hexagon:2611.00565v1

The repeated-pair symmetric nonnegative inverse eigenvalue problem in dimension five

Samuil Petkov

Department of Physics, École normale supérieure, Université PSL, Paris, France

samuil.petkov@phys.ens.psl.eu

1 October 2026

Abstract

We prove a necessary condition for a real symmetric entrywise nonnegative matrix of order five to have spectrum $(1, a, a, b, b)$, where $1 \geq a \geq b \geq -1$. Besides the trace condition $1 + 2a + 2b \geq 0$, such a realization requires $1 + 4ab \geq 0$ or $1 + a + 2b \geq 0$. Together with the known sufficiency of these conditions, this gives the repeated-pair characterization recorded as Conjecture 5.1.2 in Clark's dissertation. The proof treats arbitrary realizations. Perron-root minimization and a two-dimensional projector frame reduce a putative counterexample to a weighted five-cycle or an H support. A cycle-cone inequality excludes the first case; an exact identity for the H support excludes the second at every positive trace. No constant row sum, uniform Perron vector or restriction to irreducible matrices is imposed in the statement. On the open curved part of the boundary, where $1 + 4ab = 0$, $1 + 2a + 2b > 0$ and $1 + a + 2b < 0$, the same argument determines all realizations: up to permutation similarity they form an explicit one-parameter family of weighted five-cycles containing the Loewy–McDonald matrix.

Mathematics Subject Classification (2020). Primary 15A18; Secondary 15A29, 15B48.
Keywords. symmetric nonnegative inverse eigenvalue problem, SNIEP, nonnegative matrices, Perron root, realizable spectra, weighted cycles.

1 Introduction

The symmetric nonnegative inverse eigenvalue problem asks which real lists are the spectra of real symmetric entrywise nonnegative matrices. The order-five problem remains difficult even after imposing a fixed multiplicity pattern. We consider the list

$$\sigma(a, b) = (1, a, a, b, b), \qquad 1 \geq a \geq b \geq -1, \tag{1}$$

and write

$$S = 1 + 2a + 2b. \tag{2}$$

The normalization fixes the Perron eigenvalue at one. It does not impose a common row sum on a realizing matrix.

The repeated-pair plane has two established sufficient regions:

$$S \geq 0, \qquad \big(1 + 4ab \geq 0 \text{ or } 1 + a + 2b \geq 0\big). \tag{3}$$

The curved boundary is associated with the construction of Loewy–McDonald [6]; the linear region includes a four-dimensional realization together with a singleton eigenvalue. Clark states the complete criterion as Conjecture 5.1.2 and explicitly records that sufficiency is known [1, Chapter 5, pp. 78–79]. The question addressed here is necessity.

Earlier necessary conditions exclude substantial parts of the unrealizable region. Johnson–Marijuán–Pisonero give an exclusion on the repeated-pair plane [3], and subsequently treat

spectra with one repeated eigenvalue [4]. The latter paper also corrects a strictness issue in the earlier exclusion. The general high-trace theorem of Loewy–Spector [8] does not cover all the low-trace spectra considered here. Their later notes [9], Loewy's moment inequality and repeated-pair comparison [7], and the comparison of exclusions in Marijuán–Pisonero [10] further delimit the remaining region. The trace-zero endpoint was characterized by Spector [11].

The recent impossibility theorem of Jin–Ke–Sui [2] uses the condition $4\lambda_1 \le 9\lambda_3 - S$ in its residual region. Within its residual region, this condition becomes $5 + 2b - 7a \le 0$ on (1), so that work covers only the part of the simultaneous failures of the two alternatives in (3) where $5 + 2b - 7a \le 0$. The complementary-matrix and cycle methods in that work, and the historical H and cycle support reductions, are important antecedents of the argument below. We give the support reduction explicitly rather than importing a theorem with a trace restriction.

Our main result is the following necessity theorem. Nonnegativity always means entrywise nonnegativity, and spectra are counted with algebraic multiplicity.

Theorem 1.1. *Let $1 \ge a \ge b \ge -1$. If a real symmetric nonnegative matrix of order five has spectrum $(1, a, a, b, b)$, then (3) holds.*

Corollary 1.2 (Characterization using known sufficiency)**.** *For $1 \ge a \ge b \ge -1$, the list $(1, a, a, b, b)$ is the spectrum of a real symmetric nonnegative matrix if and only if (3) holds.*

The converse in Corollary 1.2 is the established sufficiency reconstructed in Section 6; it is not a new contribution of this manuscript. Theorem 1.1 concerns all supports and all eigenspaces of a realizing matrix. Its proof has three parts. We first minimize the Perron root while keeping the repeated eigenvalues fixed. A projector frame and a complementary commuting matrix then leave only two supports. Finally, direct inequalities rule out both supports. The essential derivations are in the main text; Appendix A records the entry map for the H identity. Section 7 describes the excluded region explicitly and, using the equality cases of the two support inequalities, classifies the realizations on the open curved boundary arc (Theorem 7.2).

2 A boundary realization with a minimal Perron root

Suppose Theorem 1.1 fails. The trace of a nonnegative matrix is nonnegative, so only the two strict inequalities

$$1 + 4ab < 0, \qquad 1 + a + 2b < 0 \tag{4}$$

need to be excluded. The first forces $a > 0 > b$. Set $t = -b$. The second gives $t > (1+a)/2 > a$, and hence

$$0 < a < t \le 1, \qquad 4at > 1, \qquad 1 + a - 2t < 0. \tag{5}$$

In particular, $a < 1$.

Lemma 2.1. *If a realization in (5) exists, then there is a symmetric nonnegative matrix A with spectrum $(\lambda, a, a, -t, -t)$ such that*

$$\lambda > t > a > 0, \quad \lambda \le 1, \quad 0 < T := \operatorname{tr} A = \lambda + 2a - 2t < a, \tag{6}$$

A is irreducible, and no orthogonal conjugate of A is entrywise strictly positive. Moreover, $\lambda^2 < 4at$.

Proof. Minimize λ over the nonempty feasible set

$$\{(\lambda, O) \in [t, 1] \times O(5) : O \operatorname{diag}(\lambda, a, a, -t, -t) O^{\mathsf{T}} \ge 0\}.$$

It is closed in a compact set, so a minimum is attained.

Every feasible realization is irreducible. Indeed, a symmetric nonnegative reducible matrix is a direct sum of irreducible blocks. A block containing $-t$ must have a nonnegative Perron

root at least t. Since $t > a$, the only available such root is λ. Both copies of $-t$ must therefore belong to its unique block. If that block were proper, it would contain at most one copy of a and have trace at most $\lambda + a - 2t < 0$, a contradiction. This argument uses no restriction on the sizes of the other blocks.

The endpoint $\lambda = t$ is also impossible. An irreducible nonnegative matrix has simple peripheral eigenvalues; here $-\lambda$ would occur twice. Thus $\lambda > t > a$. Put $s = t - a > 0$. The trace of a minimizing matrix A is $T = \lambda - 2s \geq 0$. If $T = 0$, then $\lambda = 2s$ and

$$\operatorname{tr}(A^3) = \lambda^3 + 2a^3 - 2t^3 = 6s(s^2 - at) < 0,$$

because $s^2 = \lambda^2/4 \leq 1/4 < at$. This contradicts $A^3 \geq 0$. Also $T < a$ follows from $\lambda + a - 2t < 0$.

If an orthogonal conjugate B of the minimizer were strictly positive, then $B - \varepsilon w w^{\mathsf{T}}$ would remain strictly positive for sufficiently small $\varepsilon > 0$, where $w > 0$ is its unit Perron vector. This subtracts ε from its simple Perron eigenvalue and leaves all other eigenvalues unchanged. Choosing also $\lambda - \varepsilon > t$ contradicts minimality. Finally $\lambda^2 \leq 1 < 4at$. □

3 The projector frame and complementary support

Throughout this section, A satisfies Lemma 2.1. Since A is irreducible, its unit Perron vector v is strictly positive. Let P and Q be its rank-two orthogonal spectral projectors at a and $-t$, respectively. Thus

$$A = aI + (\lambda - a)vv^{\mathsf{T}} - (a + t)Q. \tag{7}$$

Lemma 3.1. *Both P_{ii} and Q_{ii} are positive for every i. After an ordering of the five coordinates on a pentagon, every zero off-diagonal entry of A joins neighboring vertices. All nonneighbor entries of A are strictly positive.*

Proof. Since each $A_{ii} \leq T < a$, equation (7) gives $Q_{ii} > 0$. If $P_{ii} = 0$, both a-eigenvectors vanish at coordinate i. The four-dimensional principal matrix obtained by deleting i then has eigenvalues $(p, a, a, -t)$: the two copies of a remain, and interlacing retains one copy of $-t$. Its Perron root is at least t, so its trace is at least $2a > T$. But its trace is $T - A_{ii} \leq T$. This is impossible.

Choose a 5×2 orthonormal eigenvector matrix U for Q and set

$$g_i = \frac{U_{i,*}}{v_i} \in \mathbb{R}^2, \qquad w_i = v_i^2 > 0, \qquad c_0 = \frac{\lambda - a}{a + t} > 0.$$

Then

$$\sum_i w_i = 1, \qquad \sum_i w_i g_i = 0, \qquad \sum_i w_i g_i g_i^{\mathsf{T}} = I_2. \tag{8}$$

For distinct i, j, nonnegativity of A_{ij} gives $g_i \cdot g_j \leq c_0$, whereas $A_{ii} < a$ gives $\|g_i\|^2 > c_0$.

No g_i is a convex combination of the other points. Otherwise write $g_i = \sum_{j \neq i} \theta_j g_j$, with $\theta_j \geq 0$ and $\sum_{j \neq i} \theta_j = 1$. Define $y_i = 1/v_i$ and $y_j = -\theta_j / v_j$ for $j \neq i$. Then $v^{\mathsf{T}} y = U^{\mathsf{T}} y = 0$, so y belongs to the a-eigenspace. Its positive coordinate satisfies

$$a y_i = (Ay)_i = A_{ii} y_i + \sum_{j \neq i} A_{ij} y_j \leq A_{ii} y_i,$$

contradicting $A_{ii} < a$. This also excludes coincident points. The covariance in (8) has rank two; the five points are therefore the vertices of a strictly convex pentagon.

Now let $A_{ij} = 0$ with $i \neq j$; we show that g_i and g_j are adjacent vertices of the pentagon $C = \operatorname{conv}\{g_1, \ldots, g_5\}$. Put $\ell(x) = g_i \cdot x$ and $N = \{x \in \mathbb{R}^2 : \ell(x) = c_0\}$; this is a line because $g_i \neq 0$. We have $\ell(g_i) > c_0$, $\ell(g_j) = c_0$ and $\ell(g_k) \leq c_0$ for every $k \neq i$. Moreover $\sum_k w_k \ell(g_k) = 0$ by (8), so some vertex satisfies $\ell(g_k) \leq 0 < c_0$. Thus C has points strictly on both sides of N. Consequently N is not a supporting line of C, it meets the interior of

C, and $N \cap \partial C$ consists of exactly two points x_1, x_2. These points split ∂C into two arcs; the open arc lying in $\{\ell > c_0\}$ contains the vertex g_i and no other vertex. It is therefore contained in the union of the two edges of C at g_i, and x_1, x_2 lie on these two edges, one on each. Since $g_j \in N \cap \partial C = \{x_1, x_2\}$, the vertex $g_j \neq g_i$ lies on an edge at g_i. A vertex lying on an edge and distinct from one endpoint is the other endpoint. Hence g_j is a neighbor of g_i. In particular, ties $\ell(g_k) = c_0$ at several vertices cannot place a non-neighbor of g_i on N. □

The complementary commuting matrix in the next lemma is a classical tool originating in Laffey's work on extreme nonnegative matrices [5]. Loewy–Spector state its existence under the orthogonal-orbit hypothesis used here [8, Theorem 1 and Section 3]. We include the short separation argument to specify the diagonal and support conventions.

Lemma 3.2. *There is a nonzero symmetric matrix $X \geq 0$ such that*

$$AX = XA, \qquad A \circ X = 0, \qquad Xv = v. \tag{9}$$

Its off-diagonal support graph is a subgraph of the pentagon in Lemma 3.1. If that graph has k components, then the eigenspace $V_+ = \ker(X - I)$ has dimension k and is invariant under A.

Proof. Let Z be the nonempty set of zero entries of A on and above its diagonal. Consider the linear subspace

$$\mathcal{L} = \{(KA - AK)|_Z : K^{\mathsf{T}} = -K\}.$$

It does not meet the strictly positive orthant. Indeed, a tangent vector positive at every zero position would make $e^{\varepsilon K} A e^{-\varepsilon K}$ strictly positive for small $\varepsilon > 0$, contrary to Lemma 2.1. The finite-dimensional theorem of alternatives (Gordan's theorem: a linear subspace misses the open positive orthant if and only if its orthogonal complement contains a nonzero nonnegative vector) therefore supplies a nonzero nonnegative vector q annihilating $\mathcal{L}$. Define $X_{ii} = q_{ii}$ and $X_{ij} = X_{ji} = q_{ij}/2$ for $i < j$ in Z; put its remaining entries equal to zero. The factor $1/2$ accounts for the two off-diagonal terms in the trace, and $KA - AK$ is symmetric. Thus

$$0 = \operatorname{tr}\big(X(KA - AK)\big) = \operatorname{tr}\big((AX - XA)K\big) \qquad (K^{\mathsf{T}} = -K).$$

Since $AX - XA$ is skew-symmetric, the choice $K = AX - XA$ gives $\|AX - XA\|_F^2 = 0$, so $AX = XA$.

The simple Perron eigenspace of A is invariant under X, so $Xv = \rho v$. Nonnegativity, $X \neq 0$ and $v > 0$ give $\rho = v^{\mathsf{T}} X v > 0$. Divide X by ρ. Each off-diagonal connected component of X then has a positive eigenvector at one. Its Perron root is one, and that root is simple on each component. A singleton component has diagonal entry one. Hence $\dim V_+ = k$. Commutation makes V_+ and $V_+^{\perp}$ invariant under A. □

Lemma 3.3. *The off-diagonal support graph of X is connected.*

Proof. A singleton vertex i has $e_i \in V_+$. Its projections onto the λ, a and $-t$ eigenspaces are nonzero, by $v_i > 0$ and Lemma 3.1. Since V_+ is A-invariant, all three projections belong to V_+. A singleton therefore forces $k \geq 3$.

For a component C (a vertex set), write A_{CC}, X_{CC} for the principal blocks and v_C for the restriction of v. Since X is block diagonal with respect to the components, $A_{CC}X_{CC} = X_{CC}A_{CC}$ and $X_{CC}v_C = v_C$. Moreover V_+ is spanned by the vectors v_C (extended by zero), and $V_+^{\perp} = \bigoplus_C \big(\mathbb{R}^C \cap v_C^{\perp}\big)$. When $A_{CC}v_C$ is a multiple of v_C, the principal compression of A to $\mathbb{R}^C \cap v_C^{\perp}$ is the restriction of A_{CC} to that space.

We also need the following two component facts. On a two-vertex component, its positive X edge forces the corresponding A edge to vanish. Commutation makes both A diagonals equal, so its principal block is dI_2, with $d \geq 0$. On a three-vertex component, the X graph is a path. The A principal block has an isolated middle vertex and a possibly nonzero endpoint edge. Commutation makes the positive component Perron vector an eigenvector of that block, with eigenvalue $h_0 \geq 0$. Thus the middle diagonal is h_0, and the endpoint block has

eigenvalues h_0, k_0, where $|k_0| \le h_0$. On the two-dimensional space perpendicular to the component Perron vector, its compression has eigenvalues h_0, k_0.

Every disconnected pentagon subgraph has one of the following six component partitions. The component non-Perron spaces below need not be individually invariant under A; the arguments use their principal compressions inside the invariant space $V_+^\perp$.

1. For $4+1$, the singleton forces $k \ge 3$, whereas $k = 2$.
2. For $3+1+1$, the singleton and $\dim V_+ = 3$ force $\operatorname{spec}(A|_{V_+}) = (\lambda, a, -t)$. The outside spectrum is $(a, -t)$, and its compression is the three-vertex component's non-Perron block with eigenvalues (h_0, k_0). Thus $h_0 = a$, contrary to $h_0 = A_{\text{middle,middle}} \le T < a$.
3. For $2+2+1$, the same argument gives outside spectrum $(a, -t)$ and negative outside trace $a - t$. The trace of that compression is instead $d_1 + d_2 \ge 0$. Couplings between its two modes do not change the trace.
4. For $2+1+1+1$, the outside space is the single pair non-Perron mode. Its eigenvalue is $d \ge 0$, hence $d = a$. Either pair diagonal then equals $a > T$, a contradiction.
5. For $1+1+1+1+1$, normalization gives $X = I$. Complementarity makes all A diagonals zero, contradicting $T > 0$.
6. For $3+2$, the two-dimensional restriction on V_+ has nonnegative diagonals (h_0, d) and contains λ. If its other eigenvalue is a, the outside spectrum is $(a, -t, -t)$ with negative trace $a - 2t$. Its trace is also $h_0 + k_0 + d \ge d \ge 0$. If the other eigenvalue is $-t$, the outside spectrum is $(a, a, -t)$, so $A|_{V_+^\perp} - aI$ has rank one. Its principal compression on the three-vertex component's non-Perron modes has eigenvalues $h_0 - a, k_0 - a$, both strictly negative since $k_0 \le h_0 \le T < a$. This compression has rank two, a contradiction.

All disconnected possibilities are excluded. □

Proposition 3.4. *After a permutation, A has either a weighted five-cycle as its positive off-diagonal support, or the form*

$$\begin{pmatrix} d & x_0 & 0 & 0 & x_4 \\ x_0 & 0 & x_1 & z & 0 \\ 0 & x_1 & f & x_2 & 0 \\ 0 & z & x_2 & 0 & x_3 \\ x_4 & 0 & 0 & x_3 & e \end{pmatrix}, \qquad d, e, f, z \ge 0, \quad x_i > 0. \tag{10}$$

In the cycle case all five diagonal entries may be nonzero.

Proof. A connected spanning subgraph of a pentagon is the full cycle or a five-vertex path. In the full-cycle case, complementarity makes its five A positions zero; all nonneighbor positions are positive by Lemma 3.1. The complementary graph is again a five-cycle.

In the path case, label its edges $12, 23, 34, 45$ and their positive X weights $q_1, \dots, q_4$. Write its diagonals as $\ell_i \ge 0$. The A entries $13, 14, 24, 25, 35$ are positive, and the four path entries vanish. Commutation at $(1, 2)$ gives

$$q_1(A_{11} - A_{22}) + q_2 A_{13} = 0,$$

so $A_{22} > A_{11} \ge 0$ and $\ell_2 = 0$. Similarly $\ell_4 = 0$. At $(1, 3)$ and $(3, 5)$, commutation gives

$$(\ell_1 - \ell_3)A_{13} = q_3 A_{14} > 0, \qquad (\ell_5 - \ell_3)A_{35} = q_2 A_{25} > 0.$$

Hence $\ell_1, \ell_5 > 0$, forcing $A_{11} = A_{55} = 0$. The entry A_{15} is nonnegative and may vanish. The coordinate order $(4, 1, 3, 5, 2)$ now gives (10). □

4 The weighted-cycle obstruction

Lemma 4.1. *For real $y_0, \dots, y_4$, with indices taken modulo five,*

$$y_i + y_{i+2} \geq 0 \quad \textit{for all } i \quad \Longrightarrow \quad \sum_{i=0}^{4} y_i y_{i+1} \geq 0.$$

If, in addition, some y_i is negative, the inequality is strict.

Proof. Two distinct indices of the five-cycle are either adjacent or at distance two, and two coordinates at distance two cannot both be negative. Hence any two negative coordinates are adjacent, and there are at most two of them (the five-cycle has no triangle). If all coordinates are nonnegative, the conclusion is immediate. Otherwise, after a cyclic shift we are in one of the following two cases. If only $y_0 = -r < 0$, then $y_2, y_3 \geq r$ and $y_1, y_4 \geq 0$, so

$$\sum_i y_i y_{i+1} = y_1(y_2 - r) + y_4(y_3 - r) + y_2 y_3 \geq r^2 > 0.$$

If $y_0 = -r$ and $y_1 = -q$ are negative, then $y_2 \geq r$, $y_3 \geq \max(r, q)$ and $y_4 \geq q$. Consequently

$$\sum_i y_i y_{i+1} = rq + y_2(y_3 - q) + y_4(y_3 - r) \geq rq > 0.$$

□

Lemma 4.2. *Suppose a symmetric nonnegative matrix has a weighted five-cycle as its positive off-diagonal support and spectrum $(\lambda, a, a, -t, -t)$, with $\lambda > t > a > 0$. Then $\lambda^2 \geq 4at$.*

Proof. Write its cycle edges as $w_i > 0$ and its diagonals as $d_i \geq 0$, using cyclic indices modulo five. Let $v > 0$ be its unit Perron vector and put $s = t - a$ and $c = (\lambda - a)(\lambda + t) > 0$. Spectral calculus gives

$$A^2 + sA - atI = cvv^{\mathsf{T}}. \tag{11}$$

Set $\pi_i = v_i^2$, $r_i = w_i/(v_i v_{i+1})$ and $x_i = c/r_i > 0$. At the distance-two positions, equation (11) reads $w_i w_{i+1} = c v_i v_{i+2}$. Thus

$$\pi_{i+1} = \frac{x_i x_{i+1}}{c}, \qquad \sum_i x_i x_{i+1} = c. \tag{12}$$

At each edge, the two vertices have no common off-diagonal neighbor. The same equation therefore gives $x_i = d_i + d_{i+1} + s$. The Perron equation at vertex i, divided by v_i, gives

$$d_i + r_i \pi_{i+1} + r_{i-1} \pi_{i-1} = \lambda, \qquad \text{hence} \qquad d_i = \lambda - x_{i+1} - x_{i-2}, \tag{13}$$

because $r_i \pi_{i+1} = x_{i+1}$ and $r_{i-1} \pi_{i-1} = x_{i-2}$ by (12). Summing and using $\sum_i d_i = \operatorname{tr} A = \lambda - 2s$ yields

$$\sum_i x_i = 2\lambda + s. \tag{14}$$

Now set $y_i = \lambda/2 - x_i$. Then $d_i = y_{i+1} + y_{i+3}$, so diagonal nonnegativity gives $y_j + y_{j+2} \geq 0$ for all j. By (12) and (14),

$$\sum_i y_i y_{i+1} = \frac{5\lambda^2}{4} - \lambda(2\lambda + s) + (\lambda - a)(\lambda + t) = \frac{\lambda^2}{4} - at.$$

Lemma 4.1 proves the result. □

Lemma 4.2 contradicts $\lambda^2 < 4at$ from Lemma 2.1. It covers nonconstant edge weights, nonconstant diagonals and a nonuniform Perron vector.

5 An exact certificate for the H support

We divide the remaining matrix by its Perron root. To keep this normalization explicit, write

$$\mu = \frac{a}{\lambda}, \qquad \nu = -\frac{t}{\lambda}, \qquad M = \frac{A}{\lambda}.$$

Then $0 < \mu < -\nu < 1$, and M has spectrum $(1, \mu, \mu, \nu, \nu)$. Its trace is $\tau = T/\lambda \in (0, \mu)$. Both strict failures persist:

$$1 + 4\mu\nu < 0, \qquad 1 + \mu + 2\nu < 0. \tag{15}$$

The following estimate is the second support obstruction. It does not require the extra hypothesis $\tau < \mu$ used to reach this support.

Lemma 5.1. *Let M have the support (10) and spectrum $(1, \mu, \mu, \nu, \nu)$, where $0 < \mu < -\nu < 1$. Put*

$$h = -(\mu + \nu) > 0, \quad p = -\mu\nu = \mu(\mu + h) > 0, \quad \tau = \operatorname{tr} M = 1 - 2h. \tag{16}$$

Then

$$p \le \max\left\{\frac{1}{4},\ \tau(\tau + h)\right\}. \tag{17}$$

The chord z and each of d, e, f are permitted to vanish.

Proof. Let $u > 0$ be a unit Perron vector and put $\kappa = (1 - \mu)(1 - \nu) = 1 + h - p > 0$. The spectral theorem yields

$$M^2 + hM = pI + \kappa u u^{\mathsf{T}}. \tag{18}$$

Indeed, $x^2 + hx - p = (x - \mu)(x - \nu)$ vanishes on both repeated eigenspaces and has value κ on the Perron eigenspace.

Reflect the cycle in (10), if necessary, so that $d \ge e$. Define

$$\alpha = d + h, \quad \beta = e + h, \quad \gamma = f + h, \quad \delta = d + e + h. \tag{19}$$

All four quantities are positive. Moreover,

$$\tau = d + e + f, \qquad \gamma + \delta = \tau + 2h = 1. \tag{20}$$

The $(0,1), (3,4), (0,4)$ entries of (18) give

$$x_0\alpha = \kappa u_0 u_1, \qquad x_3\beta = \kappa u_3 u_4, \qquad x_4\delta = \kappa u_0 u_4. \tag{21}$$

The $(0,2), (2,4)$ entries give

$$x_0 x_1 = \kappa u_0 u_2, \qquad x_2 x_3 = \kappa u_2 u_4. \tag{22}$$

All denominators are positive, so

$$x_1 = \alpha\frac{u_2}{u_1}, \qquad x_2 = \beta\frac{u_2}{u_3}. \tag{23}$$

Set

$$R_i = \kappa u_i^2, \qquad \theta = \frac{z u_3}{\beta u_1}, \qquad \eta = \frac{z u_1}{\alpha u_3}. \tag{24}$$

The $(1,2), (2,3), (1,4), (0,3)$ entries yield

$$\begin{aligned} R_1 &= \alpha(\gamma + \beta\eta), \quad & R_3 &= \beta(\gamma + \alpha\theta), \\ R_0 &= \alpha\delta(1 - \theta), \quad & R_4 &= \beta\delta(1 - \eta). \end{aligned} \tag{25}$$

Since $R_0, R_4 > 0$, one has $0 \le \theta, \eta < 1$. The definitions also give $\theta R_1/\alpha = \eta R_3/\beta$. Substituting (25) therefore gives

$$\gamma(\theta - \eta) = (d - e)\theta\eta, \qquad \eta = \frac{\gamma\theta}{\gamma + (d - e)\theta}. \tag{26}$$

This includes $z = 0$, when both parameters vanish.

The chord entry of (18) is $hz + x_1x_2 = \kappa u_1u_3$. Together with (23)–(25), it gives

$$R_2 = (\gamma + \beta\eta)(\gamma + d\theta) = (\gamma + \alpha\theta)(\gamma + e\eta), \tag{27}$$

and hence

$$x_1^2 = \alpha(\gamma + d\theta), \qquad x_2^2 = \beta(\gamma + e\eta). \tag{28}$$

Now take the diagonal entry at vertex 2 and use the first relation in (26) in both rearrangements:

$$\begin{aligned} p &= x_1^2 + x_2^2 + f^2 + hf - R_2 \\ &= \gamma\delta + \alpha(d - \gamma)\theta + e(e - f)\eta - \alpha e\theta\eta \\ &= \gamma\delta(1 - \theta) + d\alpha\theta + e\beta\eta(1 - \theta). \end{aligned} \tag{29}$$

The second identity in (26) gives the useful form

$$p = \gamma\delta(1 - \theta) + d(d + h)\theta + \frac{\gamma e(e + h)}{\gamma + (d - e)\theta}\theta(1 - \theta). \tag{30}$$

Because $d \geq e$ and $\gamma > 0$, the ratio $\gamma/(\gamma + (d - e)\theta)$ belongs to $(0, 1]$. Using also $1 - \theta \leq 1$ in the last term of (30),

$$\begin{aligned} p &\leq (1 - \theta)\gamma\delta + \theta\big[d(d + h) + e(e + h)\big] \\ &\leq (1 - \theta)\gamma\delta + \theta(d + e)(d + e + h) \\ &\leq (1 - \theta)\frac{1}{4} + \theta\tau(\tau + h). \end{aligned} \tag{31}$$

The second step uses $(d + e)(d + e + h) - d(d + h) - e(e + h) = 2de \geq 0$. In the third, $\gamma\delta \leq 1/4$ follows from (20) and $\gamma, \delta > 0$, and $d + e \leq \tau$ with $h > 0$ gives the other bound. Since $0 \leq \theta < 1$, this proves (17). □

Completion of the proof of Theorem 1.1. Proposition 3.4 gives two possible supports. The cycle case contradicts Lemma 4.2. In the H case, the first inequality in (15) gives $p > 1/4$. The second gives $\tau < \mu$, so

$$p = \mu(\mu + h) > \tau(\tau + h),$$

because $\mu > \tau \geq 0$ and $h > 0$. Both alternatives contradict Lemma 5.1. A counterexample therefore cannot exist. The trace condition and both equality boundaries are included, since we have excluded only simultaneous strict failures. □

6 Explicit reconstruction of the known sufficiency

The converse in Corollary 1.2 predates the necessity argument. Clark [1, pp. 78–79] states known sufficiency of (3) and gives the boundary construction attributed to Loewy–McDonald [6]. Johnson–Marijuán–Pisonero [4, p. 77] describe the extension from the curved boundary by raising the Perron eigenvalue and rescaling. Loewy's Example 3.2 [7] supplies a further primary comparison of the realizable and excluded regions.

We give formulas covering every branch, including the endpoints. The parameter domain used for the displayed Loewy–McDonald family is checked directly below; the constructions are verified directly and do not depend on the original source.

Proposition 6.1 (Known sufficiency in explicit form)**.** *If* $1 \geq a \geq b \geq -1$ *and* (3) *holds, then* $(1, a, a, b, b)$ *has a symmetric nonnegative realization of order five.*

Proof. If $b \geq 0$, use the diagonal matrix. We next give the remaining three constructions.

The one-positive-eigenvalue case. Suppose $a \le 0$, so $b \le 0$. Since $S \ge 0$ and $b \le a$, one has $0 \le S \le 1 + 4a$ and hence $a \ge -1/4$. Put

$$c = \sqrt{\frac{(1+2a)(-2a-b)}{6}}, \qquad C = \begin{pmatrix} 0 & -a & -a & c & c \\ -a & 0 & -a & c & c \\ -a & -a & 0 & c & c \\ c & c & c & S/2 & (1+2a)/2 \\ c & c & c & (1+2a)/2 & S/2 \end{pmatrix}.$$

All entries are nonnegative. The subspace supported on the first three coordinates and summing to zero has eigenvalue a and dimension two. The vector $(0,0,0,1,-1)$ has eigenvalue b. On their orthogonal complement, the matrix in the orthonormal basis $(1,1,1,0,0)/\sqrt{3}$, $(0,0,0,1,1)/\sqrt{2}$ is

$$\begin{pmatrix} -2a & \sqrt{6}\,c \\ \sqrt{6}\,c & 1+2a+b \end{pmatrix}.$$

Its trace is $1+b$ and its determinant is $(-2a)(1+2a+b) - (1+2a)(-2a-b) = b$. Its roots are therefore $1, b$. This is an elementary realization of the one-positive branch of the classical symmetric sufficiency theorem; see [11, Theorem 4].

The linear branch. Suppose $a > 0 > b$ and $G := 1 + a + 2b \ge 0$. Define

$$d = \frac{G}{4}, \qquad r = \frac{1+a-2b}{4}, \qquad q = \frac{1-a}{4}, \qquad B = \begin{pmatrix} d & r & q & q \\ r & d & q & q \\ q & q & d & r \\ q & q & r & d \end{pmatrix}.$$

The matrix $B \oplus [a]$ is nonnegative: $d \ge 0$, $4r = (1-b) + (a-b) \ge 0$, and $4q = 1 - a \ge 0$. The vectors $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,0,0)$ and $(0,0,1,-1)$ have respective B eigenvalues $1, a, b, b$. This explicitly gives the old direct-sum branch [8, proof of Theorem 4].

The remaining curved branch. Suppose $a > 0 > b$ and $G < 0$. Set $t = -b$ and $s = t - a > 0$. Then $t > a$, and the allowed alternative in (3) must be $1 - 4at \ge 0$. Put

$$\rho = \max\{2s, 2\sqrt{at}\}. \tag{32}$$

The trace and curved conditions imply $0 < \rho \le 1$. We first realize $(\rho, a, a, -t, -t)$ and then raise its Perron root to one by a nonnegative rank-one update, as described in [4, p. 77].

If $s \ge \sqrt{at}$, define

$$c = \frac{s + (a+t)/\sqrt{5}}{2}, \qquad d = \frac{s - (a+t)/\sqrt{5}}{2}.$$

Here $d \ge 0$ because $5s^2 - (a+t)^2 = 4(s^2 - at) \ge 0$. The symmetric circulant with first row $(0, c, d, d, c)$ has row sum $2(c+d) = 2s = \rho$. Its other eigenvalues, each repeated twice, are

$$\frac{(\sqrt{5}-1)c - (\sqrt{5}+1)d}{2} = a, \qquad \frac{-(\sqrt{5}+1)c + (\sqrt{5}-1)d}{2} = -t.$$

These follow by the five Fourier modes. If J_5 is the all-ones matrix, adding $(1-\rho)J_5/5$ raises only the eigenvalue on the constant vector and gives the required realization. This includes the trace-zero case $\rho = 1$.

If $s \le \sqrt{at}$, set $\rho = 2\sqrt{at}$ and

$$x^2 = \frac{1}{8} + \frac{s}{4\rho}.$$

Then $1/8 < x^2 \le 1/4$. The Loewy–McDonald family displayed by Clark [1, p. 79], with equal parameters, is

$$L(x) = \begin{pmatrix} 1/2-2x^2 & 2x^2 & 1/2 & 0 & 0 \\ 2x^2 & 1/2-2x^2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & x & 0 \\ 0 & 0 & x & 1-4x^2 & x \\ 0 & 1/2 & 0 & x & 0 \end{pmatrix}. \tag{33}$$

Every entry is nonnegative on this domain. Direct multiplication gives $L(x)u = u$, where

$$u = \frac{(2x, 2x, 2x, 1, 2x)^{\mathsf{T}}}{\sqrt{1+16x^2}} > 0.$$

The permutation exchanging $1 \leftrightarrow 2$ and $3 \leftrightarrow 5$ decomposes the space into antisymmetric and symmetric subspaces. In the orthonormal bases $(e_1 - e_2)/\sqrt{2}, (e_3 - e_5)/\sqrt{2}$ and $(e_1 + e_2)/\sqrt{2}, (e_3 + e_5)/\sqrt{2}, e_4$, the two matrices are

$$\begin{pmatrix} 1/2-4x^2 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & \sqrt{2}x \\ 0 & \sqrt{2}x & 1-4x^2 \end{pmatrix}.$$

Their characteristic polynomials are $q(z)$ and $(z-1)q(z)$, respectively, where

$$q(z) = z^2 - (1/2 - 4x^2)z - 1/4.$$

Since $(a-t)/\rho = 1/2 - 4x^2$ and $-at/\rho^2 = -1/4$, the roots of q are a/ρ and $-t/\rho$. Thus $\rho L(x)$ realizes $(\rho, a, a, -t, -t)$, and

$$A = \rho L(x) + (1-\rho)uu^{\mathsf{T}}$$

is nonnegative with spectrum $(1, a, a, -t, -t)$. At $s = \sqrt{at}$, either construction applies: the circulant has $d = 0$ and the displayed family has $x = 1/2$. There is no limiting argument at their common endpoint. □

Proof of Corollary 1.2. Combine Theorem 1.1 with Proposition 6.1. □

7 The excluded region and its curved boundary

Remark 7.1 (Shape of the excluded region)**.** By Corollary 1.2, a pair with $1 \ge a \ge b \ge -1$ and $S \ge 0$ is not realizable exactly when $1 + 4ab < 0$ and $1 + a + 2b < 0$. This set is

$$\mathcal{U} = \Big\{(a,b) : \ \tfrac{\sqrt{5}-1}{4} < a < 1, \ \max\big\{-1, -\tfrac{1+2a}{2}\big\} \le b < \min\big\{-\tfrac{1}{4a}, -\tfrac{1+a}{2}\big\}\Big\}.$$

Indeed, $1 + 4ab < 0$ forces $a > 0 > b$ and then reads $b < -1/(4a)$; the other two conditions read $b < -(1+a)/2$ and $b \ge -(1+2a)/2$. For $a > 0$ the resulting interval for $b \in [-1, a]$ is nonempty exactly when $a > 1/4$, $a < 1$ and $-(1+2a)/2 < -1/(4a)$, that is $4a^2 + 2a - 1 > 0$, or $a > (\sqrt{5}-1)/4$; the last condition implies the first. The closure of $\mathcal{U}$ is a curvilinear quadrilateral with vertices

$$\big(\tfrac{1}{2}, -1\big), \quad P_1 = \big(\tfrac{\sqrt{5}-1}{4}, -\tfrac{\sqrt{5}+1}{4}\big), \quad P_2 = \big(\tfrac{\sqrt{3}-1}{2}, -\tfrac{\sqrt{3}+1}{4}\big), \quad (1, -1).$$

Its sides lie on $S = 0$ (from $(1/2, -1)$ to P_1), on the hyperbola $4ab = -1$ (from P_1 to P_2), on the line $1 + a + 2b = 0$ (from P_2 to $(1, -1)$) and on $b = -1$. The curve and the line cross at P_2, where $2a^2 + 2a - 1 = 0$, and for $(\sqrt{5}-1)/4 < a < (\sqrt{3}-1)/2$ the hyperbola lies below the line. The open arc of the hyperbola and the open segment of the line are realizable; the points of $\mathcal{U}$ on $S = 0$ and on $b = -1$ are not.

On the open curved arc we can describe every realization. Throughout this section put

$$\frac{\sqrt{5}-1}{4} < a < \frac{\sqrt{3}-1}{2}, \qquad b = -t = -\frac{1}{4a}, \qquad s = t - a.$$

Then $1 + 4ab = 0$, $S = 1 - 2s > 0$ and $1 + a + 2b = 1 + a - 2t < 0$; also $a < 1/2 < t < 1$ and $0 < s < 1/2$. For $0 \le \xi \le S/2$ put

$$x(\xi) = \left(\tfrac{1}{2} - \xi,\ \tfrac{1}{2},\ s + \xi,\ \tfrac{1}{2},\ \tfrac{1}{2}\right),$$

with indices in $\mathbb{Z}/5$, so that $x_i > 0$ and $\sum_i x_i = 2 + s$. Let C_ξ be the symmetric matrix with

$$(C_\xi)_{ii} = 1 - x_{i+1} - x_{i-2}, \qquad (C_\xi)_{i,i+1} = (C_\xi)_{i+1,i} = \sqrt{x_{i-1}x_{i+1}},$$

and all other entries zero. Explicitly,

$$C_\xi = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \sqrt{\frac{1/2-\xi}{2}} \\ \frac{1}{2} & \frac{S}{2} - \xi & \sqrt{(\frac{1}{2}-\xi)(s+\xi)} & 0 & 0 \\ 0 & \sqrt{(\frac{1}{2}-\xi)(s+\xi)} & \xi & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \sqrt{\frac{s+\xi}{2}} \\ \sqrt{\frac{1/2-\xi}{2}} & 0 & 0 & \sqrt{\frac{s+\xi}{2}} & \frac{S}{2} \end{pmatrix}. \tag{34}$$

Theorem 7.2 (Realizations on the open curved arc)**.** *With a, b, s as above, a real symmetric nonnegative 5×5 matrix has spectrum $(1, a, a, b, b)$ if and only if it is permutation similar to C_ξ for some $\xi \in [0, S/4]$. The matrices C_ξ, $0 \le \xi \le S/4$, are pairwise not permutation similar. In particular, every such realization is irreducible, its positive off-diagonal support is a five-cycle, and at least two of its diagonal entries vanish.*

Proof. The matrices C_ξ are realizations. Nonnegativity is clear, since $0 \le \xi \le S/2$ and $1/2 - \xi \ge s > 0$. Put $c = 3/4 + s$ and $v_i = (x_{i-1}x_i/c)^{1/2} > 0$. A direct expansion gives $\sum_i x_{i-1}x_i = 3/4 + s = c$, so $\|v\| = 1$. For each i,

$$\frac{(C_\xi v)_i}{v_i} = (C_\xi)_{ii} + \sqrt{x_{i-1}x_{i+1}}\sqrt{\frac{x_{i+1}}{x_{i-1}}} + \sqrt{x_{i-2}x_i}\sqrt{\frac{x_{i-2}}{x_i}} = 1,$$

so $C_\xi v = v$. We check $C_\xi^2 + sC_\xi - \frac{1}{4}I = cvv^{\mathsf{T}}$ entrywise; since every pair of distinct vertices of a five-cycle is adjacent or at distance two, three kinds of entries occur. At distance two, $(C_\xi^2)_{i,i+2} = \sqrt{x_{i-1}x_ix_{i+1}x_{i+2}} = cv_iv_{i+2}$. On an edge, the two endpoints have no common neighbor, so the $(i, i+1)$ entry of the left side is $\sqrt{x_{i-1}x_{i+1}}\,(d_i + d_{i+1} + s)$ with $d_i = (C_\xi)_{ii}$; since $d_i + d_{i+1} = 2 - \sum_k x_k + x_i = x_i - s$, it equals $x_i\sqrt{x_{i-1}x_{i+1}} = cv_iv_{i+1}$. On the diagonal the required identity is

$$d_i(d_i + s) + x_{i-1}x_{i+1} + x_{i-2}x_i - x_{i-1}x_i = \tfrac{1}{4},$$

and substitution of $x(\xi)$ verifies it for $i = 0, \dots, 4$; for instance at $i = 1$ the left side is $(\frac{1}{2} - \xi)\big[(\frac{1}{2} - s - \xi) + (s + \xi) - \frac{1}{2}\big] + \frac{1}{4}$. Consequently C_ξ maps $v^\perp$ to itself and satisfies $z^2 + sz - \frac{1}{4} = (z - a)(z + t) = 0$ there. Its eigenvalues on $v^\perp$ therefore lie in $\{a, -t\}$; their sum is $\operatorname{tr} C_\xi - 1 = S - 1 = 2a - 2t$, which forces multiplicities two and two. Thus C_ξ has spectrum $(1, a, a, -t, -t)$, and so has every permutation similar matrix.

Every realization has this form. Let A be a symmetric nonnegative realization of $(1, a, a, -t, -t)$; here $\lambda = 1$. The irreducibility and peripheral-eigenvalue arguments in the proof of Lemma 2.1 use only $\lambda \le 1$, $t > a$ and $1 + a - 2t < 0$; hence A is irreducible and $1 > t$. Its trace is $T = S$, so $0 < T < a$. No orthogonal conjugate B of A is strictly positive: otherwise, as in Lemma 2.1, $B - \varepsilon ww^{\mathsf{T}}$ would realize $(1 - \varepsilon, a, a, -t, -t)$ for small $\varepsilon > 0$, and dividing by $1 - \varepsilon$ would give a realization of $(1, a', a', -t', -t')$ with

$a' = a/(1-\varepsilon)$ and $t' = t/(1-\varepsilon) \le 1$, for which $1 - 4a't' = 1 - (1-\varepsilon)^{-2} < 0$ and $1 + a' - 2t' = (1 - \varepsilon + a - 2t)/(1-\varepsilon) < 0$, contrary to Theorem 1.1. Thus A has every property in Lemma 2.1 except $\lambda^2 < 4at$, which is replaced by $\lambda^2 = 4at$. The proofs of Lemmas 3.1, 3.2, 3.3 and Proposition 3.4 use only the former properties, so A is permutation similar to a matrix with weighted five-cycle support or of the form (10).

In the second case apply the proof of Lemma 5.1 to $M = A$: here $\mu = a$, $p = at = \frac{1}{4}$ and $\tau = S < a$, so $\tau(\tau + h) < \mu(\mu + h) = \frac{1}{4}$. If $\theta > 0$, the last line of (31) would be strictly smaller than $\frac{1}{4} = p$. Hence $\theta = 0$, that is $z = 0$, and the positive off-diagonal support of (10) is the five-cycle $0, 1, 2, 3, 4$.

So in both cases the positive off-diagonal support of A is a five-cycle. Label its vertices along the cycle and use the notation of the proof of Lemma 4.2, with $\lambda = 1$. Then $y_j + y_{j+2} \ge 0$ for all j and $\sum_i y_i y_{i+1} = \frac{1}{4} - at = 0$. By the strictness clause of Lemma 4.1 all $y_i \ge 0$, so every product $y_i y_{i+1}$ vanishes and at most two y_i are nonzero, at nonadjacent indices. After a rotation of the labels, $y = (\xi, 0, \xi', 0, 0)$ with $\xi, \xi' \ge 0$. The relation $d_i = y_{i+1} + y_{i+3}$ gives the diagonal $(0, \xi', \xi, 0, \xi + \xi')$, and the trace gives $\xi + \xi' = S/2$. Hence $x_i = \frac{1}{2} - y_i$ is the vector $x(\xi)$, and $(A)_{ii} = 1 - x_{i+1} - x_{i-2}$ by (13). By (12), $v_i^2 = x_{i-1}x_i/c$, and therefore the edge weight is $w_i = r_i v_i v_{i+1} = (c/x_i) v_i v_{i+1} = \sqrt{x_{i-1}x_{i+1}}$. Thus $A = C_\xi$ in these labels, with $0 \le \xi \le S/2$.

Reduction to $\xi \le S/4$ and distinctness. The relabeling $i \mapsto 3 - i$ preserves the five-cycle and the defining formulas for C_ξ up to the corresponding index change, and $x(S/2 - \xi)_k = x(\xi)_{2-k}$ for every k; a direct check shows that it carries C_ξ to $C_{S/2-\xi}$. Finally, permutation similar matrices have the same multiset of diagonal entries, and the diagonal multiset of C_ξ is $\{0, 0, S/2, \xi, S/2 - \xi\}$, which determines $\min\{\xi, S/2 - \xi\}$. □

Remark 7.3. On the open arc, $\rho = 2\sqrt{at} = 1$ in the curved branch of Proposition 6.1, so that construction is $L(x)$ itself, with $x^2 = \frac{1}{8} + \frac{s}{4}$. Its positive off-diagonal support is the five-cycle $1, 3, 4, 5, 2$, and listing its vertices in the order $5, 2, 1, 3, 4$ gives exactly $C_{S/4}$, since then $\frac{1}{2} - \xi = s + \xi = 2x^2$. Theorem 7.2 thus shows that the Loewy–McDonald matrix is the symmetric member of a one-parameter family, and that no other support occurs on the open arc. The theorem says nothing about the endpoints P_1, P_2 or the linear boundary segment: at P_1 the trace vanishes and the case $X = I$ of Lemma 3.3 is no longer excluded, while on the segment reducible realizations such as $B \oplus [a]$ occur.

8 Consequences and scope

Theorem 1.1 replaces several partial exclusions on the repeated-pair plane by one necessary condition. For example, take

$$a = \frac{2}{5}, \qquad b = -\frac{71}{100}.$$

Then $S = 19/50$, $1 + 4ab = -17/125$ and $1 + a + 2b = -1/50$, so the list is not realizable. The scalar tests compared in Loewy's Example 3.2 [7] do not give this exclusion: with

$$u(a, b) = 1 + a^3 + 2b^3 - (1 + a + 2b)^3,$$
$$v(a, b) = 6(a^3 - b^3 + 2a^2 b + a^2 - 2b^2 - b),$$

one has $u = 174093/500000 > 0$ and $v = 169533/500000 > 0$. Also $5 + 2b - 7a = 39/50 > 0$, so the cited residual-region condition of Jin–Ke–Sui does not apply; and $S < 1/2$ lies outside the high-trace classification. These calculations illustrate the difference between the stated tests.

The proof does not extend automatically to a split pair, or to an arbitrary five-dimensional spectrum. The two-dimensional negative projector, the five-vertex geometry and the quadratic rank-one identity all use the multiplicities in (1).

The reducibility argument identifies the linear branch's role in the strict region: when $a > 0 > b$ and $-b > a$, every reducible realization must satisfy $1 + a + 2b \ge 0$. The curved obstruction is obtained after minimizing the Perron root, and should not be read as

a statement that every realization satisfying $1 + 4ab \geq 0$ has cycle support. Only on the open curved arc, where $1 + 4ab = 0$, $S > 0$ and $1 + a + 2b < 0$, does Theorem 7.2 force a five-cycle support; there the equality case of Lemma 4.1 determines the realization up to one parameter. Realizations at the endpoints P_1, P_2 and on the linear boundary segment are not classified here.

All arguments are analytical. Exact polynomial checks can verify the identities in Sections 5 and 7, but neither grids nor failed numerical searches establish the global support reduction. The two support inequalities alone are insufficient for an arbitrary realization: Perron minimization and the complementary-support argument are what make them applicable.

AI assistance disclosure

OpenAI GPT-5.6 Sol, accessed through ChatGPT and Codex, was used to develop and audit the necessity proof, to check its algebraic identities, and for drafting. Anthropic Claude Opus 5.5, used through Claude Code, was used for editing, for an independent line-by-line check of the proof with exact symbolic computation, and to derive the classification of realizations on the open curved boundary arc (Section 7). The author reviewed and takes responsibility for every statement and proof. AI systems are not authors of this work, and the work has not been peer reviewed.

References

[1] Benjamin James Clark. *The Nonnegative Inverse Eigenvalue Problem: A Theoretical and Computational Investigation*. PhD thesis, Washington State University, 2026. URL https://rex.libraries.wsu.edu/esploro/outputs/doctoral/The-Nonnegative-Inverse-Eigenvalue-Problem-A/99901393504301842.

[2] Jiashun Jin, Zheng Tracy Ke, and Bingcheng Sui. A new impossibility region for the 5×5 symmetric nonnegative inverse eigenvalue problem, 2026. URL https://arxiv.org/abs/2608.19435. Version 1.

[3] C. R. Johnson, C. Marijuán, and M. Pisonero. Ruling out certain 5-spectra for the symmetric nonnegative inverse eigenvalue problem. *Linear Algebra and its Applications*, 512:129–135, 2017. doi: 10.1016/j.laa.2016.09.022. URL https://www.singacom.uva.es/Uploads/2017/591816d4d29c2.pdf.

[4] C. R. Johnson, C. Marijuán, and M. Pisonero. Ruling out certain 5-spectra with one repeated eigenvalue for the symmetric NIEP. *Linear Algebra and its Applications*, 612: 75–98, 2021. doi: 10.1016/j.laa.2020.12.004. URL https://www.singacom.uva.es/Uploads/2021/603c7b1e52a53.pdf.

[5] Thomas J. Laffey. Extreme nonnegative matrices. *Linear Algebra and its Applications*, 275–276:349–357, 1998. doi: 10.1016/S0024-3795(97)10059-3.

[6] R. Loewy and J. J. McDonald. The symmetric nonnegative inverse eigenvalue problem for 5×5 matrices. *Linear Algebra and its Applications*, 393:275–298, 2004. doi: 10.1016/j.laa.2003.10.018.

[7] Raphael Loewy. Some additional notes on the spectra of non-negative symmetric 5×5 matrices. *Electronic Journal of Linear Algebra*, 37:1–13, 2021. doi: 10.13001/ela.2021.5333. URL https://journals.uwyo.edu/index.php/ela/article/view/5333.

[8] Raphael Loewy and Oren Spector. A necessary condition for the spectrum of nonnegative symmetric 5×5 matrices. *Linear Algebra and its Applications*, 528:206–272, 2017. doi: 10.1016/j.laa.2016.08.006.

[9] Raphael Loewy and Oren Spector. Some notes on the spectra of nonnegative symmetric 5×5 matrices. *Linear and Multilinear Algebra*, 69(7):1363–1382, 2021. doi: 10.1080/03081087.2019.1646205.

[10] C. Marijuán and M. Pisonero. A note for the SNIEP in size 5. *Linear Algebra and its Applications*, 675:338–343, 2023. doi: 10.1016/j.laa.2023.07.001. URL https://uvadoc.uva.es/handle/10324/63643.

[11] Oren Spector. A characterization of trace zero symmetric nonnegative 5×5 matrices. *Linear Algebra and its Applications*, 434(4):1000–1017, 2011. doi: 10.1016/j.laa.2010.10.006.

A Entry map for the H certificate

This appendix records the coordinates behind the main-text identities. Indices in (10) are $0, 1, 2, 3, 4$. The independent scalar symbols d, e, f denote the diagonal entries at $0, 4, 2$, respectively. The cycle edges in their order are x_0, x_1, x_2, x_3, x_4, and z is the chord 13.

Position	Entry of $M^2 + hM$	Use in the proof
$(0,1)$	$(d+h)x_0$	$x_0\alpha = \kappa u_0 u_1$
$(3,4)$	$(e+h)x_3$	$x_3\beta = \kappa u_3 u_4$
$(0,4)$	$(d+e+h)x_4$	$x_4\delta = \kappa u_0 u_4$
$(0,2)$	x_0x_1	first identity in (22)
$(2,4)$	x_2x_3	second identity in (22)
$(1,2)$	$(f+h)x_1 + zx_2$	formula for R_1
$(2,3)$	$(f+h)x_2 + zx_1$	formula for R_3
$(1,4)$	$x_0x_4 + zx_3$	formula for R_0
$(0,3)$	$x_4x_3 + zx_0$	formula for R_4
$(1,3)$	$hz + x_1x_2$	R_2 factorization
$(2,2)$	$x_1^2 + x_2^2 + f^2 + hf$	scalar certificate for p

For example, dividing the $(1,2)$ equation by x_1 and substituting (23) gives $R_1 = \alpha(\gamma + \beta\eta)$. To obtain R_0, substitute (21) in the $(1,4)$ equation and divide by $\kappa u_1 u_4$:

$$\frac{R_0}{\alpha\delta} + \theta = 1.$$

The $(0,3)$ equation gives the analogous relation for R_4. The chord equation gives

$$R_2 = (\gamma + \alpha\theta)(\gamma + e\eta) = (\gamma + \beta\eta)(\gamma + d\theta).$$

The two forms are consistent because $\gamma(\theta - \eta) = (d - e)\theta\eta$.

Substitution in the diagonal equation gives

$$p = \gamma\delta + \alpha(d - \gamma)\theta + e(e - f)\eta - \alpha e\theta\eta.$$

Subtracting the final expression in (29) leaves a scalar multiple of $\gamma(\theta - \eta) - (d - e)\theta\eta$. Thus the passage to (30) uses an established relation, not an independent positivity assumption. The algebraic identity and the subsequent inequality have different roles: the former follows from the repeated spectrum and support; the latter additionally uses $d \geq e$, nonnegative diagonals and the positive denominators.

B Endpoint and quantifier checklist

The theorem includes all real ordered pairs in (1). If $ab \geq 0$, or if $a = 0$ or $b = 0$, the curved alternative cannot fail. If $a = 1$, the linear alternative holds because $b \geq -1$. If $b = -1$

and $a < 1$, the hypothetical minimization interval is the singleton $[t, 1] = \{1\}$; the negative peripheral eigenvalue argument in Lemma 2.1 excludes it whenever the strict failures apply. Trace zero is treated before the support reduction, by $\operatorname{tr}(A^3) \geq 0$. No division by the trace is made.

All Perron coordinates used as denominators are strictly positive after irreducibility has been established. The H chord may vanish, and its absence gives $\theta = \eta = 0$ without a limiting argument. The diagonal entries may vanish independently. The separated multiplier may have diagonal loops and may initially be disconnected; each disconnected partition is excluded in Lemma 3.3. No common eigenspace basis or uniform vector is prescribed. Finally, equality in either alternative of (3) is retained because the necessity proof assumes their simultaneous strict failure.