in this space we write

$$\Upsilon(t)[\Xi] = 2\int_{\mathbb{S}^{3N-1}} \|H_{\mathrm{rad}}^{1/2} U^*_{r\boldsymbol{\Omega}}\Xi(\boldsymbol{\Omega})\|^2_{\mathcal{F}_{\mathrm{ph}}}\mathrm{d}\boldsymbol{\Omega}, \qquad K(t)[\Xi] = \int_{\mathbb{S}^{3N-1}} \|\nabla_{\mathbb{S},\mathfrak{a}}\Xi(\boldsymbol{\Omega})\|^2\mathrm{d}\boldsymbol{\Omega}$$

for the corresponding quadratic forms, where the integrand of $K(t)[\Xi]$ is the norm of a tangent-vector-valued function with values in $\mathcal{F}_{\mathrm{ph}}$. By the Weyl relations and $\sqrt{\omega}\widehat{\chi} \in L^2$, $U_{\mathbf{X}}$ preserves $D(H_{\mathrm{rad}}^{1/2})$, and by the field bound $\mathfrak{a}$ is bounded by $(H_{\mathrm{rad}}+1)^{1/2}$. Hence the form domain of $K(t) + \Upsilon(t)$ is

$$\mathcal{Q}_{\mathbb{S}} = H^1(\mathbb{S}^{3N-1}, \mathcal{F}_{\mathrm{ph}}) \cap D(H_{\mathrm{rad}}^{1/2}),$$

with H_{rad} acting pointwise on the sphere, independently of $t > t_0$.

We write $\dot{K}(t)[\Xi]$ for the derivative of $t \mapsto K(t)[\Xi]$ with Ξ fixed.

Claim. *There is $C < \infty$, independent of t and Ξ, such that for $t > t_0$ and $\Xi \in \mathcal{Q}_{\mathbb{S}}$,*

$$|\dot{K}(t)[\Xi]| \le Cr^2 K(t)[\Xi]^{1/2}\big(\Upsilon(t)[\Xi] + \|\Xi\|^2\big)^{1/2}. \tag{56}$$

Proof of the claim. In

$$K(t)[\Xi] = \int_{\mathbb{S}^{3N-1}} \|\nabla_{\mathbb{S}}\Xi(\boldsymbol{\Omega}) + \mathrm{i}\mathfrak{a}(t,\boldsymbol{\Omega})\Xi(\boldsymbol{\Omega})\|^2\mathrm{d}\boldsymbol{\Omega}$$

the term $\nabla_{\mathbb{S}}\Xi$ does not depend on t, so only the connection is differentiated, and

$$\dot{K}(t)[\Xi] = 2\,\mathrm{Re}\langle \nabla_{\mathbb{S},\mathfrak{a}}\Xi, \mathrm{i}\dot{\mathfrak{a}}\Xi\rangle.$$

By the Cauchy–Schwarz inequality, $|\dot{K}(t)[\Xi]| \le 2K(t)[\Xi]^{1/2}\|\dot{\mathfrak{a}}\Xi\|$, so it suffices to show

$$\|\dot{\mathfrak{a}}\Xi\| \le Cr^2\big(\Upsilon(t)[\Xi] + \|\Xi\|^2\big)^{1/2}. \tag{57}$$

Let $\boldsymbol{\zeta} \in T_{\boldsymbol{\Omega}}\mathbb{S}^{3N-1}$ and $\xi \in \mathcal{F}_{\mathrm{ph}}$. By (54) and $\sum_{n=1}^N |\boldsymbol{\Omega}_n||\boldsymbol{\zeta}_n| \le |\boldsymbol{\Omega}||\boldsymbol{\zeta}| = |\boldsymbol{\zeta}|$,

$$\|\dot{\mathfrak{a}}(t,\boldsymbol{\Omega})[\boldsymbol{\zeta}]\xi\| \le er^2|\boldsymbol{\zeta}| \max_n \|\boldsymbol{\mathfrak{B}}(r\boldsymbol{\Omega}_n)\xi\|,$$

where $\|\boldsymbol{\mathfrak{B}}(\mathbf{x})\xi\|^2 = \sum_\lambda \|\mathfrak{B}^\lambda(\mathbf{x})\xi\|^2$. The form factors of $\boldsymbol{\mathfrak{B}}(\mathbf{x})$ are first derivatives of $G^\mu_{\mathbf{x}}$, which are uniformly bounded in $\mathfrak{H}_\omega$, so the field bound gives $\|\boldsymbol{\mathfrak{B}}(\mathbf{x})\eta\| \le C(\|H_{\mathrm{rad}}^{1/2}\eta\|^2 + \|\eta\|^2)^{1/2}$ uniformly in $\mathbf{x}$. Since $\boldsymbol{\mathfrak{B}}(\mathbf{x})$ commutes with $U_{r\boldsymbol{\Omega}}$, applying this to $\eta = U^*_{r\boldsymbol{\Omega}}\xi$ gives

$$\|\boldsymbol{\mathfrak{B}}(r\boldsymbol{\Omega}_n)\xi\| = \|\boldsymbol{\mathfrak{B}}(r\boldsymbol{\Omega}_n)U^*_{r\boldsymbol{\Omega}}\xi\| \le C\big(\|H_{\mathrm{rad}}^{1/2}U^*_{r\boldsymbol{\Omega}}\xi\|^2 + \|\xi\|^2\big)^{1/2}.$$

Taking $\xi = \Xi(\boldsymbol{\Omega})$, summing over an orthonormal basis of $T_{\boldsymbol{\Omega}}\mathbb{S}^{3N-1}$ and integrating over the sphere gives (57). □

Similarly, we write $\dot{\Upsilon}(t)[\Xi]$ for the derivative of $t \mapsto \Upsilon(t)[\Xi]$ with Ξ fixed.

Claim. *There is $C < \infty$, independent of t and Ξ, such that for $t > t_0$ and $\Xi \in \mathcal{Q}_{\mathbb{S}}$,*

$$|\dot{\Upsilon}(t)[\Xi]| \le Cr\,\Upsilon(t)[\Xi]^{1/2}\|\Xi\|. \tag{58}$$

Proof of the claim. Write $\Pi(h) = \mathrm{i}(a^*(h) - a(h))$, so that $[\Phi(h), H_{\mathrm{rad}}] = \mathrm{i}\Pi(\omega h)$. Since $U_{r\mathbf{\Omega}} = \mathrm{e}^{\mathrm{i}\Phi(h_{r\mathbf{\Omega}})}$ and $\Phi(h_{r\mathbf{\Omega}})$ commutes with $\Phi(\dot{h}_{r\mathbf{\Omega}})$,

$$\dot{\Upsilon}(t,\mathbf{\Omega}) = 2\mathrm{i}\, U_{r\mathbf{\Omega}}\big[\Phi(\dot{h}_{r\mathbf{\Omega}}), H_{\mathrm{rad}}\big]U_{r\mathbf{\Omega}}^* = -2U_{r\mathbf{\Omega}}\Pi(\omega\, \dot{h}_{r\mathbf{\Omega}})U_{r\mathbf{\Omega}}^*,$$

where

$$\dot{h}_{r\mathbf{\Omega}} = er\sum_{n=1}^{N}\sum_{\mu=1}^{3}\Omega_n^\mu G^\mu_{r\mathbf{\Omega}_n}.$$

Since $\sqrt{\omega}\widehat{\chi} \in L^2$, we have $\|\sqrt{\omega}\,\dot{h}_{r\mathbf{\Omega}}\| \le Cr$. With $\xi = U_{r\mathbf{\Omega}}^*\Xi(\mathbf{\Omega})$, the bound $|\langle\xi, \Pi(g)\xi\rangle| \le 2\|g/\sqrt{\omega}\|\|H_{\mathrm{rad}}^{1/2}\xi\|\|\xi\|$ gives

$$|\langle\Xi(\mathbf{\Omega}), \dot{\Upsilon}(t,\mathbf{\Omega})\Xi(\mathbf{\Omega})\rangle| \le Cr\|H_{\mathrm{rad}}^{1/2}U_{r\mathbf{\Omega}}^*\Xi(\mathbf{\Omega})\|\|\Xi(\mathbf{\Omega})\|.$$

Integrating over the sphere and using the Cauchy–Schwarz inequality gives (58). □

Thus $\dot{\Upsilon}$ is controlled by the transformed radiation energy itself, so H_{rad} can be kept in the principal part of the operator Q in (59) below.

Conjugating (55) by $r^\gamma = \mathrm{e}^{-\gamma t}$ removes the first-order term:

$$2r^{\gamma+2}UT^0u = Q(r^\gamma Uu), \qquad Q = -\partial_t^2 + K(t) + \gamma^2 + r^2\Upsilon(t). \tag{59}$$

Set

$$z(t,\mathbf{\Omega}) = \mathrm{e}^{\tau f(t)}r^\gamma U_{r\mathbf{\Omega}}u(r\mathbf{\Omega}), \qquad z(t) = z(t,\cdot) \in L^2(\mathbb{S}^{3N-1}, \mathcal{F}_{\mathrm{ph}}).$$

Then z has compact support in the cylinder $(t_0,\infty)\times\mathbb{S}^{3N-1}$. We write $\|z(t)\|$ for the norm in $L^2(\mathbb{S}^{3N-1}, \mathcal{F}_{\mathrm{ph}})$ and $\|z\|$ for the norm in $L^2((t_0,\infty)\times\mathbb{S}^{3N-1}, \mathcal{F}_{\mathrm{ph}})$ with measure $\mathrm{d}t\mathrm{d}\mathbf{\Omega}$, so that $\|z\|^2 = \int_{t_0}^\infty \|z(t)\|^2\mathrm{d}t$. The operators $K(t)$, γ^2 and $r^2\Upsilon(t)$ act at fixed t and commute with multiplication by $\mathrm{e}^{\tau f(t)}$, while

$$\mathrm{e}^{\tau f}(-\partial_t^2)\mathrm{e}^{-\tau f} = -(\partial_t - \tau\dot{f})^2 = -\partial_t^2 - \tau^2(\dot{f})^2 + 2\tau\dot{f}\partial_t + \tau\ddot{f}.$$

Hence

$$\mathrm{e}^{\tau f}Q\mathrm{e}^{-\tau f} = S + B, \qquad S = Q - \tau^2(\dot{f})^2, \qquad B = 2\tau\dot{f}\partial_t + \tau\ddot{f}.$$

By (59), $(S+B)z = 2\mathrm{e}^{\tau f}r^{\gamma+2}UT^0u$. Since $\widehat{\chi}$ has compact support, all moments $\omega^k\widehat{\chi}$ are finite, and the Weyl relations show that z, Sz and Bz are smooth, compactly supported in $(t_0,\infty)\times\mathbb{S}^{3N-1}$, and take values in $D(H_{\mathrm{rad}}^k)$ for all k, together with all field moments. On such vectors, integration by parts in t shows that S is symmetric and B is skew symmetric, so $\|(S+B)z\|^2 = \|Sz\|^2 + \|Bz\|^2 + \langle z, [S,B]z\rangle$. Since B involves only ∂_t and functions of t, the operators K and Υ enter $[S,B]$ only through $\dot{K}$ and $\partial_t(r^2\Upsilon)$, which are controlled by the claims above; no commutator of K and Υ is needed. Computing $[S,B]$ term by term, with $r = \mathrm{e}^{-t}$ so that $\partial_t(r^2) = -2r^2$ and with $\beta = \dddot{f} > 0$, gives the identity

$$\begin{aligned}\|(S+B)z\|^2 = \|Sz\|^2 + \|Bz\|^2 &+ 4\tau\int_{t_0}^\infty \beta(t)\|\partial_t z(t)\|^2\mathrm{d}t\\ &+ 4\tau^3\int_{t_0}^\infty(\dot{f}(t))^2\beta(t)\|z(t)\|^2\mathrm{d}t - \tau\int_{t_0}^\infty f^{(4)}(t)\|z(t)\|^2\mathrm{d}t\\ &- 2\tau\int_{t_0}^\infty \dot{f}(t)\dot{K}(t)[z(t)]\mathrm{d}t + 4\tau\int_{t_0}^\infty \dot{f}(t)r^2\Upsilon(t)[z(t)]\mathrm{d}t\\ &- 2\tau\int_{t_0}^\infty \dot{f}(t)r^2\dot{\Upsilon}(t)[z(t)]\mathrm{d}t.\end{aligned} \tag{60}$$

It remains to absorb the three terms in (60) that may be negative, those containing $f^{(4)}$, $\dot K$ and $\dot\Upsilon$, into the nonnegative terms. For $t \geq 2$, $\dot f(t) = 1 - \alpha t^{-\alpha-1}$ lies between $2/(\alpha+2)$ and 1, because $t^{\alpha+1} \geq 2^{\alpha+1} \geq \alpha + 2$ by Bernoulli's inequality, and $|f^{(4)}| \leq C\beta$. For any fixed $\varepsilon > 0$, (56) gives

$$\tau \int_{t_0}^{\infty} |\dot K(t)[z(t)]|\mathrm{d}t \leq \varepsilon\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t + C_\varepsilon \tau \int_{t_0}^{\infty} \frac{r^4}{\beta(t)} \big(\Upsilon(t)[z(t)] + \|z(t)\|^2\big)\mathrm{d}t.$$

Similarly, (58) gives

$$\tau \int_{t_0}^{\infty} r^2 |\dot\Upsilon(t)[z(t)]|\mathrm{d}t \leq \varepsilon\tau \int_{t_0}^{\infty} r^2 \Upsilon(t)[z(t)]\mathrm{d}t + C_\varepsilon \tau \int_{t_0}^{\infty} r^4 \|z(t)\|^2 \mathrm{d}t.$$

Since K enters (60) only through $\dot K$, none of the nonnegative terms there can absorb $\varepsilon\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t$. Instead, such a term is controlled by $\|Sz\|^2$.

Claim. *There is $C < \infty$, independent of $\tau \geq 1$ and z, such that*

$$\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t \leq C \left(\|Sz\|^2 + \tau^3 \int_{t_0}^{\infty} \beta(t) \|z(t)\|^2 \mathrm{d}t \right). \tag{61}$$

Proof of the claim. Integrating by parts in t,

$$\mathrm{Re}\langle Sz, \beta z\rangle = \int_{t_0}^{\infty} \beta(t) \big(\|\partial_t z(t)\|^2 + K(t)[z(t)] + r^2 \Upsilon(t)[z(t)] + \gamma^2 \|z(t)\|^2 \big)\mathrm{d}t - \tau^2 \int_{t_0}^{\infty} (\dot f(t))^2 \beta(t) \|z(t)\|^2 \mathrm{d}t - \frac{1}{2} \int_{t_0}^{\infty} \ddot\beta(t) \|z(t)\|^2 \mathrm{d}t.$$

Keeping the K term, discarding the other nonnegative terms and multiplying by τ gives

$$\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t \leq \tau\, \mathrm{Re}\langle Sz, \beta z\rangle + \tau^3 \int_{t_0}^{\infty} (\dot f(t))^2 \beta(t) \|z(t)\|^2 \mathrm{d}t + \frac{\tau}{2} \int_{t_0}^{\infty} |\ddot\beta(t)| \|z(t)\|^2 \mathrm{d}t.$$

By Young's inequality,

$$\tau\, \mathrm{Re}\langle Sz, \beta z\rangle \leq \frac{1}{2}\|Sz\|^2 + \frac{\tau^2}{2} \int_{t_0}^{\infty} \beta(t)^2 \|z(t)\|^2 \mathrm{d}t.$$

For $t \geq 2$, $\dot f$ and β are bounded and $|\ddot\beta| \leq C\beta$. Combining the two displays, we get for $\tau \geq 1$

$$\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t \leq \frac{1}{2}\|Sz\|^2 + C\tau^3 \int_{t_0}^{\infty} \beta(t) \|z(t)\|^2 \mathrm{d}t,$$

which is (61). □

Choose ε small enough that

$$\varepsilon\tau \int_{t_0}^{\infty} \beta(t) K(t)[z(t)]\mathrm{d}t \qquad \text{and} \qquad \varepsilon\tau \int_{t_0}^{\infty} r^2 \Upsilon(t)[z(t)]\mathrm{d}t$$

are absorbed, the first by $\|Sz\|^2$ and the τ^3 term of (60) using (61), and the second by the Υ term of (60). The remaining errors are

$$C_\varepsilon \tau \int_{t_0}^{\infty} \Big(\frac{r^4}{\beta(t)} \big(\Upsilon(t)[z(t)] + \|z(t)\|^2 \big) + r^4 \|z(t)\|^2 \Big) \mathrm{d}t.$$

Since

$$\sup_{0<r<r_0} \frac{r^2}{\beta(-\log r)} \longrightarrow 0 \qquad \text{as } r_0 \downarrow 0,$$

choosing r_0 small absorbs them into the same Υ and τ^3 terms. Finally, since $|f^{(4)}| \leq C\beta$, choosing τ_0 large absorbs the $f^{(4)}$ term. Together with (60) and (61), these choices prove

$$\begin{aligned} \tau^3 \int_{t_0}^{\infty} \beta(t) \|z(t)\|^2 \mathrm{d}t + \tau \int_{t_0}^{\infty} \beta(t) \big(\|\partial_t z(t)\|^2 + K(t)[z(t)] \big) \mathrm{d}t \\ + \tau \int_{t_0}^{\infty} r^2 \Upsilon(t)[z(t)] \mathrm{d}t \leq C \|(S+B)z\|^2. \end{aligned} \tag{62}$$

The radial derivative satisfies

$$\mathrm{e}^{\tau f} r^{\gamma+1} \partial_r (Uu) = -\partial_t z + (\tau \dot{f} - \gamma) z,$$

and since $\dot{f}$ is bounded, the contribution of $(\tau \dot{f} - \gamma)z$ is controlled by the first term in (62). Since $\mathfrak{b}$ is tangential, the radial component of $(-\mathrm{i}\boldsymbol{\nabla}_{\mathbf{X}} + \mathfrak{b})(Uu)$ is $-\mathrm{i}\partial_r(Uu)$ and its tangential component is $-\mathrm{i} r^{-1} \nabla_{\mathbb{S},\mathfrak{a}}(Uu)$. These are orthogonal, so $U \mathbf{D}_n U^* = -\mathrm{i}\boldsymbol{\nabla}_{\mathbf{x}_n} + \mathfrak{b}_n$ gives

$$\sum_{n=1}^{N} \|U \mathbf{D}_n u\|^2 = \|\partial_r (Uu)\|^2 + r^{-2} \|\nabla_{\mathbb{S},\mathfrak{a}} (Uu)\|^2.$$

Together with the angular form $K(t)[z(t)]$ and $\mathrm{d}\mathbf{X} = r^{3N} \mathrm{d}t \mathrm{d}\boldsymbol{\Omega}$, this gives the second term on the left of (53),

$$\tau \int_{B_{r_0}} \beta(t) \mathrm{e}^{2\tau f(t)} \sum_{n=1}^{N} \|\mathbf{D}_n u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X}.$$

The other terms transform as

$$\begin{aligned} \|(S+B)z\|^2 &= 4\mathcal{T}^0(u), \\ \int_{t_0}^{\infty} r^2 \Upsilon(t)[z(t)] \mathrm{d}t &= 2 \int_{B_{r_0}} \mathrm{e}^{2\tau f(t)} \|H_{\mathrm{rad}}^{1/2} u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X}. \end{aligned}$$

Step 2: Ultraviolet limit. The compact-support restriction on the cutoff can now be removed. Set $\widehat{\chi}^{\leq \Lambda} = \widehat{\chi} \mathbb{1}_{\{\omega \leq \Lambda\}}$. The constants in the preceding estimates use only the norms of $G^\mu_{\mathbf{x}}$, $\boldsymbol{\nabla}_{\mathbf{x}} G^\mu_{\mathbf{x}}$ and $\omega G^\mu_{\mathbf{x}}$ in $\mathfrak{H}_\omega$. They are therefore uniform in Λ under the assumption $\omega \widehat{\chi} \in L^2$. The limit also uses the following bound for products of two fields.

Claim. *There is $C < \infty$ such that, for all $h, g \in \mathfrak{H}_\omega$,*

$$\|\Phi(h)\Phi(g)(H_{\mathrm{rad}} + 1)^{-1}\| \leq C \|h\|_\omega \|g\|_\omega. \tag{63}$$

Proof of the claim. We write $\Phi(h)\Phi(g)$ as a sum of four products of creation and annihilation operators. On the M-photon sector, H_{rad} is multiplication by $\sum_i \omega(\mathbf{k}_i)$,

so $\sum_{i\neq j}\omega(\mathbf{k}_i)\omega(\mathbf{k}_j) \leq H_{\mathrm{rad}}^2$ there. The Cauchy–Schwarz inequality in the two annihilated photon variables therefore gives

$$\|a(h)a(g)\xi\| \leq \|h/\sqrt{\omega}\|\,\|g/\sqrt{\omega}\|\,\|H_{\mathrm{rad}}\xi\|, \qquad \xi \in D(H_{\mathrm{rad}}).$$

The other three products reduce to this estimate and to the annihilation bound of [20, Section 3.3] by the canonical commutation relations $a(h)a^*(g) = a^*(g)a(h) + \langle h, g\rangle$ and $\|a^*(g)\xi\|^2 = \|a(g)\xi\|^2 + \|g\|^2\|\xi\|^2$. □

For each fixed $u \in \mathcal{C}$,

$$T^{0,\leq\Lambda}u \longrightarrow T^0 u, \qquad \mathbf{D}_n^{\leq\Lambda}u \longrightarrow \mathbf{D}_n u \quad \text{in } L^2.$$

To prove this, we expand the covariant squares. Since $\operatorname{div}\mathfrak{A} = 0$, as noted in Step 2 of the proof of Lemma 4.7,

$$\mathbf{D}_n^2 u = -\Delta_{\mathbf{x}_n}u + 2ie\mathfrak{A}(\mathbf{x}_n)\cdot\boldsymbol{\nabla}_{\mathbf{x}_n}u + e^2|\mathfrak{A}(\mathbf{x}_n)|^2 u,$$

and the same identity holds for the cutoff $\widehat{\chi}^{\leq\Lambda}$. Put $G_{\mathbf{x}}^{\mu,>\Lambda} = G_{\mathbf{x}}^{\mu}\mathbb{1}_{\{\omega>\Lambda\}}$, so that $\mathfrak{A}^{\mu}(\mathbf{x}) - \mathfrak{A}^{\mu,\leq\Lambda}(\mathbf{x}) = \Phi(G_{\mathbf{x}}^{\mu,>\Lambda})$. Hence the components of $\mathbf{D}_n^{\leq\Lambda}u - \mathbf{D}_n u$ are $e\Phi(G_{\mathbf{x}_n}^{\mu,>\Lambda})u$. The difference $T^{0,\leq\Lambda}u - T^0 u$ is a sum of terms $\Phi(G_{\mathbf{x}_n}^{\mu,>\Lambda})\partial_{x_n^\mu}u$ and of products of two fields applied to u, each with a factor $\Phi(G_{\mathbf{x}_n}^{\mu,>\Lambda})$. The creation and annihilation bounds of [20, Section 3.3] give $\|\Phi(h)\xi\| \leq C\|h\|_\omega\|(H_{\mathrm{rad}}+1)^{1/2}\xi\|$. Together with (63), this gives, with C independent of Λ,

$$\begin{aligned}&\|\mathbf{D}_n^{\leq\Lambda}u - \mathbf{D}_n u\| + \|T^{0,\leq\Lambda}u - T^0 u\| \\ &\qquad \leq C\sup_{\mu,\mathbf{x}}\|G_{\mathbf{x}}^{\mu,>\Lambda}\|_\omega\big(\|(H_{\mathrm{rad}}+1)u\| + \|(H_{\mathrm{rad}}+1)^{1/2}\boldsymbol{\nabla}_{\mathbf{X}}u\|\big).\end{aligned}$$

The norms of u on the right are finite, since $\mathcal{F}_{\mathrm{fin}} \subset D(H_{\mathrm{rad}})$. Finally, $\|G_{\mathbf{x}}^{\mu,>\Lambda}\|_\omega$ does not depend on $\mathbf{x}$, and it tends to zero by (1) and dominated convergence. Thus, for each such u, both sides of (53) for the cutoff $\widehat{\chi}^{\leq\Lambda}$ converge to those for $\widehat{\chi}$, so (53) holds for all such u. The gauge transformation U depends on Λ but does not appear in (53), so it plays no role in this limit.

Step 3: Approximation argument. Step 2 of the proof of Lemma 4.7 uses only (1) and (10). It gives $D(T^0) = D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$ and the equivalence of the graph norms of T^0 and $L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1$. Hence, by Lemma 4.8, an annularly supported $u \in D(T^0)$ is a limit in the graph norm of T^0 of functions in $\mathcal{C}$ supported in a slightly larger annulus. Graph convergence implies form convergence, and the weights are bounded on this annulus. Consequently all terms in (53) pass to the limit. After the ultraviolet limit the approximation is performed in the original gauge, so no higher ultraviolet moments are needed for the final cutoff. The bounds on the form factors are uniform in the center. Once the radius and the lower bound on τ have been fixed, restriction to a smaller ball preserves the estimate with the same constants.

□

We use a dyadic decomposition of the punctured ball. With r, β, $\mathbf{X}_0$ and r_0 as in Lemma A.1, let, for $j \geq 0$,

$$R_j = 2^{-j}r_0, \qquad \beta_j = \beta(-\log R_j), \qquad \mathcal{A}_j = \{R_j/2 < r < 2R_j\} \cap B_{r_0}(\mathbf{X}_0),$$

and choose smooth radial functions $\eta_j \geq 0$ with $\operatorname{supp}\eta_j \subset \{R_j/2 < r < 2R_j\}$, $|\boldsymbol{\nabla}\eta_j| \leq C/R_j$, $|\Delta\eta_j| \leq C/R_j^2$ and $\sum_{j\geq 0}\eta_j^2 = 1$ on $0 < r < r_0$. As before, let

$$L = -\Delta_{\mathbf{X}} + H_{\mathrm{rad}} + 1.$$

The fractional-derivative step below plays a role similar to that of the fractional Carleman estimate in [13].

Lemma A.2 (A fractional Carleman estimate on dyadic annuli)**.** *Under the assumptions of Lemma A.1, for $\tau \geq \tau_0$ and $0 \leq s \leq 2$,*

$$\tau^{3-2s} \sum_{j\geq 0} \beta_j R_j^{2s-2} \|L^{s/2}(\eta_j \mathrm{e}^{\tau f} u)\|^2 \leq C_s \mathcal{T}^0(u). \tag{64}$$

Here $u \in D(T^0)$ has compact support in $B_{r_0}(\mathbf{X}_0)\setminus\{\mathbf{X}_0\}$, as in Lemma A.1, and the norms are on $L^2(\mathbb{R}^{3N}, \mathcal{F}_{\mathrm{ph}})$ after extension by zero. The same estimate holds, with the same constant, when η_j is replaced by any smooth radial family ζ_j supported in fixed enlargements of the annuli $\{R_j/2 < r < 2R_j\}$, with $|\zeta_j| \leq C$, $|\boldsymbol{\nabla}\zeta_j| \leq C/R_j$ and $|\Delta\zeta_j| \leq C/R_j^2$. The constant C_s is independent of $\mathbf{X}_0$ and does not increase when r_0 is decreased.

Proof. We apply Lemma A.1 on the dyadic annuli. Its zero-order term gives the endpoint $s = 0$, its covariant-gradient term together with the graph-norm equivalence for T^0 gives the endpoint $s = 2$, and the moment inequality (32) gives the intermediate values. By the graph-norm equivalence for T^0 from the proof of Lemma 4.7,

$$\|Lw\|^2 \leq C\big(\|T^0 w\|^2 + \|w\|^2\big), \qquad w \in D(T^0). \tag{65}$$

Multiplication by the smooth annular weight $\eta_j \mathrm{e}^{\tau f}$ preserves this operator domain. On its support, $r \asymp R_j$ and $\beta(t) \asymp \beta_j$, with constants independent of j. The zero-order term of (53) of Lemma A.1 gives

$$\tau^3 \sum_{j\geq 0} \beta_j R_j^{-2} \|\eta_j \mathrm{e}^{\tau f} u\|^2 \leq C\mathcal{T}^0(u). \tag{66}$$

The radial derivatives of f give

$$|\boldsymbol{\nabla}\big(\eta_j \mathrm{e}^{\tau f}\big)| \leq C\tau R_j^{-1} \mathrm{e}^{\tau f(t)}, \qquad |\Delta(\eta_j \mathrm{e}^{\tau f})| \leq C\tau^2 R_j^{-2} \mathrm{e}^{\tau f(t)}$$

on $\mathcal{A}_j$. The exponential weight is kept at its actual radial argument. The product rule

$$T^0(\eta_j \mathrm{e}^{\tau f} u) = \eta_j \mathrm{e}^{\tau f} T^0 u - \mathrm{i}\boldsymbol{\nabla}\big(\eta_j \mathrm{e}^{\tau f}\big) \cdot \mathbf{D}u - \tfrac{1}{2}\Delta(\eta_j \mathrm{e}^{\tau f}) u$$

and (65) yield

$$\begin{aligned}\|L(\eta_j \mathrm{e}^{\tau f} u)\|^2 \leq C \int_{\mathcal{A}_j} \mathrm{e}^{2\tau f(t)} \big(&\|T^0 u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \tau^2 R_j^{-2} \|\mathbf{D}u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \\ &+ \tau^4 R_j^{-4} \|u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2\big) \mathrm{d}\mathbf{X}.\end{aligned}$$

The term $\|\eta_j \mathrm{e}^{\tau f} u\|^2$ has been included in the last term, using $R_j < 1$ and $\tau \geq 1$. Multiplying by $\beta_j R_j^2/\tau$, summing over the annuli, and applying (53) of Lemma A.1 gives

$$\frac{1}{\tau} \sum_{j\geq 0} \beta_j R_j^2 \|L(\eta_j \mathrm{e}^{\tau f} u)\|^2 \leq C\mathcal{T}^0(u). \tag{67}$$

For $0 < s < 2$, the moment inequality (32) with $\theta = s/2$ gives

$$\|L^{s/2}(\eta_j \mathrm{e}^{\tau f} u)\|^2 \leq \|L(\eta_j \mathrm{e}^{\tau f} u)\|^s \|\eta_j \mathrm{e}^{\tau f} u\|^{2-s}.$$

Hölder's inequality in j combines (66) and (67), since

$$\left(\frac{\beta_j R_j^2}{\tau}\right)^{s/2} \left(\tau^3 \beta_j R_j^{-2}\right)^{1-s/2} = \tau^{3-2s} \beta_j R_j^{2s-2}.$$

The endpoints are the two preceding estimates. Every annularly supported u meets only finitely many cutoffs, and all steps apply directly in the operator domain. The proof uses only the bounded overlap and the derivative bounds of the cutoffs, and the constants come from (53) of Lemma A.1 and (65), which are uniform in $\mathbf{X}_0$ and r_0. □

We now add the external vector potential. Analogously to $\mathcal{T}^0(u)$, for u in the local operator domain of $T^{\mathbf{A}}$ with compact support in $B_{r_0}(\mathbf{X}_0) \setminus \{\mathbf{X}_0\}$, put

$$\mathcal{T}^{\mathbf{A}}(u) = \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|T^{\mathbf{A}} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X}.$$

The mixed estimates of Lemma A.2 control the first-order term $\mathbf{A}\cdot\mathbf{D}$ with $\mathbf{A} \in L^6_{\mathrm{loc}}$. Therefore Lemma A.1 carries over from T^0 to $T^{\mathbf{A}}$, with $\boldsymbol{\pi}_n$ in place of $\mathbf{D}_n$ and $\mathcal{T}^{\mathbf{A}}(u)$ in place of $\mathcal{T}^0(u)$, on a sufficiently small ball.

Lemma A.3 (A Carleman estimate with external vector potential)**.** *Assume* (1) *and* $\omega\widehat{\chi} \in L^2(\mathbb{R}^3)$. *Fix* $\mathbf{X}_0$ *and suppose near each* $(\mathbf{X}_0)_n$ *that*

$$\mathbf{A} \in L^6_{\mathrm{loc}}, \qquad \operatorname{div} \mathbf{A} \in L^p_{\mathrm{loc}}, \qquad p > 2,$$

where the divergence is distributional. With the weights of Lemma A.1, there are $r_0 > 0$, $\tau_0 \geq 1$ *and* $C < \infty$ *such that, for every* $\tau \geq \tau_0$ *and every* u *in the local operator domain of* $T^{\mathbf{A}}$ *with compact support in* $B_{r_0}(\mathbf{X}_0) \setminus \{\mathbf{X}_0\}$,

$$\begin{aligned} \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} \Big(\tau^3 \beta(t) r^{-2} \|u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \tau \beta(t) \sum_{n=1}^{N} \|\boldsymbol{\pi}_n u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \\ + \tau \|H_{\mathrm{rad}}^{1/2} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \Big) \mathrm{d}\mathbf{X} \leq C \mathcal{T}^{\mathbf{A}}(u). \end{aligned} \tag{68}$$

Moreover, the mixed estimate (64) *holds with* $\mathcal{T}^{\mathbf{A}}(u)$ *in place of* $\mathcal{T}^0(u)$ *on the right. These choices are uniform on compact sets with uniform local* L^6 *bounds on* $\mathbf{A}$ *and* L^p *bounds on* $\operatorname{div} \mathbf{A}$.

Proof. Write $T^{\mathbf{A}} = T^0 + R_{\mathbf{A}}$, where $R_{\mathbf{A}}$ consists of the first-order term $\sum_{n=1}^{N} \mathbf{A}(\mathbf{x}_n) \cdot \mathbf{D}_n$ and a zero-order multiplier $q_{\mathbf{A}}$. Using the mixed estimate (64) of Lemma A.2, we show that

$$\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|R_{\mathbf{A}} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \leq \tfrac{1}{4} \mathcal{T}^0(u)$$

for r_0 small and τ large: the first-order part is small when r_0 is small, and the zero-order part is small when τ is large. Since $T^0 u = T^{\mathbf{A}} u - R_{\mathbf{A}} u$, this gives $\mathcal{T}^0(u) \leq 4\mathcal{T}^{\mathbf{A}}(u)$, so Lemma A.1 holds with $\mathcal{T}^{\mathbf{A}}(u)$ on the right. Finally, we replace $\mathbf{D}_n$ by $\boldsymbol{\pi}_n = \mathbf{D}_n + \mathbf{A}(\mathbf{x}_n)$, using (64) of Lemma A.2 once more.

Step 1: Decomposition of $T^{\mathbf{A}}$. Fix neighborhoods of the projected center on which the local norms of $\mathbf{A}$ and $\operatorname{div}\mathbf{A}$ are finite, and write

$$M_A^2 = \sum_{n=1}^{N} \|\mathbf{A}\|^2_{L^6(B_R((\mathbf{X}_0)_n))},$$
$$M_d^2 = \sum_{n=1}^{N} \|\operatorname{div}\mathbf{A}\|^2_{L^p(B_R((\mathbf{X}_0)_n))}$$

for a sufficiently small fixed $R > 0$. We will decrease $r_0 < R$ while using these fixed upper bounds. Steps 1–3 of the proof of Lemma 4.7 involve no eigenstate, and they use the collision-free condition only to bound the electron–electron interaction. Applied without scalar potential on a neighborhood of $\mathbf{X}_0$, they show that $T^{\mathbf{A}}$ and T^0 have the same local operator domain, the compactly supported elements of $D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$. On annularly supported functions in this domain,

$$T^{\mathbf{A}} = T^0 + R_{\mathbf{A}}, \qquad R_{\mathbf{A}} = \sum_{n=1}^{N} \mathbf{A}(\mathbf{x}_n)\cdot\mathbf{D}_n + q_{\mathbf{A}},$$
$$q_{\mathbf{A}} = \frac{1}{2}\sum_{n=1}^{N} \big(|\mathbf{A}(\mathbf{x}_n)|^2 - \mathrm{i}(\operatorname{div}\mathbf{A})(\mathbf{x}_n)\big).$$

Step 2: The first-order term. This step uses only the free estimates. The Sobolev inequality and (20) give

$$\sum_{n=1}^{N} \|\mathbf{A}(\mathbf{x}_n)\cdot\mathbf{D}_n z\|^2 \le C M_A^2 \|L^{3/4} z\|^2,$$
$$\sum_{n=1}^{N} \|\mathbf{A}(\mathbf{x}_n) z\|^2 \le C M_A^2 \|L^{1/4} z\|^2 \tag{69}$$

for z supported in the configuration ball. For the field part of the first inequality, the pointwise field bound is followed by the Sobolev inequality in $\mathbf{x}_n$ applied to $(H_{\mathrm{rad}}+1)^{1/2} z$. The commuting electronic and photon spectral operators satisfy

$$\|(-\Delta_{\mathbf{x}_n})^{1/4}(H_{\mathrm{rad}}+1)^{1/2} z\| \le \|L^{3/4} z\|.$$

The ordinary derivative part is bounded by the same right-hand side. Thus no spatial derivative of the field operator is taken in (69).

Let η_j be the cutoffs introduced before Lemma A.2, and choose radial cutoffs ζ_j equal to one near $\operatorname{supp}\eta_j$, supported in fixed enlargements of its annulus, with derivative bounds of orders R_j^{-1} and R_j^{-2}. Set $z_j = \zeta_j \mathrm{e}^{\tau f(t)} u$ and $\varphi(\mathbf{X}) = f(-\log r)$. On $\operatorname{supp}\eta_j$,

$$\eta_j \mathrm{e}^{\tau f(t)} \mathbf{D} u = \eta_j \big(\mathbf{D} z_j + \mathrm{i}\tau \boldsymbol{\nabla}\varphi\, z_j\big), \qquad |\boldsymbol{\nabla}\varphi| \le C/R_j.$$

For $a > 0$, put

$$\sigma_a(r_0) = \sup_{0<r\le r_0} \frac{r^a}{\beta(-\log r)},$$

which tends to zero as $r_0 \downarrow 0$. The identity $\sum_{j\geq 0}\eta_j^2 = 1$, (69), and (64) of Lemma A.2, with ζ_j in place of η_j, therefore give

$$\begin{aligned}
&\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \Big\| \sum_{n=1}^{N} \mathbf{A}(\mathbf{x}_n)\cdot \mathbf{D}_n u(\mathbf{X})\Big\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \\
&\quad \leq C M_A^2 \sum_{j\geq 0} \Big(R_j^2 \|L^{3/4} z_j\|^2 + \tau^2 \|L^{1/4} z_j\|^2 \Big) \\
&\quad \leq C M_A^2 \sigma_1(r_0) \sum_{j\geq 0} \Big(\beta_j R_j \|L^{3/4} z_j\|^2 + \tau^2 \beta_j R_j^{-1} \|L^{1/4} z_j\|^2 \Big) \\
&\quad \leq C M_A^2 \sigma_1(r_0) \mathcal{T}^0(u). \qquad (70)
\end{aligned}$$

The second inequality uses $R_j \leq \sigma_1(r_0)\beta_j$, and the last is (64) of Lemma A.2 with $s = 3/2$ and $s = 1/2$. The constants in this inequality are independent of r_0.

Step 3: The zero-order term. The two parts of $q_{\mathbf{A}}$ are one-body multipliers in L^3 and L^p, respectively. For a one-body multiplier $b \in L^q$, $q > 2$, the Sobolev inequality and the commuting spectral operators give

$$\|b(\mathbf{x}_n)\eta_j \mathrm{e}^{\tau f} u\| \leq C_q \|b\|_q \|(-\Delta_{\mathbf{x}_n})^{3/(2q)} (\eta_j \mathrm{e}^{\tau f} u)\| \leq C_q \|b\|_q \|L^{3/(2q)} (\eta_j \mathrm{e}^{\tau f} u)\|.$$

Applying (64) of Lemma A.2 with $s = 1$ and $s = 3/p$ yields directly

$$\begin{aligned}
&\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|q_{\mathbf{A}}(\mathbf{X}) u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \\
&\quad \leq C \Big(M_A^4 \sigma_2(r_0) \tau^{-1} + M_d^2 \sigma_{4-6/p}(r_0) \tau^{-(3-6/p)} \Big) \mathcal{T}^0(u). \qquad (71)
\end{aligned}$$

Only the moduli of the multipliers enter, so the imaginary divergence term causes no change.

Step 4: Absorption. Since $\|R_{\mathbf{A}} u\|^2 \leq 2 \big\| \sum_{n=1}^{N} \mathbf{A}(\mathbf{x}_n)\cdot \mathbf{D}_n u\big\|^2 + 2\|q_{\mathbf{A}} u\|^2$, (70) and (71) give

$$\begin{aligned}
&\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|R_{\mathbf{A}} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \\
&\quad \leq C \Big(M_A^2 \sigma_1(r_0) + M_A^4 \sigma_2(r_0) \tau^{-1} + M_d^2 \sigma_{4-6/p}(r_0) \tau^{-(3-6/p)} \Big) \mathcal{T}^0(u).
\end{aligned}$$

Choose r_0 so small that $C M_A^2 \sigma_1(r_0) \leq \frac{1}{8}$. This is possible because $\sigma_1(r_0) \to 0$ as $r_0 \downarrow 0$, while C and M_A do not grow when r_0 is decreased: C comes from Lemmas A.1 and A.2, which hold on every smaller ball with the same constants, and M_A is a norm on the fixed balls $B_R((\mathbf{X}_0)_n)$. Next increase τ_0 so that the remaining two terms are at most $\frac{1}{8}$, using $3 - 6/p > 0$ and that $\sigma_a(r_0)$ does not increase as r_0 decreases. Then

$$\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|R_{\mathbf{A}} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \leq \tfrac{1}{4} \mathcal{T}^0(u).$$

Since $T^0 u = T^{\mathbf{A}} u - R_{\mathbf{A}} u$, this gives $\mathcal{T}^0(u) \leq 2\mathcal{T}^{\mathbf{A}}(u) + \frac{1}{2}\mathcal{T}^0(u)$, that is,

$$\mathcal{T}^0(u) \leq 4 \mathcal{T}^{\mathbf{A}}(u). \qquad (72)$$

This proves the assertion about the mixed estimate and bounds the left-hand side of (53) of Lemma A.1 by $C\mathcal{T}^{\mathbf{A}}(u)$.

Step 5: Replacing $\mathbf{D}_n$ by $\boldsymbol{\pi}_n$. To replace $\mathbf{D}_n$ by $\boldsymbol{\pi}_n$, use the second estimate in (69) and (64) of Lemma A.2 with $s = 1/2$ to obtain

$$\tau \int_{B_{r_0}(\mathbf{X}_0)} \beta(t) \mathrm{e}^{2\tau f(t)} \sum_{n=1}^{N} \|\mathbf{A}(\mathbf{x}_n) u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \leq C M_A^2 \frac{r_0}{\tau} \mathcal{T}^0(u).$$

Together with $\|\boldsymbol{\pi}_n u\|^2 \leq 2\|\mathbf{D}_n u\|^2 + 2\|\mathbf{A}(\mathbf{x}_n)u\|^2$ and (72), this gives (68). The fixed local norm bounds permit all choices uniformly on compact sets. The endpoint L^6 assumption is used through smallness of the radius, without requiring smallness of the external field itself. □

Lemma A.4 (A Carleman estimate with scalar potential)**.** *Under the assumptions of Lemma A.3, suppose on $B_{r_0}(\mathbf{X}_0)$ that, with the same p,*

$$W(\mathbf{X}) = \sum_{n=1}^{N} v_n(\mathbf{x}_n) + W_b(\mathbf{X}),$$
$$v_n \in L^p(B_{r_0}((\mathbf{X}_0)_n)), \qquad W_b \in L^\infty(B_{r_0}(\mathbf{X}_0)).$$

Then, after increasing τ_0, for every $\tau \geq \tau_0$ and every u in the local operator domain of $T^{\mathbf{A}} + W$ with compact support in $B_{r_0}(\mathbf{X}_0) \setminus \{\mathbf{X}_0\}$,

$$\begin{aligned} \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} \Big(\tau^3 \beta(t) r^{-2} \|u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 + \tau \beta(t) \sum_{n=1}^{N} \|\boldsymbol{\pi}_n u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \\ + \tau \|H_{\mathrm{rad}}^{1/2} u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \Big) \mathrm{d}\mathbf{X} \\ \leq C \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|T^{\mathbf{A}} u(\mathbf{X}) + W(\mathbf{X}) u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X}. \end{aligned} \tag{73}$$

The constants are uniform on compact sets under the local bounds of Lemma A.3, the indicated L^p norms, and $\|W_b\|_\infty$.

Proof. Put $s = 3/p < 3/2$, $M_p^2 = \sum_{n=1}^{N} \|v_n\|_p^2$, and $M_b = \|W_b\|_\infty$. Extend each v_n by zero outside its three-dimensional ball. The Sobolev inequality, as in (39), gives

$$\begin{aligned} \|v_n(\mathbf{x}_n) \eta_j \mathrm{e}^{\tau f} u\| &\leq C \|v_n\|_p \|(-\Delta_{\mathbf{x}_n})^{s/2} (\eta_j \mathrm{e}^{\tau f} u)\| \\ &\leq C \|v_n\|_p \|(-\Delta_{\mathbf{X}})^{s/2} (\eta_j \mathrm{e}^{\tau f} u)\|. \end{aligned}$$

The second inequality follows from the corresponding Fourier multipliers. The identity $\sum_{j\geq 0} \eta_j^2 = 1$ and $r \asymp R_j$ on $\operatorname{supp} \eta_j$ give

$$\begin{aligned} \mathcal{W}(u) &:= \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|W(\mathbf{X}) u(\mathbf{X})\|_{\mathcal{F}_{\mathrm{ph}}}^2 \mathrm{d}\mathbf{X} \\ &\leq C M_p^2 \sum_{j\geq 0} R_j^2 \|(-\Delta_{\mathbf{X}})^{s/2} (\eta_j \mathrm{e}^{\tau f} u)\|^2 + C M_b^2 \sum_{j\geq 0} R_j^2 \|\eta_j \mathrm{e}^{\tau f} u\|^2. \end{aligned}$$

By the joint spectral calculus, $\|(-\Delta_{\mathbf{X}})^{s/2} \Phi\| \leq \|L^{s/2} \Phi\|$. Using (64) of Lemma A.2 with $s = 3/p$ and $s = 0$, followed by (72), we obtain, with $\sigma_a(r_0)$ as in the proof of Lemma A.3,

$$\mathcal{W}(u) \leq C \left(M_p^2 \sigma_{4-2s}(r_0) \tau^{-(3-2s)} + M_b^2 \sigma_4(r_0) \tau^{-3} \right) \mathcal{T}^{\mathbf{A}}(u). \tag{74}$$

Both powers of τ decay because $p > 2$.

The scalar multiplier is infinitesimally bounded relative to the free electronic operator by (39) and the moment inequality (32). Thus, as in Step 3 of the proof of Lemma 4.7, which involves no eigenstate, $T^{\mathbf{A}} + W$ and $T^{\mathbf{A}}$ have the same local operator domain. All terms above are consequently defined for the vectors in the statement. Writing $T^{\mathbf{A}}u = (T^{\mathbf{A}}u + Wu) - Wu$ gives

$$\mathcal{T}^{\mathbf{A}}(u) \leq 2 \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|T^{\mathbf{A}}u(\mathbf{X}) + W(\mathbf{X})u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X} + 2\mathcal{W}(u).$$

Increase τ_0 so that, for $\tau \geq \tau_0$,

$$2C \left(M_p^2 \sigma_{4-2s}(r_0) \tau^{-(3-2s)} + M_b^2 \sigma_4(r_0) \tau^{-3} \right) \leq \frac{1}{2},$$

with C from (74). Then $2\mathcal{W}(u) \leq \frac{1}{2}\mathcal{T}^{\mathbf{A}}(u)$, and since $\mathcal{T}^{\mathbf{A}}(u) < \infty$, subtracting this term gives

$$\mathcal{T}^{\mathbf{A}}(u) \leq 4 \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|T^{\mathbf{A}}u(\mathbf{X}) + W(\mathbf{X})u(\mathbf{X})\|^2_{\mathcal{F}_{\mathrm{ph}}} \mathrm{d}\mathbf{X}.$$

Inserting this bound on the right of (68) of Lemma A.3 gives (73). All estimates use only the stated local norms, which gives uniformity in the center. □

Finally, we include electron spin. On $L^2(\mathbb{R}^{3N}, (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{F}_{\mathrm{ph}})$, let $T^{\mathbf{A}}$ and L act componentwise and set

$$\begin{aligned} V(\mathbf{X}) &= \sum_{n=1}^{N} v_n(\mathbf{x}_n) + V_b(\mathbf{X}), \\ S_{\mathrm{ext}}(\mathbf{X}) &= \frac{1}{2} \sum_{n=1}^{N} \boldsymbol{\sigma}_n \cdot \mathbf{B}(\mathbf{x}_n), \\ S_{\mathrm{ph}}(\mathbf{X}) &= -\frac{e}{2} \sum_{n=1}^{N} \boldsymbol{\sigma}_n \cdot \mathfrak{B}(\mathbf{x}_n), \\ H^{V,\mathbf{A}} &= T^{\mathbf{A}} + V + S_{\mathrm{ext}} + S_{\mathrm{ph}}, \end{aligned}$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ in the sense of distributions. By the Pauli identity (6),

$$H^{V,\mathbf{A}} = \frac{1}{2} \sum_{n=1}^{N} (\boldsymbol{\sigma}_n \cdot \boldsymbol{\pi}_n)^2 + H_{\mathrm{rad}} + V$$

in the sense of quadratic forms. The Zeeman terms S_{ext} and S_{ph} are separated because they are treated as perturbations of $T^{\mathbf{A}}$.

Lemma A.5 (A Carleman estimate with spin)**.** *In addition to the hypotheses of Lemma A.3, suppose, with the same p, that on the configuration ball under consideration*

$$v_n \in L^p(B_{r_0}((\mathbf{X}_0)_n)), \qquad V_b \in L^\infty(B_{r_0}(\mathbf{X}_0)), \qquad \mathbf{B} \in L^p(B_{r_0}((\mathbf{X}_0)_n), \mathbb{R}^3)$$

for every n. Let $E \in \mathbb{C}$ be fixed. Then, after increasing τ_0, for every $\tau \geq \tau_0$ and every u in the local operator domain of $H^{V,\mathbf{A}}$ with compact support in $B_{r_0}(\mathbf{X}_0) \setminus$

$\{\mathbf{X}_0\}$,

$$\begin{aligned}\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)}\Big(\tau^3\beta(t)r^{-2}\|u(\mathbf{X})\|^2_{\mathcal{K}_N} + \tau\beta(t)\sum_{n=1}^{N}\|\boldsymbol{\pi}_n u(\mathbf{X})\|^2_{\mathcal{K}_N} \\ + \tau\|H_{\mathrm{rad}}^{1/2}u(\mathbf{X})\|^2_{\mathcal{K}_N}\Big)\mathrm{d}\mathbf{X} \\ \le C\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2\|(H^{V,\mathbf{A}}-E)u(\mathbf{X})\|^2_{\mathcal{K}_N}\mathrm{d}\mathbf{X},\end{aligned} \tag{75}$$

where $\mathcal{K}_N = (\mathbb{C}^2)^{\otimes N}\otimes\mathcal{F}_{\mathrm{ph}}$. The constants are uniform on compact sets under the local bounds in Lemma A.3, the displayed L^p bounds, and a bound on $\|V_b - E\|_\infty$.

Proof. Expand u in an orthonormal basis of $(\mathbb{C}^2)^{\otimes N}$. Each component is an $\mathcal{F}_{\mathrm{ph}}$-valued function, and $T^{\mathbf{A}}$, $\boldsymbol{\pi}_n$, $H_{\mathrm{rad}}^{1/2}$ and L act on each component separately. The squared $\mathcal{K}_N$-norms in (68) of Lemma A.3 and (64) of Lemma A.2 are the sums of the squared $\mathcal{F}_{\mathrm{ph}}$-norms of the components. Applying Lemma A.3 to each component and summing, we obtain (68) of Lemma A.3 and the mixed estimate (64) of Lemma A.2, with $\mathcal{T}^{\mathbf{A}}(u)$ on the right, for $\mathcal{K}_N$-valued u, with the same constants. Here $\mathcal{T}^{\mathbf{A}}(u)$ is the source integral of Lemma A.3, with norms in $\mathcal{K}_N$.

It remains to replace $T^{\mathbf{A}}$ by $H^{V,\mathbf{A}} - E$. The difference $V + S_{\mathrm{ext}} + S_{\mathrm{ph}} - E$ is a perturbation of the same kind as in Lemma A.4. We show that its weighted L^2 norm is at most $\delta(\tau)\mathcal{T}^{\mathbf{A}}(u)$ with $\delta(\tau)\to 0$ as $\tau\to\infty$, and then absorb it. The classical part $V + S_{\mathrm{ext}} - E$ is handled as in Lemma A.4, using the operator norms of its matrix coefficients. The quantized Zeeman term S_{ph} is controlled by the radiation and zero-order terms of (68) of Lemma A.3.

Put

$$s = 3/p < 3/2, \qquad M_b = \|V_b - E\|_\infty,$$
$$M_p^2 = \sum_{n=1}^{N}\Big(\|v_n\|^2_{L^p(B_{r_0}((\mathbf{X}_0)_n))} + \tfrac{1}{4}\|\mathbf{B}\|^2_{L^p(B_{r_0}((\mathbf{X}_0)_n))}\Big),$$

with $\sigma_a(r_0)$ as in the proof of Lemma A.3. The Pauli identity gives $\|\boldsymbol{\sigma}_n\cdot\mathbf{B}(\mathbf{x}_n)\|_{\mathrm{op}} = |\mathbf{B}(\mathbf{x}_n)|$. Thus the Sobolev inequality gives

$$\big\|\big(v_n(\mathbf{x}_n) + \tfrac{1}{2}\boldsymbol{\sigma}_n\cdot\mathbf{B}(\mathbf{x}_n)\big)\eta_j\mathrm{e}^{\tau f}u\big\| \le C\big(\|v_n\|_p + \tfrac{1}{2}\|\mathbf{B}\|_p\big)\|L^{s/2}(\eta_j\mathrm{e}^{\tau f}u)\|.$$

Here the three-dimensional coefficients are extended by zero outside their respective balls. The identity $\sum_{j\ge 0}\eta_j^2 = 1$ and the mixed estimates with exponents s and 0 now give, exactly as in (74),

$$\begin{aligned}\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2\|(V + S_{\mathrm{ext}} - E)u(\mathbf{X})\|^2_{\mathcal{K}_N}\mathrm{d}\mathbf{X} \\ \le C\left(M_p^2\sigma_{4-2s}(r_0)\tau^{-(3-2s)} + M_b^2\sigma_4(r_0)\tau^{-3}\right)\mathcal{T}^{\mathbf{A}}(u).\end{aligned} \tag{76}$$

This argument uses the operator norm of each matrix coefficient and does not require simultaneous diagonalization of the Pauli matrices.

For the quantized term, (48) and the boundedness of the Pauli matrices give a constant C_{ph}, depending only on e, N and the cutoff norms, such that

$$\|S_{\mathrm{ph}}(\mathbf{X})\xi\|^2 \le C_{\mathrm{ph}}\big(\|H_{\mathrm{rad}}^{1/2}\xi\|^2 + \|\xi\|^2\big) \qquad \big(\xi\in D(H_{\mathrm{rad}}^{1/2})\big),$$

uniformly in $\mathbf{X}$. The radiation term of (68) of Lemma A.3 has no factor β. Its radiation and zero-order terms therefore imply

$$\begin{aligned}
&\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|S_{\mathrm{ph}} u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \\
&\quad \le C_{\mathrm{ph}} r_0^2 \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} \|H_{\mathrm{rad}}^{1/2} u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \\
&\qquad + C_{\mathrm{ph}} \sigma_4(r_0) \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} \beta(t) r^{-2} \|u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \\
&\quad \le C C_{\mathrm{ph}} \left(\frac{r_0^2}{\tau} + \frac{\sigma_4(r_0)}{\tau^3} \right) \mathcal{T}^{\mathbf{A}}(u). \qquad (77)
\end{aligned}$$

In particular, no derivative of $\mathfrak{B}$ occurs in this perturbation argument. It is carried out in the original gauge, after the covariant estimate has been proved, and needs only the existing moment assumption $\omega\widehat{\chi} \in L^2$.

As in the proof of Lemma 4.10, the local domains used here are $D(-\Delta_{\mathbf{X}}) \cap D(H_{\mathrm{rad}})$. Alternatively, the classical matrix estimate above and the moment inequality (32) give infinitesimal operator boundedness relative to L. The same conclusion for S_{ph} follows from its $H_{\mathrm{rad}}^{1/2}$ bound. Hence all the preceding terms are defined for the vectors in the statement.

Let $R_E = V + S_{\mathrm{ext}} + S_{\mathrm{ph}} - E$. Combining (76) and (77) yields

$$\int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|R_E u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} \le \delta(\tau) \mathcal{T}^{\mathbf{A}}(u),$$

where

$$\delta(\tau) = C \left(M_p^2 \sigma_{4-2s}(r_0) \tau^{-(3-2s)} + M_b^2 \sigma_4(r_0) \tau^{-3} + C_{\mathrm{ph}} \frac{r_0^2}{\tau} + C_{\mathrm{ph}} \frac{\sigma_4(r_0)}{\tau^3} \right) \longrightarrow 0.$$

Since $T^{\mathbf{A}} u = (H^{V,\mathbf{A}} - E)u - R_E u$, we obtain

$$\mathcal{T}^{\mathbf{A}}(u) \le 2 \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|(H^{V,\mathbf{A}} - E)u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X} + 2\delta(\tau) \mathcal{T}^{\mathbf{A}}(u).$$

Choose τ_0 so that $2\delta(\tau) \le 1/2$ for $\tau \ge \tau_0$. Since $\mathcal{T}^{\mathbf{A}}(u) < \infty$, subtracting the last term gives

$$\mathcal{T}^{\mathbf{A}}(u) \le 4 \int_{B_{r_0}(\mathbf{X}_0)} \mathrm{e}^{2\tau f(t)} r^2 \|(H^{V,\mathbf{A}} - E)u(\mathbf{X})\|_{\mathcal{K}_N}^2 \mathrm{d}\mathbf{X}.$$

Substitution into the componentwise version of (68) of Lemma A.3 proves (75). The construction uses only the stated local norms, which also proves uniformity in the center. □

Appendix B. Counterexamples at zero coupling

We show that the assumption $e > 0$ in Theorem 3.2 cannot be dropped, either without or with electron spin, even with a nonzero compact ultraviolet cutoff, exponential localization, and nondegenerate ground states. The examples stay within the potential spaces (4).

Proposition B.1 (A spinless common ground state for different magnetic fields)**.** *Let $N = 1$, $e = 0$, and $Z > 0$. The cutoff may be any fixed cutoff satisfying (1). Write $r = |\mathbf{x}|$. For $-1 < t < 1$, set*

$$\mathbf{a}(\mathbf{x}) = \frac{(-x_2, x_1, 0)}{1 + r^2}, \qquad \mathbf{A}_t = t\mathbf{a}, \qquad v_t(\mathbf{x}) = -\frac{Z}{r} - \frac{t^2}{2}|\mathbf{a}(\mathbf{x})|^2.$$

Every $H_{\mathrm{sl}}^{v_t,\mathbf{A}_t}$ on $\mathcal{H}_1^{\mathrm{sl}}$ has the same unique ground state, up to phase,

$$\Psi_Z = \varphi_Z \otimes \Omega, \qquad \varphi_Z(\mathbf{x}) = \left(\frac{Z^3}{\pi}\right)^{1/2} \mathrm{e}^{-Zr}, \qquad E_t = -\frac{Z^2}{2}.$$

Its density and spinless combined current density are independent of t,

$$\rho(\mathbf{x}) = \frac{Z^3}{\pi}\mathrm{e}^{-2Zr}, \qquad \mathbf{j}^{\mathrm{c}} = 0. \tag{78}$$

For $t \neq s$, the vector potentials $\mathbf{A}_t$ and $\mathbf{A}_s$ have different magnetic fields and do not differ by a gradient.

The transformed scalar potential is the same throughout this family, $u_t = v_t + |\mathbf{A}_t|^2/2 = -Z/r$. Each pair satisfies (11) for every $2 < p < 3$, and the density has an exponential moment for every $\alpha < 2Z$. Choosing a nonzero compact ultraviolet cutoff ensures $\omega\widehat{\chi} \in L^2$. Thus only the positive coupling hypothesis of the spinless version of Theorem 3.2 is missing. The same example works at $e > 0$ if $\widehat{\chi} = 0$, when $\mathfrak{A} = 0$.

Proof of Proposition B.1. We have $|\mathbf{a}| \leq r/(1+r^2) \leq 1/2$, $\operatorname{div}\mathbf{a} = 0$, and $\mathbf{a}\cdot\mathbf{x} = 0$. Thus $\mathbf{A}_t \in L^\infty$, while $v_t \in L^{3/2} + L^\infty$ by splitting the Coulomb potential at $r = 1$. Since $e = 0$, expanding the square gives

$$H_{\mathrm{sl}}^{v_t,\mathbf{A}_t} = h_t + H_{\mathrm{rad}}, \qquad h_t = -\frac{1}{2}\Delta - \frac{Z}{r} + \frac{t}{1+r^2}L_z, \qquad L_z = -\mathrm{i}(x_1\partial_{x_2} - x_2\partial_{x_1}).$$

The radial function φ_Z has $L_z\varphi_Z = 0$. It remains to check that it is the ground state for every $|t| < 1$.

We expand an electronic wavefunction ψ in its form domain in spherical harmonics,

$$\psi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} r^{-1}u_{lm}(r)Y_{lm}(\mathbf{x}/r), \qquad u_{lm}(r) = r\int_{\mathbb{S}^2} \overline{Y_{lm}(\mathbf{\Omega})}\,\psi(r\mathbf{\Omega})\mathrm{d}\mathbf{\Omega}.$$

Here Y_{lm} are the spherical harmonics. They form an orthonormal basis of $L^2(\mathbb{S}^2)$ and satisfy $-\Delta_{\mathbb{S}^2}Y_{lm} = l(l+1)Y_{lm}$ and $L_zY_{lm} = mY_{lm}$. The u_{lm} are the radial coefficients, so that $\|\psi\|^2 = \sum_{l,m}\int_0^\infty |u_{lm}(r)|^2\mathrm{d}r$. Let $w(r) = 1/r - Z$ be the logarithmic derivative of the hydrogen $1s$ radial function $r\mathrm{e}^{-Zr}$. For a radial coefficient u, completing the square and integrating by parts give

$$\begin{aligned}\mathcal{E}_Z[u] &:= \frac{1}{2}\int_0^\infty |u'|^2\mathrm{d}r - Z\int_0^\infty \frac{|u|^2}{r}\mathrm{d}r + \frac{Z^2}{2}\int_0^\infty |u|^2\mathrm{d}r \\ &= \frac{1}{2}\int_0^\infty |u' - wu|^2\mathrm{d}r \geq 0.\end{aligned} \tag{79}$$

This follows by integration by parts on a radial core and extension to the form domain. In the (l,m) channel, $-\frac{1}{2}\Delta$ acts on u_{lm} as $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{2r^2}$, and $(1+r^2)^{-1}L_z$

as multiplication by $m/(1+r^2)$. Hence

$$\Big\langle \psi, \Big(h_t + \frac{Z^2}{2}\Big)\psi \Big\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big(\mathcal{E}_Z[u_{lm}] + \int_0^\infty \Big(\frac{l(l+1)}{2r^2} + \frac{tm}{1+r^2} \Big) |u_{lm}(r)|^2 \mathrm{d}r \Big). \quad (80)$$

The weight in the last integral satisfies

$$\frac{l(l+1)}{2r^2} + \frac{tm}{1+r^2} \geq (1-|t|)\frac{l(l+1)}{2r^2} \geq 0, \quad (81)$$

because $1/(1+r^2) \leq 1/r^2$ and $|m| \leq l \leq l(l+1)/2$ for $l \geq 1$, while both sides vanish for $l = 0$. By (79) and (81), every term in (80) is nonnegative, so $h_t \geq -Z^2/2$. For $|t| < 1$, the weight in (81) is positive when $l \geq 1$, so equality forces every $l \geq 1$ channel to vanish. In the remaining channel, $\mathcal{E}_Z[u] = 0$ forces $u(r) = Cr\mathrm{e}^{-Zr}$. The electronic ground-state space is therefore one dimensional and spanned by φ_Z.

Since $H_{\mathrm{rad}} \geq 0$ and $\ker H_{\mathrm{rad}} = \mathbb{C}\Omega$, the full ground-state space is spanned by Ψ_Z. It is not separated from the rest of the spectrum by a positive gap: since $\omega(\mathbf{k}) = |\mathbf{k}|$, we have $\mathrm{spec}(H_{\mathrm{rad}}) = [0,\infty)$, so $[E_t, \infty) \subset \mathrm{spec}(H_{\mathrm{sl}}^{v_t, \mathbf{A}_t})$. The orbital is real, and at $e = 0$ the field-induced current density vanishes, proving (78). Finally, $\boldsymbol{\nabla} \times \mathbf{a}$ is clearly not identically zero, so different t give different magnetic fields. □

To obtain a spinful counterpart, we replace the spherical denominator of $\mathbf{a}$ by a cylindrical one. Its magnetic field then has a fixed direction, and a scalar correction cancels the Zeeman term in the spin-down sector.

Proposition B.2 (A spinful common ground state for different magnetic fields)**.** *Let $N = 1$, $e = 0$, and $Z > 0$. The cutoff may be any fixed cutoff satisfying (1). Write $r = |\mathbf{x}|$ and $s = (x_1^2 + x_2^2)^{1/2}$. For $0 < t < 1/2$, set*

$$\mathbf{a}_\perp(\mathbf{x}) = \frac{(-x_2, x_1, 0)}{1+s^2}, \qquad \mathbf{A}_t = t\mathbf{a}_\perp, \qquad v_t(\mathbf{x}) = -\frac{Z}{r} - \frac{t^2}{2}|\mathbf{a}_\perp(\mathbf{x})|^2 + \frac{t}{(1+s^2)^2}. \quad (82)$$

Every $H^{v_t, \mathbf{A}_t}$ on $\mathcal{H}_1$ has the same unique ground state, up to phase,

$$\Psi_Z^{\downarrow} = \begin{pmatrix} 0 \\ \varphi_Z \end{pmatrix} \otimes \Omega, \qquad \varphi_Z(\mathbf{x}) = \Big(\frac{Z^3}{\pi}\Big)^{1/2} \mathrm{e}^{-Zr}, \qquad E_t = -\frac{Z^2}{2}. \quad (83)$$

Its density, magnetization density, and combined current density are independent of t and satisfy, almost everywhere,

$$\rho(\mathbf{x}) = \frac{Z^3}{\pi}\mathrm{e}^{-2Zr}, \qquad \mathbf{m} = -\rho\mathbf{e}_3, \qquad \mathbf{j}^{\mathrm{c}} = -\frac{1}{2}\boldsymbol{\nabla} \times (\rho\mathbf{e}_3) = \frac{Z\rho}{r}(x_2, -x_1, 0). \quad (84)$$

The magnetic fields are

$$\mathbf{B}_t = \boldsymbol{\nabla} \times \mathbf{A}_t = \frac{2t}{(1+s^2)^2}\mathbf{e}_3. \quad (85)$$

Thus distinct parameters give vector potentials that do not differ by a gradient.

Here the transformed scalar potential is $u_t = v_t + |\mathbf{A}_t|^2/2 = -Z/r + t/(1+s^2)^2$. The additional term compensates the spin-down Zeeman energy $-t/(1+s^2)^2$. The potentials belong to (4) and satisfy (11) for every $2 < p < 3$, and the density has an exponential moment for every $\alpha < 2Z$. With a nonzero compact ultraviolet cutoff, only the positive coupling hypothesis of the spinful Theorem 3.2 is missing. This example also works at $e > 0$ if $\widehat{\chi} = 0$. At $t = 0$ the same state is still a ground state, but the ground-state space is two dimensional because of spin.

Proof of Proposition B.2. Set $f(s) = (1+s^2)^{-1}$ and $h_Z = -\Delta/2 - Z/r$. The vector field $\mathbf{a}_\perp$ is smooth, $|\mathbf{a}_\perp| = s/(1+s^2) \le 1/2$, and $\operatorname{div}\mathbf{a}_\perp = 0$. Direct differentiation gives (85). Hence the vector potential and its curl are bounded, and the scalar potential is Coulomb plus a bounded function. This proves the stated global admissibility and local regularity.

The Pauli identity and the scalar correction in (82) give the form decomposition

$$H^{v_t,\mathbf{A}_t} = h_t^{\mathrm{spin}} + H_{\mathrm{rad}}, \qquad h_t^{\mathrm{spin}} = \big(h_Z + tf(s)L_z\big)\mathbb{1}_{\mathbb{C}^2} + 2tf(s)^2 P_+,$$

where $P_+ = (\mathbb{1}_{\mathbb{C}^2} + \sigma^3)/2$ and $L_z = -\mathrm{i}(x_1\partial_{x_2} - x_2\partial_{x_1})$. Since $L_z\varphi_Z = 0$ and $P_+(0,\varphi_Z)^{\mathsf{T}} = 0$, the displayed spinor is an eigenstate with energy $-Z^2/2$.

For a scalar $\psi \in H^1(\mathbb{R}^3)$, define the nonnegative form

$$Q_Z[\psi] = \Big\langle \psi, \Big(h_Z + \frac{Z^2}{2}\Big)\psi\Big\rangle = \frac{1}{2}\int_{\mathbb{R}^3} \Big|\boldsymbol{\nabla}\psi(\mathbf{x}) + Z\frac{\mathbf{x}}{r}\psi(\mathbf{x})\Big|^2 \,\mathrm{d}\mathbf{x}.$$

The identity follows by integration by parts from $\operatorname{div}(\mathbf{x}/r) = 2/r$ in the sense of distributions, first on a smooth compact core and then on H^1 by form continuity. Moreover, $Q_Z[\psi] = 0$ precisely when ψ is a scalar multiple of φ_Z, since equality implies $\boldsymbol{\nabla}\big(\mathrm{e}^{Zr}\psi(\mathbf{x})\big) = 0$ locally.

In cylindrical coordinates (s,θ,z), $L_z = -\mathrm{i}\partial_\theta$. Expansion in the angular Fourier modes $\mathrm{e}^{\mathrm{i}m\theta}$ and $|m| \le m^2$ for $m \in \mathbb{Z}$ yield

$$|\langle\psi, f(s)L_z\psi\rangle| \le \int_{\mathbb{R}^3} f(s)|\partial_\theta\psi(\mathbf{x})|^2\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^3} \frac{|\partial_\theta\psi(\mathbf{x})|^2}{s^2}\mathrm{d}\mathbf{x} \le 2Q_Z[\psi].$$

For the last inequality, note that $s^{-1}\partial_\theta\psi(\mathbf{x})$ is the azimuthal component of $\boldsymbol{\nabla}\psi(\mathbf{x}) + Z(\mathbf{x}/r)\psi(\mathbf{x})$. These inequalities hold first on the smooth core and extend to H^1, since $f(s)L_z = -\mathrm{i}\mathbf{a}_\perp\cdot\boldsymbol{\nabla}$ has bounded coefficients.

Consequently, for a spinor $\psi = (\psi_+,\psi_-)^{\mathsf{T}} \in H^1(\mathbb{R}^3,\mathbb{C}^2)$,

$$\begin{aligned}\Big\langle \psi, \Big(h_t^{\mathrm{spin}} + \frac{Z^2}{2}\Big)\psi\Big\rangle &\ge (1-2t)\big(Q_Z[\psi_+] + Q_Z[\psi_-]\big) + 2t\int_{\mathbb{R}^3} f(s)^2|\psi_+(\mathbf{x})|^2\mathrm{d}\mathbf{x}\\ &\ge 0.\end{aligned}$$

Since $0 < t < 1/2$ and $f > 0$, equality forces $\psi_+(\mathbf{x}) = 0$ and $\psi_-(\mathbf{x}) = C\varphi_Z(\mathbf{x})$. Thus the electronic ground state is unique up to phase. As $H_{\mathrm{rad}} \ge 0$ and $\ker H_{\mathrm{rad}} = \mathbb{C}\Omega$, the same is true of the full ground state (83).

The orbital is real and $e = 0$, so $\mathbf{j}^{\mathrm{p}} = \mathbf{j}^{\mathrm{ph}} = 0$. The spin-down polarization gives $\mathbf{m}(\mathbf{x}) = -\rho(\mathbf{x})\mathbf{e}_3$, and $\boldsymbol{\nabla}\rho(\mathbf{x}) = -2Z\rho(\mathbf{x})\mathbf{x}/r$ gives (84). The exponential moments follow directly from the displayed density. Finally, (85) shows that distinct parameters give different curls, which excludes even a distributional gradient as their difference. □

References

[1] T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Arkiv för Matematik, Astronomi och Fysik **26B** (1939), no. 17, 1–9.

[2] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Annals of Mathematics **121** (1985), 463–494.

[3] T. H. Wolff, *Recent work on sharp estimates in second-order elliptic unique continuation problems*, Journal of Geometric Analysis **3** (1993), 621–650.

[4] K. Kurata, *A unique continuation theorem for the Schrödinger equation with singular magnetic field*, Proceedings of the American Mathematical Society **125** (1997), 853–860.

[5] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Communications on Pure and Applied Mathematics **54** (2001), 339–360.

[6] P. Hohenberg and W. Kohn, *Inhomogeneous electron gas*, Physical Review **136** (1964), B864–B871. doi 10.1103/PhysRev.136.B864.

[7] E. H. Lieb, *Density functionals for Coulomb systems*, International Journal of Quantum Chemistry **24** (1983), 243–277.

[8] M. Lewin, E. H. Lieb, and R. Seiringer, *Universal functionals in density functional theory*, in: *Density Functional Theory* (E. Cancès and G. Friesecke, eds.), Mathematics and Molecular Modeling, Springer, Cham, 2023, 115–182. doi 10.1007/978-3-031-22340-2_3.

[9] T. C. Corso, *A maximal Hohenberg–Kohn theorem for non-interacting systems via potential theory*, preprint (2026). arXiv:2607.12852.

[10] T. C. Corso, *v-representability and Hohenberg–Kohn theorem for non-interacting Schrödinger operators with distributional potentials on the one-dimensional torus*, Journal of Physics A: Mathematical and Theoretical **58** (2025), 125203. doi 10.1088/1751-8121/adc04c.

[11] T. C. Corso, *A rigorous formulation of density functional theory for spinless fermions in one dimension*, Letters in Mathematical Physics **116** (2026), 27. doi 10.1007/s11005-026-02051-1.

[12] L. Garrigue, *Unique continuation for many-body Schrödinger operators and the Hohenberg–Kohn theorem*, Mathematical Physics, Analysis and Geometry **21** (2018), 27. doi 10.1007/s11040-018-9287-z.

[13] L. Garrigue, *Unique continuation for many-body Schrödinger operators and the Hohenberg–Kohn theorem. II. The Pauli Hamiltonian*, Documenta Mathematica **25** (2020), 869–898. doi 10.4171/DM/765.

[14] D. G. de Figueiredo and J.-P. Gossez, *Strict monotonicity of eigenvalues and unique continuation*, Communications in Partial Differential Equations **17** (1992), 339–346.

[15] A. K. Rajagopal, *Time-dependent functional theory of coupled electron and electromagnetic fields in condensed-matter systems*, Physical Review A **50** (1994), 3759–3765. doi 10.1103/PhysRevA.50.3759.

[16] M. Ruggenthaler, F. Mackenroth, and D. Bauer, *Time-dependent Kohn–Sham approach to quantum electrodynamics*, Physical Review A **84** (2011), 042107. doi 10.1103/PhysRevA.84.042107.

[17] M. Ruggenthaler, J. Flick, C. Pellegrini, H. Appel, I. V. Tokatly, and A. Rubio, *Quantum-electrodynamical density-functional theory: Bridging quantum optics and electronic-structure theory*, Physical Review A **90** (2014), 012508. doi 10.1103/PhysRevA.90.012508.

[18] M. Ruggenthaler, *Ground-state quantum-electrodynamical density-functional theory*, preprint (2015). arXiv:1509.01417.

[19] I. V. Tokatly, *Time-dependent density functional theory for many-electron systems interacting with cavity photons*, Physical Review Letters **110** (2013), 233001. doi 10.1103/PhysRevLett.110.233001.

[20] M. A. Csirik, A. Laestadius, M. Penz, and E. I. Tellgren, *Density functionals for the Pauli–Fierz Hamiltonian*, preprint (2026). arXiv:2611.01208.

[21] E. H. Lieb and R. Seiringer, *The Stability of Matter in Quantum Mechanics*, Cambridge University Press, Cambridge, 2010. doi

10.1017/CBO9780511819681.

[22] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, Berlin, 1995. doi 10.1007/978-3-642-66282-9.

[23] D. Hasler and I. Herbst, *On the self-adjointness and domain of Pauli–Fierz type Hamiltonians*, Reviews in Mathematical Physics **20** (2008), 787–800. doi 10.1142/S0129055X08003389.

[24] M. Griesemer, *Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics*, Journal of Functional Analysis **210** (2004), 321–340. doi 10.1016/j.jfa.2003.06.001.

[25] M. Penz, E. I. Tellgren, M. A. Csirik, M. Ruggenthaler, and A. Laestadius, *The structure of the density-potential mapping. Part II. Including magnetic fields*, ACS Physical Chemistry Au **3** (2023), 492–511. doi 10.1021/acsphyschemau.3c00006.