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ON MEROMORPHIC FUNCTIONS SHARING THREE VALUES IM AND ONE VALUE CM

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Abstract. We prove that if two nonconstant meromorphic functions on the complex plane share three distinct values ignoring multiplicities and a fourth value counting multiplicities, then they share all four values counting multiplicities. This gives an affirmative answer to Gundersen's three-IM-one-CM problem in full generality.

1. Introduction

1.1. **Background and the main theorem.** Value-sharing problems ask how much of a meromorphic function is determined by its fibers over finitely many values. Their early history includes Pólya's work on entire functions of finite order [24]. Nevanlinna's four-value theorem brought these questions into the general theory of meromorphic functions: two distinct nonconstant meromorphic functions sharing four values with the same multiplicities are related by a Möbius transformation [23, Satz 5, p. 378].

Write $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Two meromorphic functions f, g share $a \in \widehat{\mathbb{C}}$ *ignoring multiplicities* (IM) if $f^{-1}(a) = g^{-1}(a)$. They share a *counting multiplicities* (CM) if, in addition, the local multiplicities agree at every point of this common fiber. For $a = \infty$, these multiplicities are pole orders. A value omitted by both functions is shared CM.

The extent to which multiplicities can be discarded is more subtle. Gundersen proved that three CM values and one IM value suffice [8, Theorem 1]. In the same paper he exhibited the pair

$$f(z) = \frac{e^z + 1}{(e^z - 1)^2}, \qquad g(z) = \frac{(e^z + 1)^2}{8(e^z - 1)},$$

which shares $0, 1, -1/8, \infty$ IM and shares none of these values CM [8, (1)]. He subsequently established the two-CM-two-IM theorem [9, Theorem 1], with a correction in [10]. Further accounts of these results and their proofs are given by Steinmetz [25, 26, 27] and Yang and Yi [29].

The remaining question arises from Gundersen's work [9, 10, 11] and is recorded explicitly in [12, Question 3.5].

Problem 1. If two nonconstant meromorphic functions on $\mathbb{C}$ share three distinct values IM and a fourth value CM, must all four values be shared CM?

We give an affirmative answer to Problem 1.

Theorem 1.1. *Let f, g be nonconstant meromorphic functions on $\mathbb{C}$, and let $a_1, a_2, a_3, a_4 \in \widehat{\mathbb{C}}$ be pairwise distinct. If f and g share a_1, a_2, a_3 IM and a_4 CM, then they share all four values CM.*

No assumption on the order, lower order, or periodicity of the functions is required. For distinct functions, Nevanlinna's four-value theorem then gives the usual Möbius relation. The substantive assertion of Theorem 1.1 is the equality of the local multiplicities.

Several earlier results impose additional restrictions on growth or on the distribution of the shared points. Gundersen's work [11] gives a counting criterion for the CM value. Li and Yi study the growth of sharing pairs and derive corresponding uniqueness consequences [19,

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Theorems 1–3]. Lin and Ishizaki treat periodic pairs under growth hypotheses [21, Theorem 1.1]; further finite-order criteria and counting estimates appear in [32, 17]. Li, Zhai and Yi state the finite-order case in [20, Theorem 7]. Their preprint concerns pairs for which at least one function has finite order. The proof below does not use that result; in particular, the classification needed after rescaling is proved here.

1.2. **Proof strategy.** The case $f \equiv g$ is immediate, so assume $f \not\equiv g$. After a common Möbius transformation of the target, the CM value is ∞. Put

$$P(w) = \prod_{j=1}^{3}(w - a_j), \qquad U = \frac{(f-g)f'}{P(f)}, \qquad V = \frac{(f-g)g'}{P(g)}, \qquad \Psi = UV, \qquad \Phi = U/V.$$

The function Ψ is Mues' auxiliary function [22]. The separate factors U, V and their quotient occur in Steinmetz's analysis of the four-value problem [25, Section 6]. At a common a_j-point, $1 \le j \le 3$, of multiplicities m, n, the removable value of Φ is m/n. This identity connects local multiplicity information with the global behavior of an auxiliary meromorphic function.

Rescaling by normal-family methods is a natural way to approach the problem [30, 31, 2]. Steinmetz's simultaneous rescaling lemma identifies the difficulty: shared values or unequal multiplicities may disappear in the limit, and the two limiting functions may even coincide [25, Section 7]. We therefore select a common point with unequal multiplicities *before* taking a limit. The counting estimates in Section 2 supply many such points with bounded multiplicities. The inverse-branch estimate in Section 3 then produces logarithmic coordinates on large disks, with uniform control that does not require separation between zeros and poles of the auxiliary functions.

More precisely, Proposition 4.1 gives a fixed integer $m \ge 2$, a selected common point of type $(1, m)$ at the origin, and rescaled auxiliary functions of the form

$$U_n(s) = \frac{\sqrt{q}\, e^{s/2}}{k_n(s)}, \qquad V_n(s) = \frac{q^{-1/2} e^{-s/2}}{k_n(s)}, \qquad q = 1/m,$$

where $k_n(s)/k_n(0)$ tends uniformly to 1 on expanding disks. Here type $(1, m)$ means that the first function takes the selected shared value simply and the second takes it with multiplicity m. If $k_n(0) \to 0$, a further change of scale gives a constant auxiliary ratio, which is excluded by Lemma 2.4. If $|k_n(0)| \to \infty$, both functions initially approach the marked value. Lemma 5.5 excludes this case by estimates that remain uniform along a real interval whose length tends to infinity. This step is needed because convergence on fixed compact sets does not by itself control such moving points.

When $k_n(0)$ tends to a finite nonzero limit, the limiting pair satisfies an exponential differential system. Its sectorial behavior, a local stable-manifold argument, and a covering-space argument for continuous families with fixed singular values yield a rational representation on $\mathbb{C}^*$. The final algebraic argument combines Riemann–Hurwitz, residues, and moment identities to obtain a common rotation symmetry. After taking the quotient by this symmetry, real monotonicity reduces the problem to Gundersen's two-CM-two-IM theorem. Figure 1 records the principal dependencies. The choices of selected points, inverse branches, and square-root signs need not be unique; the argument uses only the uniform estimates and the specified local multiplicities.

1.3. **Organization.** Section 2 records the auxiliary functions, local multiplicity formulas, and counting estimates. Section 3 proves the quantitative inverse-branch lemma, and Section 4 applies it to selected shared points. Section 5 establishes the compactness and vanishing-limit arguments. The exponential model is classified in Section 6, and the required result for rational functions on $\mathbb{C}^*$ is proved in Section 7. Section 8 assembles these steps to prove the main theorem. Appendix A illustrates the roles of the exceptional set and of the connected parameter interval in the covering argument.

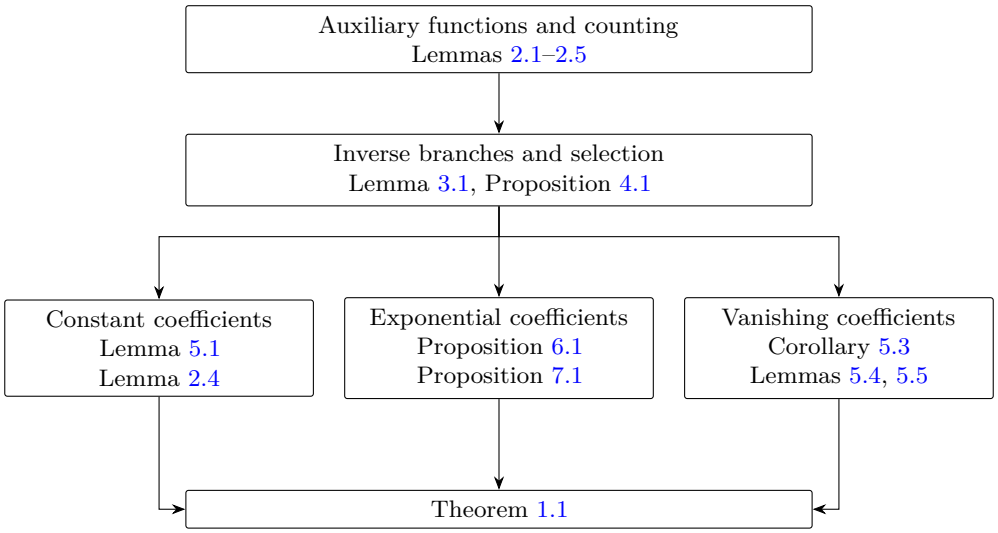


FIGURE 1. The main steps of the proof.

Formalization. Theorem 1.1 has been formalized in Lean 4 as `ThreeIMOneCM.three_IM_one_CM`. The source code is archived in the Palomar Registry as PALOMAR-2026-11-17-000010. The formal statement uses the usual local multiplicities at finite values and at infinity and imposes no growth assumption. The formal proof reorganizes some of the arguments presented here.

Statement on AI usage. The authors acknowledge the use of AI tools for exploratory discussions and language refinement during the preparation of this work. ChatGPT assisted in developing certain technical arguments in the proof of the main theorem. Harmonic's Aristotle and OpenAI's Codex assisted with the Lean 4 formalization of the main theorem.

2. Auxiliary functions and counting estimates

Assume throughout this section that $f \not\equiv g$ share distinct finite values a_1, a_2, a_3 IM and ∞ CM. Set

$$P(w) = \prod_{j=1}^{3}(w - a_j), \qquad U = \frac{(f-g)f'}{P(f)}, \qquad V = \frac{(f-g)g'}{P(g)}, \qquad \Psi = UV, \qquad \Phi = U/V.$$

For the fixed pair, write

$$T(r) = \max\{T(r,f), T(r,g)\}, \qquad L(r) = \log(e + rT(r)).$$

We use the standard notation and basic estimates of Nevanlinna theory; see [15, Sections 1.2–1.3, pp. 3–8] and [27, Sections 2.1.2–2.1.3]. Thus $N(r,h)$ counts poles and $N(r,1/h)$ counts zeros, both with multiplicities. For $a \in \widehat{\mathbb{C}}$, we set

$$N(r,a;h) = \begin{cases} N(r,1/(h-a)), & a \in \mathbb{C}, \\ N(r,h), & a = \infty. \end{cases}$$

We write $\overline{N}(r,a;h)$ for the corresponding reduced counting function. We also write

$$\rho(h) = \limsup_{r\to\infty} \frac{\log^+ T(r,h)}{\log r}$$

for the order of growth, and $h^\# = |h'|/(1+|h|^2)$ for the spherical derivative, continued at poles. Here and below, $\operatorname{ord}_{z_0} h$ is the order of a zero of h at z_0, with a negative value for a pole. A finite shared point has type (m,n) when these are its multiplicities for f and g, respectively. A marked point is a specified common point whose location, shared value, and two multiplicities are retained in the subsequent argument. We use $\log^+ x = \max(\log x, 0)$ for $x > 0$; the notation $u \sim v$ in a stated limit means $u/v \to 1$. All estimates with error $O(L(r))$ hold outside a set of

finite linear measure. A finite union of such exceptional sets is understood whenever several estimates are used together. In particular, we use the unnumbered Lemma on the Logarithmic Derivative [27, Section 2.2.1, p. 44] and the truncated second main theorem [27, Exercise 2.20, (2.15), p. 46], retaining the remainder $O(\log(e+rT(r,h)))$ outside a set of finite linear measure. For the exceptional-set and remainder statement see also [15, Theorem 2.2(i), p. 34].

Lemma 2.1. *The functions U, V, Ψ are entire and none is identically zero. At a finite shared a-point z_0 of multiplicities (m,n), put $d = \operatorname{ord}_{z_0}(f-g)$. Then*

$$\operatorname{ord}_{z_0} U = \operatorname{ord}_{z_0} V = d-1, \qquad \operatorname{ord}_{z_0} \Psi = 2d-2, \qquad \Phi(z_0) = \frac{m}{n}. \tag{2.1}$$

At a common pole z_0 of order k, let r_f, r_g be the leading Laurent coefficients of f, g. Then $\operatorname{ord}_{z_0} \Psi \geq 2k-2$, and for $k=1$,

$$\Psi(z_0) = \left(\frac{1}{r_g} - \frac{1}{r_f}\right)^2. \tag{2.2}$$

Furthermore,

$$N(r,\Phi) + N(r,1/\Phi) \leq N(r,1/\Psi), \qquad N_\infty(r) - \overline{N}_\infty(r) \leq \tfrac{1}{2} N(r,1/\Psi), \tag{2.3}$$

where $N_\infty(r) = N(r,f) = N(r,g)$ counts the common poles with multiplicities, and $\overline{N}_\infty(r)$ denotes the corresponding reduced counting function.

Proof. At a finite shared point, with $t = z - z_0$,

$$\frac{f'}{P(f)} = \frac{m}{P'(a)t} + O(1), \qquad \frac{g'}{P(g)} = \frac{n}{P'(a)t} + O(1).$$

Writing $f - g = \delta t^d(1+O(t))$, $\delta \neq 0$, gives (2.1).

At a common pole, put $t = z - z_0$ and $d = \operatorname{ord}_{z_0}(f-g) \geq -k$. Since

$$f = r_f t^{-k}(1+O(t)), \quad g = r_g t^{-k}(1+O(t)), \quad f - g = \delta t^d(1+O(t)),$$

we obtain

$$U = -\frac{k\delta}{r_f^2} t^{d+2k-1}(1+O(t)), \qquad V = -\frac{k\delta}{r_g^2} t^{d+2k-1}(1+O(t)).$$

Thus $\operatorname{ord}_{z_0} \Psi = 2d + 4k - 2 \geq 2k-2$. For $k=1$, the expansion $f - g = (r_f - r_g)t^{-1} + O(1)$ yields (2.2).

At all other points, U, V are holomorphic; neither vanishes identically. Hence U, V, Ψ are nonzero entire functions. Since

$$|\operatorname{ord}_z U - \operatorname{ord}_z V| \leq \operatorname{ord}_z U + \operatorname{ord}_z V = \operatorname{ord}_z \Psi,$$

the first inequality in (2.3) follows. The second follows from the pole estimate. □

At a shared point of type $(1,m)$ over a, the same calculation gives the useful identity

$$U(z_0) = \frac{f'(z_0)}{P'(a)}, \qquad V(z_0) = \frac{m f'(z_0)}{P'(a)}, \qquad \Phi(z_0) = 1/m.$$

Lemma 2.2. *We have $T(r,\Psi) = O(L(r))$. If Φ is nonconstant and $A = \Phi'/\Phi$, then*

$$1 + T(r,A) + T(r,1/A) + T(r,\Psi) + T(r,1/\Psi) + N(r,\Phi) + N(r,1/\Phi) = O(L(r)). \tag{2.4}$$

Proof. The Mues estimate $T(r,\Psi) = O(L(r))$ follows from [27, (3.32) and Exercise 3.25, p. 99]; see also [22] and [25, Sections 1–2]. The identity $\Phi = f'P(g)/(g'P(f))$ and the derivative estimate $T(r,h') \leq 2T(r,h) + O(L(r))$ for $h = f, g$ [27, Section 2.2.4, p. 47] yield $T(r,\Phi) \leq 10T(r) + O(L(r))$. Hence the logarithmic-derivative lemma [27, Section 2.2.1, p. 44] gives $m(r,A) = O(L(r))$. The poles of $A = \Phi'/\Phi$ are simple and occur only at zeros or poles of Φ. By (2.3),

$$N(r,A) \leq N(r,\Phi) + N(r,1/\Phi) \leq N(r,1/\Psi) = O(L(r)).$$

Thus $T(r,A)=O(L(r))$. Applying the first main theorem to the reciprocals of A and Ψ completes the proof. □

The following counting estimates are consequences of the inequalities of Mues and Rudolph in Steinmetz's Key Lemma [25, Section 4, (Ra) and (Rf)]. The related 4/5 uniqueness threshold appears in [11] and [27, Theorem 3.15, third condition, p. 103]; the quantitative estimate below follows from (Ra). We pass from reduced to full pole counting and retain a bounded-multiplicity consequence in a form convenient for the selection argument.

Lemma 2.3. *Suppose that the four values are not all shared CM. Let $D_a(r)$ be the integrated reduced counting function of common a-points at which the multiplicities differ, and put $D(r)=\sum_a D_a(r)$. Then*

$$N_\infty(r)\le\frac{4}{5}T(r)+O(L(r)),\qquad 2T(r)\le 3D(r)+O(L(r)). \tag{2.5}$$

Let $D_*(r)$ count such points outside the zero set of Ψ whose maximum multiplicity is at most 12. Then

$$D_*(r)\ge\frac{5}{21}T(r)-O(L(r)). \tag{2.6}$$

Every point counted by D_* has type $(1,m)$ or $(m,1)$, with $2\le m\le 12$.

Proof. Put $a_4=\infty$. For $j=1,\dots,4$, set

$$B_j(r)=\overline{N}(r,a_j;f)=\overline{N}(r,a_j;g),\qquad S_j(r)=N_s(r,a_j),$$

where $N_s(r,a_j)$ counts the common a_j-points that are simple for both functions. We suppress the argument r when no confusion arises. Nevanlinna's four-value counting identity [25, Section 1, (Nb)] (see also [27, Exercise 3.23, p. 97]) gives

$$\sum_{j=1}^{4}B_j=2T(r)+O(L(r)).$$

Lemmas 2.1 and 2.2 imply

$$B_4-S_4=O(L(r)),\qquad N_\infty(r)-B_4=O(L(r)),\qquad D(r)=\sum_{j=1}^{4}(B_j-S_j)+O(L(r)).$$

Indeed, a common multiple pole, or a finite shared point of equal multiplicity at least two, is a zero of Ψ. Their reduced counts are therefore $O(L(r))$, and the full excess pole count is controlled by (2.3).

Since the values are not all shared CM, Steinmetz's Key Lemma applies in its nontrivial alternative. Inequality (Ra), with the CM value as the first index, yields

$$\tfrac{3}{2}S_4+S_j\le B_4+B_j+O(L(r)),\qquad j=1,2,3.$$

Adding these inequalities gives

$$\tfrac{5}{2}B_4+\sum_{j=1}^{3}S_j\le 2T(r)+O(L(r)),$$

which proves the pole bound in (2.5). Inequality (Rf) gives $3\sum_j S_j\le 2\sum_j B_j+O(L(r))$, and consequently

$$D(r)\ge\tfrac{1}{3}\sum_{j=1}^{4}B_j-O(L(r))=\tfrac{2}{3}T(r)-O(L(r)).$$

In particular, the second estimate in (2.5) holds.

A common point of unequal multiplicities whose maximum multiplicity is greater than M contributes at least $M+2$ to the six full finite-fiber counts of f,g. Its reduced count is therefore at most $6T(r)/(M+2)+O(1)$. Taking $M=12$ and deleting the zeros of Ψ gives

$$D_*(r)\ge\left(\tfrac{2}{3}-\tfrac{6}{14}\right)T(r)-O(L(r))=\tfrac{5}{21}T(r)-O(L(r)),$$

which is (2.6). At a point retained in this count, (2.1) gives $\operatorname{ord}(f-g)=1$, so the smaller multiplicity is one. This proves the last assertion. □

The constant-ratio conclusion is also contained in the auxiliary-function analysis of [25, Section 6]. We include the argument in the present notation, with the error term needed below.

Lemma 2.4. *If Φ is constant, the four shared values are all CM.*

Proof. Write $\Phi=c\in\mathbb{C}^*$. For $c=1$ the conclusion follows from (2.1). Suppose $c\neq 1$ and the multiplicities differ at a common point. Precomposition with z_*-1+e^t, where z_* is such a point, preserves the unequal multiplicities and the constant ratio, and gives $T(r)\geq c_0 r$ for a positive constant c_0. After renaming the pair, $L(r)=o(T(r))$.

Since $U=cV$ and $UV=\Psi$, the characteristics of U,V and their reciprocals are $O(L)$. Put $s_1=a_1+a_2+a_3$ and $\sigma=f+g$. For any finite shared value a, division of the quadratic polynomial $P(w)/(w-a)$ gives

$$\frac{f'}{f-a}-c\frac{g'}{g-a}=U(\sigma-s_1+a).$$

Thus $m(r,\sigma)=O(L(r))$, and the logarithmic-derivative lemma gives $m(r,\sigma')=O(L(r))$. Put $J=\sigma'/V$; then $m(r,J)=O(L(r))$. Adding the defining equations for U,V gives

$$cP(\sigma-g)+P(g)-(\sigma-2g)J=0.$$

As a polynomial in g, its leading coefficient is $1-c\neq 0$, and its remaining coefficients have proximity $O(L)$. The elementary root bound for a monic polynomial yields $m(r,g)+m(r,f)=O(L(r))$. Consequently $T=N_\infty+O(L)\leq 4T/5+O(L)$ by (2.5), contrary to $L=o(T)$. □

For a holomorphic source map h, direct differentiation gives

$$U_{f\circ h,g\circ h}=h'U\circ h,\quad V_{f\circ h,g\circ h}=h'V\circ h,\quad \Psi_{f\circ h,g\circ h}=(h')^2\Psi\circ h,\quad \Phi_{f\circ h,g\circ h}=\Phi\circ h. \tag{2.7}$$

All multiplicities are multiplied by the same local degree of h.

Lemma 2.5. *If a counterexample to Theorem 1.1 exists, there is a $2\pi i$-periodic counterexample of infinite order, with nonconstant Φ and $T(r)\geq cr$ for all sufficiently large r.*

Proof. Normalize the CM value to ∞, select a common point z_* of unequal multiplicities, and use $h(t)=z_*-1+e^t$. The same unequal multiplicities occur at every $2\pi ik$. Counting these points gives the stated lower bound. If Ψ_0 denotes the original Mues function, then the new one is

$$\Psi_h(t)=e^{2t}\Psi_0(z_*-1+e^t).$$

It is a nonzero entire periodic function tending to zero on the negative real axis. If both composed functions had finite order, Lemma 2.2 would make Ψ_h a polynomial: the bound $T(r,\Psi_h)=O(\log r)$ extends across the finite-measure exceptional set by monotonicity. A periodic polynomial is constant, and the stated limit would make it zero. This is impossible. The ratio is nonconstant by Lemma 2.4. □

3. Quantitative inverse branches

We write $D(z,r)=\{w:|w-z|<r\}$. The following finite-disk lemma uses characteristics at one outer radius only; no growth assumption is involved.

Lemma 3.1. *Let $1\leq r<R\leq r+1$, $\Delta=R-r$, and let Φ be nonconstant and meromorphic on a neighborhood of $\overline{D(0,R)}$. Put $A=\Phi'/\Phi$, and let $\Psi\not\equiv 0$ be meromorphic there. Set*

$$\begin{aligned}M&=1+T(R,A)+T(R,1/A)+T(R,\Psi)+T(R,1/\Psi)\\&\quad+N(R,\Phi)+N(R,1/\Phi),\\B&=C_0(1+R)^3(1+\Delta^{-1})^4(1+M)^3.\end{aligned}$$

Here C_0 is a sufficiently large absolute constant, fixed throughout the proof. For every $q \in \mathbb{C}^*$, with at most CB^{26} exceptions, each distinct q-point z of Φ in $\overline{D(0,r)}$ admits a univalent map $\gamma : D(0, B^6/2) \to D(0, R)$ satisfying $\gamma(0) = z$ and

$$\Phi(\gamma(s)) = qe^s. \tag{3.1}$$

The image lies in a disk on which A and Ψ are holomorphic and zero-free. Choose a square root of Ψ on that disk and put $K = A/\sqrt{\Psi}$. Then

$$|\log|K(z)|| \le 9B^2, \qquad \sup_{|s|<B^6/2} \left|\log \frac{K(\gamma(s))}{K(z)}\right| \le 12B^{-3}, \tag{3.2}$$

where the second logarithm vanishes at $s = 0$. The constant C is absolute and, in particular, independent of q and of separations between divisor points.

Proof. All characteristics in this lemma use the classical $T = m + N$ normalization. Their monotonicity follows from [15, Corollary 1 to Theorem 1.3, p. 8]. For Poisson's and Jensen's formulas see [15, Theorem 1.1 and (1.5), pp. 1–3]. The differentiated Poisson–Jensen formula [15, (2.2), p. 36] is applied to the exponentials of the holomorphic factors below. We keep the constants explicit where an exponential bound is needed.

1. Divisors and the holomorphic factors. Choose $R_1 \in [r + 3\Delta/4, r + 7\Delta/8]$ so that its boundary avoids all zeros and poles of Φ, A, Ψ. Let n_Σ be their total multiplicity in $D(0, R_1)$, summing separately over the three functions, and put $N = \max(1, n_\Sigma)$. Each nonzero divisor point in that disk contributes at least $\log(R/R_1) \ge \Delta/(8R)$ per unit multiplicity to the appropriate counting function at R. A divisor at zero contributes $\log R \ge \log(R/R_1)$. Since $R > 1$, these contributions are nonnegative. It follows that

$$N \le 1 + 8RM/\Delta.$$

Factoring out the divisors gives

$$\Phi = e^H R_\Phi, \qquad A = e^h R_A, \qquad \Psi = e^p R_\Psi,$$

where H, h, p are holomorphic on a neighborhood of $\overline{D(0, R_1)}$. The rational factors are products of monic linear factors and their reciprocals. The sum of their numerator and denominator degrees is $n_\Sigma \le N$. Let Σ be the union of these divisor supports.

For $1 \le \rho \le R$ and $|\alpha| < \rho$, Jensen's formula gives

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|\rho e^{it} - \alpha||\, dt \le \log\rho + 2\log 2 \le 10 + \log(3R).$$

Indeed, after factoring out ρ, the remaining logarithm has mean zero and positive part at most $\log 2$. Also

$$\frac{1}{2\pi}\int_0^{2\pi} \left(|\log|A(R_1e^{it})|| + |\log|\Psi(R_1e^{it})|\right) dt \le M,$$

since $|\log|v|| = \log^+|v| + \log^+|1/v|$ and the four characteristics are nondecreasing. Put

$$P_0 = \frac{100R}{\Delta^2}\{M + N(10 + \log(3R))\}.$$

The Poisson and Schwarz formulas, on $|z| \le r + \Delta/2$, give

$$|\Re h| + |\Re p| + |h'| + |p'| \le P_0 \le B/1000. \tag{3.3}$$

For the first inequality, the Poisson kernel is bounded by $8R/\Delta$ and the derivative of the Schwarz kernel by $32R/\Delta^2$. For the numerical inequality, write $u = 1 + R$, $v = 1 + \Delta^{-1}$, and $w = 1 + M$. Then

$$N \le 9uvw, \qquad P_0 \le Cu^3v^3w.$$

Thus $P_0 \le B/1000$ and $N \le B$ once C_0 is sufficiently large.

Set $\delta = \Delta/[100(N + 1)]$. The forbidden intervals of radii within distance δ of the moduli of points of Σ have total length at most $2N\delta < \Delta/50$. Hence some $\sigma \in [r + \Delta/4, r + \Delta/2]$ has

$\operatorname{dist}(\{|z|=\sigma\},\Sigma)\ge\delta$. On this circle all linear factors have modulus between δ and $3R$. Since $H'=A-R'_\Phi/R_\Phi$ is holomorphic, the maximum principle yields

$$\max_{|z|\le r+\Delta/4}|H'(z)|\le\exp\{P_0+N\log(3R/\delta)\}+N/\delta\le e^B.$$

Indeed, $\delta^{-1}\le 1000uv^2w$, and

$$P_0+N\log(3R/\delta)+\log(2+N/\delta)\le Cu^3v^3w^2.$$

Choosing the absolute constant C_0 sufficiently large makes the last expression at most $B/10$.

2. Counting points near an arbitrary divisor cluster. Let $\eta=e^{-4B}$. Start with the closed η-disks about the points of Σ. Two intersecting disks can be replaced by a disk containing both, with radius at most the sum of their radii. Repeating this operation leaves at most N disks and does not increase the sum of radii. An arbitrarily small enlargement followed, if necessary, by further merging gives disks with disjoint closures and total radius less than $2N\eta$. Their boundaries may be chosen to contain no divisor or q-point: the relevant sets are finite on compact subsets, and the final radii have positive freedom of choice. Every retained disk meeting $\overline{D(0,r)}$ is contained in $D(0,r+\Delta/4)$, since its diameter is less than $4N\eta<\Delta/4$.

We use the following elementary form of the curvature–index inequality. If Γ is a regular closed C^2 plane curve avoiding q, then

$$2\pi|\operatorname{ind}(\Gamma,q)|\le\operatorname{TC}(\Gamma),\tag{3.4}$$

Here $\operatorname{ind}(\Gamma,q)$ is the winding number of Γ about q, and $\operatorname{TC}(\Gamma)$ is the total absolute curvature, equivalently the total variation of the tangent angle. To verify it, let $k=\operatorname{ind}(\Gamma,q)$. For almost every direction θ modulo π, the equation $\Re(e^{-i\theta}(\Gamma-q))=0$ has at least $2|k|$ solutions: the argument of $\Gamma-q$ has total increment $2\pi k$. Periodic Rolle's theorem gives at least $2|k|$ zeros of the derivative of this projection. Integrating their number over $0\le\theta<\pi$ gives the total variation of the tangent angle, by the one-dimensional change-of-variables formula. This is $\operatorname{TC}(\Gamma)$ and proves (3.4). Only real-analytic regular curves are needed below, so the exceptional directions and any constant-angle portions can also be handled by an arbitrarily small perturbation.

Let Γ_0 be a boundary circle of one of the disks meeting $\overline{D(0,r)}$. The curve $\Phi(\Gamma_0)$ is finite and regular: Φ,A are finite and nonzero on Γ_0, and $\Phi'=\Phi A$. The variation of $\arg(z-\alpha)$ along any circle avoiding α is at most 2π. For an interior point the argument is monotone with increment 2π; for an exterior point its total variation is $4\arcsin(s/|c-\alpha|)<2\pi$, where c,s are the center and radius. Using

$$\Phi'=e^{H+h}R_\Phi R_A$$

and adding the 2π variation of the source tangent, we therefore obtain

$$\operatorname{TC}(\Phi(\Gamma_0))\le 2\pi(N+1)+2\pi\operatorname{rad}(\Gamma_0)(e^B+B)<2\pi(N+2).$$

The last error is less than 1, since $\operatorname{rad}(\Gamma_0)<2Ne^{-4B}$ and $N\le B$. Adding the pole count to the index in the argument principle shows that the number of q-points inside Γ_0, counted with multiplicity, is at most

$$N+|\operatorname{ind}(\Phi(\Gamma_0),q)|\le 2N+2.$$

Thus the union of all the small disks contains at most $N(2N+2)\le 4B^2$ q-points in the inner disk.

3. Separation and packing of the remaining bad points. For a point z outside the small disks, put $d(z)=\min(1,\operatorname{dist}(z,\Sigma))$, with $d(z)=1$ if Σ is empty. Then $\eta\le d(z)\le 1$. For $z\in\overline{D(0,r)}$ and $|w-z|\le d(z)/B^4$, each divisor is at distance at least $d(z)/2$ from w. The disk in the original variable is contained in $D(0,r+\Delta/2)$. Logarithmic differentiation of the factors and (3.3) give

$$\left|\frac{A'(w)}{A(w)}\right|+\left|\frac{\Psi'(w)}{\Psi(w)}\right|\le B+\frac{2N}{d(z)}\le\frac{8B}{d(z)}.\tag{3.5}$$

At the center, $\eta \le |z-\alpha| \le 3R$ for each divisor point and $\log(3R) \le B$. Hence

$$|\log|A(z)|| \le 6B^2, \qquad |\log|\Psi(z)|| \le 6B^2. \tag{3.6}$$

This proves the first bound in (3.2) whenever the local square root is defined.

Call z good if $|A(z)| > B^{10}/d(z)$. At any other q-point outside the small disks, take a logarithm $W = \log\Phi$ on $D(z, d(z)/(100B^{11}))$. The derivative $W' = A$ is nonzero there. Integrating (3.5) shows that $A(w)/A(z)$ has positive real part throughout this disk. Integration on straight segments then proves that W is univalent. Moreover

$$|W(w) - W(z)| \le \frac{2|A(z)|d(z)}{100B^{11}} \le \frac{1}{50B} < \pi.$$

Two q-points would give logarithms differing by an integral multiple of $2\pi i$; the last bound and univalence exclude a second one.

Consequently the bad q-points with $d \ge 1/2$ have mutual separation at least $(200B^{11})^{-1}$. Packing half-sized disjoint disks gives at most $C(r+1)^2B^{22}$ such points. In a layer $2^{-j-1} \le d < 2^{-j}$, the points lie in the union of N disks of radius 2^{-j} about Σ and have mutual separation at least $2^{-j-1}/(100B^{11})$. Assigning each point to one of these disks and packing within it gives at most CNB^{22} points in the layer. Since $d \ge e^{-4B}$, there are at most CB layers. Altogether, including the small disks,

$$\#\{\text{bad } q\text{-points in } \overline{D(0,r)}\} \le 4B^2 + C(r+1)^2B^{22} + CNB^{23} \le CB^{26}.$$

No minimum separation between the divisor points was used.

4. The inverse disk at a good point. Let z be good and put $\rho = d(z)/B^4$. On a neighborhood of $\overline{D(z,\rho)}$ the functions Φ, A, Ψ are holomorphic and zero-free. With the branches normalized at z, (3.5) gives

$$\left|\log\frac{A(w)}{A(z)}\right| \le 8B^{-3}, \qquad \left|\log\frac{\Psi(w)}{\Psi(z)}\right| \le 8B^{-3}.$$

In particular $|A(w)/A(z) - 1| < 1/4$. A logarithm W of Φ is univalent there, by the same straight-segment argument, and

$$|W(w) - W(z) - A(z)(w-z)| < \tfrac{1}{4}|A(z)|\rho \quad (|w-z| = \rho).$$

Rouché's theorem, applied to $W - W(z) - s$, shows that the image of $W - W(z)$ contains $D(0, 3|A(z)|\rho/4)$. Its radius is greater than $3B^6/4$. Inverting on $D(0, B^6/2)$ gives γ and (3.1). The disk in the original variable is simply connected, so it supports a square root of Ψ. Finally

$$\log\frac{K(w)}{K(z)} = \log\frac{A(w)}{A(z)} - \frac{1}{2}\log\frac{\Psi(w)}{\Psi(z)}$$

has modulus at most $12B^{-3}$. Together with (3.6), this proves (3.2). □

4. Selection of shared points with unequal multiplicities

Proposition 4.1. *For the periodic counterexample in Lemma 2.5, there are $B_n \to \infty$, a finite shared value a, an integer $2 \le m \le 12$, and univalent maps γ_n on $D(0, B_n^6/2)$ such that, after interchanging f, g if necessary,*

$$\Phi(\gamma_n(s)) = qe^s, \qquad q = 1/m,$$

and $F_n = f \circ \gamma_n$, $G_n = g \circ \gamma_n$ have a marked $(1,m)$ point over a at zero. Their auxiliary functions satisfy

$$U_n(s) = \frac{\sqrt{q}\, e^{s/2}}{k_n(s)}, \qquad V_n(s) = \frac{q^{-1/2}e^{-s/2}}{k_n(s)}, \tag{4.1}$$

where k_n is zero-free and

$$|\log|k_n(0)|| \le 9B_n^2, \qquad \sup_{|s|<B_n^6/2}\left|\log\frac{k_n(s)}{k_n(0)}\right| \le 12B_n^{-3}. \tag{4.2}$$

Proof. Translate the source so that the origin avoids the shared fibers and all zeros and poles of $\Phi, A = \Phi'/\Phi, \Psi$. There is then $r_* > 0$ such that every marked point has modulus at least r_*. By (2.6) and $L = o(T)$, the number $n_*(r)$ of distinct marked points satisfies

$$n_*(r) \geq c\, T(r)/\log r \tag{4.3}$$

for all sufficiently large admissible radii, because $D_*(r) \leq n_*(r)\log(r/r_*)$.

Let E be the common finite-measure exceptional set in (2.4) and (4.3). The function L is continuous and nondecreasing, with $L \geq 1$. The classical Borel lemma [13, Lemma 1.1, p. 354] therefore supplies a set E_0 of finite linear measure such that

$$L(r + 1/L(r)) \leq 2L(r) \qquad (r \notin E_0).$$

The one-sided maximal set

$$E^* = \left\{ r \geq 1 : \sup_{h>0} \frac{|E \cap [r, r+h]|}{h} > \frac{1}{4} \right\}$$

has finite measure. The usual disjoint-interval covering argument gives $|E^*| \leq 20|E|$. If $r \notin E \cup E_0 \cup E^*$, the upper half of $[r, r + 1/L(r)]$ contains $R \notin E$. Consequently

$$\frac{1}{2L(r)} \leq R - r \leq \frac{1}{L(r)}, \qquad L(R) \leq 2L(r), \qquad M(R) \leq CL(r).$$

Here $M(R)$ is the coefficient expression in Lemma 3.1. Its parameter therefore satisfies

$$B \leq Cr^3 L(r)^7. \tag{4.4}$$

Infinite order supplies admissible $r_n \to \infty$ with $\log T(r_n)/\log r_n \to \infty$. This remains true outside any prescribed finite-measure set: move an initial radius to a point outside that set in $[r, r + 1]$ and use monotonicity. From (4.4),

$$B_n^{26} = O(r_n^{78} L(r_n)^{182}) = o(T(r_n)/\log r_n).$$

The marked ratio values belong to the fixed set $\{2, \ldots, 12, 1/2, \ldots, 1/12\}$. Lemma 3.1 and (4.3) therefore give good marked points. Take a subsequence with fixed value, orientation and multiplicity. Interchanging the two maps replaces Φ by $1/\Phi$ and the logarithmic coordinate by its negative, without changing the estimates.

Set $k_n = (A/\sqrt{\Psi}) \circ \gamma_n$. Choose the square-root sign so that $U(\gamma_n(s))/\sqrt{\Psi(\gamma_n(s))} = \sqrt{q}\, e^{s/2}$. The two sides have the same square, so agreement at zero gives agreement throughout. Since $\gamma_n' = 1/(A \circ \gamma_n)$, covariance (2.7) proves (4.1); (4.2) is (3.2). □

5. Normal families and vanishing limits

All sharing statements in this section concern the same four fixed values. The functions may initially be defined on disks exhausting the plane.

Lemma 5.1. *Let (f_n, g_n) be pairs of distinct nonconstant meromorphic functions sharing a_1, a_2, a_3 IM and ∞ CM. Suppose that Ψ_n is locally bounded and Φ_n converges spherically locally to ϕ. The two sequences are normal near every point where $\phi \neq 1$. They are also normal at a simple 1-point of a nonconstant ϕ, provided Ψ_n is eventually zero-free in a neighborhood of that point.*

If both limits are nonconstant, the four IM relations and the CM pole relation pass to the limits. Suppose also that a fixed point z_0 has type $(1, m)$ over a fixed finite shared value for every pair, with $m \geq 2$. If $f_n'(z_0)$ converges to a nonzero limit, then the limiting orders at z_0 are 1 and at least m.

Proof. We use the simultaneous rescaling argument in [25, Section 7, Rescaling Lemma]. First, normality of one member of a sharing pair implies normality of the other. Otherwise Zalcman's rescaling lemma [31, Section 2, pp. 216–217], in its unweighted form, applied to the nonnormal member, produces a nonconstant meromorphic function on $\mathbb{C}$, whereas the normal member tends to a constant on the same scale. Hurwitz's theorem then makes the nonconstant limit

omit at least three values, contrary to Picard's theorem. The same observation shows that a pair of global limits cannot have exactly one nonconstant member.

If a pair is nonnormal at z_0, rescale one member to obtain a nonconstant Zalcman limit. On the rescaled domains that sequence is normal, so the preceding observation gives normality of its companion. Passing to a subsequence produces simultaneous limits F, G, both nonconstant. The rescaled Mues functions tend to zero because they acquire the factor ρ_n^2. At ordinary points the defining identity gives $F = G = H$. Where $P(H)H' \neq 0$, the quotient defining Φ_n tends to 1. This contradicts $\phi(z_0) \neq 1$.

Suppose next that $\phi - 1$ has a simple zero at z_0. On a sufficiently small disk, $\Phi_n - 1$ has exactly one zero, counted with multiplicity. Zero-freeness of Ψ_n gives $\min(m, n) = 1$ at every finite shared point there. Since m/n is close to 1, both multiplicities equal 1; hence every such point is a zero of $\Phi_n - 1$. A hypothetical nonnormal sequence would again yield a common nonconstant rescaled limit H. This function takes the three finite shared values at at least two distinct finite source points. For a transcendental function this follows from Picard's theorem; for a rational function of degree d, the three fibers have degree $3d$, of which at most d lies at infinity and at most d at any single finite point. Hurwitz's theorem now produces two zeros of $\Phi_n - 1$ in the small disk, a contradiction.

IM sharing passes by Hurwitz. Near a limiting pole, apply Rouché's theorem to the reciprocals on a disk containing no other limiting poles. The total pole orders of the approximants agree, so the limiting orders agree. At a finite marked point, convergence is Euclidean; the derivative condition preserves the first simple zero and the first m vanishing Taylor coefficients of the second member. On exhausting disks, the companion argument rules out a constant second global limit. □

We need a compactness statement for a degenerate differential system.

Lemma 5.2. *Let a_n, b_n be holomorphic on a domain and converge locally uniformly to nowhere-zero functions a, b. Put $c_n = a_n - b_n$ and $c = a - b$. Every sequence of holomorphic solutions H_n of*

$$c_n H_n(H_n - 1)H_n'' - c_n' H_n(H_n - 1)H_n' - \{c_n(3H_n - 1) + b_n\}(H_n')^2 = 0 \tag{5.1}$$

is a normal family.

Proof. A nonconstant Zalcman limit h of (5.1) satisfies $b(z_0)(h')^2 = 0$ if $c(z_0) = 0$. Otherwise it satisfies

$$h(h - 1)h'' - (3h - 1 + k)(h')^2 = 0, \qquad k = b(z_0)/c(z_0). \tag{5.2}$$

The function h is entire, since a nonconstant spherical limit of holomorphic functions has no poles. At a zero of h of order p, comparison gives $k = 1/p$; at a zero of $h - 1$ of order q, it gives $k = -1 - 1/q$. Thus h cannot take both 0 and 1, and Picard's theorem prevents it from omitting both.

If h takes 0 and omits 1, its entire p-th root omits all p-th roots of unity, so $p = 1$. Equation (5.2) then gives $((h - 1)^{-2})'' = 0$. A zero-free entire affine function is constant, a contradiction. The other case similarly gives $q = 1$ and $(h^{-2})'' = 0$. This proves normality. □

We apply the scalar normality statement to a pair of equations.

Corollary 5.3. *Let a_n, b_n be holomorphic on disks centered at zero whose radii tend to infinity, and suppose that they converge locally uniformly to nowhere-zero functions. Let X_n, Y_n be holomorphic solutions of*

$$D_n X_n' = a_n X_n, \qquad D_n Y_n' = b_n Y_n, \qquad D_n = X_n - Y_n. \tag{5.3}$$

If X_n, Y_n have zeros of orders $1, m$ at zero, respectively, for one fixed integer $m \geq 2$, then a subsequence of both functions converges locally uniformly on $\mathbb{C}$ in the Euclidean metric.

Proof. Put $c_n = a_n - b_n$. After restricting to sufficiently slowly exhausting disks and omitting finitely many terms on each compact set, we may assume that a_n, b_n are nowhere zero there.

Comparison of orders in (5.3) then shows that every zero of D_n in these disks is a common zero and is simple. Hence $H_n = X_n/D_n$ extends holomorphically and

$$D_n' = c_n H_n + b_n, \qquad D_n H_n' = -c_n H_n(H_n - 1). \tag{5.4}$$

Elimination gives (5.1), so Lemma 5.2 applies. At the marked point, $H_n(0) = 1$. By Hurwitz's theorem, a locally uniform spherical limit of holomorphic functions is either holomorphic or identically infinite. The latter possibility is excluded by the value at zero. Thus a subsequence of H_n converges locally uniformly in the Euclidean metric. Since $D_n(0) = 0$, the first equation in (5.4) gives the same convergence for D_n. Finally,

$$X_n = D_n H_n, \qquad Y_n = D_n(H_n - 1)$$

give locally uniform Euclidean convergence of X_n, Y_n. □

Lemma 5.4. *Let X, Y be meromorphic on $\mathbb{C}$ and satisfy*

$$(X - Y)X' = \alpha e^{z/2} X, \qquad (X - Y)Y' = \beta e^{-z/2} Y, \qquad \alpha\beta \neq 0.$$

Apart from the zero pair, the only solutions are

$$(X, Y) = (2\alpha e^{z/2} + C, 0) \quad \text{or} \quad (X, Y) = (0, 2\beta e^{-z/2} + C).$$

Proof. Write $D = X - Y$, $a = \alpha e^{z/2}$, $b = \beta e^{-z/2}$ and $c = a - b$. If X has a pole of order k, the first equation forces D to have a simple zero. Thus Y has the same principal part, and $c = D(\log(X/Y))'$ vanishes to order at least $k + 1$. This contradicts the simplicity of the zeros of c. The same argument applies to Y. Thus X, Y are entire. If $D \equiv 0$, the equations give the zero pair. Otherwise a zero of D is a common zero of X, Y, and comparison of orders in the equation for a nonzero member gives $\operatorname{ord} D = 1$. Hence $H = X/D$ extends holomorphically across every zero of D and is entire.

If H is nonconstant, (5.1) becomes

$$H(H-1)\frac{H''}{(H')^2} - \frac{c'}{c}\frac{H(H-1)}{H'} - 3H + 1 - \frac{b}{c} = 0. \tag{5.5}$$

For a polynomial H of degree $d \geq 1$, division by H and passage along the positive real axis gives an unbounded term $-x/(2d)$, with all other terms bounded.

For transcendental H, write $M(r) = \max_{|z|=r} |H(z)|$. If $H(z) = \sum_{j\geq 0} h_j z^j$, its central index is

$$\nu(r) = \max\{j : |h_j| r^j = \max_{\ell \geq 0} |h_\ell| r^\ell\}.$$

The local Wiman–Valiron theorem [16], in the form of [18, Lemma 2.2, (17)–(19)], gives an exceptional set of finite logarithmic measure such that, for all sufficiently large r outside this set, there is z_r with $|z_r| = r$ for which

$$\nu := \nu(r) \leq (\log M(r))^2, \qquad |H(z_r)| \sim M(r),$$

and

$$H(z) = H(z_r)(z/z_r)^\nu (1 + o(1)) \qquad (|\log(z/z_r)| \leq \nu^{-2/3}).$$

On strictly smaller logarithmic disks, Cauchy's estimates on disks of radius comparable to r/ν give

$$H'/H \sim \nu/z, \qquad HH''/(H')^2 \longrightarrow 1.$$

Put $R_r = r\nu^{-2/3}$ and consider the arc

$$\mathcal{A}_r = \{z_r e^{it} : |t| \leq \tfrac{1}{4}\nu^{-2/3}\}.$$

All the preceding estimates hold uniformly on this arc, and $|H(z)| \sim M(r)$ there. The zeros of $c = \alpha e^{z/2} - \beta e^{-z/2}$ form a vertical arithmetic progression with spacing 2π. The arc therefore contains a point at distance at least $\eta \min(1, R_r)$ from this zero set, for a constant $\eta > 0$ independent of r. Indeed, take a subarc of length comparable to $\min(1, R_r)$. If this subarc were covered by the small pairwise disjoint disks of that radius about the zeros, connectedness

would place it in a single disk, contradicting its chord length when η is sufficiently small. At the selected point,

$$|c'/c| + |b/c| \le C(1 + R_r^{-1}), \qquad b/(cH) \longrightarrow 0.$$

The last limit follows from $\nu \le (\log M(r))^2$. Dividing (5.5) by H yields

$$\frac{c'}{c}\frac{z}{\nu} \longrightarrow -2. \tag{5.6}$$

It follows that $\nu \le Cr + C\nu^{2/3}$, and hence $\nu = O(r)$. In particular, $R_r \to \infty$.

We can now choose $w_r \in \mathcal{A}_r$ with $|\Re w_r| \to \infty$. If $|\Re z_r| \ge r/2$, take $w_r = z_r$; otherwise the horizontal projection of the arc has length at least $R_r/8$ for large r. Along a subsequence, either $\Re w_r \to +\infty$ or $\Re w_r \to -\infty$. In these cases $c'/c \to 1/2$ or $-1/2$, respectively. Repeating the derivation of (5.6) at w_r gives $w_r/\nu \to -4$ or 4, respectively, contradicting the sign of its real part. Thus H is constant. The second equation in (5.4) gives $H = 0$ or 1, and integration completes the proof. □

Lemma 5.5. *Let (F_n, G_n) share $0, a_2, a_3, \infty$, with ∞ CM, and have a marked zero of type $(1, m)$ at 0, where $m \ge 2$. Suppose that $\epsilon_n \downarrow 0$, $A_0, B_0 \in \mathbb{C}^*$, and*

$$\frac{U_n(z)}{\epsilon_n A_0 e^{z/2}} \longrightarrow 1, \qquad \frac{V_n(z)}{\epsilon_n B_0 e^{-z/2}} \longrightarrow 1, \qquad A_0/B_0 = 1/m, \tag{5.7}$$

uniformly, for each fixed R, on

$$-R \le \Re z \le 2\log(1/\epsilon_n) + R, \qquad |\Im z| \le R. \tag{5.8}$$

Assume these rectangles are contained in the domains of the pairs. No sequence of pairs satisfying these assumptions exists.

Proof. In this proof, $P(w) = w(w - a_2)(w - a_3)$, and U_n, V_n are the corresponding auxiliary functions. On compact sets, $\Psi_n \to 0$ and $\Phi_n \to m^{-1}e^z$. Lemma 5.1 applies, since (5.7) ensures local zero-freeness. Two nonconstant limits would coincide by the Mues identity and force the limiting ratio to be 1; a single nonconstant limit is excluded by the companion argument. The marked value therefore gives $F_n, G_n \to 0$ locally uniformly in the Euclidean metric.

Put $p = P'(0)$ and $Q(w) = P(w)/w$. On slowly exhausting disks the normalized functions $X_n = F_n/\epsilon_n$, $Y_n = G_n/\epsilon_n$ satisfy (5.3), with

$$a_n = (U_n/\epsilon_n)Q(F_n) \longrightarrow pA_0e^{z/2}, \qquad b_n = (V_n/\epsilon_n)Q(G_n) \longrightarrow pB_0e^{-z/2}.$$

Corollary 5.3 and Lemma 5.4, together with $X_n'(0) \to pA_0 \ne 0$, give the unique limits

$$F_n/\epsilon_n \longrightarrow 2pA_0(e^{z/2} - 1), \qquad G_n/\epsilon_n \longrightarrow 0. \tag{5.9}$$

Write $\alpha = pA_0$, $D_n = F_n - G_n$, $h_n = G_n/D_n$, and $\mathcal{A}_n = U_nQ(F_n)$, $\mathcal{B}_n = V_nQ(G_n)$. Where the maps are small and $D_n \ne 0$, direct differentiation gives

$$D_n' = \mathcal{A}_n + (\mathcal{A}_n - \mathcal{B}_n)h_n, \qquad \frac{h_n'}{h_n} = -\frac{(\mathcal{A}_n - \mathcal{B}_n)(1 + h_n)}{D_n}. \tag{5.10}$$

The function h_n cannot vanish in this region: a zero would give $G_n = 0$, hence $F_n = 0$ by sharing, contrary to $D_n \ne 0$.

For fixed $R_0 > 0$, (5.9) gives $h_n(R_0) \to 0$ and

$$\frac{D_n(R_0)}{\epsilon_n e^{R_0/2}} \longrightarrow 2\alpha(1 - e^{-R_0/2}).$$

On the real segment starting at R_0, put $\tau = \epsilon_n e^{x/2}$ and $W_n = D_n/\tau$. We continue to $\tau = \delta$, where $\delta > 0$ will be chosen small and independent of n. Let $\eta_n \to 0$ bound the relative errors in (5.7) on a fixed-height rectangle containing this segment. On a fixed neighborhood of $(W_n, h_n) = (2\alpha, 0)$, we have $F_n = \tau W_n(1 + h_n)$ and $G_n = \tau W_n h_n$. Consequently

$$\frac{\mathcal{A}_n}{\tau} = \alpha + O(\delta + \eta_n), \qquad \frac{\mathcal{B}_n}{\tau} = pB_0e^{-x} + O(e^{-x}(\delta + \eta_n)).$$

Equations (5.10) therefore give, uniformly on the segment,

$$W_n' = -\tfrac{1}{2}(W_n - 2\alpha) + E_n, \qquad |E_n| \le C(\delta + |h_n| + \eta_n),$$
$$\frac{h_n'}{h_n} = -\tfrac{1}{2} + O(\delta + |h_n| + |W_n - 2\alpha| + e^{-R_0} + \eta_n).$$

Fix the constants on that neighborhood. Choose a radius $0 < r_0 < |\alpha|$ sufficiently small, then $b_0 > 0$ sufficiently small relative to r_0, and then R_0 large, δ small, and n large. On the region $|W_n - 2\alpha| \le r_0$, $|h_n| \le b_0$, these choices give

$$\Re(h_n'/h_n) \le -\tfrac{1}{4}, \qquad |E_n| \le r_0/4, \qquad |W_n(R_0) - 2\alpha| \le r_0/4, \quad |h_n(R_0)| \le b_0/2.$$

Thus

$$|h_n(x)| \le |h_n(R_0)| e^{-(x-R_0)/4} \le b_0/2,$$

and variation of constants yields

$$|W_n(x) - 2\alpha| \le e^{-(x-R_0)/2}|W_n(R_0) - 2\alpha| + \int_{R_0}^{x} e^{-(x-s)/2}|E_n(s)|\,ds$$
$$\le \frac{r_0}{4}e^{-(x-R_0)/2} + \frac{r_0}{2}(1 - e^{-(x-R_0)/2}) \le r_0/2.$$

Both bounds are strict relative to the chosen region. In particular, $D_n = \tau W_n$ stays nonzero and F_n, G_n stay bounded and small. The solution therefore continues to $L_n = 2\log(\delta/\epsilon_n)$ without a pole or a zero of D_n. The estimate is independent of the segment length, since the integral of the exponential kernel is at most 2. At this endpoint,

$$G_n(L_n) \to 0, \qquad F_n(L_n) \text{ stays near } 2\alpha\delta \neq 0, \qquad |F_n'(L_n)| \ge c_\delta > 0. \tag{5.11}$$

The derivative bound follows from $F_n' = U_n Q(F_n)(1 + h_n)$.

After translation by L_n, the domains still exhaust the plane and

$$U_n(L_n + z) \longrightarrow A_0\delta e^{z/2}, \qquad V_n(L_n + z) = \frac{B_0\epsilon_n^2}{\delta}e^{-z/2}(1 + o(1)) \longrightarrow 0.$$

Thus $\Psi_n \to 0$ and $\Phi_n \to \infty$ locally. Lemma 5.1, followed by a diagonal subsequence argument, gives meromorphic limits on the whole plane. The first is finite at the origin, so convergence there is Euclidean and the derivative bound in (5.11) makes that limit nonconstant. If the second were nonconstant, the Mues identity and the limiting ratio would be incompatible. It is therefore constant, necessarily 0, contradicting the companion argument. □

6. The exponential model

Proposition 6.1. *Suppose that distinct nonconstant meromorphic functions f, g share a_1, a_2, a_3 IM and ∞ CM, and that*

$$U = \kappa e^t, \qquad V = \kappa e^{-t}, \qquad \kappa \neq 0. \tag{6.1}$$

Then, for some positive integer q, there are rational functions R, S such that $f(t) = R(e^{t/q})$ and $g(t) = S(e^{t/q})$.

We first establish the growth and asymptotic information needed in its proof. A proper right sector is a closed subsector of $|\arg t| < \pi/2$; a proper left sector is defined similarly.

Lemma 6.2. *Under (6.1), $T(r) = O(r)$. For some $a \in \{a_1, a_2, a_3\}$,*

$$f(t) \to a, \qquad g(t) \sim -\kappa^{-1}e^t, \qquad \frac{f'}{P(f)} \to \kappa^2$$

uniformly on proper right sectors. On proper left sectors,

$$f(t) \sim -\kappa^{-1}e^{-t}, \qquad g/f \to 0, \qquad \frac{g'}{P(g)} \to -\kappa^2. \tag{6.2}$$

Moreover $\mu = -\kappa^2 P'(a)$ is a positive real number.

Proof. Here $\Psi = \kappa^2$ and $\Phi = e^{2t}$. Lemma 2.1 shows that at a finite shared point $\min(m,n) = 1$, every common pole is simple, and all critical values lie among the three finite shared values. The points with $\max(m,n) \le 10$ lie over the nineteen ratio values $1, 2, \dots, 10, 1/2, \dots, 1/10$. At each remaining point the sum of the two multiplicities is at least 12, so their reduced counting function is at most $T(r)/2 + O(1)$. The truncated second main theorem with three values [27, Exercise 2.20, (2.15), p. 46] consequently gives

$$T(r) \le 19T(r, e^{2t}) + \tfrac{1}{2}T(r) + O(\log(e + rT(r))).$$

Absorbing the logarithm yields $T(r) = O(r)$ outside a finite-measure set; monotonicity removes the exception.

At a shared point of modulus r in a proper right or left sector, the multiplicity ratio has exponentially large or small modulus. But an a_j-point of multiplicity m satisfies $m \log 2 \le T(2r, f) + O(1) = O(r)$, and similarly for g. Thus all three finite values are eventually omitted in those sectors. Montel's theorem and Marty's criterion, applied to the dilates, give

$$f^{\#}(t) + g^{\#}(t) = O(|t|^{-1}). \tag{6.3}$$

At a common simple pole, (2.2) would imply $|\kappa| = O(|t|^{-1})$. Hence the sectors are also eventually pole-free.

We claim that $g \to \infty$ on proper right sectors. Otherwise a sequence of dilates has limits F, G on the right half-plane, with G finite on a small disk. For $F_r(\zeta) = f(r\zeta)$ and $G_r(\zeta) = g(r\zeta)$,

$$(F_r - G_r)F_r'/P(F_r) = r\kappa e^{r\zeta}.$$

A nonconstant F, or a finite constant outside the shared set, is impossible on an ordinary subdisk. If $F \equiv \infty$, the substitution $X_r = 1/F_r$ makes the left side $-(1 - G_r X_r)X_r'/\prod_j (1 - a_j X_r)$, which tends to zero. Thus F is a finite shared value. Both dilates are then bounded on a smaller disk, so $\log^+ |f'/P(f)| \ge cr$ on a fixed angular arc for every radius in $[\alpha r, \beta r]$, where $0 < \alpha < \beta$. This contradicts the estimate $m(r, f'/P(f)) = O(\log r)$, obtained by partial fractions and the logarithmic-derivative lemma, at a radius outside its finite-measure exceptional set.

For $|f|$ large, (6.3) gives $|f'/P(f)| \le C/(|t||f|)$. Since $|U|$ grows exponentially, it follows that $f/g \to 0$. The same conclusion is immediate where f stays bounded. With $y = 1/g$ the exact equation is

$$y' = \kappa e^{-t} \frac{\prod_j (1 - a_j y)}{1 - fy}.$$

Integrating along sector rays towards infinity gives $y \sim -\kappa e^{-t}$, uniformly on proper sectors. Substitution into $U = (f - g)f'/P(f)$ yields $f'/P(f) \to \kappa^2$. Reflection and interchange, $(f, g) \mapsto (g(-t), f(-t))$, preserve (6.1); this proves (6.2).

For the right dilates, $F_r'/P(F_r) \sim r\kappa^2$ excludes every nonconstant limit and every finite constant except a root of P. The limit $F \equiv \infty$ is also excluded: with $X_r = 1/F_r$, one has $F_r'/P(F_r) = -X_r X_r'/\prod_j (1 - a_j X_r) \to 0$ on compact subsets. The cluster set of the continuous dilation path is connected, whereas its possible limits form a finite set. Hence $f \to a$ for one fixed root a. Thus $f'/(f - a) \to \kappa^2 P'(a)$. Integration along every ray in the full right half-plane gives

$$\log |f(re^{i\theta}) - a| = r\Re(\kappa^2 P'(a)e^{i\theta}) + o(r).$$

Its coefficient is nonpositive for $|\theta| < \pi/2$. Since it is nonzero, $\kappa^2 P'(a)$ is negative real. □

Lemma 6.3. *There are holomorphic functions $\mathcal{H}, \mathcal{V}$ near $(0,0)$, with $\mathcal{H}(X, 0) = 0$, $\mathcal{H}_W(0,0) = 1$ and $\mathcal{V}(0,0) = -\kappa$, and a constant $C \ne 0$, such that on an entire sufficiently far right half-plane*

$$\begin{aligned} f(t) &= a + \mathcal{H}(e^{-t}, Ce^{-\mu t}), \\ \frac{1}{g(t)} &= e^{-t}\mathcal{V}(e^{-t}, \mathcal{H}(e^{-t}, Ce^{-\mu t})). \end{aligned} \tag{6.4}$$

Proof. Put $X = e^{-t}$, $u = f - a$ and $1/g = Xv$. The exact analytic system is

$$\begin{aligned} X' &= -X, \\ u' &= -\frac{\kappa v P(a+u)}{1-(a+u)Xv}, \\ v' &= v + \kappa \frac{\prod_j (1 - a_j Xv)}{1-(a+u)Xv}. \end{aligned}$$

At the equilibrium $(0, 0, -\kappa)$, the eigenvalues are $-1, -\mu, 1$, and the stable subspace projects isomorphically onto the (X, u)-plane. The local time-one map has multipliers $e^{-1}, e^{-\mu}, e$. The holomorphic stable-manifold theorem [1, Theorem 5.4(i)] therefore gives a two-dimensional local stable manifold, which is a holomorphic graph $v = \mathcal{V}(X, u)$ with $\mathcal{V}(0, 0) = -\kappa$. The local flow commutes with the time-one map. By the characterization of the stable manifold in terms of forward orbits, the graph is also invariant under the local flow, after shrinking the neighborhood. The trajectory supplied by Lemma 6.2 tends to the equilibrium along the positive real axis; its tail therefore lies on this graph.

On the graph, the factor $P(a + u)$ in the second equation gives

$$X' = -X, \qquad u' = (-\mu + \mathcal{R}(X, u))u, \qquad \mathcal{R}(0, 0) = 0.$$

Fix $0 < \sigma < \min(1, \mu)$. For small initial data, the positive real-time flow stays in the graph and satisfies

$$|X(s)| + |u(s)| \le e^{-\sigma s}(|X_0| + |u_0|), \qquad s \ge 0,$$

after shrinking the graph so that $|\mathcal{R}| \le \mu - \sigma$. Thus

$$I(X, u) = \int_0^\infty \mathcal{R}(X(s), u(s))\, ds$$

converges locally uniformly in the initial data and is holomorphic. Set $W(X, u) = u \exp I(X, u)$. The flow property gives $I' = -\mathcal{R}$, so $W' = -\mu W$, while $W(X, 0) = 0$ and $W_u(0, 0) = 1$. Its local inverse in the second variable is $u = \mathcal{H}(X, W)$, with $\mathcal{H}(X, 0) = 0$ and $\mathcal{H}_W(0, 0) = 1$.

Along the original trajectory, $W = Ce^{-\mu t}$ for some $C \neq 0$; otherwise $f \equiv a$. This proves (6.4) near the positive real ray. Since $\mu > 0$ is real, both e^{-t} and $Ce^{-\mu t}$ are uniformly small on a sufficiently far right half-plane. The identity theorem extends the formulas there. Finally, $\mathcal{H}(X, W) = W(1 + O(|X| + |W|))$ gives $f(t) - a \sim Ce^{-\mu t}$ uniformly as $\Re t \to \infty$. □

The next observation concerns continuous families, not arbitrary sequences of functions with prescribed singular values.

For this lemma, $S(F)$ denotes the closure in $\widehat{\mathbb{C}}$ of the critical and asymptotic values of F; omitted values belong to this set. The restriction of F over $\widehat{\mathbb{C}} \setminus S(F)$ is an unbranched covering. We shall use the covering criterion in [5, Proposition 1]. No bound on the order is part of the lemma.

Lemma 6.4. *Let I be a real interval and let $s \mapsto F_s$ be continuous for locally uniform spherical convergence, where every F_s is a transcendental meromorphic function on $\mathbb{C}$. Suppose that*

$$S(F_s) \subset E \qquad (s \in I)$$

for one finite set $E \subset \widehat{\mathbb{C}}$. For every $s_0 \in I$, there is an interval neighborhood $J \subset I$ and continuous functions $\alpha : J \to \mathbb{C}^*$, $\beta : J \to \mathbb{C}$ such that

$$F_s(\alpha(s)z + \beta(s)) = F_{s_0}(z) \quad (z \in \mathbb{C},\ s \in J), \qquad \alpha(s_0) = 1, \quad \beta(s_0) = 0.$$

The equality is an equality of meromorphic maps to the sphere.

Proof. Translate the parameter so that $s_0 = 0$ and enlarge E to contain ∞. Put

$$Y = \widehat{\mathbb{C}} \setminus E, \qquad X_s = \mathbb{C} \setminus F_s^{-1}(E).$$

The sets removed from the source are closed and discrete. Thus each X_s is path-connected. There are no critical or asymptotic values over Y, so the restriction $p_s = F_s|_{X_s} : X_s \to Y$ is an

unbranched covering. It is onto: a value omitted by F_s is a singular value and belongs to E. These facts also follow directly from the inverse continuation criterion in [5, Proposition 1 and its proof].

1. A continuously marked sheet. Choose $b \in Y$ and $z_0 \in F_0^{-1}(b)$. The value b is finite and regular. On a sufficiently small closed disk D about z_0, the map F_0 is holomorphic and $F_0 - b$ has exactly one zero, which is simple. Spherical convergence to a finite holomorphic function on D implies Euclidean convergence there for s close to zero; in particular there are no poles of F_s on D. Rouché's theorem gives a unique simple b-point $z_s \in D$. On one fixed connected interval J about zero, $s \mapsto z_s$ is continuous. For example,

$$z_s = \frac{1}{2\pi i} \int_{\partial D} z \frac{F_s'(z)}{F_s(z) - b}\, dz$$

proves continuity directly, after making the boundary disk choice strict. All subsequent loop comparisons use this same interval J and this same basepoint branch.

2. Stability of the lift of a fixed loop. Fix a piecewise smooth loop $\lambda : [0,1] \to Y$ based at b. Write $\widetilde{\lambda}_s$ for its lift under p_s starting at z_s. We claim that its endpoint is locally continuous at every $s_* \in J$. The lift $\widetilde{\lambda}_{s_*}([0,1])$ is compact and contains no critical point. Cover it by finitely many inverse charts of F_{s_*} and subdivide $[0,1]$ so that each piece of the loop is contained in a relatively compact target disk of one such chart.

Here persistence of a chart requires no control at infinity. If $v : V \to U$ is an inverse chart, shrink V and enlarge its compact source image slightly inside U. On the resulting source boundary, $F_{s_*}(z) - w$ is bounded away from zero uniformly for w in the smaller closed target disk. Rouché's theorem therefore gives, for s close to s_*, a unique simple solution of $F_s(z) = w$ in that source domain. These inverses are holomorphic in w, continuous in s, and converge uniformly on the closed target disk. At adjacent subdivision endpoints the continued inverses agree by uniqueness of the nearby preimage. Applying this construction to the finite chain of charts proves the claimed endpoint continuity.

If the lift closes at s_*, its endpoint and z_s are, for nearby s, the unique b-point near z_{s_*} and therefore coincide. If it does not close, its endpoint and z_{s_*} are distinct and stay separated under a sufficiently small perturbation. Consequently

$$\chi_\lambda(s) = \begin{cases} 1, & \widetilde{\lambda}_s(1) = z_s, \\ 0, & \widetilde{\lambda}_s(1) \neq z_s \end{cases}$$

is locally constant on J. Connectedness of J makes it constant on the whole interval.

This argument applies to each fixed loop separately. It does not take an intersection of infinitely many parameter neighborhoods. Hence it gives equality of the entire based covering subgroups:

$$(p_s)_*\pi_1(X_s, z_s) = (p_0)_*\pi_1(X_0, z_0) \quad \text{in } \pi_1(Y, b), \quad s \in J.$$

Indeed, membership of a loop class in such a subgroup is exactly the condition that its lift closes.

3. Extending the covering isomorphism. The isomorphism criterion for based connected coverings [14, Proposition 1.37, p. 67] now gives an isomorphism $L_s : X_0 \to X_s$, uniquely determined by

$$F_s \circ L_s = F_0, \qquad L_s(z_0) = z_s.$$

It is biholomorphic: in inverse charts it is a composition of a local inverse of F_s with F_0, and its inverse has the same property.

Let $a \in \mathbb{C} \setminus X_0$. In a punctured disk about a, L_s is holomorphic and injective. The isolated singularity cannot be essential, since the great Picard theorem would contradict injectivity. A pole would have order one, again by injectivity. In that case L_s has an inverse branch v_s on $|w| > R_s$, with $v_s(w) \to a$, and

$$F_s(w) = F_0(v_s(w)) \qquad (|w| > R_s).$$

The right side extends meromorphically to $w = \infty$, because F_0 is meromorphic at the finite point a. This would make F_s rational, contrary to the hypothesis. Thus the singularity of L_s at a is removable. Applying the same argument to L_s^{-1}, with the roles of F_s and F_0 reversed, gives entire extensions of both maps. Their compositions equal the identity on nonempty open sets and hence everywhere. Therefore L_s is an entire automorphism, $L_s(z) = \alpha(s)z + \beta(s)$.

Finally, differentiating the intertwining identity at z_0 gives

$$\alpha(s) = \frac{F_0'(z_0)}{F_s'(z_s)}, \qquad \beta(s) = z_s - \alpha(s)z_0.$$

Both coefficients are continuous on J, with values $1, 0$ at zero. This proves all assertions. □

We are now ready to present the following.

Proof of Proposition 6.1. Integer imaginary translates $(f(t + 2\pi ik), g(t + 2\pi ik))$ satisfy the same coefficient equations. Lemma 5.1 makes this family normal, since $\Psi = \kappa^2$ and every 1-point of $\Phi = e^{2t}$ is simple. Lemma 6.3, through (6.4), shows that its global limits are distinct nonconstant sharing pairs and retain the coefficient identities.

Suppose μ is irrational. The phases $e^{-2\pi ik\mu}$ are dense on the unit circle. For each $|\eta| = 1$, the closure under locally uniform spherical convergence therefore contains a unique pair (F_η, G_η) obtained by replacing C with $C\eta$ in (6.4). Uniqueness on the right half-plane and normality make this family continuous in η. Lemma 6.2, applied to each pair, gives order at most one, only the three possible finite critical values, and the two distinct asymptotic values a and ∞. The exponential asymptotics also show that each F_η is transcendental. Bergweiler–Eremenko's theorem [3, Corollary 3] states that a transcendental meromorphic function of finite order with finitely many critical values has at most 2ρ asymptotic values, where ρ is its order; the count includes infinity. Thus all inverse singular values of every F_η lie in $\{a_1, a_2, a_3, \infty\}$.

By Lemma 6.4, on a short phase interval about 1,

$$F_\eta(\alpha_\eta t + \beta_\eta) = F_1(t), \qquad \alpha_\eta \to 1, \quad \beta_\eta \to 0.$$

As $t \to -\infty$ along the real axis, the arguments remain in a proper left sector. The universal asymptotic (6.2) implies $e^{-(\alpha_\eta - 1)t - \beta_\eta} \to 1$. Hence $\alpha_\eta = 1$, and $\beta_\eta \in 2\pi i\mathbb{Z}$; continuity gives $\beta_\eta = 0$. This contradicts the different nonzero leading coefficients $C\eta$ on the right. Therefore $\mu = p/q \in \mathbb{Q}_{>0}$.

Formula (6.4) gives the common period $2\pi iq$, first on a half-plane and then globally. Hence $f = R(e^{t/q})$ and $g = S(e^{t/q})$, where R, S are meromorphic on $\mathbb{C}^*$. If R had an essential singularity at either puncture, one generic finite value b would have infinitely many distinct preimages x_j there. Each gives a disjoint progression $q \log x_j + 2\pi iq\mathbb{Z}$ of b-points of f. A fixed progression $z_0 + 2\pi iq\mathbb{Z}$ has $r/(\pi q) + O(1)$ points in $|z| \le r$, so its integrated counting function is $r/(\pi q) + O(\log r)$. Distinct values of x_j give disjoint progressions. Thus, for each fixed positive integer M, any M of them give

$$N(r, b; f) \ge \frac{M}{\pi q} r - C_M \log r$$

for all sufficiently large r, with a constant C_M. If $T(r, f) \le Cr$ for large r, choose $M > \pi qC$ and then let $r \to \infty$, a contradiction. Thus both punctures are meromorphic and R is rational. The same argument applies to S. □

7. Rational functions on the punctured plane

Proposition 7.1. *Let $R, S \in \mathbb{C}(t)$ be nonconstant. If they share four distinct values on $\mathbb{C}^*$, one CM, then all four are shared CM on $\mathbb{C}^*$. No sharing condition is imposed at 0 or ∞.*

We prove Proposition 7.1 in the following three subsections. All degrees, multiplicities, and divisors in this section are those of rational maps on the sphere; the sharing hypotheses are imposed only on $\mathbb{C}^*$.

7.1. **Degrees, endpoint behavior, and residues.** We may suppose $R \neq S$ and make the CM value infinity. Denote the three finite shared values by a_1, a_2, a_3 and put $P(w) = \prod_{j=1}^{3}(w - a_j)$. Let C_0 be the number of distinct points in the four common fibers in $\mathbb{C}^*$. The Riemann–Hurwitz formula gives total ramification $2d - 2$ for a rational map of degree d; see [6, Section 17.14, Riemann–Hurwitz formula]. Its four specified fibers therefore contain at least $4d - (2d - 2) = 2d + 2$ distinct points, and at least $2d$ remain after deleting $0, \infty$. The coincidence divisor of R, S has degree $\deg R + \deg S$. Indeed, for homogeneous representations $R = [R_0 : R_1]$ and $S = [S_0 : S_1]$ without common factors, the nonzero homogeneous polynomial $R_0S_1 - R_1S_0$ has that degree and defines this divisor. At a common pole, its local multiplicity is computed using $1/R - 1/S$. Consequently,

$$2\max(\deg R, \deg S) \leq C_0 \leq \deg R + \deg S.$$

Thus both maps have degree d, $C_0 = 2d$, and equality holds throughout. Every critical point lies over a shared value, both endpoints map into the shared set, and every common point is a simple coincidence. There are no endpoint coincidences: the common points in $\mathbb{C}^*$ already account for the full degree $2d$ of the coincidence divisor, so every such point contributes exactly one and neither endpoint contributes. At finite common points, $\min(m, n) = 1$; at common poles both orders are one and the residues differ.

Put $D = t\,d/dt$ and use the rational auxiliaries

$$U = (R - S)\frac{DR}{P(R)}, \qquad V = (R - S)\frac{DS}{P(S)}. \tag{7.1}$$

They have no zeros or poles on $\mathbb{C}^*$, hence are monomials. If at either endpoint both maps are finite, local expansion shows that both monomials are constants, so both maps are finite at the other endpoint as well. There is at least one common point over a finite shared value, because the $2d$ common points include at most d poles. At such a point the constant ratio $q = U/V$ equals the positive multiplicity ratio m/n. A finite shared value unused at the endpoints by S has total S-multiplicity d, whence $qd \leq d$; a value unused by R gives $d \leq qd$. Such values exist because there are three finite values and two endpoints. Therefore $q = 1$ and the conclusion follows in this case.

Otherwise the maps have poles at opposite endpoints. The same map cannot have a pole at both endpoints, since its associated monomial would then have a zero at both 0 and ∞. Comparing the monomial orders shows that the two endpoint pole orders agree; denote this order by $\rho \geq 1$. Interchange the maps and rescale t to obtain

$$\begin{gathered} R(\infty) = S(0) = \infty, \quad R(0) = a, \quad S(\infty) = b, \\ U = \kappa t^{-\rho}, \qquad V = \kappa t^{\rho}, \\ R(t) \sim (\rho/\kappa)t^{\rho}\ (t \to \infty), \qquad S(t) \sim (\rho/\kappa)t^{-\rho}\ (t \to 0), \\ \kappa^2 = -\rho\sigma/P'(a) = -\rho\tau/P'(b), \end{gathered} \tag{7.2}$$

where σ, τ are the orders at the finite endpoints. In particular,

$$t^{2\rho} = n/m > 0, \qquad \min(m, n) = 1 \tag{7.3}$$

at every finite shared point. The number of common poles is $d - \rho$.

If $a \neq b$, normalize $a = 0$, $b = 1$, and write the third value as c. Equation (7.2) gives $c = \sigma/(\sigma + \tau) \in (0, 1)$ and $\kappa^2 < 0$. Consider

$$\omega = t^{\rho}\, d\log\frac{R}{R - 1}.$$

Its finite nonzero residues are mt_0^{ρ} at $R = 0$ and $-mt_0^{\rho}$ at $R = 1$, all real by (7.3). It is regular at 0 and at finite poles of R, while $\operatorname{Res}_{\infty}\omega = \kappa$. The residue theorem forces $\kappa \in \mathbb{R}$, a contradiction. Hence $a = b$. Normalize this value to 0 and the others to $1, c$. The same residue calculation and (7.2) give

$$c < 0, \quad \kappa \in \mathbb{R} \setminus \{0\}, \quad \sigma = \tau, \quad P(w) = w(w - 1)(w - c). \tag{7.4}$$

Assume that the multiplicities still differ over one of the shared values, and put $f(z) = R(e^z)$, $g(z) = S(e^z)$. The exponential map is surjective and unramified onto $\mathbb{C}^*$, so this pair has the same sharing properties and retains a point of unequal multiplicities. Valiron's lemma [27, Section 2.1.4, p. 37] (see also [7, Chapter 1, Section 6, Theorem 6.1]) and $T(r, e^z) = r/\pi$ give

$$T(r) = \frac{d}{\pi} r + O(1).$$

Each of the $d - \rho$ common simple poles in $\mathbb{C}^*$ lifts to a progression $\log t_0 + 2\pi i\mathbb{Z}$, whose integrated counting function is $r/\pi + O(\log r)$. Hence

$$N_\infty(r) = \frac{d-\rho}{\pi} r + O(\log r).$$

Lemma 2.3, with $L(r) = O(\log r)$, now yields

$$d - \rho \le \tfrac{4}{5} d, \qquad d \le 5\rho,$$

by letting $r \to \infty$ outside the exceptional set.

7.2. **Moments and rotation invariance.** Put $N = 2\rho$. For distinct finite shared values v, w, direct substitution of (7.1) gives

$$t^N \, d\log \frac{R - v}{R - w} - d\log \frac{S - v}{S - w} = \kappa(v - w) t^{\rho - 1} \, dt.$$

For a finite shared value v, let $m_v(x)$ be the multiplicity of $R - v$ at $x \in \mathbb{C}^*$, with $m_v(x) = 0$ when $R(x) \neq v$. The poles of $R - v$ and $R - w$ cancel in their quotient, so

$$\frac{d}{dt} \log \frac{R(t) - v}{R(t) - w} = \frac{\operatorname{ord}_0(R - v) - \operatorname{ord}_0(R - w)}{t} + \sum_{x \in \mathbb{C}^*} \frac{m_v(x) - m_w(x)}{t - x},$$

where the sum is finite. For $\ell \ge 1$, the coefficient of $t^{-\ell-1}$ in the expansion at infinity is therefore $\sum_{x \in \mathbb{C}^*} (m_v(x) - m_w(x)) x^\ell$. The logarithmic derivative involving S is $O(t^{-1})$ at infinity. Comparing the coefficients of $t^{N-\ell-1}$ in the preceding differential identity gives

$$M_{0,\ell} = M_{1,\ell} = M_{c,\ell}, \qquad M_{v,\ell} := \sum_{\substack{x \in \mathbb{C}^* \\ R(x) = v}} m_v(x) x^\ell, \quad 1 \le \ell < N, \quad \ell \neq N/2. \tag{7.5}$$

The index $\ell = N/2 = \rho$ is excluded because it corresponds to the nonzero term on the right-hand side. We give the algebraic step turning these moments into rotation invariance. We use elementary facts about cyclotomic integers and their Galois conjugates; background may be found in [28].

Write $\zeta_h = e^{2\pi i/h}$ for every positive integer h. By (7.3), the possible types and locations are

type	(m, n)	t
1	$(1, 1)$	ζ_N^j
$k+$	$(1, k)$	$k^{1/N} \zeta_N^j$
$k-$	$(k, 1)$	$k^{-1/N} \zeta_N^j \quad (k \ge 2)$.

For $v = 1$ or c, compare its fiber with the zero fiber. At type T, let $F_T(X) = \sum_{j=0}^{N-1} \Delta_{T,j} X^j$, where $\Delta_{T,j}$ is the indicator of the v-fiber minus that of the zero fiber at the indicated location. Thus $\Delta_{T,j} \in \{-1, 0, 1\}$. If $B_T = \sum_j |\Delta_{T,j}|$ and $k = \max(m, n)$, then

$$k B_T \le 2d - \sigma < 5N, \qquad B_1 \le N.$$

Equation (7.5) says

$$\sum_T m_T r_T^\ell F_T(\zeta_N^\ell) = 0 \quad (1 \le \ell < N, \ \ell \neq N/2), \tag{7.6}$$

where r_T is the positive radius of type T and m_T is its R-multiplicity, as specified in the table.

We claim that $F_T(\zeta_h) = 0$ for every divisor $h \mid N$, $h \ge 3$, and every T. Use descending induction over these divisors. Denote the cyclotomic polynomials by $\mathcal{C}_e$ to distinguish them

from the ratio Φ. By the induction hypothesis, $\prod_{e|N,\ e>h} \mathcal{C}_e$ divides every F_T. At ζ_h this product is (N/h) times a unit in $\mathbb{Z}[\zeta_h]$. Indeed, multiplication by the factors with $e < h$, $e \nmid h$, gives $(X^N-1)/(X^h-1)$, whose value is N/h. Each added factor is a unit: since $e < h$ and $e \nmid h$, the integral Bézout identity in [4, Lemma 6, p. 1463] gives $U, V \in \mathbb{Z}[X]$ such that $U\mathcal{C}_e + V\mathcal{C}_h = 1$. Evaluating at ζ_h gives $U(\zeta_h)\mathcal{C}_e(\zeta_h) = 1$.

Consequently $F_T(\zeta_h) \in (N/h)\mathbb{Z}[\zeta_h]$. If this value is nonzero, its algebraic norm shows that some conjugate has modulus at least N/h. All conjugates have modulus at most B_T. Therefore

$$F_T(\zeta_h) \neq 0 \quad \Longrightarrow \quad k < 5h. \tag{7.7}$$

Since each F_T has integer coefficients, vanishing at ζ_h is equivalent to vanishing at all primitive h-th roots. Thus only these active types occur in the primitive h-th-root equations.

Set $E = \mathbb{Q}(\zeta_h)$ and adjoin the positive h-th roots of the active integers. The resulting finite Galois group over E acts on each radical by a character. In (7.6), with $\ell = Ns/h$ and $\gcd(s,h) = 1$, distinct characters separate: their eigenvectors are linearly independent over E. For completeness, let K be the resulting field and $G = \mathrm{Gal}(K/E)$. Every radical is an eigenvector for G, since E contains all h-th roots of unity. For a character χ that occurs, the operator

$$\frac{1}{|G|} \sum_{\gamma \in G} \chi(\gamma)^{-1} \gamma$$

retains exactly the terms with that character. Each character class therefore satisfies the moment identity separately. Raising characters to a primitive exponent s does not change the partition, because $\gcd(s,h) = 1$. Two exterior radicals, or an exterior and an interior radical, have the same character precisely when their quotient belongs to E.

We use the following elementary restriction. If $\beta > 0$, $\beta^h \in \mathbb{Q}$ and $\beta \in E$, then $\beta \in \mathbb{Q}$ for odd h, and $\beta^2 \in \mathbb{Q}$ for even h. In fact $\mathbb{Q}(\beta)$ is a real Galois subfield of the abelian extension $E/\mathbb{Q}$, so all its conjugates are real roots of $X^h - \beta^h$.

Except for $h = 3, 4, 8$, the bound (7.7) implies that a character class has at most one exterior type and one interior type, and the unit type has a class of its own. For odd $h \geq 5$ use $5h < 2^h$, and for even $h \geq 12$ use $5h < 2^{h/2}$. For $h = 6$, the real subfield is $\mathbb{Q}$. For $h = 10$, the only additional possibility would require $\sqrt{2} \in \mathbb{Q}(\zeta_5)$, whereas its unique quadratic subfield is $\mathbb{Q}(\sqrt{5})$. Indeed, $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is cyclic of order 4, and $x = \zeta_5 + \zeta_5^{-1}$ satisfies $x^2 + x - 1 = 0$; hence $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{5})$ is that subfield.

A singleton class vanishes by its first primitive moment. For a class $a+, b-$, put $\theta = (ab)^{1/h} \in E \cap \mathbb{R}$, $\theta > 1$, $A = F_{a+}(\zeta_h)$, $B = F_{b-}(\zeta_h)$. The moments for $s = 1, h-1$, using complex conjugation in the latter, give

$$A + (b/\theta)B = 0, \qquad aA + \theta B = 0.$$

Their determinant is $\theta - \theta^{h-1} \neq 0$.

For $h = 3, 4$, the real subfield of $\mathbb{Q}(\zeta_h)$ is $\mathbb{Q}$, so two characters agree exactly when the corresponding quotient of positive radicals is rational. For $h = 8$, the real subfield is $\mathbb{Q}(\sqrt{2})$. Under $k < 40$, two distinct exterior types can agree only when their ratio is 16 or $1/16$, and an exterior-interior pair can agree only when its product is 16 or 256. These tests, including the unit type, give exactly the following nonsingleton classes permitted by (7.7):

h	classes
3	$\{1, 8+, 8-\}$, $\{2+, 4-\}$, $\{2-, 4+\}$, $\{3+, 9-\}$, $\{3-, 9+\}$
4	$\{1, 16+, 16-\}$, $\{2+, 8-\}$, $\{2-, 8+\}$, $\{4+, 4-\}$, $\{9+, 9-\}$
8	$\{1, 16+, 16-\}$, $\{2+, 32+, 8-\}$, $\{2-, 32-, 8+\}$, $\{4+, 4-\}$.

Here $k+$ and $k-$ denote the ordered multiplicity types in the preceding table, not new arithmetic operations. The two-type classes have already been handled. For $h = 3$, set $X_3 = F_1(\zeta_3)$, $Y_3 = F_{8+}(\zeta_3)$ and $Z_3 = F_{8-}(\zeta_3)$. The triple relations are $X_3 + 2Y_3 + 4Z_3 = X_3 + 4Y_3 + 2Z_3 = 0$, whence $Y_3 = Z_3 = A$ and $X_3 = -6A$. A nonzero $A \in (N/3)\mathbb{Z}[\zeta_3]$ has modulus at least $N/3$, contradicting $|X_3| \leq N$. For $h = 4$, define X_4, Y_4, Z_4 in the same way using the ordered types

$1, 16+, 16-$ at $\zeta_4 = i$. The relations $X_4 + 2Y_4 + 8Z_4 = X_4 + 8Y_4 + 2Z_4 = 0$ give $Y_4 = Z_4 = A$, $X_4 = -10A$, and $A \in (N/4)\mathbb{Z}[i]$, again impossible.

For $h = 8$, the first two triples have the form $\{a+, (16a)+, b-\}$, with $(a, b) = (1, 16)$ or $(2, 8)$, interpreting $1+$ as the unit type. Divide the primitive moments $s = 1, 3, 5$ by the common radical and apply the inverse cyclotomic automorphism $\zeta_8 \mapsto \zeta_8^{s^{-1}}$, where s^{-1} is taken modulo 8. For the ordered unknowns $F_{a+}(\zeta_8)$, $F_{(16a)+}(\zeta_8)$ and $F_{b-}(\zeta_8)$, the resulting coefficient matrix is

$$\begin{pmatrix} 1 & \sqrt{2} & b/\sqrt{2} \\ 1 & -2\sqrt{2} & -b/(2\sqrt{2}) \\ 1 & -4\sqrt{2} & -b/(4\sqrt{2}) \end{pmatrix}, \qquad \det = -15b/4 \neq 0.$$

For the inverse triple $\{2-, 32-, 8+\}$, with the unknowns ordered accordingly, the matrix is

$$\begin{pmatrix} 1 & 8\sqrt{2} & \sqrt{2}/2 \\ 1 & -4\sqrt{2} & -\sqrt{2} \\ 1 & -2\sqrt{2} & -2\sqrt{2} \end{pmatrix}, \qquad \det = 30.$$

This completes the induction.

Since $\deg F_T < N$, divisibility by all cyclotomic factors except those of orders $1, 2$ gives

$$F_T(X) = (u_T + v_T X)(1 + X^2 + \cdots + X^{N-2}).$$

Thus, for each fixed type, the difference of the two fiber indicators is constant on each parity class. Fix such a class and write the three indicators as I_0, I_1, I_c. At a given location, at most one is nonzero. Both $I_1 - I_0$ and $I_c - I_0$ are constant on the class, and these differences determine the triple uniquely:

$$(I_0, I_1, I_c) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

gives four distinct pairs of differences. Each individual indicator is therefore constant on the parity class. Each finite shared fiber, with its ordered multiplicities, is invariant under $t \mapsto \zeta_\rho t$. Therefore the zero and pole divisors of $(R - 1)/(R - c)$ and $(S - 1)/(S - c)$ are rotation invariant. The quotient of a rotated function by the original is constant; its value at infinity is 1. Hence

$$R(t) = r(t^\rho), \qquad S(t) = s(t^\rho) \quad \text{for some } r, s \in \mathbb{C}(t).$$

The power map is unramified on $\mathbb{C}^*$, so unequal multiplicities are preserved. In the descended identities the endpoint pole order is 1 and the coefficient is κ/ρ.

7.3. **Reduction to real rational functions.** It remains to consider (7.4) with $\rho = 1$. All finite shared points are real by (7.3). The quotients $(R - 1)/(R - c)$ and $(S - 1)/(S - c)$ have real divisors and real nonzero values at infinity, so R, S are real rational functions. Replace t by $-t$ if necessary to make $\kappa > 0$. Every critical point of R in $\mathbb{C}^*$ has type $(m, 1)$ with $m > 1$, and hence $|t| < 1$; every critical point of S in $\mathbb{C}^*$ has type $(1, m)$ with $m > 1$, and hence $|t| > 1$. The endpoint expansions and the fact that a simple real pole preserves the sign of the derivative on its two sides imply

$$R'(t) > 0 \quad (t \in \mathbb{R},\ |t| \geq 1), \qquad S'(t) < 0 \quad (t \in \mathbb{R},\ 0 < |t| \leq 1) \tag{7.8}$$

at points where the respective maps are finite.

A common zero of type $(1, m)$, $m > 1$, has $|t| > 1$ and $R'(t) = \kappa c/t^2 < 0$ by (7.1), a contradiction. A zero of type $(m, 1)$ gives $|t| < 1$ and $S'(t) = -\kappa c > 0$, also impossible. Type $(1, 1)$ occurs at $t = \pm 1$ and gives $R'(t) - S'(t) = \kappa c < 0$, again contradicting (7.8). Thus 0 is omitted by both maps on $\mathbb{C}^*$ and is therefore shared CM. The functions $R(e^z)$ and $S(e^z)$ now share $0, \infty$ CM and $1, c$ IM. The two-CM-two-IM theorem [9, 10] makes all four values CM. Since the exponential map is surjective and unramified onto $\mathbb{C}^*$, the same conclusion holds for R, S, contrary to the assumed unequal multiplicities. This completes the proof of Proposition 7.1. □

8. Proof of the main theorem

We now prove Theorem 1.1.

Proof. Assume that the multiplicities differ at a common point over one of the four shared values. Normalize the CM value to infinity and use Lemma 2.5 to obtain a periodic counterexample of infinite order with nonconstant Φ. Proposition 4.1 supplies a fixed marked value a, a type $(1, m)$ with $m \geq 2$, and inverse branches for which (4.1)–(4.2) hold. Write $q = 1/m$ and pass to a subsequence so that $k_n(0)$ converges in $\widehat{\mathbb{C}}$.

If $k_n(0) \to 0$, make the source change $s = k_n(0)\zeta$. The new coefficients tend locally to $\sqrt{q}, q^{-1/2}$, and their ratio tends to $q \neq 1$. Their domains exhaust $\mathbb{C}$. Lemma 5.1 gives global limits. At the marked point the first derivative tends to $P'(a)\sqrt{q} \neq 0$, so both limits are nonconstant and retain the unequal multiplicities at the marked point. The auxiliary ratio U/V of the limiting pair is the constant $q \neq 1$, contradicting Lemma 2.4.

If $k_n(0) \to k \in \mathbb{C}^*$, the original logarithmic coordinates give limiting coefficients $k^{-1}\sqrt{q}\, e^{s/2}$ and $k^{-1}q^{-1/2}e^{-s/2}$. The Mues limit is nonzero and the ratio qe^s has only simple 1-points. Lemma 5.1 again gives a nonconstant sharing pair with unequal multiplicities at the marked point on $\mathbb{C}$. The change $s = 2t - \log q$ makes its coefficients $\kappa e^t, \kappa e^{-t}$, where $\kappa = 2/k$. Proposition 6.1 writes it as a rational pair composed with an exponential covering. Sharing and the unequal multiplicities descend through this surjective unramified covering, contradicting Proposition 7.1.

Finally suppose $|k_n(0)| \to \infty$. Set $\epsilon_n = |k_n(0)|^{-1}$ and pass to a subsequence such that $\epsilon_n \downarrow 0$ and $|k_n(0)|/k_n(0) \to \omega$, $|\omega| = 1$. Then (5.7) holds with $A_0 = \omega\sqrt{q}$ and $B_0 = \omega q^{-1/2}$. Its relative convergence is uniform on every rectangle (5.8): indeed

$$2\log(1/\epsilon_n) = 2\log|k_n(0)| \leq 18B_n^2 = o(B_n^6),$$

and (4.2) holds on $|s| < B_n^6/2$. Translate the marked target value to 0 and apply Lemma 5.5. This excludes the last case. The contradiction proves equality of the multiplicities over each of the four shared values. Undoing the common target Möbius transformation gives the asserted conclusion. □

Appendix A. Examples illustrating the hypotheses

The following examples illustrate the exceptional set in Lemma 3.1 and the parameter and singular-value assumptions in Lemma 6.4.

A.1. **Fast oscillation and clustered divisors.** Consider $\Phi(z) = qe^{\lambda z}$, with $q, \lambda \neq 0$, and $\Psi \equiv 1$. Here $A = \lambda$, $\Sigma = \varnothing$, and the q-points are $z_k = 2\pi ik/\lambda$. For every one of them the exact inverse logarithmic map is

$$\gamma_k(s) = z_k + s/\lambda, \qquad K \equiv \lambda.$$

At a good point, $|\lambda| > B^{10}$. Therefore

$$|\gamma_k(s) - z_k| < \frac{B^6}{2|\lambda|} < \frac{1}{2B^4} < \Delta \qquad (|s| < B^6/2).$$

The long inverse disk is thus contained in $D(0, R)$, even for a center on the inner boundary. If $|\lambda| \leq B^{10}$, the number of q-points in $\overline{D(0, r)}$ is at most $1 + r|\lambda|/\pi \leq 1 + rB^{10}$; they may all be exceptions. This example explains why neither an assertion about every q-point nor a bound independent of the quantity M is being made.

For an example with coalescing divisor points, let $\epsilon \neq 0$ and

$$\Phi_{n,\epsilon}(z) = \left(\frac{z-\epsilon}{z+\epsilon}\right)^n, \qquad A_{n,\epsilon}(z) = \frac{2n\epsilon}{z^2 - \epsilon^2}, \qquad n \geq 1.$$

For $q \neq 0, 1$, its n distinct q-points are exactly

$$z = \epsilon\frac{1+\omega}{1-\omega}, \qquad \omega^n = q.$$

They can be arbitrarily tightly clustered. At $q = 1$ there are $n - 1$ finite points, since the root $\omega = 1$ represents the point at infinity. For $0 < |\epsilon| < R$, the counting terms alone give

$$N(R, \Phi_{n,\epsilon}) + N(R, 1/\Phi_{n,\epsilon}) = 2n \log(R/|\epsilon|).$$

Thus collision does not leave the budget fixed. The cluster argument in Lemma 3.1 does not require the points to remain separated as $\epsilon \to 0$.

A critical q-point is deliberately excluded from the good set. For example, $\Phi(z) = qe^{z^m}$, $m \geq 2$, has a q-point at zero but $A(z) = mz^{m-1}$ vanishes there. The equation $\Phi(\gamma(s)) = qe^s$, $\gamma(0) = 0$, would imply $\gamma(s)^m = s$ near zero, which has no holomorphic solution. This point belongs to Σ and is counted among the exceptions.

A.2. **The location of singular values.** The continuous family $F_s(z) = e^z + s$, $s \in \mathbb{R}$, has no finite critical values, but its finite asymptotic value is s. Distinct members cannot differ by affine precomposition: such a precomposition is surjective and leaves the omitted-value set unchanged, whereas $F_s(\mathbb{C}) = \mathbb{C} \setminus \{s\}$. A fixed bound on the number of singular values is likewise insufficient. Lemma 6.4 fixes their locations in one finite set.

A.3. **A sequence with singular values in a fixed finite set.** There is also a concrete obstruction to replacing the parameter interval in Lemma 6.4 by an arbitrary convergent sequence. Let T_n be the Chebyshev polynomial, defined by $T_n(\cos u) = \cos(nu)$, and put

$$F_n(z) = T_n\left(1 - \frac{e^{2z}}{2n^2}\right), \qquad F_\infty(z) = \cos(e^z). \tag{A.1}$$

Every one of these functions is transcendental entire, and all its spherical inverse singular values belong to $E = \{-1, 1, \infty\}$. Moreover $F_n \to F_\infty$ locally uniformly, but no F_n is an affine precomposition of F_∞.

Here are the details. Set $P_n(w) = T_n(1 - w^2/(2n^2))$. The critical values of this polynomial belong to $\{-1, 1\}$: the point $w = 0$ maps to 1, and the other critical points arise from the zeros $\cos(k\pi/n)$ of T_n'. The restriction

$$P_n : P_n^{-1}(\mathbb{C} \setminus \{-1, 1\}) \longrightarrow \mathbb{C} \setminus \{-1, 1\}$$

is a finite unbranched covering, and its source omits zero. Composing with the restriction of the exponential covering shows that F_n is a covering over $\mathbb{C} \setminus \{-1, 1\}$. The corresponding statement for F_∞ follows from the cosine covering over the same twice-punctured plane: $\cos^{-1}(\mathbb{C} \setminus \{-1, 1\})$ also omits zero. The critical and asymptotic values of cosine used here are recalled in [5, Section 2, examples on p. 4].

On compact w-sets and for sufficiently large n,

$$T_n\left(1 - \frac{w^2}{2n^2}\right) = \cos\left(2n \arcsin \frac{w}{2n}\right) \longrightarrow \cos w,$$

with the local branch of arcsin at zero. Substituting $w = e^z$ proves local uniform convergence in (A.1). Each F_n is a nonconstant polynomial in e^{2z} and has order one. On the other hand,

$$F_\infty(x + i\pi/2) = \cosh(e^x), \qquad x \to +\infty,$$

so F_∞ has infinite order. Order is invariant under affine precomposition with nonzero slope, proving the assertion.

Thus the family indexed by $\{0\} \cup \{1/n : n \geq 1\}$, with value F_∞ at 0, is continuous, and all its inverse singular values lie in the same fixed finite set E. Nevertheless, the family is not locally equivalent by affine precomposition. This parameter space is not an interval. The fixed-loop indicators in the proof of Lemma 6.4 need not be constant on a disconnected parameter space. This illustrates the role of connectedness in the fixed-loop argument.

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