Inclusions and invariant forms of Svinin's recurrences

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Abstract

Svinin's two adjacent-order inclusions hold at every order for sequences with all entries nonzero. We give explicit coefficient maps and disprove both inclusions at every order for unrestricted polynomial solutions. We also prove Svinin's conjectured equivalence of the associated recurrence and its invariant form for every $g \geq 1$ on the nonzero-coordinate domain.

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1 Introduction

The Somos–4 and Somos–5 equations are bilinear recurrences. Their iterates are Laurent polynomials in the initial values, a property that Fomin and Zelevinsky [1, Theorem 1.8] established for the two-term Gale–Robinson family. Hone [2, Proposition 2.3] gave explicit Somos–5 coefficients for a Somos–4 sequence in terms of its conserved quantity. Svinin [3, Theorem 5.1] subsequently showed that Somos–4 sequences satisfy Gale–Robinson recurrences of each type with suitable coefficients.

Hone, Roberts and Vanhaecke [4, Section 4] constructed a family of integrable maps associated with the Volterra lattice and obtained Somos–5 solutions from the genus-one map. Svinin [5] used discrete polynomials to express the associated recurrences in higher genus and exhibited the expression that later appeared in the conjectured invariant form [5, Section 5]. He then proved Laurentness for a family of odd-order homogeneous recurrences and defined its even-order counterpart [6, Sections 5.3–5.4]. Tang [10] established Laurentness for the even orders. In Conjectures 5.2 and 5.3 of [6], Svinin asked whether each even-order sequence satisfies the next odd-order recurrence and each odd-order sequence satisfies the next even-order recurrence, in each case with suitable constant coefficients.

A second question concerns the recurrence in ratio coordinates. Svinin [7, Theorem 2.12] constructed its invariants and in Conjecture 3.1 proposed an equivalent equation with those invariants as coefficients. He verified the first two cases. In earlier work, Svinin [8, Section 5.3] derived the corresponding equation and parameter formulas under a conjectured discrete-polynomial identity, since proved in the $(1,1)$ specialization by the Trureturing project [9]. Section 5 gives the precise parameter correspondence. The earlier paper also established solution-space inclusions for the recurrences in its S-polynomial form [8, Theorem 1].

We prove both adjacent-order inclusions for nonzero sequences, with explicit coefficient formulas. For unrestricted polynomial solutions we give counterexamples at every order. We also prove the invariant recurrence equivalence, with explicit recovery of the original coefficients.

2 Svinin's recurrences

Work over a field $\mathbb{K}$. For a sequence $(t_n)_{n\in\mathbb{Z}}$ with all entries nonzero, put

$$w_n = \frac{t_n t_{n+2}}{t_{n+1}^2}, \qquad u_n = w_n w_{n+1} = \frac{t_n t_{n+3}}{t_{n+1} t_{n+2}}. \tag{1}$$

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For integers p, q and $k \geq 1$, let

$$E_k[p,q] = \sum_{\substack{p \leq i_1 < \cdots < i_k \leq q \\ i_{r+1} - i_r \geq 2 \ (1 \leq r < k)}} u_{i_1} \cdots u_{i_k}. \tag{2}$$

The summation sets are the independent sets of size k in the path on the integers from p to q, with edges between successive integers. Set $E_0[p,q] = 1$, also for an empty interval, and set $E_k[p,q] = 0$ for $k < 0$ or when no admissible set exists. Write $U[p,q] = \prod_{i=p}^{q} u_i$, with an empty product equal to one. For $p \leq q$, cancellation in (1) gives

$$U[p,q] = \frac{t_p t_{q+3}}{t_{p+2} t_{q+1}}. \tag{3}$$

We use the convention that an empty sum is zero. For $s, k \geq 0$ and an integer n, Svinin's polynomial $T_s^k(n)$ sums the monomials $\prod_{r=1}^{k} u_{n+\lambda_r+r-1}$ over $0 \leq \lambda_1 < \cdots < \lambda_k \leq s-1$, with $T_s^0(n) = 1$. The substitution $i_r = n + \lambda_r + r - 1$ gives

$$T_s^k(n) = E_k[n, n+s+k-2], \qquad B_s^k(n) = T_s^k(n) \prod_{r=1}^{s+k} t_{n+r}. \tag{4}$$

After substituting (1), the denominator pairs $\{i_r + 1, i_r + 2\}$ in each monomial are disjoint and lie in $\{n+1, \ldots, n+s+k\}$. Cancelling them in the product defines $B_s^k(n)$ as a polynomial in the t_i, including at zero coordinates.

Every recurrence below is imposed for all $n \in \mathbb{Z}$. For $g \geq 1$, $N \in \{2g+2, 2g+3\}$, and constants $a_0, \ldots, a_g \in \mathbb{K}$, the polynomial equation R_N is

$$\prod_{i=3}^{N-3} t_{n+i}(t_n t_{n+N} - a_g t_{n+1} t_{n+N-1} - a_0 t_{n+2} t_{n+N-2}) = \sum_{j=1}^{g-1} a_j B_{N-j-3}^j(n+1). \tag{5}$$

In the term with index j, $B_{N-j-3}^j(n+1) = E_j[n+1, n+N-4] \prod_{i=2}^{N-2} t_{n+i}$. Dividing by $\prod_{i=3}^{N-3} t_{n+i}$ on nonzero sequences gives

$$t_n t_{n+N} = t_{n+2} t_{n+N-2} \sum_{j=0}^{g-1} a_j E_j[n+1, n+N-4] + a_g t_{n+1} t_{n+N-1}. \tag{6}$$

For $N = 2g+3$, divide by $t_{n+2} t_{n+2g+1}$. The left side is $U[n, n+2g]$ by (3), and the ratio multiplying a_g telescopes as

$$\frac{t_{n+1} t_{n+2g+2}}{t_{n+2} t_{n+2g+1}} = \prod_{\ell=1}^{g} u_{n+2\ell-1}.$$

This is $E_g[n+1, n+2g-1]$, since a path on $2g-1$ vertices has a unique independent set of size g: its $g-1$ gaps are at least two, so they are all exactly two and the set includes both ends. Thus the odd recurrence is

$$U[n, n+2g] = \sum_{j=0}^{g} a_j E_j[n+1, n+2g-1]. \tag{7}$$

3 The two inclusions

Theorem 3.1. *Let $g \geq 1$ be an integer. Over any field, every R_{2g+2} sequence with all entries nonzero satisfies R_{2g+3} with suitable constant coefficients. Likewise, every R_{2g+3} sequence with all entries nonzero satisfies R_{2g+4} with suitable constant coefficients.*

For every $g \geq 1$, both inclusions fail if the sequence classes consist of all solutions of (5), including those with zero entries.

For $p \le q$ and $k \ge 1$, partitioning the independent sets by whether they contain the indicated endpoint gives

$$E_k[p,q] = E_k[p+1,q] + u_p E_{k-1}[p+2,q], \tag{8}$$
$$E_k[p,q] = E_k[p,q-1] + u_q E_{k-1}[p,q-2]. \tag{9}$$

They give the following identity for integers $a < b$ and $j \ge 0$:

$$u_a E_j[a+2,b] - u_b E_j[a,b-2] = E_{j+1}[a,b-1] - E_{j+1}[a+1,b]. \tag{10}$$

For $b \ge a+2$, expanding the right side at a and b cancels $E_{j+1}[a+1,b-1]$ and leaves

$$u_a E_j[a+2,b-1] - u_b E_j[a+1,b-2].$$

For $j \ge 1$, expanding the left side at b and a gives the same difference, since both expansions add the term $u_a u_b E_{j-1}[a+2,b-2]$. For $j = 0$, this term is zero and the two expressions are both $u_a - u_b$. For $b = a+1$, the case $j = 0$ reads $u_a - u_b = u_a - u_b$, and all terms vanish for $j \ge 1$.

Lemma 3.2. *Let $g \ge 1$, and suppose a nonzero sequence satisfies R_{2g+2} with coefficients $a_0, \dots, a_g$. Then*

$$I_n^{\mathrm{e}} = U[n, n+2g] + \sum_{j=0}^{g-1} a_j E_{j+1}[n+1, n+2g-1] \tag{11}$$

is independent of n. The same sequence satisfies R_{2g+3} with

$$b_0 = I_0^{\mathrm{e}}, \qquad b_j = -a_{j-1} \quad (1 \le j \le g). \tag{12}$$

Proof. Define

$$K_n = \prod_{i=n}^{n+2g} w_i, \qquad v_n = \prod_{i=n+1}^{n+2g-1} w_i, \qquad A_n = \sum_{j=0}^{g-1} a_j E_j[n+1, n+2g-2].$$

The even recurrence is exactly

$$K_n = a_g + \frac{A_n}{v_n}. \tag{13}$$

Indeed, pairing $w_i w_{i+1} = u_i$ gives $K_n v_n = U[n, n+2g-1] = t_n t_{n+2g+2}/(t_{n+2} t_{n+2g})$ and $v_n = t_{n+1} t_{n+2g+1}/(t_{n+2} t_{n+2g})$, so this follows directly from (6). Since $K_n K_{n+1} = U[n, n+2g]$,

$$\begin{aligned} U[n+1, n+2g+1] - U[n, n+2g] &= K_{n+1}(K_{n+2} - K_n) \\ &= u_{n+1} A_{n+2} - u_{n+2g} A_n \\ &= \sum_{j=0}^{g-1} a_j \big(E_{j+1}[n+1, n+2g-1] - E_{j+1}[n+2, n+2g] \big). \end{aligned}$$

In the second line we used $K_{n+1}/v_{n+2} = u_{n+1}$ and $K_{n+1}/v_n = u_{n+2g}$; the last line is (10). This is $I_{n+1}^{\mathrm{e}} = I_n^{\mathrm{e}}$. Rearranging (11) gives (7) with the coefficients (12). □

Lemma 3.3. *Let $g \ge 1$, and suppose a nonzero sequence satisfies R_{2g+3} with coefficients $a_0, \dots, a_g$. Then*

$$I_n^{\mathrm{o}} = U[n, n+2g+1] + \sum_{j=0}^{g-1} a_j E_{j+1}[n+1, n+2g] \tag{14}$$

is independent of n. The same sequence satisfies R_{2g+4} with

$$b_0 = I_0^{\mathrm{o}}, \qquad b_j = -a_{j-1} \quad (1 \le j \le g), \qquad b_{g+1} = 0. \tag{15}$$

Proof. Apply (7) at $n+2$ and at n. Identity (10) gives

$$\begin{aligned} U[n+1,n+2g+2]-U[n,n+2g+1] &= u_{n+1}U[n+2,n+2g+2]-u_{n+2g+1}U[n,n+2g] \\ &= \sum_{j=0}^{g} a_j\bigl(E_{j+1}[n+1,n+2g]-E_{j+1}[n+2,n+2g+1]\bigr). \end{aligned}$$

Each path in the last line consists of g adjacent pairs, with at most one vertex from each pair in an independent set. The term with $j=g$ is therefore zero. The remaining identity states that I_n^{o} is constant. Thus

$$U[n,n+2g+1]=\sum_{j=0}^{g} b_j E_j[n+1,n+2g].$$

Multiplying by $t_{n+2}t_{n+2g+2}$, and using $U[n,n+2g+1]=t_nt_{n+2g+4}/(t_{n+2}t_{n+2g+2})$, gives (6) for $N=2g+4$, with genus parameter $g+1$ and $b_{g+1}=0$. □

Proof of Theorem 3.1. Lemmas 3.2 and 3.3 give the inclusions on nonzero sequences.

For the polynomial inclusions, fix a target order $M\geq 5$. Set $t_i=1$ for $i=0,M$ and for $3\leq i\leq M-3$, and set $t_i=0$ elsewhere. Take all coefficients of R_{M-1} equal to zero. There is no pair of supported indices at distance $M-1$: within $[0,M]$ the only possible pairs are $(0,M-1)$ and $(1,M)$. Thus $t_nt_{n+M-1}=0$ for every n, and (5) holds for the source order.

For R_M at $n=0$, the leading product $t_0t_M\prod_{i=3}^{M-3}t_i$ is 1. The terms involving t_1t_{M-1} and t_2t_{M-2} vanish. Each monomial of the remaining terms is formed from $\prod_{i=2}^{M-2}t_i$ by substituting the ratios in (1) at an independent set in $[1,M-4]$ and cancelling denominators. If the set omits 1, the product's factor t_2 remains. If it contains 1, the numerator of u_1 contributes t_1, which no denominator contains. Thus all the remaining terms vanish for any target coefficients. Taking $M=2g+3$ and $M=2g+4$ gives the two counterexamples for each $g\geq 1$. □

4 A path convolution identity

For integers a, $s\geq 1$, and $k\geq 0$, define Svinin's second polynomial family

$$S_s^k(a)=\sum_{0\leq\lambda_1\leq\cdots\leq\lambda_k\leq s-1}\prod_{j=1}^{k}u_{a+\lambda_j+k-j},\qquad S_s^0(a)=1. \tag{16}$$

Throughout this section and the next, the variables u_i may be taken as independent. The identities may be specialized to any field with all displayed denominators nonzero.

Lemma 4.1. *For every $h\geq 0$ and every integer a,*

$$\sum_{s=0}^{h}(-1)^s S_{h-s+2}^{h-s}(a+s)E_s[a,a+2h-1]=U[a+h+1,a+2h]. \tag{17}$$

This is a polynomial identity over the integers.

Proof. For $h=0$, both sides equal one. Suppose $h\geq 1$, and put $c=a+h$. Let J be the matrix over $\mathbb{Z}[u_a,\dots,u_{a+2h-1}]$ on vertices $a,a+1,\dots,a+2h$ with entries $J_{i,i+1}=1$ and $J_{i+1,i}=u_i$ for $a\leq i<a+2h$, with all other entries zero. Writing I for the identity matrix and z for an indeterminate, we have

$$\det(zI-J)=\sum_{s=0}^{h}(-1)^s E_s[a,a+2h-1]z^{2h+1-2s}. \tag{18}$$

In the determinant expansion, the nonzero permutations have only fixed points and adjacent transpositions, since a longer cycle in the permutation would give a cycle in the undirected

path. Each selected adjacent transposition contributes $-u_i$; the selected edges must be pairwise nonadjacent.

For $d \geq 0$ and vertices i, j, the entry $(J^d)_{i,j}$ is the sum of the weights of length-d walks from i to j in the finite path, by matrix multiplication. The weight of a walk is the product of its step weights. Consider walks on the unrestricted integer line, with weight one for a right step and weight u_i for the left step from $i+1$ to i. A walk of length $2k+1$ from c to $c+1$ with $k \geq 0$ has k left steps and $k+1$ right steps. For $1 \leq j \leq k$, if λ_j counts the right steps before its j-th left step, then $0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq k+1$, and the left-step weight is $u_{c+\lambda_j-j}$. For $k > 0$, insert λ_1 right steps before the first left step, $\lambda_{j+1} - \lambda_j$ between left steps j and $j+1$, and $k+1-\lambda_k$ after the last. For $k = 0$ the walk is one right step. Thus the total unrestricted weight is $S^k_{k+2}(c-k)$.

If $k < h$, every such walk stays in the finite interval. Leaving to the left requires $h+1$ steps to reach $a-1$ and $h+2$ more to reach $c+1$. Leaving to the right requires $h+1$ steps to reach $a+2h+1$ and h more to reach $c+1$. For $k = h$, exactly one walk leaves: $h+1$ right steps followed by h left steps. Its weight is $U[a+h+1, a+2h]$. Thus

$$S^h_{h+2}(a) = (J^{2h+1})_{c,c+1} + U[a+h+1, a+2h],$$

and, for $1 \leq s \leq h$,

$$S^{h-s}_{h-s+2}(a+s) = (J^{2h+1-2s})_{c,c+1}.$$

Taking the $(c, c+1)$ entry of the Cayley–Hamilton identity associated with (18) proves (17). □

5 The invariant recurrence

Fix an integer $g \geq 1$, a sequence $(u_n)_{n\in\mathbb{Z}}$ of nonzero elements of $\mathbb{K}$, and coefficients $\alpha_0, \ldots, \alpha_g \in \mathbb{K}$. The associated recurrence of [7, Equation (0.6)] is

$$U[n, n+2g] = \sum_{j=0}^{g} \alpha_j T^j_{2g-j}(n+1). \tag{19}$$

To reproduce exactly the invariants of [7, Theorem 2.12], define, at the indices used below,

$$P^k_s(a) = T^k_s(a) - u_a u_{a+s+k-2} T^{k-2}_{s-2}(a+2) \quad (k \geq 2,\ s \geq k+1),$$

with $P^0_s(a) = 1$ and $P^1_s(a) = T^1_s(a)$. For $0 \leq r \leq g$ and $a \in \mathbb{Z}$, put

$$\mathcal{P}_r(a) = \sum_{q=0}^{g-r} \alpha_{r+q} P^q_{2g-2r-q+1}(a), \tag{20}$$

$$\mathcal{T}_r(a) = \sum_{q=0}^{g-r} \alpha_{r+q} T^q_{2g-2r-q}(a). \tag{21}$$

For the window $u_0, \ldots, u_{2g-1}$, set $H_0 = 1$ and

$$H_s = T^s_{2g-s+1}(0) + \sum_{r=0}^{s-1} T^{s-r-1}_{2g-s-r}(r+1) \frac{\mathcal{P}_r(r)}{U[r, 2g-r-1]}, \qquad 1 \leq s \leq g. \tag{22}$$

These are the functions in [7, Equation (2.22)]. Write $H_s(n)$ for the same function with each coordinate index shifted by n, and put

$$C_r(n) = \frac{\mathcal{P}_r(n+r)}{U[n+r, n+2g-r-1]}, \qquad 0 \leq r < g.$$

Lemma 5.1. *For every $g \geq 1$, the following identity holds in the Laurent polynomial ring in independent variables $u_0, \ldots, u_{2g}$ over $\mathbb{Z}[\alpha_0, \ldots, \alpha_g]$.*

$$u_g \sum_{s=0}^{g} (-1)^s S_{g-s+2}^{g-s}(s) H_s - (-1)^g \alpha_g = \frac{U[0,2g] - \mathcal{T}_0(1)}{U[0,g-1]}. \tag{23}$$

Proof. Using (4) on each T in (22) gives

$$H_s = E_s[0, 2g-1] + \sum_{r=0}^{s-1} E_{s-r-1}[r+1, 2g-r-2] C_r(0). \tag{24}$$

Let $F = \sum_{s=0}^{g} (-1)^s S_{g-s+2}^{g-s}(s) H_s$. For the free term of (24), apply Lemma 4.1 with $(h, a) = (g, 0)$. In the coefficient of $C_r(0)$, set $s = r + 1 + j$, where $0 \leq j \leq g - r - 1$. After taking out $(-1)^{r+1}$, the sum is the convolution with $(h, a) = (g - r - 1, r + 1)$. Thus

$$\begin{aligned} F &= U[g+1, 2g] + \sum_{r=0}^{g-1} (-1)^{r+1} C_r(0) U[g+1, 2g-r-1] \\ &= U[g+1, 2g] + \sum_{r=0}^{g-1} (-1)^{r+1} \frac{\mathcal{P}_r(r)}{U[r,g]}. \end{aligned} \tag{25}$$

For $v \geq k+1$ and $k \geq 0$, the endpoint decompositions are

$$\begin{aligned} P_v^k(x) &= T_{v-1}^k(x+1) + u_x T_{v-2}^{k-1}(x+2) \\ &= T_{v-1}^k(x) + u_{x+v+k-2} T_{v-2}^{k-1}(x+1), \end{aligned} \tag{26}$$

Terms of negative degree are omitted. The two expansions partition the independent sets according to the left or right endpoint, with simultaneous selection of both endpoints excluded. Sum the first expansion with the coefficients in (20), replacing q by $q-1$ as the summation index in the second contribution. This gives

$$\mathcal{P}_r(r) = \mathcal{T}_r(r+1) + u_r \mathcal{T}_{r+1}(r+2) \quad (0 \leq r < g), \qquad \mathcal{T}_g(a) = \alpha_g \quad (a \in \mathbb{Z}). \tag{27}$$

Multiplication of (25) by $U[0,g]$ now gives the telescoping sum

$$\begin{aligned} U[0,g]F &= U[0,2g] + \sum_{r=0}^{g-1} (-1)^{r+1} U[0, r-1] \mathcal{P}_r(r) \\ &= U[0,2g] - \mathcal{T}_0(1) + (-1)^g U[0, g-1] \alpha_g. \end{aligned}$$

For $0 \leq r < g-1$, the term containing $u_r \mathcal{T}_{r+1}(r+2)$ from $\mathcal{P}_r(r)$ cancels the term containing $\mathcal{T}_{r+1}(r+2)$ from $\mathcal{P}_{r+1}(r+1)$. Division by $U[0, g-1]$ proves (23). □

Invariance of the coefficients

Set $A_s(n) = T_{2g-s+1}^s(n)$ for $1 \leq s \leq g$.

Lemma 5.2. *For $1 \leq s \leq g$, the following identity holds in the Laurent polynomial ring in $u_n, \ldots, u_{n+2g}$ over $\mathbb{Z}[\alpha_0, \ldots, \alpha_g]$.*

$$H_s(n+1) - H_s(n) = (A_s(n+1) - A_s(n)) \left(1 - \frac{\mathcal{T}_0(n+1)}{U[n, n+2g]}\right). \tag{28}$$

In particular, (19) preserves every H_s.

Proof. For $v \geq k \geq 1$, subtracting the endpoint decompositions on the intervals shifted by one gives

$$T_v^k(x) - T_v^k(x+1) = u_x T_{v-1}^{k-1}(x+2) - u_{x+v+k-1} T_{v-1}^{k-1}(x+1). \tag{29}$$

For $0 \leq m \leq g$, put

$$L_m(n) = \frac{\mathcal{T}_{g-m}(n+1)}{U[n, n+2m]}.$$

Thus $L_0(n) = \alpha_g / u_n$. Applying (29) to the terms of positive degree in $\mathcal{T}_r(x) - \mathcal{T}_r(x+1)$ gives

$$\mathcal{T}_r(x) - \mathcal{T}_r(x+1) = u_x \mathcal{T}_{r+1}(x+2) - u_{x+2g-2r-1} \mathcal{T}_{r+1}(x+1) \quad (0 \leq r < g).$$

With $r = g - m$ and $x = n + 1$, divide by $U[n+1, n+2m]$, for $1 \leq m \leq g$, and rearrange to obtain

$$L_{m-1}(n+2) - L_{m-1}(n+1) = u_n L_m(n) - u_{n+2m+1} L_m(n+1). \tag{30}$$

For $0 \leq s \leq m \leq g$, define

$$J_{m,s}(n) = \sum_{q=0}^{s} P_{2m-s-q+2}^{s-q}(n+q) L_{m-q}(n+q).$$

We claim that

$$J_{m,s}(n+1) - J_{m,s}(n) = T_{2m-s+1}^{s}(n+1)\big(L_m(n+1) - L_m(n)\big). \tag{31}$$

For $s = 0$ this is $J_{m,0} = L_m$. For the induction step, separate the $q = 0$ term to obtain

$$J_{m,s}(n) = P_{2m-s+2}^{s}(n) L_m(n) + J_{m-1,s-1}(n+1).$$

By the induction hypothesis at $n+1$ and (30), the difference is

$$\begin{aligned} &\big(P_{2m-s+2}^{s}(n+1) - u_{n+2m+1} T_{2m-s}^{s-1}(n+2)\big) L_m(n+1) \\ &\quad - \big(P_{2m-s+2}^{s}(n) - u_n T_{2m-s}^{s-1}(n+2)\big) L_m(n). \end{aligned}$$

The right and left endpoint expansions in (26), respectively, reduce both coefficients to $T_{2m-s+1}^{s}(n+1)$. This proves (31), including the boundary $m = s$, where the induction ends at $J_{0,0} = L_0$.

The first identity in (26) gives

$$C_r(n) = u_{n+2g-r} L_{g-r}(n+r) + L_{g-r-1}(n+r+1).$$

Substitute this into the n-shift of (22) and collect the coefficient of $L_{g-q}(n+q)$. For $1 \leq q < s$, that coefficient is

$$T_{2g-s-q+1}^{s-q}(n+q) + u_{n+2g-q} T_{2g-s-q}^{s-q-1}(n+q+1) = P_{2g-s-q+2}^{s-q}(n+q).$$

For $q = s$ it is one. For $q = 0$ it is $u_{n+2g} T_{2g-s}^{s-1}(n+1) = P_{2g-s+2}^{s}(n) - A_s(n)$. Hence

$$H_s(n) = J_{g,s}(n) + A_s(n)\big(1 - L_g(n)\big). \tag{32}$$

Take the difference and use (31). The terms involving $L_g(n+1)$ cancel, leaving

$$H_s(n+1) - H_s(n) = \big(A_s(n+1) - A_s(n)\big)\big(1 - L_g(n)\big).$$

This is (28), since $L_g(n) = \mathcal{T}_0(n+1)/U[n, n+2g]$. □

Theorem 5.3. *For every integer $g \geq 1$, on sequences with all u_n nonzero, the associated recurrence (19) is equivalent to*

$$u_{n+g} \sum_{s=0}^{g} (-1)^s S_{g-s+2}^{g-s}(n+s) H_s = (-1)^g \alpha_g, \qquad H_0 = 1, \tag{33}$$

where $H_1, \ldots, H_g \in \mathbb{K}$ are constant. For a fixed nonzero initial window, the parameters $(\alpha_0, \ldots, \alpha_{g-1}; \alpha_g)$ and $(H_1, \ldots, H_g; \alpha_g)$ correspond by an invertible triangular change given by evaluating (22) at that window.

Proof. The forward implication follows from Lemma 5.1 and Lemma 5.2. For the reverse implication, fix $u_0, \ldots, u_{2g-1}$, all nonzero, and prescribe $H_1, \ldots, H_g, \alpha_g$. Recover $C_r(0)$ successively by

$$C_r(0) = H_{r+1} - E_{r+1}[0, 2g-1] - \sum_{j=0}^{r-1} E_{r-j}[j+1, 2g-j-2] C_j(0) \quad (0 \leq r < g). \tag{34}$$

Next recover $\alpha_{g-1}, \ldots, \alpha_0$, in that order, by

$$\alpha_r = U[r, 2g-r-1] C_r(0) - \sum_{q=1}^{g-r} \alpha_{r+q} P_{2g-2r-q+1}^{q}(r) \quad (0 \leq r < g). \tag{35}$$

These formulas solve (24) and (20) exactly. Apply this inversion at each window $u_n, \ldots, u_{n+2g-1}$ of a nonzero sequence satisfying (33). With the coefficients recovered at that window, the residual identity gives (19) at n. Lemma 5.2, applied to this step with those coefficients held fixed, gives the same prescribed values of $H_1, \ldots, H_g$ at the next window. Uniqueness of the triangular inversion there implies the recovered coefficients agree at consecutive windows. They are therefore constant, and the sequence satisfies (19) with those coefficients. □

Comparison with Svinin's earlier formulas

In [8, Equations (87)–(90)], take the indices $k = g$, $s = g+1$, and $i = n$, and its coefficients $c_j = \alpha_{g-j}/\alpha_g$ for $1 \leq j \leq g$ and $\alpha_g \neq 0$. On (19), the denominator in its equation (89) is $U[n, n+2g]/\alpha_g$, and the numerator polynomial is $\mathcal{T}_r(n+r+1)/\alpha_g$. The two end products cancel the first and last r factors, giving $L_{g-r}(n+r)$. Reindexing its equation (90) by $q = r - j$ gives $J_{g,r}(n)$, which equals $H_r(n)$ by (32) on the recurrence. Its equation (87) then reduces to (33), since the two products on its right multiply to $U[n, n+2g]/u_{n+g}$.

The forward equation also follows at $\alpha_g = 0$ by specializing the resulting algebraic identity. Indeed, over $\mathbb{Z}$, impose (19) at $n = 0$ in the Laurent polynomial ring in $u_0, \ldots, u_{2g}$ with polynomial parameters $\alpha_0, \ldots, \alpha_g$. The relation eliminates α_0 with coefficient 1 on the right side, leaving a Laurent polynomial domain in the coordinates with independent parameters $\alpha_1, \ldots, \alpha_g$. The difference of the two sides of (33), with H_s given by (22), lies in this domain and vanishes in its fraction field by the preceding comparison. It therefore vanishes under every specialization with nonzero coordinates.

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