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A SHARP LOWER BOUND FOR ENTROPY FIRST-PASSAGE SKEWNESS IN A THREE-STATE MARKOV RING

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Abstract. We study the time at which the accumulated medium entropy of a continuous-time three-state Markov ring first reaches or exceeds a positive threshold. For six positive finite transition rates and positive cycle affinity, the skewness scaled by the square root of the threshold has a finite large-threshold limit. We prove that its infimum at fixed affinity A is $\min\{s(A), s(A/2), s(A/3)\}$, where s is the one-stage biased-walk expression. This establishes the three-state case of the lower envelope conjectured by Wampler and Barato. An exact passage decomposition on the lifted integer chain controls the cumulants, including the initial-phase mixture. Two Hermitian entropy tilts constrain the characteristic coefficients to an outer region on which derivative-sign arguments reduce the bound to three endpoint values. Homogeneous rates attain the third value; positive finite-rate families approach the first two. An appendix records which steps extend to rings with any number of states.

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1. Entropy observation and the main result

A first-passage experiment fixes a threshold for an accumulated current and measures the time needed to reach it. In a Markov ring, medium entropy is a dimensionless current with increments determined by the directed transitions. Wampler and Barato [1] introduced threshold-scaled first-passage skewness and conjectured an affinity-dependent lower envelope for unicyclic networks. The envelope is prior work; the result proved here is its three-state case for the medium-entropy stopping time defined below. Bounds on the skewness and kurtosis of the cycle completion time in unicyclic enzymatic schemes, depending on the number of intermediate states, were obtained earlier by Barato and Seifert [4]. Bounds on the skewness and kurtosis of steady-state currents, including those of unicyclic Markov networks, were derived or conjectured by Ptaszyński [5].

The relation between current and first-passage fluctuations is established in the large-deviation framework of Gingrich and Horowitz [2], including inversion of their scaled cumulant-generating functions. For the present ring, we give an exact passage decomposition that controls the initial-phase mixture and justifies the required cumulant limits. Affinity- and topology-dependent bounds on current fluctuations were proved by Pietzonka, Barato and Seifert [3]. Their pointwise bound on the scaled cumulant-generating function does not, by differentiation alone, yield the third standardized passage-cumulant bound considered here. Our proof instead constrains the characteristic coefficients and optimizes the resulting exact skewness formula.

States are $i = 0, 1, 2$, with indices read modulo three. We index reverse rates by the edge they reverse:

$$q_{i,i+1} = u_i > 0, \qquad q_{i+1,i} = w_i > 0, \qquad q_{ii} = -(u_i + w_{i-1}). \tag{1}$$

Thus the anticlockwise departure rate from state i is w_{i-1}. There is one transition channel in each directed edge. The six rates are finite and have units of inverse time. Their products and the logarithmic cycle affinity are

$$U = \prod_{i=0}^{2} u_i, \qquad V = \prod_{i=0}^{2} w_i, \qquad A = \log(U/V) > 0. \tag{2}$$

Entropy is measured in units of Boltzmann's constant. A clockwise crossing of edge i contributes

$$\ell_i = \log(u_i/w_i), \qquad \sum_{i=0}^{2} \ell_i = A, \tag{3}$$

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and a reverse crossing contributes $-\ell_i$. Individual ℓ_i may be negative.

Let $\Sigma(t)$ be the sum of these increments up to time t, with $\Sigma(0) = 0$. The initial phase has the unique stationary distribution π. For a dimensionless threshold $q > 0$, define

$$T_q = \inf\{t \geq 0 : \Sigma(t) \geq q\}. \tag{4}$$

Write $\kappa_r(T_q)$ for its cumulants. The scaled skewness of the rate matrix Q is

$$S(Q) = \lim_{q\to\infty} \sqrt{q}\, \frac{\kappa_3(T_q)}{\kappa_2(T_q)^{3/2}}, \tag{5}$$

once the limit and its denominator have been justified. The square-root factor distinguishes this observable from the usual unscaled skewness. Define

$$s(h) = 2\sqrt{h}\, \frac{1 + 4e^h + e^{2h}}{\sqrt{(e^h - 1)(1 + e^h)^3}}, \qquad h > 0. \tag{6}$$

Theorem 1. *For every rate matrix satisfying* (1)–(2)*, T_q is almost surely finite and has an exponential moment for every $q > 0$. The limit* (5) *exists and is finite. Moreover,*

$$S(Q) \geq \min\{s(A), s(A/2), s(A/3)\}. \tag{7}$$

If $\mathcal{R}_A$ is the set of all such rate matrices with fixed affinity $A > 0$, then

$$\inf_{Q\in\mathcal{R}_A} S(Q) = \min\{s(A), s(A/2), s(A/3)\}. \tag{8}$$

The value $s(A/3)$ is realized by homogeneous rates. Each of $s(A)$ and $s(A/2)$ is approached by a family in $\mathcal{R}_A$ with positive finite rates at every finite parameter value.

The proof begins with the medium-entropy passage cumulants (Section 2): a bounded difference between two current observations does not by itself justify differentiation of a limiting passage transform. We then derive an exact skewness formula (Section 3) and a spectral outer region for its coefficients (Section 4). Sections 5 and 6 establish the lower bound on that region and construct the families that prove sharpness. Appendix A records which of these steps hold for rings with $N \geq 3$ states.

2. From entropy thresholds to passage cumulants

2.1. **A lifted chain and a deterministic entropy boundary.** Conditional on initial phase i, lift the trajectory to a nearest-neighbor chain X_t on $\mathbb{Z}$, with $X_0 = i$. At site n, the forward rate is $u_{n \bmod 3}$ and the backward rate is $w_{n-1 \bmod 3}$. These rates are bounded, so the chain is nonexplosive. Define $F : \mathbb{Z} \to \mathbb{R}$ by

$$F(0) = 0, \qquad F(n + 1) - F(n) = \ell_{n \bmod 3}. \tag{9}$$

The entropy is exactly

$$\Sigma(t) = F(X_t) - F(i), \qquad F(n + 3) = F(n) + A, \qquad F(n) = \frac{A}{3}n + \beta_{n \bmod 3}, \tag{10}$$

where β is a fixed three-periodic sequence. Let $\Delta = \max\beta - \min\beta$.

Lemma 2. *For $q > \Delta$, the integer*

$$m_i(q) = \min\{n \in \mathbb{Z} : F(n) - F(i) \geq q\} \tag{11}$$

exists and exceeds i, and

$$T_q = \inf\{t \geq 0 : X_t = m_i(q)\}, \qquad m_i(q + A) = m_i(q) + 3. \tag{12}$$

Proof. $F(n)$ tends to the two corresponding infinities as $n \to \pm\infty$, and $F(n) - F(i) \leq \Delta$ when $n \leq i$. Hence the minimum exists and exceeds i. Every site below $m_i(q)$ fails the threshold. A nearest-neighbor path cannot reach a higher site without visiting $m_i(q)$, which gives the first identity. The second follows from $F(n + 3) - F(i) = F(n) - F(i) + A$. □

This remains valid when F is not monotone. At a crossing, the entropy overshoot is at most the largest positive edge increment, but the exact identity (12) will give the stronger control needed below.

2.2. **Exponential moments of upward passages.** Stationarity implies that the oriented edge flux

$$J = \pi_i u_i - \pi_{i+1} w_i \tag{13}$$

is independent of i. Since

$$e^A = \prod_i \frac{\pi_i u_i}{\pi_{i+1} w_i} = \prod_i \left(1 + \frac{J}{\pi_{i+1} w_i}\right) > 1, \tag{14}$$

we have $J > 0$. The stationary displacement rate of the lift is therefore $3J > 0$.

Lemma 3. *For every site n, the time τ_n to hit $n+1$ starting from n has an exponential moment, with a positive exponent that does not depend on n. For every $q > 0$, T_q is almost surely finite and has an exponential moment.*

Proof. For real η, let K_η be the backward phase generator whose clockwise and anticlockwise off-diagonal rates are multiplied by e^η and $e^{-\eta}$, respectively, while its diagonal is unchanged. Its Perron eigenvalue $\mu(\eta)$ is analytic near zero, with

$$\mu(0) = 0, \qquad \mu'(0) = 3J > 0. \tag{15}$$

Choose $\eta < 0$ sufficiently close to zero that $\mu(\eta) < 0$, and a positive right eigenvector v. The tilted semigroup gives

$$\mathbb{E}_n \left[e^{\eta(X_t - n)} \frac{v_{X_t \bmod 3}}{v_{n \bmod 3}} \right] = e^{\mu(\eta) t}; \tag{16}$$

the left side is finite because the number of jumps up to time t is dominated by a Poisson variable. On $\{\tau_n > t\}$, the nearest-neighbor property gives $X_t \le n$. Thus

$$\mathbb{P}_n(\tau_n > t) \le \frac{v_{n \bmod 3}}{\min_j v_j} e^{\mu(\eta) t}. \tag{17}$$

All three phase passage laws have an exponential moment with any exponent strictly between 0 and $-\mu(\eta)$. Strong Markov at successive upper levels gives the same conclusion for every finite upward passage. For an arbitrary $q > 0$, choose an upper site whose entropy exceeds the threshold. Its hitting time dominates T_q, proving finiteness and an exponential moment even when (12) does not apply. □

2.3. **The initial-phase mixture.** Let $M_i(s) = \mathbb{E}_i e^{s\tau_i}$ be the moment-generating function (MGF) of a one-level passage, and put

$$B(s) = M_0(s) M_1(s) M_2(s), \qquad b_T(s) = \log B(s), \tag{18}$$

using the analytic logarithm that vanishes at zero. Successive one-level passages are independent by strong Markov, and their laws have period three. The MGF of a three-level passage is therefore $B(s)$ from every initial phase: a cyclic reordering of the three scalar factors leaves their product unchanged.

Proposition 4. *For each fixed integer $n \ge 1$,*

$$\kappa_n(T_q) = \frac{q}{A} b_T^{(n)}(0) + O(1), \qquad q \to \infty. \tag{19}$$

Proof. Fix $q_0 > \Delta$. For $q \ge q_0$, write $q = k_q A + r$ with $k_q \ge 0$ an integer and $r \in [q_0, q_0 + A)$. Equation (12) gives

$$m_i(q) - i = 3k_q + d_i(r), \qquad d_i(r) = m_i(r) - i > 0. \tag{20}$$

As r varies in this bounded interval, the integers $d_i(r)$ take only finitely many values. Conditional passage MGFs and their stationary mixture have the exact form

$$\mathbb{E}[e^{sT_q} \mid X_0 = i] = B(s)^{k_q} C_{i,r}(s), \qquad \mathbb{E} e^{sT_q} = B(s)^{k_q} G_r(s), \tag{21}$$

where $C_{i,r}$ is a finite product of one-level passage MGFs and $G_r = \sum_i \pi_i C_{i,r}$. The functions G_r belong to a finite collection of analytic MGFs, all equal to one at zero. Their logarithms thus have uniformly bounded derivatives in a common neighborhood of zero. Since $k_q = (q - r)/A$ with r bounded, (19) follows. □

The bound holds for real thresholds, with no need for the bounded remainder to converge. It also shows why conditional cumulants must not simply be averaged over the initial phase.

2.4. **Local inversion of the entropy transform.** Define the entropy scaled cumulant-generating function (SCGF) with the positive counting convention

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} e^{z\Sigma(t)}. \tag{22}$$

The entropy-tilted backward generator is

$$(L_z)_{ij} = q_{ij} e^{z \log(q_{ij}/q_{ji})} \quad (i \neq j), \qquad (L_z)_{ii} = q_{ii}. \tag{23}$$

Its simple eigenvalue through zero is $\lambda(z)$, analytic near zero, with positive eigenvector r_z for real z near zero. This follows from the finite-dimensional tilted semigroup and analytic perturbation of the irreducible generator. In particular,

$$j_1 := \lambda'(0) = JA > 0. \tag{24}$$

Proposition 5. *Let $j_n = \lambda^{(n)}(0)$. Then $j_2 > 0$, the limit* (5) *exists, and*

$$S(Q) = \frac{3j_2^2 - j_1 j_3}{\sqrt{j_1}\, j_2^{3/2}}. \tag{25}$$

Proof. For small real $z > 0$, also $\lambda(z) > 0$, and

$$\mathcal{M}_t = e^{z[F(X_t)-F(i)]-\lambda(z)t} \frac{(r_z)_{X_t \bmod 3}}{(r_z)_i} \tag{26}$$

is a martingale. Fixed-time integrability follows from bounded rates and bounded jump increments. Stop it at the first hit τ of $i + 3$. The stopped process is uniformly bounded: before this hit $X_t \leq i + 3$, F is bounded above on that half-line, and $\lambda(z) > 0$. Since τ is almost surely finite, dominated convergence in optional stopping gives

$$1 = e^{Az} \mathbb{E}_i e^{-\lambda(z)\tau}, \qquad B(-\lambda(z)) = e^{-Az}. \tag{27}$$

The exponential moments already proved make both sides analytic near zero. The identity therefore extends to a complex neighborhood. Taking local logarithms and inverting the branch with $\lambda'(0) > 0$ yields

$$\lambda\left(-\frac{b_T(s)}{A}\right) = -s. \tag{28}$$

Three differentiations give

$$\frac{b_T'(0)}{A} = \frac{1}{j_1}, \qquad \frac{b_T''(0)}{A} = \frac{j_2}{j_1^3}, \qquad \frac{b_T'''(0)}{A} = \frac{3j_2^2 - j_1 j_3}{j_1^5}. \tag{29}$$

The three-level passage includes an independent first exponential holding time, so $b_T''(0) > 0$ and hence $j_2 > 0$. Combining (19) and (29) proves existence of (5) and gives (25). □

The same argument works for any fixed initial-phase distribution, although only the stationary case is needed in Theorem 1.

3. An exact formula in two characteristic coefficients

Set $D = U - V > 0$, $C = U + V$, and

$$a = \sum_i (u_i + w_i), \qquad b = \sum_i \big(u_i u_{i+1} + w_i w_{i+1} + u_i w_{i+1}\big) > 0. \tag{30}$$

The coefficients have units $[a] = \text{time}^{-1}$, $[b] = \text{time}^{-2}$ and $[D] = \text{time}^{-3}$. Expanding the three-by-three determinant in (23) gives

$$\det(\xi I - L_z) = \xi^3 + a\xi^2 + b\xi + C - Ue^{Az} - Ve^{-Az}. \tag{31}$$

Entropy factors cancel on every two-edge backtrack. The two oriented three-cycles have increments A and $-A$, giving the last two terms. Thus this determinant concerns the medium entropy itself.

Introduce dimensionless parameters

$$c = \frac{C}{D} = \coth(A/2), \qquad \alpha = \frac{a}{\sqrt{b}}, \qquad \delta = \frac{D}{b^{3/2}}. \tag{32}$$

Proposition 6. *The derivatives $j_n = \lambda^{(n)}(0)$ are*

$$j_1 = \frac{AD}{b}, \qquad j_2 = \frac{A^2C - 2aj_1^2}{b}, \qquad j_3 = \frac{A^3D - 6aj_1j_2 - 6j_1^3}{b}, \tag{33}$$

and

$$\frac{S(Q)}{\sqrt{A}} = \mathcal{F}_\alpha(\delta) := \frac{3c^2 - 1 - 6c\alpha\delta + 6\delta^2}{(c - 2\alpha\delta)^{3/2}}. \tag{34}$$

For a feasible rate matrix, the denominator of (34) *is positive.*

Proof. By (31), the eigenvalue through zero satisfies

$$\lambda^3 + a\lambda^2 + b\lambda = U(e^{Az} - 1) + V(e^{-Az} - 1). \tag{35}$$

Implicit differentiation at zero gives (33). Substitution into (25) yields (34). Since $j_2 = A^2D(c - 2\alpha\delta)/b$, the denominator is positive by $j_2 > 0$. □

Multiplying every rate by a common positive constant leaves c, α, δ unchanged and rescales every passage time by the reciprocal constant. Both descriptions confirm that S is invariant under the time-unit choice.

For homogeneous rates $u_i = u > w = w_i$, put $h = \log(u/w) = A/3$. The lifted forward and backward jumps have constant rates, so

$$\lambda(z) = u(e^{hz} - 1) + w(e^{-hz} - 1). \tag{36}$$

Equation (25) then gives

$$S(Q) = 2\sqrt{h}\,\frac{u^2 + 4uw + w^2}{\sqrt{(u - w)(u + w)^3}} = s(A/3). \tag{37}$$

This calculation fixes the entropy normalization of the homogeneous branch without assuming it minimizes the heterogeneous problem.

4. A spectral outer region

The lower bound will be proved on a region containing the normalized coefficients of every admissible rate matrix. Define

$$k = \tanh(A/4) \in (0, 1), \qquad v = e^{A/6} > 1, \qquad c = \frac{1 + k^2}{2k}. \tag{38}$$

For a real phase θ, evaluate (23) at $z = -1/2 + \mathrm{i}\theta/A$. Its off-diagonal entries become

$$(L_z)_{ij} = \sqrt{q_{ij}q_{ji}}\exp\left(\frac{\mathrm{i}\theta}{A}\log\frac{q_{ij}}{q_{ji}}\right). \tag{39}$$

The reverse entries are complex conjugates and the diagonal is real. Hence the entire matrix is Hermitian; no continuation of a selected Perron eigenvalue to this complex tilt is needed. At $\theta = 0$ and $\theta = \pi$, (31), with $\xi = \sqrt{b}\,y$, gives two real-rooted cubics:

$$y^3 + \alpha y^2 + y + k\delta, \qquad y^3 + \alpha y^2 + y + \delta/k. \tag{40}$$

We have $\alpha > \sqrt{3}$. One direct check, writing $h_i = u_i + w_{i-1}$, is

$$a^2 - 3b = \frac{1}{2}\sum_i (h_i - h_{i+1})^2 + 3\sum_i u_i w_i > 0. \tag{41}$$

We may therefore parameterize α by a unique $x \in (0, 1/3)$:

$$\sqrt{x} = \frac{\alpha - \sqrt{\alpha^2 - 3}}{3}, \qquad \alpha = \frac{1 + 3x}{2\sqrt{x}}. \tag{42}$$

Define

$$d_k(x) := \frac{k\sqrt{x}(1 - x)}{2}, \qquad x_0 = \frac{v^2 - v + 1}{3(v^2 + v + 1)} \in (1/9, 1/3). \tag{43}$$

Lemma 7. *Every feasible rate matrix maps into the outer region*

$$0 < x \le x_0, \qquad 0 < \delta \le d_k(x). \tag{44}$$

Proof. For $g(y) = y^3 + \alpha y^2 + y$, the local minimum and maximum occur at $-\sqrt{x}$ and $-1/(3\sqrt{x})$, with values

$$g(-\sqrt{x}) = -\frac{\sqrt{x}(1-x)}{2}, \qquad g\left(-\frac{1}{3\sqrt{x}}\right) = \frac{1-9x}{54x^{3/2}}. \tag{45}$$

A translated cubic has three real roots, counted with multiplicity, precisely when its local minimum is nonpositive and its local maximum is nonnegative. Apply the minimum condition to the larger constant δ/k in (40). It gives

$$0 < \delta \le d_k(x). \tag{46}$$

If $x > 1/9$, the maximum of g is negative. Applying the maximum condition to the smaller constant $k\delta$ also gives

$$\delta \ge \frac{9x-1}{54kx^{3/2}}, \qquad R(x) := \frac{9x-1}{27x^2(1-x)} \le k^2. \tag{47}$$

On $(1/9, 1/3)$,

$$R'(x) = \frac{2(3x-1)^2}{27x^3(1-x)^2} > 0. \tag{48}$$

Direct substitution gives $R(x_0) = ((v^3-1)/(v^3+1))^2 = k^2$. Thus (47) implies $x \le x_0$ when $x > 1/9$, and the same conclusion is automatic when $x \le 1/9$. □

We have dropped the lower constraint on δ in (47). The result is an outer region, and no converse assertion of rate realizability will be used.

The objective remains well defined even on the added points and at $\delta = 0$. Indeed, for $0 \le \delta \le d_k(x)$,

$$\alpha\delta \le \frac{k(1+2x-3x^2)}{4} \le \frac{k}{3}, \qquad c - 2\alpha\delta \ge \frac{3-k^2}{6k} > 0. \tag{49}$$

This positivity permits a direct derivative argument throughout the enlarged interval.

5. Two endpoint reductions

Lemma 8. *For $0 < x \le x_0$ and $0 \le \delta \le d_k(x)$,*

$$\mathcal{F}_\alpha(\delta) \ge \min\left\{\frac{3c^2-1}{c^{3/2}}, H_k(x)\right\}, \qquad H_k(x) = \mathcal{F}_\alpha(d_k(x)), \tag{50}$$

and the first value equals $s(A)/\sqrt{A}$.

Proof. At fixed α and c, differentiation of (34) gives

$$\mathcal{F}'_\alpha(\delta) = \frac{3\alpha(c^2-1) + 6c(2-\alpha^2)\delta - 6\alpha\delta^2}{(c-2\alpha\delta)^{5/2}}. \tag{51}$$

The numerator is positive at zero and strictly decreasing for $\delta \ge 0$, because $\alpha^2 > 3$. The function therefore increases and may subsequently decrease. Its minimum on $[0, d_k(x)]$ lies at an endpoint. The value at $\delta = 0$ is $(3c^2-1)/c^{3/2}$, which equals $s(A)/\sqrt{A}$ by $c = \coth(A/2)$. □

On the other endpoint curve, the products involving $\sqrt{x}$ cancel:

$$H_k(x) = \frac{N_k(x)}{V_k(x)^{3/2}}, \tag{52}$$

where

$$N_k(x) = \frac{6k^4x^3 - 3k^4x^2 + 9k^2x^2 - 6k^2x - k^2 + 3}{4k^2}, \tag{53}$$

$$V_k(x) = \frac{3k^2x^2 - 2k^2x + 1}{2k} > 0. \tag{54}$$

Here $V_k > 0$ because the discriminant of its numerator, $4k^2(k^2-3)$, is negative. This expression has a differentiable extension to $x = 0$. That extension is essential: α itself diverges as $x \downarrow 0$ on this curve, while $\alpha\delta \to k/4$. It is distinct from the fixed-α endpoint $\delta = 0$ already used in (50).

Lemma 9. *For $0 \le x \le x_0$,*

$$H_k(x) \ge \min\{H_k(0), H_k(x_0)\}, \tag{55}$$

where $H_k(0) = s(A/2)/\sqrt{A}$ and $H_k(x_0) = s(A/3)/\sqrt{A}$.

Proof. The sign of H_k' is that of $2N_k'V_k - 3N_kV_k'$, and

$$2N_k'V_k - 3N_kV_k' = -\frac{3(3x-1)}{4k}G_k(x), \tag{56}$$

$$G_k(x) = k^2(k^2+3)x^2 - 4k^2x + 1 - k^2. \tag{57}$$

On $0 \le x \le x_0 < 1/3$, the prefactor in (56) is positive, $G_k(0) = 1 - k^2 > 0$, and

$$G_k'(x) = 2k^2\big((k^2+3)x - 2\big) < 0. \tag{58}$$

Thus H_k also increases and may subsequently decrease, and its minimum lies at one of the two endpoints. The first value is

$$H_k(0) = \frac{3-k^2}{\sqrt{2k}} = \frac{s(A/2)}{\sqrt{A}}. \tag{59}$$

For the second, direct substitution identifies the normalized coefficients of the homogeneous rates $u_i = v^2$, $w_i = 1$:

$$\alpha(x_0) = \frac{3(v^2+1)}{\sqrt{3(v^4+v^2+1)}}, \qquad d_k(x_0) = \frac{v^6-1}{[3(v^4+v^2+1)]^{3/2}}. \tag{60}$$

Equation (37) gives $H_k(x_0) = s(A/3)/\sqrt{A}$. □

Proof of the lower bound (7). By Proposition 6 and Lemma 7, $S(Q)/\sqrt{A} = \mathcal{F}_\alpha(\delta)$ with (x, δ) in the outer region (44). Lemmas 8 and 9 give $S(Q)/\sqrt{A} \ge \min\{s(A), s(A/2), s(A/3)\}/\sqrt{A}$. No ordering of the three branch values is required, so the argument covers their crossings without locating them. □

6. Sharpness at fixed affinity

To prove (8), it remains to realize each candidate as a value or limit of admissible three-state chains. All rate families below are written in a fixed time unit; a common reference frequency may multiply every displayed rate.

Let $\Lambda(z)$ be the SCGF of the signed cycle count, normalized by one unit per clockwise winding. Its edge increments and the entropy increments differ, after division by A, by a phase potential. Their tilted matrices are diagonally similar, so $\lambda(z) = \Lambda(Az)$. Accordingly

$$\Lambda^3 + a\Lambda^2 + b\Lambda = U(e^z - 1) + V(e^{-z} - 1). \tag{61}$$

If $c_r = \Lambda^{(r)}(0)$, then (25) becomes

$$S = \sqrt{A}\,\frac{3c_2^2 - c_1c_3}{\sqrt{c_1}\,c_2^{3/2}}. \tag{62}$$

For an integer $n \ge 1$ and positive rates $p > q_* > 0$, consider a branch

$$\Lambda(z) = p(e^{z/n} - 1) + q_*(e^{-z/n} - 1), \qquad p/q_* = e^{A/n} > 1. \tag{63}$$

Its first three cumulant rates are $(p - q_*)/n$, $(p + q_*)/n^2$, and $(p - q_*)/n^3$. Substitution into (62) gives $S = s(A/n)$. A common positive prefactor on Λ cancels from this ratio.

Proposition 10. *There are families $Q_M \in \mathcal{R}_A$, $M > 0$, with $S(Q_M) \to s(A)$ and $S(Q_M) \to s(A/2)$ as $M \to \infty$, respectively, and $S(Q) = s(A/3)$ for homogeneous $Q \in \mathcal{R}_A$.*

Proof. For the one-stage limit choose $p/q_* = e^A$ and

$$u = (M, M, p), \qquad w = (M, M, q_*), \qquad M > 0. \tag{64}$$

The affinity is exactly A, and the coefficients are

$$U = M^2p, \quad V = M^2q_*, \quad a = 4M + p + q_*, \quad b = 3M^2 + 3M(p + q_*). \tag{65}$$

Writing $\varepsilon = 1/M$ and dividing (61) by M^2 gives

$$\varepsilon^2\Lambda^3 + [4\varepsilon + (p+q_*)\varepsilon^2]\Lambda^2 + [3 + 3(p+q_*)\varepsilon]\Lambda = p(e^z - 1) + q_*(e^{-z} - 1). \tag{66}$$

At $\varepsilon = 0$ the root through zero is

$$\Lambda_\infty(z) = \tfrac{1}{3}[p(e^z - 1) + q_*(e^{-z} - 1)]. \tag{67}$$

The derivative of the limiting implicit equation with respect to Λ at the origin is 3. The analytic implicit-function theorem therefore gives a root jointly analytic in (ε, z) near $(0, 0)$. For small positive ε it is the physical eigenvalue branch through zero. Its first three derivatives converge to those of Λ_∞, with a positive limiting skewness denominator. Hence $S(Q_M) \to s(A)$.

For the two-stage limit choose $p/q_* = e^{A/2}$ and

$$u = (M, 2p, p), \qquad w = (M, q_*, 2q_*). \tag{68}$$

Now

$$U = 2Mp^2, \quad V = 2Mq_*^2, \quad a = 2M + 3(p + q_*), \quad b = 4M(p + q_*) + 2(p + q_*)^2. \tag{69}$$

After division by $2M$, the limiting equation is

$$\Lambda^2 + 2(p + q_*)\Lambda = p^2(e^z - 1) + q_*^2(e^{-z} - 1). \tag{70}$$

Its analytic root through zero is

$$\Lambda_\infty(z) = p(e^{z/2} - 1) + q_*(e^{-z/2} - 1). \tag{71}$$

The derivative with respect to Λ at the origin is $2(p + q_*) > 0$. The same implicit-function argument proves convergence of the first three cumulants and gives $S(Q_M) \to s(A/2)$.

Finally, $u_i = e^{A/3}$ and $w_i = 1$ give $S = s(A/3)$ exactly by (37). Every finite M in the preceding families is positive and finite, and has the required affinity. The large-threshold limit defining $S(Q_M)$ is taken before $M \to \infty$; no exchange of those limits is used. □

Proof of Theorem 1. Finiteness and the exponential moment are Lemma 3. Existence and finiteness of the limit are Proposition 5. The lower bound (7) was proved at the end of Section 5. Proposition 10 gives the reverse inequality for the infimum, proving (8). □

7. Scope and outlook

The proof applies to a three-state ring with a single channel in each directed edge, six positive finite rates, and positive finite affinity. It establishes the scaled medium-entropy passage observable, its lower bound, and the sharp infimum. It does not treat exact equilibrium $A = 0$, classify all minimizers, locate the branch crossings, or extend the bound to multicyclic networks.

For a ring with $N \geq 3$ states, Appendix A proves that the passage-cumulant formula (25) still holds, that S depends on the rates only through A and the three lowest coefficients of the characteristic polynomial, that the two Hermitian tilts still confine these coefficients, and that homogeneous rates give $S = s(A/N)$. What is missing for $N \geq 4$ is the minimization of the skewness functional over the resulting coefficient region. The natural N-state form of (7) is the inequality $S \geq \min_{1 \leq n \leq N} s(A/n)$. We do not prove it for any $N \geq 4$; numerical exploration for $N = 4$ and $N = 5$ found no violation, which is evidence and not a proof.

AI assistance disclosure

OpenAI GPT-5.6 Sol, accessed through ChatGPT and Codex, was used to develop and audit the $N = 3$ proof and for drafting. Anthropic Claude Opus 5.5, used through Claude Code, was used for editing, independent symbolic verification of the identities, and to derive Proposition 11 (the general-N cumulant formula). The author reviewed and takes responsibility for every statement and proof. AI systems are not authors of this work, and the work has not been peer reviewed.

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Appendix A. Rings with $N \geq 3$ states

Let $N \geq 3$ and read state indices modulo N. Define the rates by (1), and U, V, A, ℓ_i, Σ, T_q, $S(Q)$, L_z and $\lambda(z)$ by (2)–(5), (22) and (23), with products and sums over $i = 0, \dots, N-1$. As before $C = U + V$, $D = U - V > 0$, $c = \coth(A/2)$ and $k = \tanh(A/4)$. Since L_0 is a generator, its characteristic polynomial has no constant term:

$$P(\xi) := \det(\xi I - L_0) = \xi^N + \cdots + a_3\xi^3 + a_2\xi^2 + a_1\xi. \tag{72}$$

For $N = 3$, $a_1 = b$, $a_2 = a$ and $a_3 = 1$. Put

$$\Phi(p, r) = \frac{3c^2 - 1 - 6cp + 6r}{(c - 2p)^{3/2}}. \tag{73}$$

Proposition 11. *For every ring with $N \geq 3$ states, positive finite rates and $A > 0$:*

(1) *T_q is almost surely finite and has an exponential moment for every $q > 0$; the limit* (5) *exists, $j_2 = \lambda''(0) > 0$, and* (25) *holds.*
(2) $\det(\xi I - L_z) = P(\xi) + C - Ue^{Az} - Ve^{-Az}$.
(3) *$a_1 > 0$, the quantity $p = a_2 D/a_1^2$ satisfies $c - 2p > 0$, and with $r = a_3 D^2/a_1^3$,*

$$S(Q) = \sqrt{A}\,\Phi(p, r). \tag{74}$$

(4) *For every $t \in [kD, D/k]$, the polynomial $P(\xi) + t$ has N real roots, counted with multiplicity, and all of them are negative.*
(5) *For homogeneous rates $u_i = u > w = w_i$, $S(Q) = s(A/N)$.*

Proof. (1) The proofs of Lemmas 2 and 3 and Propositions 4 and 5 apply with 3 replaced by N. The lifted chain has forward rate $u_{n \bmod N}$ and backward rate $w_{n-1 \bmod N}$ at site n; then $F(n+N) = F(n) + A$, $F(n) = An/N + \beta_{n \bmod N}$ with β N-periodic, and $m_i(q + A) = m_i(q) + N$. The flux identity reads $e^A = \prod_{i=0}^{N-1}(1 + J/(\pi_{i+1}w_i))$, so $J > 0$ and $\mu'(0) = NJ$. The block MGF is $B(s) = \prod_{i=0}^{N-1} M_i(s)$, the MGF of an N-level passage from every initial phase, and $q = k_q A + r$ gives $m_i(q) - i = Nk_q + d_i(r)$. The martingale is stopped at the first hit of $i + N$, where $F(i + N) - F(i) = A$, which again gives (27). The remaining steps use only these facts.

(2) Each row of $\xi I - L_z$ has nonzero entries only in the columns $i - 1$, i and $i + 1$. A permutation σ of $\mathbb{Z}/N\mathbb{Z}$ with $\sigma(i) \in \{i - 1, i, i + 1\}$ for all i is either a product of disjoint transpositions of neighbors or one of the two rotations $i \mapsto i \pm 1$; these are distinct because $N \geq 3$. For a transposition of i and $i + 1$, the product $(L_z)_{i,i+1}(L_z)_{i+1,i} = u_i w_i$ does not depend on z. The rotation $i \mapsto i + 1$ has sign $(-1)^{N-1}$ and contributes $(-1)^{N-1}\prod_i(-u_i e^{z\ell_i}) = -Ue^{Az}$; likewise the other rotation contributes $-Ve^{-Az}$. Setting $z = 0$ identifies the z-independent part as $P(\xi) + U + V$.

(3) The eigenvalue 0 of the irreducible generator L_0 is simple and its other eigenvalues have negative real parts, so $a_1 = P'(0)$ is the product of their negatives, which is positive. By (2), $P(\lambda(z)) = U(e^{Az} - 1) + V(e^{-Az} - 1)$. Differentiating three times at $z = 0$, with $P'(0) = a_1$, $P''(0) = 2a_2$ and $P'''(0) = 6a_3$, gives

$$a_1 j_1 = AD, \qquad 2a_2 j_1^2 + a_1 j_2 = A^2 C, \qquad 6a_3 j_1^3 + 6a_2 j_1 j_2 + a_1 j_3 = A^3 D. \tag{75}$$

Hence $j_2 = A^2 D(c - 2p)/a_1$, so $c - 2p > 0$ by (1), and substitution into (25) gives (74). For $N = 3$ this is (34), with $p = \alpha\delta$ and $r = \delta^2$.

(4) At $z = -1/2 + \mathrm{i}\theta/A$ the matrix $H_\theta := L_z$ is Hermitian, exactly as in Section 4. Since $Ue^{-A/2} = Ve^{A/2} = \sqrt{UV}$, (2) gives $\det(\xi I - H_\theta) = P(\xi) + C - 2\sqrt{UV}\cos\theta$. As θ varies, the constant $C - 2\sqrt{UV}\cos\theta$ covers $[(\sqrt{U} - \sqrt{V})^2, (\sqrt{U} + \sqrt{V})^2] = [kD, D/k]$, so each $P + t$ with t in this interval is real-rooted. For negativity, recall that for every real z the matrix L_z is irreducible with nonnegative off-diagonal entries, so by Perron–Frobenius applied to the tilted semigroup e^{tL_z} the limit (22) exists and equals the Perron eigenvalue of L_z; this eigenvalue is simple, hence analytic in z. In particular $H_0 = L_{-1/2}$ is real symmetric, and its largest eigenvalue is $\lambda(-1/2)$. The function λ is convex on $\mathbb{R}$, being a limit of the functions $z \mapsto t^{-1}\log \mathbb{E}e^{z\Sigma(t)}$, which are convex by Hölder's inequality. Moreover $\lambda(0) = 0$, and $\lambda(-1) = 0$ because $L_{-1-z} = L_z^{\mathsf{T}}$. Convexity gives $\lambda(-1/2) \leq 0$. Equality would force $\lambda = 0$ on $[-1, 0]$ and hence, by analyticity, $\lambda'(0) = 0$, contradicting $\lambda'(0) = j_1 > 0$. For general θ,

H_θ has the diagonal of H_0 and off-diagonal entries of the same moduli, so $x^* H_\theta x \le |x|^{\mathsf{T}} H_0 |x|$ for every $x \in \mathbb{C}^N$, where $|x|$ is taken entrywise. Every eigenvalue of H_θ is therefore at most $\lambda(-1/2) < 0$.

(5) For homogeneous rates L_z is circulant, and the constant vector gives $\lambda(z) = u(e^{hz} - 1) + w(e^{-hz} - 1)$ with $h = \log(u/w) = A/N$. The computation leading to (37) gives $S = s(h)$. □

Remark 12. By Proposition 11, the N-state problem reduces to minimizing $\Phi(p, r)$ over the data (a_1, a_2, a_3, D) of polynomials P satisfying item (4). Since Φ is increasing in r, only the lower boundary of the attainable set of (p, r) matters. For $N = 3$ this set is contained in the region of Lemma 7, and Sections 4–5 carry out the minimization. For $N \ge 4$ the constraint of item (4) involves coefficients that do not appear in Φ, and we have not determined the corresponding lower boundary.

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