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COUNTEREXAMPLES, SPECTRAL OBSTRUCTIONS, AND DELETION STABILITY FOR WOW-284

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Abstract. WOW-284 is a Graffiti conjecture asserting that every connected graph of order at least three and girth at least five has minimum dual degree at most $-\lambda_{\min}(D)$, where D is the distance matrix and the dual degree of a vertex is the average degree of its neighbors. We refute it with exact counterexamples on 38, 39, 40, 42 and 50 vertices and study the structure of regular counterexamples. For a connected k-regular graph G of girth at least five and diameter three, we prove that $\delta^*(G) + \lambda_{\min}(D) = 2k - 2 - \max_\theta(\theta+1)^2$, the maximum being over the nonprincipal adjacency eigenvalues, so strict failure is a spectral window condition. The closed window $|\theta + 1| \le \sqrt{2k-2}$ alone, without any diameter assumption, forces $|V(G)| < (k+2)(k^2+3)/6$ when $k \ge 4$, and an integral excess matrix derived from the optimal polynomial sharpens this bound for diameter-three counterexamples. Every regular strict counterexample has degree at least six and diameter at most four, and in degree six it has at most 50 vertices.

Under vertex deletion, the Hoffman–Singleton graph remains a strict counterexample after the removal of any five vertices, while an explicit six-vertex deletion destroys strictness. A degree-57 Moore graph, if one exists, remains a connected strict counterexample after the removal of any 56 vertices. Every strict counterexample has at least 15 vertices, and an exhaustive search excludes orders up to 19. The strict induced subgraphs of the Hoffman–Singleton graph of orders 36 to 50 are classified up to automorphism; among them are three further irregular counterexamples of order 40.

1. WOW-284 and the main results

Let G be a finite simple connected graph of order $n = |V(G)| \ge 2$. Let $A = A(G)$ and $D = D(G)$ be its adjacency and distance matrices, let I and J denote the identity and all-ones matrices of order n, and let $\mathbf{1}$ be the all-ones vector. Write $d(v)$ for the degree of v, $N(v)$ for its open neighborhood, and, for $S \subseteq V(G)$, let $G - S$ denote the subgraph induced by $V(G) \setminus S$. For $U \subseteq V(G)$, $G[U]$ is the induced subgraph on U, and for a matrix indexed by $V(G)$ the suffix $[U]$ denotes its principal submatrix on U. Define

$$d^*(v) = \frac{1}{d(v)} \sum_{u \in N(v)} d(u), \qquad \delta^*(G) = \min_{v \in V(G)} d^*(v).$$

Here $d_G(u,v)$ is the length of a shortest path from u to v, so $D = (d_G(u,v))_{u,v \in V(G)}$, and $\lambda_{\min}(D)$ is its least eigenvalue. Let e_v denote the standard coordinate vector at v. For a square matrix M, write $\chi_M(t) = \det(tI - M)$ for its characteristic polynomial and $E_\lambda(M) = \ker(M - \lambda I)$ for its λ-eigenspace, with I of the matching order. We list spectra with eigenvalue multiplicities as superscripts in parentheses, and order adjacency eigenvalues decreasingly. Orthogonality refers to the Euclidean inner product; the orthogonal complement of the constant vector is the zero-sum space. The girth $g(G)$ is the length of a shortest cycle, with infinite girth when there is no cycle. The diameter is the largest distance between two vertices. For a connected regular graph, we call the adjacency eigenvalues other than its degree eigenvalue nonprincipal.

Aouchiche and Hansen [1, Conjecture 7.16] record the following Graffiti conjecture and attribute it to Fajtlowicz's 1998 *Written on the Wall* report [2].

Conjecture (WOW-284). *If G has order at least three and girth at least five, then*

$$\delta^*(G) \le -\lambda_{\min}(D(G)).$$

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We measure the failure of the conjectured inequality by a scalar score: for every finite simple connected graph of order at least two, put

$$\Phi(G) = \delta^*(G) + \lambda_{\min}(D(G)).$$

Among graphs satisfying the conjecture's hypotheses, G is a strict counterexample precisely when $\Phi(G) > 0$.

1.1. **Main results.** The first counterexample comes from a degree-k Moore graph of diameter two [3]: it has $\delta^* = k$ and $\lambda_{\min}(D) = -(3 + \sqrt{4k-3})/2$ (Theorem 2.2). The conjecture holds on these graphs exactly for $k \le 3$, with equality at $k = 3$, and fails for every realizable $k > 3$. The degree-seven Hoffman–Singleton graph therefore gives a gap of three. From the Hoffman–Singleton coordinates, Section 4 extracts further strict counterexamples on 38, 39, 40 and 42 vertices (Theorem 4.2); the second subconstituent of a degree-K Moore graph is strict exactly when $K \ge 6$ (Theorem 4.1).

The paper follows two lines of argument. For regular graphs of diameter three, the distance-polynomial identity turns strict failure of WOW-284 into an interval condition on the nonprincipal adjacency eigenvalues (Theorem 3.1). Taking the trace of a polynomial in the adjacency matrix then gives $n < B_k = (k+2)(k^2+3)/6$ under the closed window alone when $k \ge 4$, and no polynomial of higher degree improves this bound (Proposition 6.1, Theorem 6.4). Evaluating the optimal polynomial at the adjacency matrix, subtracting a constant multiple of the all-ones matrix and adjusting the diagonal gives the integral excess matrix of Theorem 6.5. Its integer entries, together with the nonnegative principal minors of the associated slack matrix, give stronger order and cycle constraints. These lead to the three-to-one bound (Theorem 6.6) and to $n \le 50$ in degree six (Theorem 7.2), where the case $n = 50$ remains unresolved. The polynomial bound in Section 6 is optimal within the stated class of polynomials; we do not claim that the resulting graph-order bound is best possible.

The degree and diameter restrictions narrow the search: every regular strict counterexample has degree at least six and diameter at most four, and a diameter-four example would have degree at least ten (Theorems 7.1, 5.2, and 5.3).

The second argument concerns deletion. Deleting vertices can increase the distances between those left behind, so taking a principal submatrix of the old distance matrix is not enough. For small deletions from a Moore graph, we identify exactly which distances increase. This correction gives general stability bounds (Section 8); Appendix A gives the exact one- and two-deletion spectra. For Hoffman–Singleton, the bounds prove strictness through three deletions, and exact orbit certificates handle four and five. A six-vertex deletion then shows that five is the universal radius, the largest s such that deleting any s or fewer vertices leaves a strict counterexample (Theorem 8.6). For a hypothetical degree-57 Moore graph, the same corrections give strictness after any 56 deletions (Proposition 8.7). Theorem 3.3 characterizes equality on the window boundary; Jørgensen's 96-vertex graph attains it.

We do not claim that our examples exhaust all counterexamples, nor that the order-38 example is minimal. Section 9 proves that a strict counterexample has at least 15 vertices, shows by exhaustive computation that none has at most 19, classifies the strict induced subgraphs of the Hoffman–Singleton graph of orders 36 to 50, and gives three further, irregular counterexamples of order 40.

1.2. **Related work.** The Hoffman–Singleton counterexample is also recorded in a public Capy Build transcript [4] and in a verification note by Wilbanks and Annie [5].

Howlader and Panigrahi already give the distance spectra of Moore graphs, including the Hoffman–Singleton graph [6, Theorem 2.5(1)]. The distance-polynomial framework is developed in [6, 7]. Nonbacktracking walks never immediately reverse an edge. Linear-programming bounds based on their counts are due to Nozaki [8]; related spectral-Moore bounds appear in [9]. We use these methods to derive order restrictions from the WOW-284 spectral window and to study vertex deletion.

1.3. **Organization.** Section 2 computes the score of every Moore graph of diameter two and gives the Hoffman–Singleton graph in coordinates. Section 3 proves the diameter-three identity and identifies its equality case, and Section 4 uses it to certify the smaller counterexamples. Section 5 bounds the diameter of regular counterexamples, Section 6 proves the polynomial order bounds, and Section 7 combines them into the degree and order restrictions. Section 8 treats vertex deletion, Section 9 treats small orders, and Section 10 lists open questions.

The appendices contain the longer proofs and the finite certificates, in the order in which the main text first needs them: the deletion spectra of Moore graphs and the proof of Theorem 4.1 (Appendix A), the certificates for the explicit examples (Appendix B), three technical steps in the polynomial order bounds (Appendix C), the proof of the three-to-one bound (Appendix D), the degree-five case of the degree bound (Appendix E), necessary conditions at degree six and order fifty (Appendix F), further deletions inside a neighborhood (Appendix G), the methods and further results for small orders (Appendix H), and

two construction families excluded by the spectral window (Appendix I). Appendix J records the exact computations and the scope of the Lean verification. Identities and finite case checks inside the analytic proofs are also reproduced in exact arithmetic (Appendix J); such checks are mentioned below only where a proof depends on them.

2. Dual degree, Moore graphs, and the Hoffman–Singleton graph

This section computes the score of every Moore graph of diameter two (Theorem 2.2). WOW-284 holds on these graphs exactly for degree at most three, so the Hoffman–Singleton graph is a strict counterexample on 50 vertices. We also give this graph in coordinates (Proposition 2.3); the smaller counterexamples of Section 4 are its induced subgraphs.

2.1. Dual degree and the size of a radius-two ball.

When triangles and four-cycles are absent, dual degree measures the size of a radius-two ball relative to the degree of its center. The same count explains why a regular graph of girth at least five and diameter two is a Moore graph.

A distance sphere consists of the vertices at a specified distance from its center. For $v \in V(G)$, write $\Gamma_i(v)$ for the distance-i sphere and $B_2(v) = \Gamma_0(v) \cup \Gamma_1(v) \cup \Gamma_2(v)$ for the radius-two ball.

Proposition 2.1 (Second-degree identity)**.** *If G contains no triangle and no 4-cycle, then*

$$|B_2(v)| = 1 + \sum_{u \in N(v)} d(u), \qquad d^*(v) = \frac{|B_2(v)| - 1}{d(v)}.$$

Proof. For distinct neighbors u, w of v, the sets $N(u) \setminus \{v\}$ and $N(w) \setminus \{v\}$ are disjoint; an intersection would form a 4-cycle, and a member in $N(v)$ would form a triangle. These sets partition $\Gamma_2(v)$, so

$$|\Gamma_2(v)| = \sum_{u \in N(v)} (d(u) - 1).$$

Adding the center and its neighbors proves the first identity, which is Backelin's Lemma 2.1 [10]. Dividing by $d(v)$ gives the second identity. □

For a k-regular graph of girth at least five, Proposition 2.1 gives $|B_2(v)| = k^2 + 1$ for every v; hence $n \geq k^2 + 1$, with equality exactly when the diameter is at most two (the *Moore bound*). A degree-k *Moore graph of diameter two* is a k-regular graph of diameter two on $k^2 + 1$ vertices. At most $1 + k + k(k-1)$ vertices lie within distance two of a vertex, with equality only when no triangle or 4-cycle is present; so in such a graph adjacent vertices have no common neighbor and nonadjacent vertices have exactly one.

2.2. When Moore graphs violate WOW-284.

Theorem 2.2. *Let M be a degree-k Moore graph of diameter two, $k \geq 2$. Then*

$$|V(M)| = k^2 + 1, \qquad g(M) = 5, \qquad \delta^*(M) = k,$$

$$\lambda_{\min}(D(M)) = -\frac{3 + \sqrt{4k-3}}{2},$$

and

$$\Phi(M) = k - \frac{3 + \sqrt{4k-3}}{2}.$$

Thus M satisfies WOW-284 exactly for $k \leq 3$, with equality exactly at $k = 3$.

Proof. Equality in the Moore bound fixes the common-neighbor counts: adjacent vertices have no common neighbor, and nonadjacent vertices have exactly one. These counts also give a five-cycle through every edge. Given an edge uv, choose $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$. The vertices x, y cannot be adjacent, since that would create a four-cycle; their unique common neighbor therefore completes a five-cycle through uv. The common-neighbor counts give the matrix identity

$$A^2 = (k-1)I - A + J.$$

On $\mathbf{1}^\perp$, the nonprincipal adjacency eigenvalues are the roots

$$r, s = \frac{-1 \pm \sqrt{4k-3}}{2}.$$

If m_r, m_s are their multiplicities, then

$$m_r + m_s = k^2, \qquad k + m_r r + m_s s = 0.$$

Consequently

$$m_r = \frac{k(k\sqrt{4k-3}+k-2)}{2\sqrt{4k-3}}, \qquad m_s = \frac{k(k\sqrt{4k-3}-k+2)}{2\sqrt{4k-3}},$$

and both roots occur. To obtain the distance eigenvalues, use the fact that every nonedge has distance two. Thus $D = 2J - 2I - A$, and the least distance eigenvalue is $-2 - r = -(3 + \sqrt{4k-3})/2$. Since regularity gives $\delta^* = k$, the sign of $\Phi(M)$ is that of $2k - 3 - \sqrt{4k-3}$. For $k \geq 2$ both terms are positive, so it is the sign of $(2k-3)^2 - (4k-3)$, and comparison with the dual degree reduces to

$$(2k-3)^2 - (4k-3) = 4(k-1)(k-3)$$

This gives the stated threshold. □

2.3. **The Hoffman–Singleton graph in coordinates.** We use two families of vertices, with coordinates taken modulo five. All subscripts below lie in $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$. Let

$$V(M) = \{P_{i,j} : i, j \in \mathbb{F}_5\} \dot\cup \{Q_{k,\ell} : k, \ell \in \mathbb{F}_5\},$$

with edges

$$P_{i,j} \sim P_{i,j\pm1},$$
$$Q_{k,\ell} \sim Q_{k,\ell\pm2},$$
$$P_{i,j} \sim Q_{k,ik+j}.$$

This is Hafner's affine-coordinate form of the Hoffman–Singleton graph after a minor reindexing [11].

Proposition 2.3. *The coordinate construction is a simple connected* 7*-regular graph on* 50 *vertices. Adjacent pairs have no common neighbor and nonadjacent pairs have exactly one. Consequently it has girth five, diameter two, and*

$$\operatorname{Spec} D(M) = \{91^{(1)}, 1^{(21)}, (-4)^{(28)}\}.$$

In particular, $\delta^*(M) = 7$ *and* $\Phi(M) = 3$.

Proof. The neighborhoods are

$$N(P_{i,j}) = \{P_{i,j-1}, P_{i,j+1}\} \cup \{Q_{k,ik+j} : k \in \mathbb{F}_5\},$$
$$N(Q_{k,\ell}) = \{Q_{k,\ell-2}, Q_{k,\ell+2}\} \cup \{P_{i,\ell-ik} : i \in \mathbb{F}_5\}.$$

Each neighborhood contains seven distinct vertices. For the common-neighbor count, first consider two P-vertices in the same layer, meaning that their first coordinates agree. The 5-cycle contributes no common neighbor for an adjacent pair and one for a nonadjacent pair; a common Q-neighbor would force their second coordinates to agree. For vertices in distinct P-layers, the equation $(i - i')k = j' - j$ determines a unique common Q-neighbor. The Q-vertices satisfy the same rule: within a layer the 5-cycle uses steps ±2, and between layers there is a unique common P-neighbor. Finally, for a cross pair $P_{i,j}, Q_{k,\ell}$, put $r = \ell - (ik + j)$. The pair is adjacent when $r = 0$; it has one common P-neighbor when $r \in \{\pm1\}$, and one common Q-neighbor when $r \in \{\pm2\}$. These cases exhaust the five residues. Since every nonadjacent pair has a common neighbor, the graph is connected of diameter two; it is 7-regular on $50 = 7^2 + 1$ vertices, hence a degree-7 Moore graph of diameter two. The girth and diameter assertions now follow from Theorem 2.2. For degree 7, the multiplicity equations in its proof give adjacency multiplicities 28 at 2 and 21 at -3, and hence the displayed distance spectrum. □

3. Distance spectra from adjacency eigenvalues

We now express the distance matrix in terms of the adjacency matrix. Let A_i have entry one on pairs of vertices at distance i, and zero elsewhere. In diameter three, the girth assumption expresses the distance-two matrix as a quadratic in the adjacency matrix. The distance-three matrix is what remains of the all-ones matrix after the closer pairs are removed. The nonprincipal distance eigenvalues therefore follow from the adjacency eigenvalues. We use this reformulation to derive spectral obstructions to WOW-284: it gives the distance spectra of the regular examples in Section 4 and underlies the order bounds of Section 6. The end of this section identifies its equality case.

Theorem 3.1 (Diameter-three score formula)**.** *Let G be connected, k-regular, of girth at least five and diameter three, with adjacency matrix A and order n. Then*

$$D = 3J + (k-3)I - 2A - A^2,$$
$$D + kI = 3J + (2k-2)I - (A+I)^2.$$

The principal distance eigenvalue is $3n - k^2 - k - 3$*, and a nonprincipal adjacency eigenvalue* θ *gives the distance eigenvalue*

$$\mu(\theta) = k - 2 - (\theta + 1)^2.$$

Consequently

$$\boxed{\Phi(G) = 2k - 2 - \max_{\theta \neq k}(\theta + 1)^2.}$$

Thus G *is a strict counterexample exactly when every nonprincipal adjacency eigenvalue* θ *satisfies*

$$|\theta + 1| < \sqrt{2k - 2}$$

Proof. We first count pairs at distance two. Girth at least five means that distinct vertices have at most one common neighbor, and adjacent vertices have none. The diagonal of the squared adjacency matrix counts the degree, so $A_2 = A^2 - kI$. Because the diameter is three, the other distinct nonadjacent pairs are at distance three; thus $A_3 = J - I - A - A_2$. Putting these expressions into $D = A + 2A_2 + 3A_3$ proves both identities.

To compute the spectrum, apply these identities to adjacency eigenvectors. Regularity and the identities make the constant vector an eigenvector of D. Its eigenvalue is positive and is the Perron root, because all off-diagonal entries of the distance matrix are positive. On $\mathbf{1}^\perp$, J acts as zero, so each nonprincipal adjacency eigenvalue gives the stated distance eigenvalue. Taking the least of these values and using $\delta^* = k$ gives the score formula. □

Corollary 3.2 (Bipartite obstruction)**.** *A connected* k*-regular bipartite graph of girth at least five and diameter three is not a strict counterexample for* $k \geq 3$*.*

Proof. The nonprincipal eigenvalue $-k$ satisfies

$$(-k + 1)^2 - (2k - 2) = (k - 1)(k - 3) \geq 0 \qquad (k \geq 3).$$

Thus the eigenvalue $-k$ lies on the boundary of the shifted window when $k = 3$, and outside it when $k > 3$. In either case the strict window condition fails. □

3.1. **The equality boundary and a graph attaining it.** The boundary of the shifted adjacency window corresponds to equality in WOW-284. The following criterion identifies this boundary, and an explicit graph shows that it can be attained.

Theorem 3.3 (Equality boundary)**.** *Let* G *be connected,* k*-regular, of girth at least five and diameter three. Then*

$$\Phi(G) = 0 \quad \Longleftrightarrow \quad \max_{\theta \neq k} |\theta + 1| = \sqrt{2k - 2}.$$

Equivalently, $D + kI$ *is positive semidefinite and singular. When* $\Phi(G) = 0$ *and* $2k - 2$ *is not a square, the two boundary adjacency eigenvalues occur with equal multiplicity, so the distance eigenvalue* $-k$ *has even multiplicity. If* $2k - 2$ *is a square, then* $k = 2r^2 + 1$ *for some integer* r*.*

Proof. This is immediate from Theorem 3.1. In the nonsquare case, the two boundary values are algebraic conjugates, so their multiplicities in the integral characteristic polynomial agree. In the square case, $2k - 2 = s^2$ forces s even. □

Jørgensen's 9-regular order-96 graph of girth five and diameter three [12] is an exact equality case: $\delta^* = 9$ and $\lambda_{\min}(D) = -9$ with multiplicity eight. Here $2k - 2 = 16$ is a square, and the boundary adjacency eigenvalues 3 and -5 occur with multiplicities 7 and 1, so the equal-multiplicity conclusion of Theorem 3.3 needs the nonsquare hypothesis.

4. Smaller counterexamples from the Hoffman–Singleton graph

We obtain smaller counterexamples from these coordinates in two ways. First we delete a Petersen subgraph, followed by one or two further vertices. We also take the graph induced by a distance-two sphere; this second subconstituent gives another regular example.

Theorem 4.2 shows that these graphs, on 38, 39, 40 and 42 vertices, are strict counterexamples. The regular ones, of orders 40 and 42, have diameter three, so Theorem 3.1 computes their distance spectra from their adjacency spectra; the irregular ones are certified by exact computation. We first treat the second subconstituent of an arbitrary Moore graph (Theorem 4.1).

Let

$$\mathcal{P} = \{P_{0,j}, Q_{0,j} : j \in \mathbb{F}_5\}.$$

The induced graph $M[\mathcal{P}]$ is a Petersen graph. Put

$$R = M - \mathcal{P}, \quad H_{39} = R - P_{1,0}, \quad H_{38} = R - \{P_{1,0}, P_{1,1}\},$$

and let X_{42} be the second subconstituent of $P_{0,0}$, namely the graph induced by the vertices at distance two from it.

4.1. **Distance-two spheres in Moore graphs.**

Theorem 4.1 (Moore second subconstituents)**.** *Let M be a degree-K Moore graph of diameter two, $K \geq 3$, and let X be the graph induced by $\Gamma_2(v)$ for a fixed vertex v. Then X has order $K(K-1)$, degree $K-1$, girth at least five, diameter three, and*

$$\lambda_{\min}(D(X)) = -\frac{5+\sqrt{4K-3}}{2}.$$

Consequently

$$\Phi(X) = K - 1 - \frac{5+\sqrt{4K-3}}{2},$$

which is positive exactly for integers $K \geq 6$.

Each vertex in the second sphere loses its unique neighbor in the first sphere. The order, degree and girth assertions follow directly. For the spectrum, we group second-sphere vertices by their neighbors in the first sphere and split the adjacency operator accordingly. We then check that the induced graph has diameter three. Appendix A gives the complete proof and the multiplicity calculation.

4.2. **The explicit counterexamples.**

Theorem 4.2. *The following are strict counterexamples.*

G	$\lvert V(G)\rvert$	$\delta^*(G)$	$\lambda_{\min}(D(G))$
H_{38}	38	17/3	$-3-\sqrt{7}$
H_{39}	39	35/6	$> -35/6$
R	40	6	-5
X_{42}	42	6	-5
M	50	7	-4

The entry for H_{39} records an exact strict lower bound obtained from positive definiteness of $6D + 35I$; it is not a decimal approximation to the least eigenvalue. Moreover, all 40 labeled singleton deletions of R and all 120 labeled deletions of the endpoints of an edge of R are strict counterexamples; within each family, the distance characteristic polynomial is constant.

Proof. For the regular examples, we obtain the distance spectra from their adjacency spectra. The graph R is the Anstee–Robertson graph, the unique $(6,5)$-cage. Here a cage is a regular graph of the prescribed degree and girth with the fewest possible vertices. O'Keefe and Wong proved minimality, and Wong proved uniqueness [13, 14]; its realization as a Petersen deletion of the Hoffman–Singleton graph also appears in [15, pp. 262–263]. We derive its spectrum by separating the deleted and surviving vertices. Write B for the adjacency matrix of the deleted Petersen graph, $A_R = A(R)$, and C for the 10×40 matrix recording edges between $\mathcal{P}$ and R. A vertex of R has at most one neighbor in $\mathcal{P}$: two adjacent neighbors would make a triangle, while two nonadjacent Petersen vertices already have their unique common neighbor in $\mathcal{P}$. There are $10(7-3) = 40$ edges between the two parts. Thus each vertex of R has exactly one neighbor in $\mathcal{P}$, and R is 6-regular.

The Petersen identity $B^2 = 2I - B + J$ and the blocks of the Moore identity give

$$CC^{\mathsf{T}} = 4I, \qquad A_R C^{\mathsf{T}} = J_{40,10} - C^{\mathsf{T}} - C^{\mathsf{T}} B.$$

The first identity shows that the image of C^{T} has dimension ten. The second transfers Petersen eigenvectors to this image and determines the adjacency action there. Indeed, if $Bz = \beta z$ and $z \perp \mathbf{1}_{10}$, then $J_{40,10} z = 0$, so $A_R C^{\mathsf{T}} z = -(1+\beta) C^{\mathsf{T}} z$, and $C^{\mathsf{T}} z \neq 0$ because $CC^{\mathsf{T}} = 4I$; since $C^{\mathsf{T}} \mathbf{1}_{10} = \mathbf{1}_{40}$, the same identity gives $A_R \mathbf{1}_{40} = (10 - 1 - 3)\mathbf{1}_{40}$. The Petersen eigenvalues $1^{(5)}$ and $(-2)^{(4)}$ give eigenvalues $(-2)^{(5)}$ and $1^{(4)}$ on this image; the constant vector gives the eigenvalue 6. Its orthogonal complement is $\ker C$. Since $\mathbf{1}_{40} = C^{\mathsf{T}} \mathbf{1}_{10}$, J vanishes on $\ker C$, and $\ker C$ is A_R-invariant because the symmetric matrix A_R preserves the image of C^{T}. On this 30-dimensional space, the block $A_R^2 + C^{\mathsf{T}} C = 6I - A_R + J$ gives

$$A_R^2 + A_R - 6I = 0.$$

Hence its eigenvalues are 2 and -3. The dimension and trace equations give multiplicities 18 and 12, respectively: if a, b are these multiplicities, then $a + b = 30$ and $6 + 4 \cdot 1 + 5 \cdot (-2) + 2a - 3b = \operatorname{tr} A_R = 0$. We obtain

$$\operatorname{Spec} A_R = \{6^{(1)}, 2^{(18)}, 1^{(4)}, (-2)^{(5)}, (-3)^{(12)}\}.$$

To apply the distance polynomial, we must also establish the diameter. Let $x, y \in V(R)$ be nonadjacent. If their common neighbor in M survives, they remain at distance two. Otherwise they share a deleted Petersen neighbor. Choose $a \in N_R(x)$. Girth five makes a, y nonadjacent, and makes the Petersen neighbor of a different from that of y. Their unique common neighbor therefore lies in R, giving a path of length three from x to y. Each deleted vertex has four neighbors in R, so pairs whose common neighbor was deleted do occur. Thus R is connected with diameter three. Theorem 3.1 now gives

$$\operatorname{Spec} D(R) = \{75^{(1)}, 3^{(5)}, 0^{(16)}, (-5)^{(18)}\}, \qquad \Phi(R) = 1.$$

For X_{42}, the proof of Theorem 4.1 (Appendix A) gives diameter three and $\operatorname{Spec} A(X_{42}) = \{6^{(1)}, 2^{(21)}, (-1)^{(6)}, (-3)^{(14)}\}$, so Theorem 3.1 gives

$$\operatorname{Spec} D(X_{42}) = \{81^{(1)}, 4^{(6)}, 0^{(14)}, (-5)^{(21)}\}.$$

The classical second-subconstituent identification and adjacency spectrum are recorded in [16, Table 3, p. 265]. For the irregular examples, we check strictness directly. A degree count for H_{38} gives $\delta^* = 17/3$. The characteristic-polynomial factor $x^2 + 6x + 2$ has the root $-3 - \sqrt{7}$, and exact Sturm isolation shows that no smaller root occurs. For H_{39}, the exact matrix $6D + 35I$ is positive definite, giving the required strict eigenvalue bound. The exact checks verify the graph hypotheses, distances and dual degrees. Sturm counts isolate the real roots, while rational LDL^{T} factorizations determine definiteness by an invertible triangular change of coordinates. These certificates are checked by exact computation. The characteristic polynomials for the two labeled deletion families are recorded in Appendix B. The labeled deletion families are enumerated exhaustively by computer. No transitivity or numerical root ordering is used. Fixed graph6, adjacency-list, and edge-list records of these graphs are part of the data described in Appendix J. □

5. The diameter of regular counterexamples

Theorem 3.1 applies only in diameter three. This section shows that diameter three is the main case: a regular strict counterexample has diameter two, three or four, and diameter four requires degree at least ten (Corollary 5.4). Since Theorem 2.2 settles Moore graphs, the order bounds of Section 6 concentrate on diameter three. A geodesic, or shortest path, gives an upper bound on the least distance eigenvalue. We begin with the geodesic bound.

Lemma 5.1. *For every connected graph,*

$$\lambda_{\min}(D(G)) \le -\operatorname{diam}(G).$$

Proof. For a diametral pair u, v, the Rayleigh quotient of $e_u - e_v$ is $-d_G(u, v)$. □

Theorem 5.2 (Endpoint-neighborhood obstruction)**.** *Let G be any connected finite simple graph, and let u, v be vertices at distance $\ell = d_G(u, v) \ge 5$. Put $p = d(u)$ and $q = d(v)$. Then*

$$\boxed{\lambda_{\min}(D(G)) \le p + q - 2 - \sqrt{(p-q)^2 + pq(\ell-2)^2}.}$$

If δ is the ordinary minimum degree, then

$$\boxed{\lambda_{\min}(D(G)) \le -\delta(\ell - 4) - 2.}$$

Consequently every strict WOW-284 counterexample satisfies

$$\Delta > \delta(\ell - 4) + 2,$$

where Δ is the ordinary maximum degree. In particular, every regular strict counterexample has diameter at most four.

Proof. Choose a test vector x with opposite signs on the two endpoint neighborhoods, which are disjoint. Give weight $a > 0$ to $N(u)$, weight $-b < 0$ to $N(v)$, and zero elsewhere. Distances within a neighborhood are at most two; distances between the neighborhoods are at least $\ell - 2$. The cross-neighborhood terms have negative weights. Thus both distance estimates give an upper bound on the Rayleigh quotient:

$$x^{\mathsf{T}} D(G) x \le 2p(p-1)a^2 + 2q(q-1)b^2 - 2(\ell-2)pq\,ab, \qquad \|x\|^2 = pa^2 + qb^2,$$

and $(y_1, y_2) = (a\sqrt{p}, b\sqrt{q})$ gives a Rayleigh quotient for

$$\begin{pmatrix} 2(p-1) & -(\ell-2)\sqrt{pq} \\ -(\ell-2)\sqrt{pq} & 2(q-1) \end{pmatrix}.$$

The least eigenvalue gives the first bound. Its eigenvector can be chosen with both coordinates positive, as required by our sign assignment. To replace the endpoint degrees by the minimum degree, write $p = \delta + \alpha$, $q = \delta + \beta$, with $\alpha, \beta \geq 0$, and put $t = \ell - 2$. The right-hand side of the identity

$$(p-q)^2 + pqt^2 - (p+q+\delta(t-2))^2 = (t-2)\{\delta t(\alpha+\beta) + (t+2)\alpha\beta\}$$

is nonnegative. Since $p + q + \delta(t-2) > 0$, comparison of the nonnegative square roots gives the second bound. Finally $\delta^*(G) \leq \Delta$. For a k-regular graph $\Delta = \delta = k$, and $k > k(\ell-4) + 2$ fails for every $\ell \geq 5$; hence no two vertices of a regular strict counterexample are at distance five or more. □

Theorem 5.3 (Diameter four)**.** *Let G be connected, k-regular, of girth at least five and diameter four. Then*

$$\boxed{\lambda_{\min}(D(G)) \leq -\frac{7+\sqrt{16k+1}}{2}.}$$

Consequently no such graph of degree $2 \leq k \leq 9$ is a strict counterexample.

Proof. Choose u, v at distance four and put $U = N(u)$, $V = N(v)$. For fixed $a \in U$, every $b \in V$ with $d_G(a,b) = 2$ has a common neighbor in $N(a) \setminus \{u\}$. Distinct such vertices b, b' require distinct common neighbors, since otherwise $b - c - b' - v - b$ is a 4-cycle. Thus at most $k-1$ vertices of V are at distance two from each a. No pair in $U \times V$ is adjacent, and every remaining pair has distance at least three. Therefore

$$\sum_{a\in U, b\in V} d_G(a,b) \geq 2k^2 + k.$$

Use this cross-distance estimate in a test vector. For $\alpha, \beta > 0$, assign weights $\alpha, \beta, -\alpha, -\beta$ to u, U, v, V, respectively. Count each unordered pair with the product of its two weights, then double the sum. The result is

$$\frac{x^{\mathsf{T}} D(G) x}{x^{\mathsf{T}} x} \leq \frac{-4\alpha^2 - 4k\alpha\beta - 3k\beta^2}{\alpha^2 + k\beta^2}.$$

After setting $y_1 = \alpha$ and $y_2 = \sqrt{k}\,\beta$, the right-hand side is the Rayleigh quotient of

$$\begin{pmatrix} -4 & -2\sqrt{k} \\ -2\sqrt{k} & -3 \end{pmatrix},$$

whose least eigenvalue is the displayed value. A minimizing vector can be chosen with positive coordinates. The strict comparison with $-k$ holds for $2 \leq k \leq 9$; for $4 \leq k \leq 9$, where $2k - 7 > 0$, it is $k^2 - 11k + 12 < 0$. □

Corollary 5.4 (Regular trichotomy)**.** *Every regular strict counterexample has one of the following forms:*

(1) diameter two, hence a Moore graph;

(2) diameter three, with $|\theta + 1| < \sqrt{2k-2}$ for every nonprincipal adjacency eigenvalue θ;

(3) diameter four, with degree at least ten.

There are no regular strict counterexamples of diameter at least five.

Proof. Girth excludes diameter one; in diameter two, Proposition 2.1 gives $n = k^2 + 1$; diameters three, four and at least five are Theorems 3.1, 5.3 and 5.2. □

6. Order bounds from the spectral window

For a regular graph of girth at least five whose nonprincipal adjacency eigenvalues lie in the window of Theorem 3.1, traces of polynomials in the adjacency matrix bound the order. This section proves four such bounds. The fourth spectral moment gives $n < B_k$ (Proposition 6.1). No polynomial of higher degree in the nonbacktracking basis improves this bound, which holds under the closed window without any diameter assumption (Theorem 6.4). Evaluating the optimal polynomial at the adjacency matrix gives an integral matrix and a sharper bound in diameter three (Theorem 6.5); for degree at least six, a case analysis of this matrix gives the three-to-one bound (Theorem 6.6). Section 7 applies these bounds in low degree.

6.1. The fourth-moment bound.

We first derive an order bound from the fourth spectral moment. Let G satisfy the hypotheses of Theorem 3.1, and write its nonprincipal adjacency eigenvalues as θ_i. Put $y_i = \theta_i + 1$.

Proposition 6.1 (Fourth-moment identity)**.** *One has*

$$\sum_{i=1}^{n-1}(2k-2-y_i^2)(y_i+1)^2=(k+2)\big((k+2)(k^2+3)-6n\big).$$

Every strict counterexample therefore satisfies

$$\boxed{n<B_k:=\frac{(k+2)(k^2+3)}{6}.}$$

Proof. The shifted WOW window makes every summand nonnegative, so the identity will bound the order. With $y=\theta+1$, each summand is a quartic in θ:

$$(2k-2-y^2)(y+1)^2=-\theta^4-6\theta^3+(2k-15)\theta^2+(8k-20)\theta+8k-12.$$

Girth at least five gives $\operatorname{tr}A=\operatorname{tr}A^3=0$, $\operatorname{tr}A^2=nk$ and $\operatorname{tr}A^4=nk(2k-1)$, since a closed 4-walk must retrace an edge. Explicitly, a closed 4-walk at u either returns to u after two steps (k^2 walks) or has the form $u\,a\,c\,a\,u$ with $c\neq u$ ($k(k-1)$ walks); any other closed 4-walk would traverse a 4-cycle. Summed over all n eigenvalues, the quartic gives

$$-nk(2k-1)+(2k-15)nk+(8k-12)n=-6n(k+2);$$

the principal eigenvalue $\theta=k$, that is $y=k+1$, contributes $-(k+2)^2(k^2+3)$, and subtracting it gives the identity. In a strict counterexample, each factor $2k-2-y_i^2$ is positive. To obtain a strict order bound, we must also exclude a zero sum: otherwise every nonprincipal adjacency eigenvalue would equal -2, and $\operatorname{tr}A=0$ would give $k-2(n-1)=0$, or $n=(k+2)/2$, incompatible with the elementary bound $n\geq k+1$ for a simple k-regular graph. □

6.2. **Nonbacktracking polynomials and the optimal bound.** The summand in the proof of Proposition 6.1 is $-6(k+2)f_*(\theta)$ for the quartic f_* of Theorem 6.4. We now ask whether a higher-degree polynomial can improve the fourth-moment bound. We minimize the order bound subject to the coefficient and sign conditions below; this is the one-variable nonbacktracking linear program (LP). To prove its optimum, we give an attaining polynomial and a positive weighted sum of point evaluations, the dual measure, that bounds every admissible polynomial. The same measure determines when equality can hold.

The linear program uses the following basis of nonbacktracking polynomials; they also express the distance matrix in higher diameter.

Proposition 6.2 (Higher-diameter transfer)**.** *Let G be connected and k-regular, with diameter d and girth at least $2d-1$. Let A_i denote the distance-i matrix, and define*

$$F_0=1,\quad F_1=x,\quad F_2=x^2-k,\quad F_i=xF_{i-1}-(k-1)F_{i-2}\quad(i\geq 3).$$

Then $A_i=F_i(A)$ for $0\leq i\leq d-1$, and

$$D=dJ+q_d(A),\qquad q_d(x)=\sum_{i=0}^{d-1}(i-d)F_i(x).$$

In particular,

$$q_3(x)=k-3-2x-x^2,$$
$$q_4(x)=-x^3-2x^2+(2k-4)x+2k-4.$$

Proof. The polynomial matrices count nonbacktracking walks: walks that never immediately reverse an edge at an internal step [8, Theorem 1]. Under the girth condition, these counts agree with the distance matrices. For $1\leq i\leq d-1$, a repeated vertex in such a walk would produce a cycle of length at most i, so the walk is a path. A shorter path between the same endpoints would produce a cycle of length at most $2i-1$; two distinct paths of length i would produce one of length at most $2i$. These lengths are below $2d-1$, so neither is possible. Hence there is exactly one such walk when the endpoints have distance i, and none otherwise. The case $i=0$ is immediate, giving $A_i=F_i(A)$ for $0\leq i\leq d-1$. In the sum $D=\sum_{i=0}^{d}iA_i$, the identity $A_d=F_d(A)$ need not hold. Eliminate A_d using $J=\sum_{i=0}^{d}A_i$. This proves the formula within the established distance-polynomial framework [6, 7]. □

Definition 6.3. Let $F_i=F_i^{(k)}$ be the nonbacktracking polynomials from Proposition 6.2, and set

$$I_k=[-1-\sqrt{2k-2},-1+\sqrt{2k-2}].$$

A finite polynomial $f=\sum_i f_iF_i$ is *admissible* if

$$f_0>0,\qquad f_i\geq 0\quad(i\geq 5),\qquad f(x)\leq 0\quad(x\in I_k).$$

For a connected k-regular graph of girth at least five whose nonprincipal spectrum lies in I_k, one has $\operatorname{tr} F_i(A) = 0$ for $1 \le i \le 4$, while $\operatorname{tr} F_i(A) \ge 0$ for $i \ge 5$, since these traces count closed nonbacktracking walks [8, Theorem 1 and Corollary 1]. The coefficient conditions therefore give $nf_0 \le \operatorname{tr} f(A)$. On the other hand, $f(\theta) \le 0$ for every nonprincipal eigenvalue, so $\operatorname{tr} f(A) \le f(k)$. Thus

$$nf_0 \le \operatorname{tr} f(A) \le f(k), \qquad n \le \frac{f(k)}{f_0}.$$

Theorem 6.4 (Exact LP ceiling and rigidity)**.** *For every integer $k \ge 4$ and every admissible f,*

$$\boxed{\frac{f(k)}{f_0} \ge B_k = \frac{(k+2)(k^2+3)}{6}.}$$

Equality holds if and only if f is a positive scalar multiple of

$$\boxed{f_*(x) = \frac{(x+2)^2(x^2+2x-(2k-3))}{6(k+2)}.}$$

Thus increasing the polynomial degree cannot improve this one-variable LP bound. Consequently, any connected k-regular graph of girth at least five whose nonprincipal spectrum lies in I_k satisfies $n < B_k$.

Proof. First expand the stated quartic in the nonbacktracking basis to check that it attains the bound:

$$\begin{aligned} 6(k+2)f_*(x) = {} & 6(k+2)F_0(x) + 2(2k+7)F_1(x) \\ & + (k+13)F_2(x) + 6F_3(x) + F_4(x). \end{aligned}$$

The constant coefficient is positive, and there are no coefficients above degree four. On I_k, the factor $(x+1)^2 - (2k-2)$ is nonpositive, while $(x+2)^2 \ge 0$; hence f_* is admissible and $f_*(k) = B_k$.

To prove that no admissible polynomial does better, we use a positive measure supported on the two window endpoints and the interior zero of the quartic. Put $\omega = \sqrt{2k-2}$, $\xi_\pm = -1 \pm \omega$, and $\xi_0 = -2$, and define

$$\begin{aligned} w_- &= \frac{k(k+2)(2k^2-6-3(k-1)\omega)}{24(2k-3)}, \\ w_0 &= \frac{k(k-1)(k^2+3)}{6(2k-3)}, \\ w_+ &= \frac{k(k+2)(2k^2-6+3(k-1)\omega)}{24(2k-3)}. \end{aligned}$$

$w_-, w_0, w_+ > 0$. For the only nontrivial inequality, this follows from

$$(2k^2-6)^2 - 18(k-1)^3 = 2(k-3)(2k-3)(k^2+3) > 0.$$

Each delta denotes a unit point mass at its subscript. Integration against the measure $\mu = w_-\delta_{\xi_-} + w_0\delta_{\xi_0} + w_+\delta_{\xi_+}$ is therefore a weighted sum of three evaluations. This measure satisfies

$$\mu(1) = B_k - 1, \qquad \mu(F_i) = -F_i(k) \quad (1 \le i \le 4).$$

To check these identities, put $y = x + 1$. The support becomes $\{-\omega, -1, \omega\}$, so

$$\int y^j \, d\mu = (-1)^j w_0 + \big(w_+ + (-1)^j w_-\big)\omega^j,$$

$$w_+ + w_- = \frac{k(k+2)(k^2-3)}{6(2k-3)}, \qquad w_+ - w_- = \frac{k(k-1)(k+2)}{4(2k-3)}\,\omega.$$

Using $\omega^2 = 2k-2$, one finds $\int d\mu = k(k^2+2k+3)/6 = B_k - 1$ and

$$\int y \, d\mu = \tfrac{1}{6}k(k-1)(k+3), \qquad \int y^2 \, d\mu = \tfrac{1}{6}k(k-1)(k+1)(k+3),$$

$$\int y^3 \, d\mu = \tfrac{1}{6}k(k-1)(3k^2+4k-3), \qquad \int y^4 \, d\mu = \tfrac{1}{6}k(k-1)(2k^3+5k^2-2k-9).$$

Writing $F_1, \dots, F_4$ in powers of y and using $F_i(k) = k(k-1)^{i-1}$ gives $\mu(F_i) = -F_i(k)$. The measure cancels the first four nonconstant basis terms. For the remaining terms, define the slack $a_i = \mu(F_i) + F_i(k)$. We need this slack to be strictly positive for $i \ge 5$. Appendix C verifies this positivity: the first five values follow from exact polynomial expansions, and a Chebyshev bound handles every later term. For an arbitrary admissible polynomial, expansion in the nonbacktracking basis gives

$$\int f \, d\mu = B_k f_0 - f(k) + \sum_{i \ge 5} f_i a_i \ge B_k f_0 - f(k).$$

Here $\int f\,d\mu = \sum_i f_i \mu(F_i)$; by the identities above, $F_0(k) = 1$ and $\mu(F_i) = a_i - F_i(k)$ for $i \geq 5$, the terms $-f_i F_i(k)$, $i \geq 0$, add up to $-f(k)$. The coefficient condition and the slack positivity in Appendix C make the last sum nonnegative. Also, $f \leq 0$ on $\operatorname{supp}\mu$, so $\int f\,d\mu \leq 0$. These inequalities prove the lower bound for every admissible polynomial. Equality must hold in both inequalities. Since every high-degree slack is strictly positive, $f_i = 0$ for $i \geq 5$. The measure has positive weight at each support point, so f must vanish at $\xi_-, -2, \xi_+$. Its interior zero -2 has even multiplicity, because $f \leq 0$ on I_k. A polynomial of degree at most four with these zeros is a scalar multiple of f_*, and $f_0 > 0$ makes that scalar positive.

The polynomial bound includes the endpoints of I_k, so the trace bound gives only $n \leq B_k$. Equality $n = B_k$ would force every nonprincipal eigenvalue to be a zero of f_*, that is, to lie in $\{-2, -1 \pm \omega\}$; the trace and second-moment equations then give a negative or nonintegral multiplicity. Appendix C carries out this computation, so $n < B_k$ throughout the closed window. □

6.3. **The integral excess matrix.**

Theorem 6.5 (Integral optimal-slack bound)**.** *Let G be connected and k-regular with $k \geq 4$, of girth at least five and diameter three, and suppose*

$$|\theta + 1| < \sqrt{2k-2} \qquad (\theta \neq k)$$

for every adjacency eigenvalue θ. Define

$$g_k(x) = (x+2)^2\big((x+1)^2 - (2k-2)\big), \quad C_k = (k+2)^2(k^2+3), \quad h_k = 6(k+2),$$

$$\mathcal{S}_k = -g_k(A) + \frac{C_k}{n} J.$$

Then

$$\mathcal{S}_k \succeq 0, \qquad \mathcal{S}_k \mathbf{1} = 0, \qquad \operatorname{tr}\mathcal{S}_k = h_k(B_k - n).$$

Moreover,

$$\mathcal{E}_k = g_k(A) - (h_k+1)J + I$$

is a nonzero, symmetric, entrywise nonnegative integral matrix with zero diagonal and constant row sum

$$\varepsilon_{k,n} = C_k - (h_k+1)n + 1.$$

On $\mathbf{1}^\perp$, one has $\mathcal{S}_k = I - \mathcal{E}_k$. In particular,

$$\boxed{n \leq \left\lfloor \frac{(k+2)^2(k^2+3)}{6(k+2)+1} \right\rfloor}.$$

Proof. Apply the scaled optimal LP polynomial $g_k = 6(k+2)f_*$ from Theorem 6.4 to the adjacency matrix. The correction by the all-ones matrix makes $\mathcal{S}_k$ vanish on $\langle \mathbf{1} \rangle$. On a nonprincipal θ-eigenspace, its eigenvalue is

$$\big(2k - 2 - (\theta+1)^2\big)(\theta+2)^2 \geq 0.$$

To compute the diagonal, use the nonbacktracking expansion of g_k, whose constant coefficient is h_k. Girth at least five implies $(F_i(A))_{uu} = 0$ for every u and $1 \leq i \leq 4$. Thus

$$\operatorname{tr}\mathcal{S}_k = C_k - h_k n = h_k(B_k - n), \qquad (g_k(A))_{uu} = h_k.$$

We obtain a stronger order restriction from the integer off-diagonal entries. Put $z_{uv} = (g_k(A))_{uv} \in \mathbb{Z}$ for $u \neq v$, and set $a = C_k/n - h_k = (\mathcal{S}_k)_{uu}$. Nonnegativity of the 2×2 principal minor on $\{u, v\}$ gives

$$\left| \frac{C_k}{n} - z_{uv} \right| \leq a.$$

We must exclude $C_k/n - z_{uv} = a$, equivalently $z_{uv} = h_k$. This would imply $(e_u - e_v)^{\mathsf{T}} \mathcal{S}_k (e_u - e_v) = 0$, and positive semidefiniteness would then put $e_u - e_v \in \ker \mathcal{S}_k$. The strict spectral window gives

$$\ker \mathcal{S}_k = \langle \mathbf{1} \rangle \oplus E_{-2}(A).$$

Since $e_u - e_v \perp \mathbf{1}$, it would be a (-2)-eigenvector of A. Yet the u-coordinate of $A(e_u - e_v)$ is -1 if $u \sim v$ and 0 otherwise, never -2. Therefore

$$z_{uv} \geq h_k + 1.$$

This proves the stated entrywise properties of $\mathcal{E}_k$. Its row sum follows from $g_k(A)\mathbf{1} = C_k\mathbf{1}$; on $\mathbf{1}^\perp$, the term J vanishes and gives the claimed identity.

It remains to prove $\mathcal{E}_k \neq 0$. Otherwise the adjacency operator on the rational space $\mathbf{1}^\perp$ is annihilated by the quartic $g_k(x) + 1$, which is irreducible over $\mathbb{Q}$; this forces $4 \mid n - 1$, and $\operatorname{tr} A = 0$ then gives $n - 1 = 2k/3 < k$, which is impossible. Appendix C gives the details. Thus $\mathcal{E}_k \neq 0$. Its constant row sum is consequently a positive integer, so $\varepsilon_{k,n} \geq 1$, which is equivalent to the displayed order bound. □

6.4. **The three-to-one excess bound.** Up to a positive factor, the trace of $\mathcal{S}_k$ is the gap between the graph order and the scalar LP bound. Its entries contain further information: 2×2 principal minors constrain individual entries, and larger minors relate several of them. In the integral excess matrix, these constraints become arithmetic restrictions. For degree at least six they give the following bound.

Theorem 6.6 (Three-to-one excess bound)**.** *Under the hypotheses of Theorem 6.5, assume $k \geq 6$ and write $\varepsilon = \varepsilon_{k,n}$. The integral parameter*

$$r = 2\varepsilon - n - 2 = 2(k+2)^2(k^2+3) - (12k+27)n$$

satisfies

$$r > 0, \qquad n \leq 3r.$$

Consequently

$$\boxed{n \leq \left\lfloor \frac{3(k+2)^2(k^2+3)}{18k+41} \right\rfloor}.$$

Outline of the proof. Appendix D gives the complete case analysis. The 2×2 principal minors of $\mathcal{S}_k = I - \mathcal{E}_k + \rho J$, where $\rho = (\varepsilon_{k,n} - 1)/n$, bound the off-diagonal entries of $\mathcal{E}_k$. If $r \leq 0$, or if $n > 3r$, these bounds, together with a kernel argument or with a Gram-determinant estimate and a divisor check, make $\mathcal{E}_k$ the adjacency matrix of a simple graph whose complement is regular with least eigenvalue at least -2. Since $n \geq k^2 + 2 \geq 38$, the classification of such graphs [17], together with counting arguments, reduces the complement and its components to complete graphs, cocktail-party graphs and line graphs. The remaining parameter families are excluded by the radius-two bound, by divisibility conditions on $4k+9$ and $18k+41$, by the parity of kn, or because a rational invariant space on which the adjacency operator satisfies an irreducible quartic would have dimension not divisible by four.

7. Degree at least six and order at most fifty

We now combine the diameter restrictions of Section 5 with the order bounds of Section 6. Small degree gives another obstruction, because too few vertices are available to satisfy the spectral window: every regular strict counterexample has degree at least six (Theorem 7.1), and in degree six it has at most 50 vertices (Theorem 7.2). For the low-degree argument we also use the fourth-moment bound of Proposition 6.1.

7.1. Degree at least six.

Theorem 7.1. *Every connected regular strict counterexample to WOW-284 has degree at least six.*

Proof. Let the degree of G be k. Lemma 5.1 and strictness first give $\operatorname{diam}(G) < k$. This already excludes $k \leq 2$: the order satisfies $|V(G)| \geq 3$, and the girth hypothesis excludes completeness, so $\operatorname{diam}(G) \geq 2 \geq k$. For $k = 3$, the radius-two lower bound and $\operatorname{diam}(G) \leq 2$ force equality in the Moore bound. Theorem 2.2 then gives $\Phi(G) = 0$, contradicting the required $\Phi(G) > 0$.

For $k = 4$, the two possible diameters lead to different obstructions. Diameter two would require a degree-four Moore graph, but its adjacency multiplicities are nonintegral. For diameter three, Proposition 6.1 gives $n < B_4 = 19$. The radius-two ball has 17 vertices, and the third layer is nonempty, so $n = 18$. Every vertex then has exactly one vertex at distance three: the distance-three matrix is a perfect matching. Its -1-eigenspace $W = \ker(A_3 + I)$ is a nine-dimensional rational subspace of $\mathbf{1}^\perp$, spanned by the nine vectors $e_u - e_{u'}$, one for each pair $\{u, u'\}$ at distance three. By the proof of Theorem 3.1, A_3 is a polynomial in A and J, and $AJ = JA$. Since A commutes with A_3, the space W is A-invariant, and

$$A_3 = J + 3I - A - A^2 \quad \Longrightarrow \quad (A^2 + A - 4I)|_W = 0.$$

The polynomial $x^2 + x - 4$ is irreducible over $\mathbb{Q}$, so the characteristic polynomial of the restricted operator must be a power of this quadratic. The dimension of the rational space must therefore be even, a contradiction.

For $k = 5$, diameter two again fails the Moore multiplicity condition. Diameter four can be excluded directly: a diametral geodesic gives the principal submatrix $D(P_5)$. Its characteristic-polynomial factor $x^2 + 6x + 4$ supplies the eigenvalue $-3 - \sqrt{5} < -5$. Cauchy interlacing therefore contradicts strictness. In diameter three, Meringer's enumeration [18, p. 142] (the $(5,5)$-cages, of order 30, are the smallest 5-regular graphs of girth five) and Proposition 6.1 ($B_5 = 98/3$) leave $n \in \{30, 31, 32\}$. The handshake identity $5n = 2|E(G)|$ gives $n \equiv 0 \pmod 2$, leaving $n \in \{30, 32\}$. Both orders are excluded in Appendix E: at $n = 32$ by compressing the adjacency matrix to the distance layers of a vertex, which forces $\theta_2(A) > -1 + \sqrt{8}$ for the second-largest adjacency eigenvalue $\theta_2(A)$, contrary to Theorem 3.1; and at $n = 30$ by the four $(5,5)$-cages, each of which has a distance eigenvalue at most -5. □

7.2. **Order bounds in degrees six through nine.**

Theorem 7.2. *Every connected* 6*-regular strict counterexample to WOW-284 has order at most* 50*.*

Proof. First establish diameter three, as required by the order bound. Theorem 5.2 excludes diameter at least five, and Theorem 5.3 excludes diameter four. Diameter two would attain the Moore bound, giving order 37 and adjacency characteristic polynomial $(x-6)(x^2+x-5)^{18}$; its root sum is -12, contradicting $\operatorname{tr} A = 0$. Diameter one is excluded by the girth hypothesis. Thus the diameter is three. Strictness and Theorem 3.1 give $|\theta+1| < \sqrt{10}$ for every nonprincipal adjacency eigenvalue, so the hypotheses of Theorem 6.5 hold with $k = 6$, $C_6 = 8^2 \cdot 39 = 2496$ and $h_6 + 1 = 49$; already that theorem gives

$$n \le \left\lfloor \frac{8^2 \cdot 39}{49} \right\rfloor = 50;$$

Theorem 6.6 gives the same value, so this bound does not depend on the line-graph classification. □

Corollary 7.3 (Low-degree order windows)**.** *There is no regular strict counterexample of degree at most five. In degrees six through nine, every regular strict counterexample satisfies*

$$\begin{aligned} k = 6: &\quad n \le 50, \\ k = 7: &\quad n = 50 \text{ in diameter two, or } n \le 74 \text{ in diameter three}, \\ k = 8: &\quad n \le 108, \\ k = 9: &\quad n \le 150. \end{aligned}$$

Proof. The degree exclusion is Theorem 7.1, Theorem 5.2 excludes diameter at least five, and Theorem 5.3 removes diameter four for $k \le 9$. For diameter three, Theorem 6.6 gives, for $k = 6, 7, 8, 9$, respectively,

$$n \le 50,\ 75,\ 108,\ 150.$$

Since $7n = 2|E(G)|$, the degree-seven bound improves to 74. The diameter-two alternatives are determined by the Moore multiplicities: only the degree-seven, order-50 Hoffman–Singleton case survives. (For $k \in \{6, 8, 9\}$, $4k - 3 \in \{21, 29, 33\}$ is not a square, so the multiplicities in the proof of Theorem 2.2 are irrational.) □

Appendix D lists the corresponding order bounds through degree twenty, including the parity adjustment for odd degrees.

The degree-six bound still permits $n = 50$. Appendix F restricts the adjacency multiplicities and cycle counts at this order, and studies the signed graph obtained from the excess matrix. It also bounds the number of five-cycles through an edge, which gives a second, local proof that a 6-regular strict counterexample cannot have 51 vertices. These restrictions are necessary conditions; they leave the existence of a strict counterexample unresolved.

8. Vertex deletion from Moore graphs

Deleting vertices can increase the distances between those that remain. We identify these increases and use them to bound the change in the least distance eigenvalue. With these bounds we prove that the Hoffman–Singleton graph remains a strict counterexample after the deletion of any five vertices, while some six-vertex deletion is not strict (Theorem 8.6), and that a degree-57 Moore graph, if one exists, remains one after the deletion of any 56 vertices (Proposition 8.7). The full one- and two-deletion spectra are given in Appendix A.

Let M be a degree-k Moore graph of diameter two.

8.1. **Changed distances and a stability bound.**

Theorem 8.1 (Small-puncture normal form)**.** *Let M be a degree-k Moore graph of diameter two, let $S \subseteq V(M)$ have size $s \le k-1$, and put $H = M - S$. Then H is connected, has diameter at most three, and*

$$\boxed{\delta^*(H) = k - \frac{s}{k}.}$$

Let B be the surviving-vertex by deleted-vertex incidence matrix, with $B_{xz} = 1$ exactly when $x \sim_M z$. Then

$$\boxed{D(H) = 2(J - I) - A(H) + BB^{\mathsf{T}} - \operatorname{diag}(BB^{\mathsf{T}}).}$$

Proof. The case $s = 0$ is immediate, so assume $1 \le s \le k-1$. Let $x, y \in V(H)$ be nonadjacent in M, and suppose their unique common neighbor z lies in S. For every $a \in N_M(x) \setminus \{z\}$, the vertices a, y are nonadjacent and have a unique common neighbor b_a. Thus

$$x - a - b_a - y$$

is a length-three path. The $k-1$ paths are internally vertex-disjoint: if $b_a = b_{a'}$, there would be a four-cycle, and if $b_a = a'$, there would be a triangle through x; also $b_a \neq z$, since $z \sim x \sim a$. Apart from z, only $s - 1 \le k - 2$ vertices have been deleted. Each can meet at most one of the internally disjoint paths, so at least one path survives. Thus every pair that loses its length-two path is at distance exactly three in the surviving graph, giving the matrix formula.

For $x \in V(H)$, let $t_x = |N_M(x) \cap S|$. A deleted neighbor of x is adjacent to no surviving neighbor of x, by triangle-freeness; each other deleted vertex has at most one common neighbor with x. Thus

$$\sum_{y \in N_H(x)} t_y \le s - t_x.$$

Since $d_H(y) = k - t_y$, $d_H^*(x) = k - (k - t_x)^{-1} \sum_{y \in N_H(x)} t_y$. Consequently,

$$d_H^*(x) \ge k - \frac{s - t_x}{k - t_x} \ge k - \frac{s}{k},$$

where the last inequality follows from

$$\left(k - \frac{s-t}{k-t}\right) - \left(k - \frac{s}{k}\right) = \frac{t(k-s)}{k(k-t)} \ge 0.$$

The intersection bound

$$\left| \bigcap_{z \in S} \Gamma_2(z) \right| \ge k^2 + 1 - s(k+1) \ge 2$$

provides a surviving vertex x at distance two from every deleted vertex. This vertex attains the lower bound. For $z \in S$, let y_z be the unique common neighbor of x, z. It cannot be deleted: $y_z \in S$ would, by the choice of x, require $d_M(x, y_z) = 2$, contrary to $x \sim y_z$. All these common neighbors therefore survive. They need not be distinct: each deleted vertex contributes once to the sum of the neighbor-degree deficits. The lower bound is consequently attained. □

Proposition 8.2 (Deletion stability)**.** *Let G be connected with at least two vertices, let $S \subseteq V(G)$, and suppose $H = G - S$ is connected with at least two vertices. Put*

$$D_0 = D(G)[V(H)], \qquad E_S = D(H) - D_0,$$
$$a = \delta^*(G), \quad b = \delta^*(H), \quad \gamma = \Phi(G).$$

Then

$$\boxed{\Phi(H) \ge \gamma - (a - b) + \lambda_{\min}(E_S).}$$

Proof. The score $\Phi(H)$ is the least eigenvalue of $D(H) + bI$. We write this matrix as

$$D(H) + bI = (D_0 + aI) + E_S - (a - b)I.$$

The first term is a principal submatrix of $D(G) + aI$, whose least eigenvalue is γ. Principal-submatrix interlacing [19, Corollary 2.2] therefore gives $\lambda_{\min}(D_0 + aI) \ge \gamma$. Weyl's inequality bounds the least eigenvalue after adding E_S, and subtracting $(a - b)I$ shifts every eigenvalue by $-(a - b)$. This gives the stated bound. For Moore punctures, E_S is the explicit distance-increase matrix specified in Theorem 8.1. □

Corollary 8.3 (Uniform deletion stability)**.** *Under the hypotheses of Theorem 8.1, put*

$$t_x = |N_M(x) \cap S|, \qquad \tau(S) = \max_{x \in V(M-S)} t_x.$$

Then

$$\boxed{\Phi(M - S) \ge k - \frac{3 + \sqrt{4k-3}}{2} - \frac{s}{k} - \tau(S).}$$

In particular, the deletion is a strict counterexample whenever the right-hand side is positive. Since $\tau(S) \le s$, every deletion of s vertices is strict whenever

$$\boxed{s\left(1 + \frac{1}{k}\right) < k - \frac{3 + \sqrt{4k-3}}{2}.}$$

For $k > 3$, equivalently, every deletion of at most

$$r_k = \min\left\{k-1,\ \left\lceil \frac{k}{k+1}\left(k - \frac{3+\sqrt{4k-3}}{2}\right)\right\rceil - 1\right\}$$

vertices is strict. In particular, $r_7 = 2$; for a hypothetical degree-57 Moore graph, whose existence remains open [20], this estimate gives $r_{57} = 47$. Proposition 8.7 strengthens that guarantee to every deletion of at most 56 vertices.

Proof. By Theorem 8.1, the distance-increase matrix is

$$E_S = BB^{\mathsf{T}} - \operatorname{diag}(t_x).$$

Since $BB^{\mathsf{T}} \succeq 0$ and $\operatorname{diag}(t_x) \preceq \tau(S)I$, one has $\lambda_{\min}(E_S) \geq -\tau(S)$. The parent Moore graph has score

$$\Phi(M) = k - \frac{3+\sqrt{4k-3}}{2},$$

and Theorem 8.1 shows that deletion lowers the minimum dual degree by s/k. Proposition 8.2 now gives the first bound. The second follows from $\tau(S) \leq s$. The largest integer s within the allowed deletion range that lies strictly below the endpoint of this inequality is r_k. □

8.2. **Deletions inside a neighborhood.** The stability bound of Corollary 8.3 handles many deletion sets without computing their full distance spectra. When the deleted vertices share a surviving neighbor, the distance correction has a more specific form, which gives a stronger bound. This proves stability through three deletions for Hoffman–Singleton. Exact orbit certificates then cover sizes four and five and determine the universal radius.

Each deleted vertex contributes a correction clique on its surviving neighbors. If the deleted vertices have a common surviving neighbor, these cliques meet only at that vertex. This intersection pattern lets us compute the least eigenvalue of the correction explicitly.

Proposition 8.4 (Neighborhood punctures)**.** *Let M be a degree-k Moore graph of diameter two, $k \geq 2$. Let $w \in V(M)$, let $\varnothing \neq S \subseteq N_M(w)$, and put $s = |S| \leq k-1$ and $H = M - S$. The distance-increase matrix*

$$E_S = D(H) - D(M)[V(H)]$$

is the adjacency matrix of s copies of K_k meeting only at w, together with $k(k-s)$ isolated vertices. In particular,

$$\lambda_{\min}(E_S) = \frac{k-2-\sqrt{(k-2)^2+4s(k-1)}}{2},$$

and

$$\Phi(H) \geq k - \frac{3+\sqrt{4k-3}}{2} - \frac{s}{k} + \frac{k-2-\sqrt{(k-2)^2+4s(k-1)}}{2}.$$

Proof. The deleted set is independent, since a pair of its vertices joined by an edge would form a triangle with w. Thus each deleted vertex has all k of its neighbors in H. For distinct $z, z' \in S$, the Moore property gives

$$N_M(z) \cap N_M(z') = \{w\}.$$

By Theorem 8.1, every destroyed length-two path is replaced by a length-three path. Hence $(E_S)_{xy} = 1$, for distinct surviving vertices x, y, precisely when they belong to $N_M(z)$ for some $z \in S$. There is at most one such z. These neighborhoods therefore give the asserted cliques, with $k(k-s)$ remaining vertices.

We take the coordinate vector at the center and the indicator vector of all noncentral clique vertices. Their span is invariant under the correction matrix. After normalizing both vectors to have length one, the matrix on this span is the normalized symmetric quotient

$$\begin{pmatrix} 0 & \sqrt{s(k-1)} \\ \sqrt{s(k-1)} & k-2 \end{pmatrix}.$$

The other eigenvalues are -1 on the zero-sum space of the $k-1$ noncentral vertices of each clique, $k-2$ on the differences of their constant sums, and 0 on the isolated vertices. These spaces have dimensions $s(k-2)$, $s-1$, and $k(k-s)$, respectively, so the decomposition is complete. The smaller quotient root is at most -1, because its characteristic polynomial has value $(k-1)(1-s) \leq 0$ at -1. It is therefore the least eigenvalue of E_S. The final inequality follows from Proposition 8.2 and $\delta^*(H) = k - s/k$. □

8.3. **The Hoffman–Singleton radius.**

Corollary 8.5 (Analytic three-vertex stability)**.** *Let M be the Hoffman–Singleton graph. For every $S \subseteq V(M)$ with $|S| \le 3$,*

$$\Phi(M - S) > \frac{1}{7}.$$

In particular, every such puncture is a strict counterexample to WOW-284.

Proof. The graph M has score 3. If $|S| \le 2$, Corollary 8.3 gives $\Phi(M - S) \ge 3 - |S|/7 - |S| \ge 5/7$. Now let $|S| = 3$, and put

$$\tau = \max_{x \in V(M-S)} |N_M(x) \cap S|.$$

If $\tau \le 2$, the same corollary gives $\Phi(M - S) \ge 3 - 3/7 - 2 = 4/7$. If $\tau = 3$, some surviving vertex w is adjacent to all three deleted vertices. Proposition 8.4 gives

$$\Phi(M - S) \ge \frac{71 - 7\sqrt{97}}{14} > \frac{1}{7},$$

where the strict comparison follows from $49 \cdot 97 = 4753 < 4761 = 69^2$. □

Theorem 8.6 (Hoffman–Singleton robustness radius)**.** *Let M be the Hoffman–Singleton graph. Every induced graph $M - S$ with $|S| \le 5$ is a strict counterexample to WOW-284. This is sharp in the universal sense: there exists a six-vertex set whose deletion is not strict. Hence the universal vertex-deletion robustness radius is exactly five.*

Proof. Theorem 8.1 gives

$$\delta^*(M - S) = \frac{49 - |S|}{7}.$$

Corollary 8.5 proves strictness analytically for every set with $|S| \le 3$. Only sets of sizes four and five remain to be certified. We reduce the finite check using automorphisms, which are vertex permutations preserving adjacency. Deletion sets related by these permutations give isomorphic surviving graphs. Two explicitly stored permutations are verified edge by edge on the coordinate graph, and they generate a group of order 252000. The full orbit inventory on deletion sets of sizes $0, 1, 2, 3, 4, 5$ has counts

$$1, 1, 2, 4, 11, 33.$$

Thus the remaining sizes require 44 representatives. For each one, exact fraction arithmetic gives

$$7D(M - S) + (49 - |S|)I \succ 0.$$

To show that the radius cannot be increased, delete

$$\{P_{2,4}, P_{3,1}, P_{3,4}, Q_{2,1}, Q_{3,4}, Q_{4,4}\}.$$

The resulting graph has $\delta^* = 43/7$, while an exact LDL^{T} decomposition of $7D(M - S) + 43I$ has exactly one negative and no zero pivot. Since L is invertible, Sylvester's law of inertia makes $7D(M - S) + 43I$ indefinite, so the graph is not strict. □

Some six-vertex deletions remain strict: by Proposition G.1 in Appendix G, deleting any six neighbors of a vertex leaves a strict counterexample. Such deletions do not increase the universal radius, since the deletion in Theorem 8.6 fails.

8.4. **Arbitrary deletions from a degree-57 Moore graph.** For a hypothetical degree-57 Moore graph, Corollary G.2 in Appendix G uses Proposition 8.4 to show $\Phi(M - S) > 685/57$ whenever S lies in the neighborhood of a vertex and $|S| \le 56$. For an arbitrary deletion set, we group the correction cliques that pass through one surviving vertex. The remaining cliques can be bounded separately, giving the following improvement.

Proposition 8.7 (Splitting the distance correction at a vertex)**.** *Let M be a degree-k Moore graph of diameter two, $k \ge 2$, and let $S \subseteq V(M)$ have size $s \le k - 1$. Put $H = M - S$,*

$$E_S = D(H) - D(M)[V(H)], \qquad \tau = \max_{x \in V(H)} |N_M(x) \cap S|,$$

and, for $0 \le r \le k - 1$, define

$$f_k(r) = \frac{k - 2 - \sqrt{(k-2)^2 + 4r(k-1)}}{2}.$$

Then

$$\lambda_{\min}(E_S) \ge \max\{-\tau,\ f_k(\tau) - s + \tau\},$$

and

$$\Phi(H) \geq k - \frac{3+\sqrt{4k-3}}{2} - \frac{s}{k} + \max\{-\tau,\ f_k(\tau) - s + \tau\}.$$

In particular, if M has degree 57, then every $S \subseteq V(M)$ with $|S| \leq 56$ satisfies

$$\Phi(M - S) \geq \frac{286}{57} > 0.$$

Thus every such puncture is a strict counterexample to WOW-284.

Proof. For $s = 0$, one has $\tau = 0$, $f_k(0) = 0$, and $E_S = 0$, so the claims follow from Theorem 2.2. Assume $s \geq 1$. For $z \in S$, let b_z be the indicator vector of $N_M(z) \cap V(H)$, and put

$$C_z = b_z b_z^{\mathsf{T}} - \operatorname{diag}(b_z).$$

The matrix C_z records the distance increases caused by deleting z: it is the adjacency matrix of a clique on the surviving neighbors of z, with all other vertices isolated. Theorem 8.1 gives $E_S = \sum_{z \in S} C_z$. Each clique matrix has least eigenvalue at least -1, since

$$C_z + I = b_z b_z^{\mathsf{T}} + \operatorname{diag}(\mathbf{1} - b_z) \succeq 0.$$

The Gram representation also gives the uniform bound: $E_S \succeq -\tau I$.

To obtain the second bound, choose $w \in V(H)$ attaining τ, and put $Z = N_M(w) \cap S$. Every deleted vertex has a surviving neighbor, since $s < k$, so $1 \leq |Z| = \tau \leq k - 1$. We group the τ clique matrices indexed by Z and estimate the remaining $s - \tau$ separately. The correction matrix for deleting Z alone is

$$F_Z = D(M - Z) - D(M)[V(M - Z)].$$

By the small-puncture formula, its principal submatrix on $V(H)$ is exactly $\sum_{z \in Z} C_z$. Proposition 8.4 gives $\lambda_{\min}(F_Z) = f_k(\tau)$, so interlacing bounds the least eigenvalue of this grouped sum below by $f_k(\tau)$. Each of the remaining clique matrices has least eigenvalue at least -1. Weyl's inequality therefore yields

$$\lambda_{\min}(E_S) \geq f_k(\tau) - (s - \tau).$$

Together with $E_S \succeq -\tau I$, this proves the first inequality. The score bound follows from Proposition 8.2 and $\delta^*(H) = k - s/k$.

For $k = 57$, the original Moore graph has score 48. If $\tau \leq 42$, the first bound gives $\lambda_{\min}(E_S) \geq -42$. If $\tau \geq 43$, then

$$(27 + 2\tau)^2 - (55^2 + 224\tau) = 4\big((\tau - 43)(\tau + 14) + 28\big) > 0.$$

Since $27 + 2\tau > 0$, this implies $f_{57}(\tau) > 14 - \tau$, and hence

$$\lambda_{\min}(E_S) > 14 - s \geq -42.$$

Consequently, for every $s \leq 56$,

$$\Phi(M - S) \geq 48 - \frac{s}{57} - 42 \geq 48 - \frac{56}{57} - 42 = \frac{286}{57} > 0.$$

□

8.5. **One and two vertex deletions.** For one deleted vertex, and for the endpoints of an edge when $k \geq 3$, the least distance eigenvalue is $-2 - \sqrt{k}$. Combining this with the dual-degree formula gives, respectively,

$$\Phi(M - v) = k - \frac{1}{k} - 2 - \sqrt{k}, \qquad \Phi(M - \{u, v\}) = k - \frac{2}{k} - 2 - \sqrt{k} \quad (uv \in E(M)).$$

Both scores are positive for every integer $k \geq 5$. Theorems A.1 and A.2 give the complete spectra in Appendix A. For two nonadjacent deletions, Theorem A.3 also gives a full factorization and proves strictness for every realizable integer $k \geq 6$. The stability bound above avoids these diagonalizations when only strictness is needed.

9. Small orders and induced subgraphs of the Hoffman–Singleton graph

This section treats graphs that need not be regular. Every strict counterexample has at least 15 vertices (Theorem 9.2), an exhaustive search excludes orders up to 19 (Proposition 9.3), and the strict induced subgraphs of the Hoffman–Singleton graph of order at least 36 are classified up to automorphism (Proposition 9.4); among them are three further irregular counterexamples of order 40 (Proposition 9.5).

Here and in Appendix H, each statement carries one of four labels: *proved* (a written proof is given in the text), *computer-assisted* (an exhaustive computation reproduced by the code described in Appendix J), *Lean-checked* (with the Lean names and the scope stated in Appendix J), or *conjectured*. In the computer-assisted results, floating-point computations only propose candidates and integer test vectors; every decision rests on an exact certificate. For a connected graph with $\delta^*(G) = p/q$, an integer vector x

with $x^{\mathsf{T}}(qD + pI)x < 0$ certifies $\Phi(G) < 0$, and every graph without such a vector is decided by exact fraction-free elimination of $qD + pI$. None of these results decides whether 38 is the minimum order of a strict counterexample, and no infinite family of strict counterexamples is claimed.

Throughout, M is the coordinate Hoffman–Singleton graph of Proposition 2.3, and R, H_{38}, H_{39} and X_{42} are the graphs defined before Theorem 4.2.

9.1. **A lower bound of fifteen vertices.** We use the radius-two count of Proposition 2.1, $|B_2(v)| = 1+d(v)\,d^*(v)$, together with the following path constants, obtained from the exact characteristic polynomials

$$\chi_{D(P_4)} = (x^2 - 4x - 6)(x^2 + 4x + 2), \qquad \lambda_{\min}(D(P_4)) = -2 - \sqrt{2},$$
$$\chi_{D(P_6)} = (x + 1)(x^2 + 8x + 4)(x^3 - 9x^2 - 36x - 20), \qquad \lambda_{\min}(D(P_6)) = -4 - 2\sqrt{3},$$

and $\lambda_{\min}(D(P_3)) = -2$. A shortest path is an isometric subgraph, so its distance matrix is a principal submatrix of D; Cauchy interlacing then bounds $\lambda_{\min}(D)$ from above.

Lemma 9.1 (proved)**.** *Let G be connected, of order at least three and girth at least five. If* $\operatorname{diam}(G) \le 2$ *and $\Phi(G) > 0$, then G is a Moore graph of degree* 7 *or* 57*; in particular $|V(G)| \ge 50$.*

Proof. Diameter one would make G complete, hence not triangle-free, so $\operatorname{diam}(G) = 2$. If G is a tree, it is a star $K_{1,m}$ with $m \ge 2$: its center has $d^* = 1$, and a leaf, the center and another leaf form a shortest path, so $\Phi(G) \le 1 - 2 < 0$.

Suppose that G contains a cycle. Let u, w be nonadjacent. They have a unique common neighbor c. Each $x \in N(u) \setminus \{c\}$ is not adjacent to w, since otherwise u, x, w, c is a four-cycle, so x and w have a unique common neighbor $f(x) \in N(w)$. We have $f(x) \ne c$, since otherwise u, x, c is a triangle, and f is injective, since $f(x) = f(x')$ with $x \ne x'$ gives the four-cycle $u, x, f(x), x'$. Hence $d(u) \le d(w)$, and by symmetry nonadjacent vertices have equal degree. If w were a leaf with neighbor u, every vertex would be adjacent to u and G would be a star; so every degree is at least two. Let $u \sim w$, and choose $x \in N(u) \setminus \{w\}$ and $y \in N(w) \setminus \{u\}$; then $x \ne y$ and $x \nsim y$. If every vertex were adjacent to u or to w, a common neighbor z of x and y would satisfy $z \notin \{u, w\}$ and would be adjacent to u or w, closing a triangle. So some vertex z is adjacent to neither, and $d(u) = d(z) = d(w)$. Thus G is k-regular, every ball $B_2(v)$ is all of $V(G)$, and Proposition 2.1 gives $|V(G)| = k^2 + 1$: G is a Moore graph of diameter two. Theorem 2.2 and the Hoffman–Singleton theorem [3] then force $k \in \{7, 57\}$. □

Theorem 9.2 (proved)**.** *Every strict counterexample to WOW-284 has at least* 15 *vertices. If G is a strict counterexample of order $n < 50$, then* $\operatorname{diam}(G) \ge 3$*,*

$$\delta^*(G) > 2 + \sqrt{2}, \qquad \Delta(G) < \frac{n - 1}{2 + \sqrt{2}},$$

where $\Delta(G)$ is the maximum degree.

Proof. By Lemma 9.1, a strict counterexample of diameter at most two has at least 50 vertices. Otherwise $\operatorname{diam}(G) \ge 3$, and the first four vertices of a diametral shortest path span an isometric P_4, so $\lambda_{\min}(D) \le -2 - \sqrt{2}$ and $\delta^*(G) > -\lambda_{\min}(D) \ge 2 + \sqrt{2}$. Let w have degree $\Delta(G)$. Since $d^*(w) \ge \delta^*(G)$,

$$n \ge |B_2(w)| = 1 + \Delta(G)\,d^*(w) > 1 + \Delta(G)(2 + \sqrt{2}),$$

which is the degree bound. Since $\Delta(G) \ge d^*(v) \ge \delta^*(G) > 3$ for every v, we have $\Delta(G) \ge 4$ and $n > 9 + 4\sqrt{2} > 14.6$. □

For orders at most 37, the same count gives $\delta^*(G) \le \min\{\Delta(G), 36/\Delta(G)\} \le 6$, with equality only for a 6-regular graph on 37 vertices, that is, a Moore graph of degree six, which does not exist [3]. Hence $\delta^*(G) < 6$, an isometric P_6 is excluded because $-4 - 2\sqrt{3} < -6$, and a strict counterexample of order at most 37 has diameter three or four.

9.2. **Exhaustive search on at most nineteen vertices.**

Proposition 9.3 (computer-assisted)**.** *No connected graph of girth at least five on at most* 19 *vertices is a strict counterexample.*

The graphs were generated, one per isomorphism class, by `geng` from nauty [21]. For $15 \le n \le 19$, the maximum degree was capped by Theorem 9.2, and only graphs with $\delta^* > 2 + \sqrt{2}$ were kept; every remaining graph was decided exactly as described above. Appendix H gives the options and the counts.

9.3. **Induced subgraphs of the Hoffman–Singleton graph.**

Proposition 9.4 (computer-assisted)**.** *For $0 \le s \le 14$, the $\mathrm{Aut}(M)$-orbits of s-subsets $S \subseteq V(M)$ with $M - S$ connected, and those with $\Phi(M - S) > 0$, number as follows ($|S| = 50 - |V(M - S)|$).*

$\lvert V(M-S)\rvert$	*connected*	*strict*	$\lvert V(M-S)\rvert$	*connected*	*strict*
50	1	1	42	2,521	108
49	1	1	41	10,799	22
48	2	2	40	42,575	4
47	4	4	39	151,802	1
46	11	11	38	488,298	1
45	33	33	37	1,419,003	0
44	128	110	36	3,736,168	0
43	561	185			

Every other connected orbit has $\Phi < 0$, except one orbit of order 43 with $\Phi = 0$ (Proposition H.3). In particular, M has no strict induced subgraph of order 36 or 37, and the strict induced subgraphs of orders 38 and 39 are, up to automorphisms of M, exactly H_{38} and H_{39}.

The orbits were enumerated by orderly generation over the 252,000 automorphisms of M, and each connected representative was decided in the same way. Appendix H describes the enumeration and the certificate for the equality case.

Proposition 9.5 (computer-assisted, Lean-checked)**.** *The graphs*

$$X_{40} = M - \{P_{0,j}, Q_{k,0} : j, k \in \mathbb{F}_5\} = X_{42} - \{P_{0,2}, P_{0,3}\},$$
$$Y_{40} = M - \{P_{0,j} : j \in \mathbb{F}_5\} - \{P_{1,0}, Q_{0,0}, Q_{0,2}, Q_{2,2}, Q_{4,4}\},$$
$$Z_{40} = M - \{P_{0,j} : j \in \mathbb{F}_5\} - \{P_{1,0}, Q_{0,0}, Q_{0,2}, Q_{1,1}, Q_{1,4}\}$$

are strict counterexamples of order 40, of girth five and diameter three:

G	*edges*	*degree counts*	*certificate*
X_{40}	115	$5^{10}\,6^{30}$	$\delta^* = 17/3,\ 3D + 17I \succ 0$
Y_{40}	118	$5^{9}\,6^{26}\,7^{5}$	$\delta^* = 39/7,\ 7D + 39I \succ 0$
Z_{40}	118	$5^{10}\,6^{24}\,7^{6}$	$\delta^* = 39/7,\ 7D + 39I \succ 0$

Together with R, they represent the four strict orbits of order 40 in Proposition 9.4.

The four graphs are pairwise nonisomorphic: R is regular, X_{40} has 115 edges, and Y_{40}, Z_{40} have different degree counts. The positive definiteness was checked twice, by fraction-free elimination and by an independent rational LDL^{T} decomposition with all pivots positive. Lean checks the same data and the displayed positive-definite shifts (Appendix J).

10. Open questions on spectral constraints and stability

We list the questions that the results above leave open. The necessary conditions at degree six and order 50 do not decide whether a strict counterexample with these parameters exists. The minimum order of a strict counterexample is also unknown; by the computations of Section 9 it is at least 20, and no induced subgraph of the Hoffman–Singleton graph of order 36 or 37 is strict. A further direction is to extend the order bounds for regular graphs to irregular graphs.

Appendix I defines a balanced prime-field family of girth-five graphs and rules out strict counterexamples in that family when the diameter is three. It also rules out every layer-respecting perfect-matching deletion of Hoffman–Singleton considered there. These calculations test the window criterion but are not needed for the counterexamples or the deletion bounds above.

Disclosure and reproducibility. OpenAI ChatGPT-5.6 Sol Pro assisted with adversarial proof checking, proof exploration, and Lean formalization. Codex also assisted with supplementary proof arguments. The author assumes full responsibility for the mathematics, attribution, and conclusions. The source, exact certificates, and build instructions are available at `github.com/SamPetkov/wow284`; Appendix J describes the code and data.

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Appendix A. Distance spectra of Moore graphs after vertex deletion

This appendix proves the exact spectra quoted in Section 8 and, using the one-vertex decomposition, the second-subconstituent spectrum of Theorem 4.1. Here M is any degree-k Moore graph of diameter two, not only the coordinate graph of Proposition 2.3, and $\Delta = \sqrt{4k-3}$.

Every distance matrix below uses shortest-path distances in the surviving graph. The correction to the original distances is given by Theorem 8.1: a surviving nonadjacent pair moves from distance two to distance three exactly when its unique common neighbor was deleted.

A.1. **One deleted vertex and the endpoints of an edge.**

Theorem A.1 (One deleted vertex)**.** *For $v \in V(M)$, put $H = M - v$. Then*

$$|V(H)| = k^2, \qquad \delta^*(H) = k - \frac{1}{k}, \qquad \lambda_{\min}(D(H)) = -2 - \sqrt{k}.$$

The complete distance spectrum is

$$\operatorname{Spec} D(H) = \{\rho_+, \rho_-\} \cup \{(-2+\sqrt{k})^{(k-1)}, (-2-\sqrt{k})^{(k-1)}\} \cup \left\{\left(-\frac{\Delta+3}{2}\right)^{(m_+)}, \left(\frac{\Delta-3}{2}\right)^{(m_-)}\right\},$$

where

$$\rho_\pm = k^2 - 2 \pm \sqrt{k(k^3 - 2k^2 + 3k - 1)},$$
$$m_\pm = \frac{k(k-2)(\Delta \pm 1)}{2\Delta}.$$

Thus

$$\Phi(H) = k - \frac{1}{k} - 2 - \sqrt{k},$$

which is positive exactly for integers $k \geq 5$.

Theorem A.2 (Endpoints of an edge)**.** *Let $uv \in E(M)$ and $H = M - \{u, v\}$. For $k \geq 3$,*

$$|V(H)| = k^2 - 1, \qquad \delta^*(H) = k - \frac{2}{k}, \qquad \lambda_{\min}(D(H)) = -2 - \sqrt{k}.$$

The complete distance spectrum is

$$\operatorname{Spec} D(H) = \{\sigma_+, \sigma_-, k-4\} \cup \{(-2+\sqrt{k})^{(2k-4)}, (-2-\sqrt{k})^{(2k-4)}\} \cup \left\{\left(-\frac{\Delta+3}{2}\right)^{(a_+)}, \left(\frac{\Delta-3}{2}\right)^{(a_-)}\right\},$$

where

$$\sigma_\pm = k^2 - 3 \pm \sqrt{k^4 - 2k^3 + 3k^2 - 8k + 7},$$
$$a_+ = \frac{(k-2)(k + (k-2)\Delta)}{2\Delta}, \quad a_- = \frac{(k-2)((k-2)\Delta - k)}{2\Delta}.$$

Thus

$$\Phi(H) = k - \frac{2}{k} - 2 - \sqrt{k},$$

which is positive exactly for integers $k \geq 5$.

Proof of Theorems A.1 and A.2. The dual-degree formulas are the cases $s = 1, 2$ of Theorem 8.1. In each case we split $\mathbb{R}^{V(H)}$ into the cell-constant space, invariant planes pairing a zero-sum vector on a neighbor cell with its vector of neighbor sums, and a residual space on which $D(H)$ is $-2I$ minus an adjacency block satisfying the Moore relation.

For one deleted vertex, put $A = N(v)$, $B = \Gamma_2(v)$, let C be the A-by-B incidence matrix, and let B_0 be the adjacency matrix on B. Only pairs inside A lose their unique length-two path. By Theorem 8.1, their new distance is three. All other distances are unchanged, so

$$D(H) = \begin{pmatrix} 3(J-I) & 2J - C \\ 2J - C^{\mathsf{T}} & 2(J-I) - B_0 \end{pmatrix}.$$

The normalized constant quotient is

$$Q_1 = \begin{pmatrix} 3(k-1) & (2k-1)\sqrt{k-1} \\ (2k-1)\sqrt{k-1} & 2k^2 - 3k - 1 \end{pmatrix},$$

whose eigenvalues are $\rho_\pm$. Block comparison in the Moore identity gives

$$CC^{\mathsf{T}} = (k-1)I, \qquad CB_0 = J - C, \qquad C^{\mathsf{T}}\mathbf{1}_A = \mathbf{1}_B,$$
$$B_0^2 + C^{\mathsf{T}}C = (k-1)I - B_0 + J.$$

Thus C^{T} is injective and

$$\mathbb{R}^B = \langle \mathbf{1}_B \rangle \perp C^{\mathsf{T}}(\mathbf{1}_A^\perp) \perp K, \qquad K = \ker C, \qquad \dim K = k(k-2).$$

Choose an orthogonal basis $\mathcal{B}$ of $\mathbf{1}_A^\perp$, and for $z \in \mathcal{B}$ put

$$W_z = \operatorname{span}\{(z,0),(0,C^\mathsf{T}z)\}.$$

The quotient acts on the cell-constant space W_{const}. The remaining directions split into the two-dimensional spaces W_z for $z \in \mathcal{B}$ and the residual space $0 \oplus K$. The identities above show that these spaces are invariant and mutually orthogonal. Put

$$W_{\text{inc}} = \bigoplus_{z\in\mathcal{B}} W_z.$$

Consequently

$$\mathbb{R}^{V(H)} = W_{\text{const}} \perp W_{\text{inc}} \perp (0 \oplus K), \qquad 2 + 2(k-1) + k(k-2) = k^2.$$

For $z \in \mathcal{B}$, $D(H)(z,0) = -(3z, C^\mathsf{T}z)$ and, since $B_0C^\mathsf{T}z = (CB_0)^\mathsf{T}z = -C^\mathsf{T}z$, $D(H)(0,C^\mathsf{T}z) = -((k-1)z, C^\mathsf{T}z)$. On each zero-sum incidence space, the matrix of the distance operator in the chosen generators is $\left(\begin{smallmatrix} -3 & -(k-1) \\ -1 & -1 \end{smallmatrix}\right)$, with eigenvalues $-2 \pm \sqrt{k}$. The restriction to K satisfies $B_0^2 + B_0 - (k-1)I = 0$ and $D = -2I - B_0$, so the remaining distance eigenvalues are obtained by negating the two Moore adjacency roots and subtracting two. Their multiplicities are determined by the trace of B_0: after removing its principal eigenvalue $k-1$ and its $k-1$ copies of -1 on $C^\mathsf{T}(\mathbf{1}_A^\perp)$, the remaining trace is zero. This trace and the residual dimension give $m_\pm$.

For an adjacent deleted pair, put

$$A = N(u) \setminus \{v\}, \qquad B = N(v) \setminus \{u\}, \qquad C = V(M) \setminus \big(\{u,v\} \cup A \cup B\big).$$

Only pairs inside A or inside B lose a length-two path. Since $k \geq 3$, Theorem 8.1 applies to the two deleted vertices and makes the new distance of each such pair three. Vectors with opposite constant values on the two neighbor cells and zero on the third cell form an invariant line with eigenvalue $k-4$. On the remaining cell-constant directions, the two neighbor-cell values agree, and the normalized symmetric quotient is

$$Q_2 = \begin{pmatrix} 5k-8 & \sqrt{(k-1)(2k-3)(4k-6)} \\ \sqrt{(k-1)(2k-3)(4k-6)} & 2k^2-5k+2 \end{pmatrix},$$

with eigenvalues $\sigma_\pm$. Let R_A, R_B be the incidence matrices from A, B to C, and let T be the adjacency matrix on C. The Moore identity gives

$$R_AR_A^\mathsf{T} = R_BR_B^\mathsf{T} = (k-1)I, \qquad R_AR_B^\mathsf{T} = J,$$
$$R_AT = J - R_A, \qquad R_BT = J - R_B,$$
$$R_A^\mathsf{T}R_A + R_B^\mathsf{T}R_B + T^2 = (k-1)I - T + J.$$

The transpose maps of R_A and R_B, restricted to their cell zero-sum spaces, are injective and have orthogonal images. Pair each zero-sum vector in its original cell with the vector obtained from its neighbor sums. Each pair spans an invariant two-dimensional space carrying $\left(\begin{smallmatrix} -3 & -(k-1) \\ -1 & -1 \end{smallmatrix}\right)$, as in the one-vertex calculation. The residual space

$$K = \ker R_A \cap \ker R_B, \qquad \dim K = (k-2)^2,$$

is invariant and satisfies

$$T^2 + T - (k-1)I = 0, \qquad D = -2I - T.$$

These spaces and the three-dimensional cell-constant space are mutually orthogonal and complete, since

$$3 + 2(2k-4) + (k-2)^2 = k^2 - 1.$$

The constant direction of T has eigenvalue $k-2$, while its two zero-sum incidence images contribute trace $-2(k-2)$. Since $\operatorname{tr} T = 0$, one has $\operatorname{tr}(T|_K) = k-2$, and dimension and trace give $a_\pm$. The residual negative root is greater than $-2-\sqrt{k}$. For $k \geq 3$, the antisymmetric constant root also satisfies

$$(k-4) - (-2-\sqrt{k}) = k - 2 + \sqrt{k} > 0.$$

It remains to compare the incidence root with the smaller roots of the constant quotients. We shift each quotient by subtracting the candidate root from its diagonal. If the shifted matrices are positive definite, then every quotient eigenvalue lies strictly above that root. In each case the leading diagonal entry is positive, and

$$\det(Q_1 + (2+\sqrt{k})I) = k(2k^2 + 2k^{3/2} - 3k + 2) > 0,$$
$$\det(Q_2 + (2+\sqrt{k})I) = (\sqrt{k}-1)(\sqrt{k}+1)(2k^2 + 2k^{3/2} - 3k + 2\sqrt{k} + 6) > 0.$$

The shifted symmetric quotients are therefore positive definite. The constant-quotient eigenvalues lie strictly above the incidence root, completing the comparison. Exact quotient normalization, trace-to-multiplicity equations, replacement paths, and all least-root comparisons are independently checked by exact computation. □

A.2. **The distance spectrum of a Moore second sphere.** To prove Theorem 4.1, we use the decomposition of the second-sphere adjacency matrix from the proof of Theorem A.1. It gives the eigenvalues and their multiplicities, after which we recover the distance spectrum.

Proof of Theorem 4.1. Each vertex of the second sphere loses exactly one neighbor when we pass to its induced graph. Put $m = K(K-1)$; the Moore property gives $|\Gamma_2(v)| = m$. Each vertex of $\Gamma_2(v)$ has its single parent in $N(v)$, while each vertex of $N(v)$ has $K-1$ neighbors in $\Gamma_2(v)$. The induced graph X consequently has order m and degree $(K-1)$. It contains no triangle or four-cycle, since M contains neither. To obtain the distance spectrum we now calculate the adjacency spectrum and then prove that the diameter is three.

The graph X is the graph on $\Gamma_2(v)$ in the proof of Theorem A.1 with $k = K$; its adjacency matrix is B_0 there. That proof shows that B_0 has eigenvalue $K-1$ on the constant vector, -1 with multiplicity $K-1$ on $C^{\mathsf{T}}(\mathbf{1}_A^{\perp})$, and on $\ker C$, of dimension $K(K-2)$, only $r, s = (-1 \pm \sqrt{4K-3})/2$, with the multiplicities $m_+, m_- > 0$ of Theorem A.1; both roots occur. It remains to check that X has diameter three. If the unique common neighbor in M of nonadjacent vertices $x, y \in X$ lies in X, their distance in X is two. The only other possibility is their common parent in $N(v)$. Choose $b \in N_X(x)$, and let p be the common parent of x, y. The parent of b is not p, since otherwise p, x, b would form a triangle. Moreover $b \not\sim y$, and the unique common neighbor c of b, y belongs to X: it cannot be v, and it cannot lie in $N(v)$, since b and y have different parents there. Thus $x - b - c - y$ is a path in X. Each parent has $K-1 \geq 2$ children, and pairs of distinct children have no length-two path in X. Thus X has diameter three and Theorem 3.1 applies. Among the nonprincipal adjacency eigenvalues, $(-1+\sqrt{4K-3})/2$ maximizes $|\theta+1|$; substitution gives

$$\lambda_{\min}(D(X)) = -\frac{5+\sqrt{4K-3}}{2}.$$

For $K = 3, 4, 5$, direct comparison gives a nonpositive score. For $K \geq 6$, both sides of the relevant comparison are positive, and the threshold reduces to

$$(2K-7)^2 - (4K-3) = 4(K^2 - 8K + 13) > 0$$

because $K > 4 + \sqrt{3}$. The finite $K = 7$ instance is also checked by exact computation. □

A.3. **Two nonadjacent deleted vertices.**

Theorem A.3 (Two nonadjacent deleted vertices)**.** *Let u, v be nonadjacent vertices of M, $k \geq 5$, and put $H = M - \{u, v\}$. Define*

$$\begin{aligned} R_k(x) = {} & x^4 + (10-2k^2)x^3 + (2k^3 - 17k^2 - 2k + 36)x^2 \\ & + (12k^3 - 49k^2 - 4k + 53)x \\ & - 2k^4 + 17k^3 - 38k^2 + 5k + 20, \end{aligned}$$

$$M_- = \frac{k(k-2) + (k^2-4k+2)\Delta}{2\Delta}, \quad M_+ = \frac{-k(k-2) + (k^2-4k+2)\Delta}{2\Delta}.$$

Then

$$\begin{aligned} \chi_{D(H)}(x) = {} & (x-k+3)R_k(x)(x^2+4x-k+3)^{k-2} \\ & \cdot (x^2+4x-k+5)^{k-2}\left(x + \frac{\Delta+3}{2}\right)^{M_-} \\ & \cdot \left(x - \frac{\Delta-3}{2}\right)^{M_+}. \end{aligned}$$

Moreover,

$$\delta^*(H) = k - \frac{2}{k},$$

and H is a strict counterexample for every realizable integer $k \geq 6$.

Proof. The deleted vertices have a unique common neighbor w. With $A = N(u) \setminus \{w\}$, $B = N(v) \setminus \{w\}$, $C = N(w) \setminus \{u, v\}$, and

$$Z = V(M) \setminus \big(\{u, v, w\} \cup A \cup B \cup C\big),$$

the surviving graph has cell sizes, in the order $\{w\}, A, B, C, Z$,

$$1, \quad k-1, \quad k-1, \quad k-2, \quad (k-1)(k-2).$$

The Moore common-neighbor rule gives a perfect matching between A and B, and the five-cell partition is equitable. The distance rule from Theorem 8.1, stated at the start of this section, applies because $2 \le k-1$. We now restrict the distance matrix to vectors constant on each cell. The row-sum quotient is

$$\begin{pmatrix} 0 & 3k-3 & 3k-3 & k-2 & 2(k-1)(k-2) \\ 3 & 3k-6 & 2k-3 & 2k-4 & 2k^2-7k+6 \\ 3 & 2k-3 & 3k-6 & 2k-4 & 2k^2-7k+6 \\ 1 & 2k-2 & 2k-2 & 2k-6 & 2k^2-7k+5 \\ 2 & 2k-3 & 2k-3 & 2k-5 & 2k^2-7k+5 \end{pmatrix}.$$

Its characteristic polynomial is $(x-k+3)R_k(x)$.

Identify $\mathbb{R}^A$ with $\mathbb{R}^B$ through the perfect matching, and let R_A, R_B, R_C be the incidence matrices from the three neighbor cells to Z. Let T be the adjacency matrix on Z. The Moore identity gives

$$\begin{gathered} R_A R_A^{\mathsf{T}} = R_B R_B^{\mathsf{T}} = (k-2)I, \quad R_C R_C^{\mathsf{T}} = (k-1)I, \\ R_A R_B^{\mathsf{T}} = J - I, \quad R_A R_C^{\mathsf{T}} = R_B R_C^{\mathsf{T}} = J, \\ R_A T + R_B = J - R_A, \quad R_B T + R_A = J - R_B, \quad R_C T = J - R_C, \\ R_A^{\mathsf{T}} R_A + R_B^{\mathsf{T}} R_B + R_C^{\mathsf{T}} R_C + T^2 = (k-1)I - T + J. \end{gathered}$$

Moreover, every vertex of Z has one neighbor in each of A, B, C, so

$$R_A^{\mathsf{T}}\mathbf{1}_A = R_B^{\mathsf{T}}\mathbf{1}_B = R_C^{\mathsf{T}}\mathbf{1}_C = \mathbf{1}_Z, \qquad T\mathbf{1}_Z = (k-3)\mathbf{1}_Z.$$

Use the coordinate order $(\{w\}, A, B, C, Z)$. Choose orthogonal bases $\mathcal{B}_A$ of $\mathbf{1}_A^{\perp}$ and $\mathcal{B}_C$ of $\mathbf{1}_C^{\perp}$. For $\xi \in \mathcal{B}_A$ and $\eta \in \mathcal{B}_C$, put

$$\begin{aligned} u_\xi^{\pm} &= (0, \xi, \pm\xi, 0, 0), \quad & v_\xi^{\pm} &= (0, 0, 0, 0, (R_A^{\mathsf{T}} \pm R_B^{\mathsf{T}})\xi), \\ u_\eta^{C} &= (0, 0, 0, \eta, 0), \quad & v_\eta^{C} &= (0, 0, 0, 0, R_C^{\mathsf{T}}\eta), \end{aligned}$$

and define

$$W_\xi^{\pm} = \operatorname{span}\{u_\xi^{\pm}, v_\xi^{\pm}\}, \qquad W_\eta^{C} = \operatorname{span}\{u_\eta^{C}, v_\eta^{C}\}.$$

The incidence identities make the relevant maps injective, so each displayed span is two-dimensional. They also show that these spaces are mutually orthogonal and orthogonal to the five-dimensional cell-constant space W_{const}. For the residual space

$$K = \ker R_A \cap \ker R_B \cap \ker R_C$$

one has

$$K \subseteq \mathbf{1}_Z^{\perp}, \qquad \dim K = (k-2)(k-4), \qquad T(K) \subseteq K.$$

Thus W_{const}, the spaces $W_\xi^{\pm}, W_\eta^{C}$, and $0 \oplus 0 \oplus 0 \oplus 0 \oplus K$ form an orthogonal direct sum. Their dimensions account for every surviving vertex, since

$$5 + 4(k-2) + 2(k-3) + (k-2)(k-4) = k^2 - 1.$$

In the ordered generator pairs (u_ξ^{+}, v_ξ^{+}), (u_ξ^{-}, v_ξ^{-}), and (u_η^{C}, v_η^{C}), respectively, the distance operator has matrices

$$\begin{pmatrix} -4 & -(k-3) \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -2 & -(k-1) \\ -1 & -2 \end{pmatrix}, \qquad \begin{pmatrix} -2 & -(k-1) \\ -1 & -1 \end{pmatrix}.$$

These give the two displayed quadratic factors and $k-3$ copies of each Moore linear factor. On K, the Z-block identity gives

$$T^2 + T - (k-1)I = 0, \qquad D(H)|_K = -2I - T.$$

The constant direction of Z has T-eigenvalue $k-3$. The Z-images of the symmetric, antisymmetric and common-neighbor spaces have eigenvalues $-2, 0, -1$, with dimensions $k-2, k-2, k-3$, respectively. Since $\operatorname{tr} T = 0$,

$$\operatorname{tr}(T|_K) = 2(k-2).$$

The residual adjacency operator has just two possible eigenvalues, so its dimension and trace determine both multiplicities. We add the Moore-root copies from the common-neighbor spaces to obtain the full exponents M_- and M_+.

This completes the spectral factorization. The value $\delta^*(H) = k - \frac{2}{k}$ again follows from the $s = 2$ case of Theorem 8.1. For strictness we do not need the least root of the quartic. Since $u, v \in N_M(w)$, Proposition 8.4 with $s = 2$ gives

$$\Phi(H) \ge \frac{3k - 5 - \Delta - \sqrt{k^2 + 4k - 4}}{2} - \frac{2}{k}.$$

For $k \geq 7$, one has

$$\sqrt{4k-3} < k-1, \qquad \sqrt{k^2+4k-4} < k+2.$$

Indeed, the corresponding differences of squares are $k(k-6)+4 > 0$ and $8 > 0$. The preceding lower bound is therefore greater than

$$\frac{k-6}{2} - \frac{2}{k} \geq \frac{1}{2} - \frac{2}{7} = \frac{3}{14} > 0.$$

For $k = 6$, use $\sqrt{21} < 23/5$ and $\sqrt{56} < 15/2$, which follow from $529 > 525$ and $225 > 224$. The same lower bound is then greater than

$$\frac{13 - 23/5 - 15/2}{2} - \frac{1}{3} = \frac{7}{60} > 0.$$

Thus $\Phi(H) > 0$ for every realizable integer $k \geq 6$. The direct-sum decomposition, injectivity, orthogonality, all recomputed distances, and the strict comparison are independently checked by two exact computations. □

Appendix B. Finite spectral computations and deletion checks

We record the finite certificates behind the examples and deletion families of Theorem 4.2, and the orbit check of Theorem 8.6. The graph encodings are part of the data described in Appendix J; the coordinate definitions of the small counterexamples are given above Theorem 4.2.

For H_{38}, the factor $x^2 + 6x + 2$, together with the exact Sturm count, identifies the least distance root as $-3 - \sqrt{7}$. An independent rational LDL^{T} decomposition of $3D(H_{38}) + 17I$ has inertia $(38, 0, 0)$, where the entries count positive, negative and zero eigenvalues. For H_{39}, the decomposition of $6D(H_{39}) + 35I$ has inertia $(39, 0, 0)$. Thus each shift is positive definite and proves strictness. For R and X_{42}, the exact spectra in Theorem 4.2 identify the least eigenvalue -5. These graph, metric, dual-degree and matrix checks are reproduced by exact computation.

The two labeled deletion families of R have the following common distance characteristic polynomials. For every $v \in V(R)$,

$$\begin{aligned}\det(xI - D(R-v)) = P_{39}(x) := {} & x^9(x+5)^{12}(x^2+6x+3) \\ & \cdot (x^3+3x^2-15x-7)^2(x^3+3x^2-15x-3)^2 \\ & \cdot (x^4-78x^3+303x^2-70x-450),\end{aligned}$$

whereas for every $uv \in E(R)$,

$$\begin{aligned}\det(xI - D(R-\{u,v\})) = P_{38}(x) := {} & x^4(x-2)(x+5)^8(x^2+6x+2)^2 \\ & \cdot (x^3+3x^2-15x-3) \\ & \cdot (x^4+5x^3-7x^2-23x-6) \\ & \cdot (x^5+9x^4+7x^3-77x^2-54x-4) \\ & \cdot (x^9-67x^8-404x^7+1772x^6+7205x^5 \\ & \qquad -18489x^4-17018x^3+20288x^2+16824x+1680).\end{aligned}$$

The least root of P_{38} is $-3 - \sqrt{7}$; that of P_{39} is the least root of $x^3 + 3x^2 - 15x - 3$, about -5.588, above $-35/6$.

All 40 singleton and 120 edge-endpoint deletions are enumerated and checked by computer; the root and positive-definiteness checks are recorded with each family.

For the orbit check in Theorem 8.6, the orbit cardinalities on deletion sets of each size $s \leq 5$ sum to $\binom{50}{s}$, so the 52 representatives cover every deletion set. Each representative with $s = 4, 5$ gives

$$\operatorname{inertia}\big(7D(M-S) + (49-s)I\big) = (50-s, 0, 0);$$

the six-vertex set of that theorem gives $(43, 1, 0)$ for $7D(M-S) + 43I$.

Appendix C. Technical steps in the polynomial order bounds

This appendix completes three steps in the proofs of Theorems 6.4 and 6.5: the positivity of the dual slack, the exclusion of the graph order $n = B_k$, and the nonvanishing of the excess matrix.

C.1. **Positivity of the dual LP slack.** Use the measure, support points, and slack notation from the proof of Theorem 6.4. We verify the positivity of every slack beyond degree four, as used in its optimality and rigidity argument.

We check the first five values exactly and then bound the tail. For $5 \le i \le 9$, removing the common positive factor $k(k-1)(k+2)(k^2+3)/6$ leaves, respectively,

$$2, \quad 5k-13, \quad 2(3k^2-17k+25),$$
$$6k^3-47k^2+139k-150, \quad 2(3k^4-27k^3+106k^2-219k+194),$$

all positive for $k \ge 4$; for the nontrivial residual factors this follows after writing $k = m+4$, when all coefficients are nonnegative and the constant terms are positive. For $i \ge 10$, put $r = k-1 \ge 3$. The support lies in $[-2\sqrt{r}, 2\sqrt{r}]$, because $1+\sqrt{2r} \le 2\sqrt{r}$. For $|z| \le 1$, the recurrence gives

$$F_i(2\sqrt{r}\,z) = r^{i/2}U_i(z) - r^{(i-2)/2}U_{i-2}(z),$$

where U_j is the Chebyshev polynomial of the second kind. Using $|U_j(z)| \le j+1$, the total mass $B_k - 1$, and $F_i(k) = kr^{i-1}$, we obtain

$$\begin{aligned}\frac{|\mu(F_i)|}{F_i(k)} &\le \frac{r^2+4r+6}{6}\, r^{-i/2}\big((i+1)r+i-1\big)\\ &\le \frac{2i+1}{3}\, r^{3-i/2} \le \frac{2i+1}{3}\, 3^{3-i/2}.\end{aligned}$$

The second inequality is $(r^2+4r+6)\big((i+1)r+i-1\big) \le 2(2i+1)r^3$; the difference of the two sides is $(r-3)\big((3i+1)r^2+4ir+2i-2\big) \ge 0$. The third uses $r \ge 3$ and $3-i/2 < 0$. At $i = 10$ the right-hand side is $7/9$, and it decreases thereafter because $3(2i+1)^2-(2i+3)^2 = 8i^2-6 > 0$. Hence

$$\frac{|\mu(F_i)|}{F_i(k)} < 1 \qquad (i \ge 10).$$

The principal contribution is positive and exceeds the absolute value of the measure's contribution. Their sum is therefore positive, proving the required positivity for every remaining slack.

C.2. **Equality in the closed-window order bound.**

Completion of the proof of Theorem 6.4. We use the notation of that proof and of the trace bound before it. The polynomial bound includes the endpoints of I_k. We must therefore rule out graph equality separately. Suppose $n = B_k$. Equality in the trace bound for the optimal quartic forces every nonprincipal eigenvalue to belong to

$$\{-2, -1-\omega, -1+\omega\}, \qquad \omega = \sqrt{2k-2}.$$

Let m_0, m_-, m_+ be their respective multiplicities. If ω is irrational, the trace equation

$$0 = k - (m_- + m_+) - 2m_0 + \omega(m_+ - m_-)$$

forces $m_+ = m_-$. Counting the nonprincipal eigenvalues then gives $m_0 = k-n+1 < 0$, since

$$B_k - (k+1) = \frac{k(k-1)(k+3)}{6} > 0.$$

This is impossible. Thus ω is rational and, as the square root of an integer, must be an integer.

The second moment of $A+I$ now determines m_0. The principal eigenvalue contributes $(k+1)^2$. Each nonprincipal adjacency eigenvalue $-1 \pm \omega$ contributes $2k-2$ to this moment, while each copy of -2 contributes 1. Hence

$$(2k-2)(n-1-m_0) + m_0 = n(k+1) - (k+1)^2,$$

and substituting $n = B_k$ gives

$$m_0 = \frac{k(k-1)(k^2+3)}{6(2k-3)}.$$

Since m_0 is an integer, $2k-3$ divides $k(k-1)(k^2+3)$. The identity

$$16k(k-1)(k^2+3) = (2k-3)(8k^3+4k^2+30k+21) + 63$$

therefore implies $2k-3 \mid 63$. With $k \ge 4$, this leaves $k \in \{5, 6, 12, 33\}$. Only $k = 33$ has integral $\sqrt{2k-2}$, but then $m_0 = 9152/3$, again impossible. Consequently $n < B_k$ throughout the closed window. □

C.3. **The excess matrix is nonzero.**

Completion of the proof of Theorem 6.5. We use the notation of that theorem and its proof. We prove $\mathcal{E}_k \neq 0$. Suppose instead that the excess matrix is zero. Then

$$g_k(A) = (h_k + 1)J - I,$$

so $p_k(A) = 0$ on $\mathbf{1}^\perp$, where $p_k(x) = g_k(x) + 1$. After $y = x + 2$,

$$p_k(y - 2) = y^4 - 2y^3 + (3 - 2k)y^2 + 1.$$

There are two possible ways to factor this polynomial over $\mathbb{Q}$. A rational linear factor would give a root among ± 1, whose values are $3 - 2k$ and $7 - 2k$. For a factorization into monic integer quadratics, the constant terms would be both 1 or both -1. The cubic and linear coefficients would then be equal or opposite, respectively; their actual values are -2 and 0. Neither factorization is possible.

On the rational zero-sum space, substituting the adjacency operator into this irreducible quartic gives zero. Its characteristic polynomial must therefore be a power of the quartic:

$$\chi_{A|_{\mathbf{1}^\perp}}(x) = p_k(x)^m, \qquad n - 1 = 4m.$$

The four roots of p_k sum to -6, so $\operatorname{tr} A = 0$ gives

$$0 = k - 6m.$$

Hence $n - 1 = 4m = 2k/3$, contradicting $n \geq k + 1$. Thus $\mathcal{E}_k \neq 0$. □

Appendix D. Proof of the three-to-one bound

This appendix proves Theorem 6.6, determines the parameters at which its bound can be attained, and tabulates the bound.

Proof of Theorem 6.6. Put $\mathcal{E} = \mathcal{E}_k$, $C = C_k$, $h = h_k$, and

$$\rho = \frac{\varepsilon - 1}{n} = \frac{1 + r/n}{2}.$$

The optimal slack matrix has the form

$$\mathcal{S}_k = I - \mathcal{E} + \rho J \succeq 0.$$

We use two standard graph constructions in the proof. The line graph has one vertex for each edge of the original graph, with two vertices adjacent when those edges meet. The original graph is called a root of the line graph. A cocktail-party graph is a complete graph with a perfect matching removed.

First suppose $r \leq 0$. We show that $\mathcal{E}$ is the adjacency matrix of a simple graph, then exclude its complement by classifying graphs with bounded least eigenvalue and checking rational eigenspace dimensions. After proving $r > 0$, we suppose $n > 3r$ and exclude the following four cases:

(1) an entry $\mathcal{E}_{uv} = 2$, which forces $n = 3r + 2$ and is excluded by divisibility and parity;
(2) a simple $\mathcal{E}$ whose complement has at least three components, which forces $n \leq 3r + 6$ and is excluded by fixed remainders;
(3) a connected complement, whose cocktail-party case contradicts $r > 0$ and whose line-graph roots are excluded by size and divisibility;
(4) a complement with two components, excluded by their classified types and the remaining fixed remainders.

Here diameter three and the radius-two count give $n \geq k^2 + 2 \geq 38$. The classification therefore applies whenever the complement is connected. In the final two-component case we first exclude $k = 6, 7, 8$; for $k \geq 9$, each component has order greater than 28. The cases also use two divisibility conditions. The first follows from the identity

$$128(2C - r) = (4k + 9)(64k^3 + 112k^2 + 196k + 327) + (129 - 128r).$$

Since $n = (2C - r)/(12k + 27)$ is integral and $4k + 9$ is odd,

$$4k + 9 \mid 129 - 128r. \tag{1}$$

The second condition will be used when the order is close to the desired bound. If $n = 3r + t$, the defining equation for r gives $2(18k + 41)\, r = 2C - (12k + 27)t$. Since $12k + 27$ is odd, t is even, and

$$18k + 41 \mid 2C - (12k + 27)t.$$

Since $\gcd(18, 18k + 41) = 1$, Euclidean division after multiplication by 18^4 yields the necessary condition

$$18k + 41 \mid 132650 + 34992t. \tag{2}$$

We first prove positivity of r. Suppose $r \le 0$. A 2×2 principal minor of $\mathcal{S}_k$ gives

$$\mathcal{E}_{uv} \le 2 + \frac{r}{n} \qquad (u \ne v).$$

Thus $\mathcal{E}$ is the adjacency matrix of a simple graph when $r < 0$. If $r = 0$ and $\mathcal{E}_{uv} = 2$, positivity puts $e_u + e_v$ in $\ker \mathcal{S}_k$. After subtracting its projection onto $\mathbf{1}$, the strict spectral window gives a (-2)-eigenvector of A. Its u-coordinate would require

$$A_{uv} = -2 + \frac{2k+4}{n},$$

which is impossible because $n \ge k^2 + 2$ and $A_{uv} \in \{0, 1\}$. Hence $\mathcal{E}$ is simple also when $r = 0$.

Let X be the graph with adjacency matrix $J - I - \mathcal{E}$. It is regular of degree

$$d = n - 1 - \varepsilon = \frac{n - r - 4}{2},$$

and $A(X) + 2I = I - \mathcal{E} + J = \mathcal{S}_k + \frac{n-r}{2n} J \succeq 0$ because $r \le 0$. Each component has at least $d + 1$ vertices, and $n \ge 3(d + 1)$ would give $n \le 3r + 6 \le 6$; so X has at most two components. We exclude the remaining possibilities by examining eigenspaces of $\mathcal{E}$. This matrix is a polynomial in A and J, so it commutes with A. Each rational eigenspace of $\mathcal{E}$ is therefore A-invariant.

The graph X is now simple and regular, and its order is $n \ge 38$. If it is connected, the classification of connected regular graphs of order greater than 28 and least eigenvalue at least -2 makes it a line graph or a cocktail-party graph [17]; see also [22, Theorems 4.1.1 and 4.1.5] and [23, Introduction].

Suppose first that X is connected. The complete case would give $\mathcal{E} = 0$, excluded in Theorem 6.5. In the cocktail-party case, $\mathcal{E}$ is a perfect matching and $r = -n$. The identity

$$1296C = (6k + 13)(216k^3 + 396k^2 + 654k + 1175) + 277$$

forces $k = 44$ and $n = 14812$. On the 7406-dimensional (-1)-eigenspace of $\mathcal{E}$, the rational operator A is annihilated by $g_{44}(x) + 2$.

If $X = L(Y)$, choose a connected root Y. Since X is d-regular, every edge ab of Y satisfies $\deg_Y(a) + \deg_Y(b) = d + 2$. Degrees alternate along paths. An odd cycle forces them to coincide; otherwise Y is bipartite, with a constant degree on each part. Thus Y is regular or semiregular bipartite. Here semiregular means that the degree is constant within each part, though the two degrees may differ.

If Y is q-regular, then $q \ge n/4$ and $|V(Y)| \le 8$, contradicting $q \le |V(Y)| - 1$ because $n \ge 38$. If Y is semiregular bipartite with part sizes $a \ge b \ge 2$, then

$$\frac{1}{a} + \frac{1}{b} = \frac{n - r}{2n} \ge \frac{1}{2}, \qquad n \le ab.$$

The cases $b \ge 3$ have $ab < 38$. For $b = 2$, connectedness forces $Y = K_{a,2}$, hence $r = -4$. Equation (1) leaves $k = 158$ and $n = 664748$. The resulting (-1)-eigenspace of $\mathcal{E}$ has dimension 332373.

If X has two components, write their orders as $d + 1 + a_1$ and $d + 1 + a_2$. Then

$$a_1 + a_2 = r + 2,$$

so $r \in \{-2, -1, 0\}$. Checking the divisors in (1), together with $n \ge k^2 + 2$ and kn even, leaves only

$$(r, k, n) = (-1, 62, 40875).$$

The components then have orders 20437 and 20438. Their common degree is one less than the smaller order, so the smaller component is complete. In the larger component, each vertex has exactly one nonneighbor; these missing edges form a perfect matching, giving the cocktail-party graph. Thus $\mathcal{E}$ has eigenvalue -1 with multiplicity 10219.

In each of the three exceptional cases, the eigenspace just found is rational, orthogonal to the constant vector and preserved by the adjacency operator. The excess matrix acts there as minus the identity. Its defining relation therefore makes $g_k(x) + 2$ vanish on that space. This quartic is irreducible over $\mathbb{Q}$ for every integer $k \ge 6$, except $k = 7$. To check irreducibility, substitute $y = x + 2$, giving

$$y^4 - 2y^3 + (3 - 2k)y^2 + 2.$$

Gauss's lemma reduces a quadratic factorization to monic integer quadratics whose constant terms multiply to 2; coefficient comparison leaves only $k = 4$ or $k = 7$, while the rational-root alternatives give $k = 2$ or $k = 4$. The quartic is therefore irreducible at degrees $44, 62, 158$, but the corresponding dimensions $7406, 10219, 332373$ are not divisible by four. Since an invariant rational space annihilated by an irreducible quartic is a vector space over the degree-four field $\mathbb{Q}[x]/(g_k(x) + 2)$, its rational dimension must be divisible by four. This contradiction proves $r > 0$.

Now suppose, for a contradiction, that $n > 3r$, and put $x = r/n \in (0, 1/3)$. To apply the graph classification again, we first show that the excess matrix is the adjacency matrix of a simple graph. The

2×2 minors give $\mathcal{E}_{uv} \le 2$. Suppose $\mathcal{E}_{uv} = 2$, and for $w \notin \{u, v\}$ set $s_w = \mathcal{E}_{uw} + \mathcal{E}_{vw}$. Cauchy–Schwarz, applied to $e_u + e_v$ and e_w in the Gram matrix $\mathcal{S}_k$, gives

$$(1 + x - s_w)^2 \le x(3 + x),$$

Thus the nonnegative integer s_w belongs to $\{1, 2\}$. Summing over the rows shows that exactly r vertices satisfy $s_w = 2$. We collect these vertices in W, let $y = \sum_{w \in W} e_w$, and put

$$\eta_W = \sum_{\{w,z\} \subseteq W} \mathcal{E}_{wz}.$$

Then

$$(e_u + e_v)^{\mathsf{T}} \mathcal{S}_k (e_u + e_v) = 2x,$$
$$(e_u + e_v)^{\mathsf{T}} \mathcal{S}_k y = r(x - 1),$$
$$y^{\mathsf{T}} \mathcal{S}_k y = r + \frac{r^2(1 + x)}{2} - 2\eta_W.$$

Positivity of this 2×2 Gram determinant yields

$$2xr + r^2(3x - 1) - 4x\eta_W \ge 0.$$

Since $\eta_W \ge 0$, this forces $n \le 3r + 2$; as t is even, $n = 3r + 2$. By (2), $18k + 41$ divides $202634 = 2 \cdot 71 \cdot 1427$, and the only such divisor with $k \ge 6$ is 1427, giving $(k, n, r) = (77, 77831, 25943)$; but kn is odd, contrary to the handshake lemma. Therefore $\mathcal{E}$ is simple.

It follows that X, with adjacency matrix $J - I - \mathcal{E}$, is again a simple graph. It is d-regular and has least eigenvalue at least -2. If X had three or more components, then $n \ge 3(d + 1)$, or $n \le 3r + 6$. This leaves $t \in \{4, 6\}$, since $t = 2$ was excluded above; the remainders 272618 and 342602 have no divisor $18k + 41$ with $k \ge 6$. Hence X has at most two components.

For connected X, the cocktail-party case contradicts $r > 0$. We are left with $X = L(Y)$, with the same regular/semiregular dichotomy as above. If Y is q-regular on v vertices, then

$$q = \frac{n - r}{4} > \frac{n}{6}, \qquad v = \frac{2n}{q} < 12.$$

Simplicity and $n \ge 38$ leave only

$$(q, v; n, r) \in \{(8, 10; 40, 8), (9, 10; 45, 9),$$
$$(8, 11; 44, 12), (10, 11; 55, 15)\}.$$

The radius-two lower bound forces $k \le 7$, and direct substitution in the defining formula for r excludes all four cases.

For a semiregular bipartite root with part sizes $a \ge b \ge 2$, write $p = n/a \le q = n/b$ for its two degrees. Then

$$p + q = \frac{n - r}{2} > \frac{n}{3}, \qquad (b - 2)n = b(r + 2p),$$

so $b < 6$. Connectedness excludes $p = 1$. For $b = 2$, the left side of the displayed identity is zero while the right side is positive. For $b = 5$, the degree-sum inequality makes the sum of the reciprocals of the part sizes greater than one third. The larger part therefore has at most seven vertices, and the root has at most thirty-five edges, below the required line-graph order thirty-eight. The case $b = 4$ leaves only

$$(p, q, a, b; n, r) = (4, 10, 10, 4; 40, 12),\ (4, 11, 11, 4; 44, 14),$$

both excluded by the radius-two bound and the formula for r. For $b = 3$, one has $p = 2$ or 3, giving $n = 3r + 12$ or $n = 3r + 18$. Equation (2) gives the remainders $552554 = 2 \cdot 276277$ and $762506 = 2 \cdot 381253$. Their odd cofactors are prime and congruent to 13 (mod 18), whereas $18k + 41 \equiv 5 \pmod{18}$. Thus X cannot be connected.

It remains to exclude two components. The conditions $r > 0$ and $n > 3r$ mean $3C/(18k + 41) < n < 2C/(12k + 27)$. For $k = 6, 7, 8$ these intervals, $(50.26, 50.42)$, $(75.66, 75.89)$ and $(108.65, 108.94)$, contain no integer; for $k \ge 9$ the left endpoint exceeds 150. Both components then have order greater than 28, so we can apply the same classification to each component. Suppose a component of order N is the line graph of a simple q-regular graph Y on v vertices. Its degree is $d = 2q - 2$, and the number of edges of the root is $N = vq/2$. Hence

$$q = \frac{d + 2}{2} = \frac{n - r}{4} > \frac{n}{6} > 25, \qquad v = \frac{2N}{q} \le \frac{2n}{q} < 12.$$

Simplicity gives $q \le v - 1 \le 10$, a contradiction. If a component of order N has a semiregular bipartite root, its part sizes satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{d+2}{N} \ge \frac{n-r}{n+r+2} > \frac{49}{100}.$$

If $b \ge 3$, the five possible pairs (a, b) have $ab \le 18 < N$. If $b = 2$, connectedness forces the root $K_{d,2}$, whose line graph has order $2d$. Thus every component is K_{d+1}, a cocktail-party graph of order $d + 2$, or $L(K_{d,2})$. The first two types alone force $r \le 0$. If exactly one component has order $2d$, then $n = 3r + 8$ or $n = 3r + 10$. If both do, then $n = 2r + 8$. Together with $n > 3r$, this gives $r < 8$, hence $n < 24$, contrary to $n > 150$. For the two remaining mixed cases, the divisor checks at the end of this appendix exclude $n = 3r + 8$ and leave only $k = 123$ for $n = 3r + 10$. In that case kn is odd. This final contradiction proves $n \le 3r$.

All cases under the contrary assumption have been excluded. Substitute the definition of r into the three-to-one inequality to obtain

$$(18k + 41)n \le 3(k + 2)^2(k^2 + 3),$$

which is the displayed bound. □

Corollary D.1 (Rigidity at equality)**.** *Under the hypotheses and notation of Theorem 6.6, equality in $n \le 3r$ can occur only for*

$$\boxed{(k, n, r) = (103, 185220, 61740).}$$

This is an arithmetic parameter classification; it does not assert the existence of a graph with these parameters.

Proof. Equality gives

$$(18k + 41)r = (k + 2)^2(k^2 + 3) =: C_k.$$

Exact Euclidean division yields

$$18^4 C_k = (18k + 41)(5832k^3 + 10044k^2 + 17946k + 29107) + 66325,$$

where $66325 = 5^2 \cdot 7 \cdot 379$. Hence $18k + 41 \mid 66325$. Since $k \ge 6$, this divisor is at least 149 and is congruent to 5 modulo 18. Among the positive divisors of 66325, only 1895 has these properties. Thus $k = 103$, after which direct substitution gives $r = 61740$ and $n = 3r = 185220$. □

D.1. **Order comparisons and divisor checks.** This subsection tabulates the bound of Theorem 6.6 and completes the two-component case of its proof. For comparison across degrees, the diameter-three bounds before parity adjustment in degrees 6 through 20 are

k	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n	50	75	108	150	201	263	336	422	521	635	765	911	1075	1257	1459.

When k is odd, $kn = 2|E(G)|$ requires n to be even; in particular, the entries for $k = 7, 11, 15, 17, 19$ improve to $74, 262, 634, 910, 1256$, respectively.

In the two-component case of that proof, $n = 2r + 8$ is excluded directly, leaving $n = 3r + t$ with $t \in \{8, 10\}$, and (2) requires $18k + 41 \mid 132650 + 34992t$. The middle column below gives this remainder; enumerating its positive divisors of the form $18k + 41$, with $k \ge 6$, gives the last column.

t	remainder	candidate k
8	412586	none
10	482570	123

For $t = 10$, $k = 123$ gives $r = 104847$ and $n = 314551$, so kn is odd, contradicting the handshake identity. These divisions and candidates are checked over the integers by computer.

Appendix E. The degree-five case

This appendix completes the proof of Theorem 7.1. That proof reduced the case of a 5-regular strict counterexample G to diameter three and order $n \in \{30, 32\}$; we exclude both orders.

Completion of the proof of Theorem 7.1 for $k = 5$. At $n = 32$, restrict the adjacency quadratic form to vectors constant on each distance layer about a vertex. The layer sizes are $1, 5, 20, 6$. Write a for the average internal degree in the distance-two layer. Each vertex of that layer has one neighbor in the first layer, so the edges from the second to the third layer satisfy $20(4 - a) \le 6 \cdot 5$, giving $a \ge 5/2$. We use the layer indicators normalized to have length one. Restricting the adjacency quadratic form to their span gives the following symmetric compression; the argument does not require an equitable layer partition:

$$Q(a) = \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 \\ \sqrt{5} & 0 & 2 & 0 \\ 0 & 2 & a & (4-a)\sqrt{10/3} \\ 0 & 0 & (4-a)\sqrt{10/3} & 5 - \frac{10}{3}(4-a) \end{pmatrix}.$$

The derivative of its only a-dependent block is

$$\begin{pmatrix} 1 & -\sqrt{10/3} \\ -\sqrt{10/3} & 10/3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{10/3} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{10/3} \end{pmatrix} \succeq 0.$$

Because the derivative is positive semidefinite, no ordered eigenvalue of $Q(a)$ decreases as a increases. Write $5 = \mu_1(a) \ge \mu_2(a) \ge \mu_3(a) \ge \mu_4(a)$. Here $\mu_1(a) = 5$: $(1, \sqrt{5}, \sqrt{20}, \sqrt{6})$ is an eigenvector of $Q(a)$ for every a, and a compression of A has no eigenvalue above 5. It is therefore enough to bound the second eigenvalue at the smallest feasible value,

$$\chi_{Q(5/2)}(x) = \frac{1}{4}(x-5)p_{5,6}(x), \qquad p_{5,6}(x) = 4x^3 + 10x^2 - 16x - 30, \qquad p_{5,6}(11/6) = -29/27 < 0.$$

Since $p_{5,6}(11/6) < 0$ and $p_{5,6}(x) \to +\infty$ as $x \to +\infty$, its largest root satisfies $\mu_2(5/2) > 11/6 > -1 + \sqrt{8}$. Monotonicity and Poincaré separation give

$$\theta_2(A) \ge \mu_2(a) \ge \mu_2(5/2),$$

where $\theta_2(A)$ is the second-largest adjacency eigenvalue; this contradicts the bound $\theta_2(A) < -1 + \sqrt{8}$ given by Theorem 3.1. At $n = 30$, Meringer's isomorph-free enumeration leaves exactly four $(5, 5)$-cages [18, p. 142]; each fixed record has an exact distance eigenvalue at most -5. The characteristic-polynomial factors and their roots are listed below; the classification ensures that the four records cover every case; the factors give the distance-spectral obstruction. The accompanying release contains an independent exact check of the complete case split and of the four fixed graph6 records. □

For the four degree-five cages, the displayed factor divides $\det(xI - D)$ over $\mathbb{Z}[x]$. Its negative root is an eigenvalue, which suffices for $\lambda_{\min}(D) < -5$; no assertion that it is the least eigenvalue is needed.

fixed graph6 record	distance factor	eigenvalue
Foster	$x^2 + 6x - 11$	$-3 - 2\sqrt{5}$
Meringer	$x + 6$	-6
Robertson–Wegner	$x + 6$	-6
Wong	$x^2 + 6x - 11$	$-3 - 2\sqrt{5}$

The records have order 30, degree five, girth five and diameter three, and are pairwise nonisomorphic. Meringer's classification [18, p. 142] supplies completeness. The integer characteristic polynomials, graph properties and nonisomorphism checks are reproduced by exact computation.

Appendix F. Local cycle counts and the degree-six, order-fifty case

We collect necessary conditions for the degree-six, order-fifty case left open by the main order bound. The first subsection holds in every degree: it applies the principal-minor method of Section 6 to the number of five-cycles through an edge. The second bounds the multiplicity of the adjacency eigenvalue minus two without assuming strictness. The remaining subsections assume a strict counterexample and derive local cycle and signed-graph restrictions.

F.1. **Five-cycle counts through an edge.**

Proposition F.1 (Edge-local cycle bounds)**.** *Let G be connected and k-regular, where $k \ge 4$, of girth at least five and diameter three, and suppose*

$$|\theta + 1| \le \sqrt{2k - 2} \qquad (\theta \ne k)$$

for every adjacency eigenvalue θ. For an edge uv, let σ_{uv} be the number of 5-cycles containing that edge. Then

$$2k-2 \le \sigma_{uv} \le \frac{2(k+2)^2(k^2+3)}{n} - 10k - 26.$$

If $n = k^2+1+c$, then also

$$\sigma_{uv} \ge (k-1)^2 - c.$$

Proof. Let $\mathcal{S}_k = -g_k(A) + (C_k/n)J$ be as in Theorem 6.5; its eigenvalues $(2k-2-(\theta+1)^2)(\theta+2)^2$ and 0 show $\mathcal{S}_k \succeq 0$ under the closed window. By the expansion in the proof of Theorem 6.4, $g_k = 6(k+2)F_0 + (4k+14)F_1 + (k+13)F_2 + 6F_3 + F_4$. For an edge uv, girth at least five gives $(F_2(A))_{uv} = (F_3(A))_{uv} = 0$, while $(F_4(A))_{uv}$ counts nonbacktracking walks $u\,a\,b\,c\,v$; by girth these are paths, which close up with uv to the five-cycles through uv. Hence $(g_k(A))_{uu} = 6(k+2)$ and $(g_k(A))_{uv} = 4k+14+\sigma_{uv}$, so the diagonal and edge entries of $\mathcal{S}_k$ are

$$a = \frac{C_k}{n} - 6(k+2), \qquad b = \frac{C_k}{n} - (4k+14) - \sigma_{uv}.$$

The principal submatrix on $\{u,v\}$ is $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, so $a \ge 0$ and $-a \le b \le a$. The inequality $b \le a$ gives $\sigma_{uv} \ge 2k-2$, while $b \ge -a$ gives the stated upper bound.

For the final bound, every radius-two ball has size k^2+1. The set

$$\{u,v\} \cup (N(u) \setminus \{v\}) \cup (N(v) \setminus \{u\})$$

contains $2k$ vertices and lies in $B_2(u) \cap B_2(v)$. Every further intersection vertex is at distance two from both endpoints, so it is the middle vertex b of exactly one path $u\,a\,b\,c\,v$ above; conversely, the middle vertex of each such path is such an intersection vertex. Hence

$$|B_2(u) \cap B_2(v)| = 2k + \sigma_{uv}.$$

Inclusion–exclusion and $n = k^2+1+c$ now give $\sigma_{uv} \ge (k-1)^2 - c$. □

Strict form of Proposition F.1. Under the same hypotheses, with a strict spectral window, both endpoint bounds improve to

$$2k-1 \le \sigma_{uv} < \frac{2(k+2)^2(k^2+3)}{n} - 10k - 26.$$

For the lower bound, the proof of Theorem 6.5 gives $(g_k(A))_{uv} \ge h_k + 1 = 6(k+2)+1$, whereas the edge calculation above gives $(g_k(A))_{uv} = 4k+14+\sigma_{uv}$. For the upper bound, use $\mathcal{S}_k, a, b$ from the proof above. For a positive semidefinite matrix, a vector with zero quadratic form lies in the kernel. Thus equality $b = -a$ would put $e_u + e_v$ in $\ker \mathcal{S}_k$. In the strict window,

$$\ker \mathcal{S}_k = \langle \mathbf{1} \rangle \oplus \ker(A + 2I),$$

so $w = e_u + e_v - (2/n)\mathbf{1}$ would satisfy $Aw = -2w$. At the u-coordinate, this gives $1 - 2k/n = -2 + 4/n$, hence $n = 2(k+2)/3 < k+1$, contradicting the order bound for a simple k-regular graph. Thus $b > -a$, proving the strict upper bound.

For $\ell \ge 3$, let N_ℓ denote the number of ℓ-cycles in G.

Corollary F.2 (Edge–cycle divisibility sieve)**.** *Under the hypotheses of Proposition F.1, put*

$$L_{k,n} = \max\{2k-2,\ 2k^2-2k+2-n\}, \qquad U_{k,n} = \frac{2(k+2)^2(k^2+3)}{n} - 10k - 26.$$

Necessarily

$$\boxed{\lceil L_{k,n} \rceil \le \lfloor U_{k,n} \rfloor.}$$

If both sides equal an integer s, then every edge of G lies in exactly s five-cycles and

$$\boxed{5 \mid \frac{skn}{2}.}$$

Under the strict spectral window, put $L^{\mathrm{str}}_{k,n} = \max\{2k-1,\ 2k^2-2k+2-n\}$. Then the stronger necessary condition is

$$\lceil L^{\mathrm{str}}_{k,n} \rceil \le \lceil U_{k,n} \rceil - 1.$$

If these two integer bounds coincide at s, the same divisibility conclusion holds.

Proof. Writing $n = k^2 + 1 + c$, the two lower bounds in Proposition F.1 combine to

$$\sigma_{uv} \geq \max\{2k - 2, (k-1)^2 - c\} = L_{k,n},$$

while its upper bound is $U_{k,n}$. Since σ_{uv} is an integer, the first conclusion follows. If the two integer bounds coincide at s, then $\sigma_{uv} = s$ for every edge. Counting edge–five-cycle incidences gives

$$5N_5 = \sum_{uv \in E(G)} \sigma_{uv} = s|E(G)| = \frac{skn}{2},$$

which proves the divisibility condition. The strict-window version follows from the strengthened edge bounds above: for integral σ_{uv}, the inequality $\sigma_{uv} < U_{k,n}$ is equivalent to $\sigma_{uv} \leq \lceil U_{k,n} \rceil - 1$. □

The first excluded order in degree six. For a connected 6-regular strict counterexample, the proof of Theorem 7.2 shows that the diameter is three, and Theorem 3.1 gives the spectral window, so Proposition F.1 applies. This gives a short local obstruction to the first order excluded by Theorem 7.2. If $n = 51$, Corollary F.2 gives $\lceil L_{6,51} \rceil = \lfloor U_{6,51} \rfloor = 11$, forcing

$$5N_5 = 153 \cdot 11 = 1683, \qquad 5 \nmid 1683,$$

which is impossible.

F.2. **The adjacency multiplicity bound at order fifty.**

Proposition F.3 (The order-50 (-2)-multiplicity bound)**.** *Let G be 6-regular, of order 50, and of girth at least five. The multiplicity $m_{-2}(A)$ of the adjacency eigenvalue -2 satisfies*

$$m_{-2}(A) \leq 20.$$

Proof. We isolate the eigenvalue whose multiplicity is to be bounded. Put $m = m_{-2}(A)$, remove the principal eigenvalue 6 and the m copies of -2, and write $\mu_0, \ldots, \mu_4$ for the moments of the remaining eigenvalues. Girth at least five gives

$$\operatorname{tr} A = 0, \quad \operatorname{tr} A^2 = 300, \quad \operatorname{tr} A^3 = 0, \quad \operatorname{tr} A^4 = 3300,$$

and therefore

$$(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) = (49 - m, -6 + 2m, 264 - 4m, -216 + 8m, 2004 - 16m).$$

The moment matrix

$$\mathsf{H} = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix}$$

is the Gram matrix obtained by evaluating the constant, linear, and quadratic monomials at the remaining eigenvalues, counted with multiplicity. It is therefore positive semidefinite, so its determinant is nonnegative. Expanding that determinant gives

$$\det \mathsf{H} = 3600(1625 - 81m) \geq 0.$$

Since m is an integer, $m \leq 20$. □

F.3. **Cycle counts at order fifty.** The order bound still allows the degree-six boundary case. We examine its local cycle counts and a representation of its signed matrix by inner products of vectors. These necessary conditions restrict the possible structure without settling existence.

Proposition F.4 (An integer-sensitive layer bound)**.** *Let G be a finite simple connected 6-regular graph of order 50, girth at least five and diameter three. If $\tau(v)$ counts five-cycles through v, then, for every vertex v,*

$$\Phi(G) \leq \frac{795 - 21\tau(v)}{377}.$$

In particular, $\tau(v) \geq 38$ implies $\Phi(G) \leq -3/377 < 0$.

Proof. Fix v and put $L_i = \Gamma_i(v)$. Girth and diameter give the four layer sizes $1, 6, 30, 13$. Every vertex of L_2 has one parent in L_1, and an edge inside L_2 corresponds to exactly one five-cycle through v. Thus, writing $\tau = \tau(v)$,

$$e(L_2) = \tau, \qquad e(L_2, L_3) = 150 - 2\tau.$$

For $u \in L_2$, let s_u count its neighbors in L_2; for $w \in L_3$, let e_w count its neighbors in L_3. Counting degrees within and between these layers gives

$$\sum_{u \in L_2} s_u = 2\tau, \qquad \sum_{w \in L_3} e_w = 2\tau - 72 \geq 0.$$

Since these degrees are integers, the inequalities $(s_u-2)(s_u-3)\ge 0$ and $e_w(e_w-1)\ge 0$ imply

$$\sum_{u\in L_2} s_u^2 \ge 10\tau - 180, \qquad \sum_{w\in L_3} e_w^2 \ge 2\tau - 72.$$

To use these degree sums in a Rayleigh bound, take f constant on L_0, L_1, L_2, L_3 with values $28, 9, -1, -4$. Then $f \perp \mathbf{1}$, $\|f\|^2 = 1508$, and $(A+I)f$ has respective entries $82, 32, 3s_u - 12 = -1 + 9 - s_u - 4(5-s_u)$ and $-10 - 3e_w = -4 - (6-e_w) - 4e_w$. Since $\sum_u (3s_u-12)^2 = 9\sum_u s_u^2 - 144\tau + 4320$ and $\sum_w (10+3e_w)^2 = 1300 + 60\sum_w e_w + 9\sum_w e_w^2$, the degree-sum bounds give

$$\begin{aligned}\|(A+I)f\|^2 &= 82^2 + 6\cdot 32^2 + \sum_{u\in L_2}(3s_u-12)^2 + \sum_{w\in L_3}(10+3e_w)^2 \\ &\ge 12868 + (2700-54\tau) + (138\tau - 3668) = 11900 + 84\tau.\end{aligned}$$

The distance identity is $D + 6I = 3J + 10I - (A+I)^2$. Since G is regular, $\Phi(G) = \lambda_{\min}(D+6I)$; the Rayleigh principle applied to this nonzero f gives

$$\Phi(G) \le \frac{10\|f\|^2 - \|(A+I)f\|^2}{\|f\|^2} \le \frac{3180 - 84\tau}{1508} = \frac{795 - 21\tau}{377}.$$

The calculation uses the fact that the degrees within each layer are integers; no equitable-partition assumption is used. □

Theorem F.5. *Let G be a connected 6-regular graph of order 50 and girth at least five, and suppose $\Phi(G) > 0$. Then G has diameter three. For an edge e, let σ_e be the number of five-cycles containing e. Then $\sigma_e \in \{12, 13\}$. Call e low or high according as $\sigma_e = 12$ or 13, respectively, and let H be the spanning subgraph of high edges. Put $m = |E(H)|$, and let $\tau(v)$ be the number of five-cycles through v. Then*

$$\tau(v) \in \{36, 37\}, \qquad d_H(v) = 2\tau(v) - 72 \in \{0, 2\},$$
$$m \equiv 0 \pmod 5, \qquad N_5 = 360 + \frac{m}{5}.$$

For a two-edge path $u - v - w$, put

$$R_{uvw} = 6\alpha_{uvw} + \beta_{uvw},$$

where α and β count the five- and six-cycles containing the path. The allowed values are

types of the two incident edges	R_{uvw}
low–low	30, 31, 32
mixed	30, 31, 32
high–high	30, 31.

Writing $S_2 = \sum_v d_H(v)^2$, one has

$$1950 - m \le N_6 \le 2200 - \frac{5m}{6} - \frac{S_2}{12},$$
$$\frac{43m^2 - 70200m + 119632500}{58500} \le N_6 < \frac{4220000 - 2200m - 7m^2}{2000}.$$

Writing n_i for the number of vertices v with $d_H(v) = i$, where $i \in \{0, 2\}$, one has $n_0 + n_2 = 50$ and $m = n_2 \equiv 0 \pmod 5$, so

$$(n_0, n_2) = (50 - 5j, 5j) \qquad \text{for some } j \in \{0, 1, \dots, 10\}.$$

Each of these eleven pairs admits at least one integer N_6 satisfying all four displayed bounds, including the strict upper moment bound. Thus the bounds restrict N_6, but do not eliminate any further degree profiles.

Proof. Theorems 5.2 and 5.3 exclude diameter at least four. Diameter at most two is also impossible: Proposition 2.1 would give $50 = 1 + 6^2 = 37$. Thus the diameter is three. Theorem 3.1 now gives $|\theta + 1| < \sqrt{10}$ for every nonprincipal adjacency eigenvalue. Corollary F.2 gives

$$L_{6,50} = 12, \qquad U_{6,50} = \frac{346}{25} < 14.$$

Since σ_e is integral, $\sigma_e \in \{12, 13\}$ for every edge e. For a fixed vertex, the layer degree sums in Proposition F.4 give $\tau \ge 36$. Its Rayleigh bound and $\Phi(G) > 0$ give $\tau < 265/7$, hence $\tau \le 37$. Thus $\tau \in \{36, 37\}$. Finally, $\sum_{e\ni v} \sigma_e = 2\tau(v)$, so the high-edge degree is $2\tau(v) - 72$; counting edge–five-cycle incidences gives $5N_5 = 12\cdot 150 + m$.

The next restrictions come from a two-path $u - v - w$. Girth at least five gives $(A^3)_{uw} = \alpha_{uvw}$ and $(A^4)_{uw} = 16 + \beta_{uvw}$. Here $16 = 3\cdot 6 - 2$ counts the length-four walks with an immediate reversal: the three reversal positions give six choices each, with two overlaps. The remaining walks correspond to six-cycles containing $u - v - w$. Let $h_e = 0$ for a low edge and $h_e = 1$ for a high edge. Here

$g_6(x) = x^4 + 6x^3 + 3x^2 - 28x - 36$ and $C_6/50 = 2496/50$. Girth five gives $(A^2)_{uu} = 6$, $(A^4)_{uu} = 66$ and, on an edge, $(A^3)_{uv} = 11$, $(A^4)_{uv} = \sigma_{uv}$. Hence $g_6(A)$ has diagonal entries 48, edge entries $38+\sigma_{uv} = 50+h_{uv}$, and entry $19 + R_{uvw}$ at $\{u, w\}$. Substitution into the centered slack matrix gives

$$25\mathcal{S}_6[\{u, v, w\}] = \begin{pmatrix} 48 & -2 - 25h_{uv} & 773 - 25R_{uvw} \\ -2 - 25h_{uv} & 48 & -2 - 25h_{vw} \\ 773 - 25R_{uvw} & -2 - 25h_{vw} & 48 \end{pmatrix} \succeq 0.$$

The 2×2 minors give $R_{uvw} \in \{29, 30, 31, 32\}$. At $R_{uvw} = 29$, the Gram norm of $e_u - e_w$ vanishes, so this vector lies in the kernel of $\mathcal{S}_6$. On $\mathbf{1}^\perp$, $\mathcal{S}_6 = -g_6(A)$. The strict shifted window excludes the endpoint zeros of g_6, leaving only its double zero at -2. Thus $A(e_u - e_w) = -2(e_u - e_w)$, which is impossible: its u-coordinate would have to be both zero and -2. If both edges are high and $R_{uvw} = 32$, the displayed determinant is -33750, also impossible. This proves the local table.

To pass from the local table to global cycle bounds, sum over unoriented two-edge paths. Each five-cycle contains five such paths and each six-cycle contains six, so

$$\sum_{u-v-w} R_{uvw} = 30N_5 + 6N_6 = 10800 + 6m + 6N_6.$$

There are $50\binom{6}{2} = 750$ paths, of which

$$\sum_v \binom{d_H(v)}{2} = \frac{S_2 - 2m}{2}$$

have two high edges. Every local value is at least 30 and at most 32, with upper bound 31 for a high–high path. Hence

$$22500 \le 10800 + 6m + 6N_6 \le 24000 - \frac{S_2 - 2m}{2}.$$

Rearranging gives the first pair of N_6 bounds. Since $d_H(v) \in \{0, 2\}$, $S_2 = 4m$, and the first upper bound is $2200 - 7m/6$. For the second pair, we evaluate polynomials on the shifted nonprincipal spectrum. A positive semidefinite Gram matrix gives the lower bound. For the strict upper bound, the spectral window gives positive weights, and diameter three ensures that the weighted Gram matrix is positive definite. The last subsection of this appendix gives this support argument, both matrices and their determinant calculations. For each of the eleven profiles the second pair is sharper; it leaves $2045 \le N_6 \le 2109$ at $m = 0$ and $1987 \le N_6 \le 2046$ at $m = 50$. The exact matrix calculations for the four N_6 bounds, the layer-vector calculation and the eleven-profile check are reproduced by exact computation. These eleven profiles are necessary data, not constructions of graphs. □

Corollary F.6 (Eleven high-edge profiles)**.** *Under the hypotheses of Theorem F.5, the high-edge graph H is a disjoint union of cycles of length at least five, covering $5j$ vertices for some $j \in \{0, 1, \dots, 10\}$, and $50 - 5j$ isolated vertices.*

Proof. All degrees of H are zero or two, so every nontrivial component is a cycle. As a subgraph of G, it has no triangle or four-cycle. The cycles cover $n_2 = m$ vertices, a multiple of five by Theorem F.5.

The intrinsic degree and girth conditions on H alone cannot eliminate any of these eleven values of j: for $j \ge 1$, the auxiliary graph $C_{5j} \sqcup (50 - 5j)K_1$ has exactly these data; for $j = 0$, use the empty graph on fifty vertices. These auxiliary graphs are not asserted to occur as high-edge subgraphs of suitable graphs G. Further exclusion must use their relation to G, beyond the abstract degree and girth conditions. □

F.4. **A signed graph from the excess matrix.** We next compare eigenspaces of the adjacency and excess matrices. In a signed adjacency matrix, a nonzero off-diagonal entry records an edge and gives its sign.

Proposition F.7 (Signed-complement bridge)**.** *Under the hypotheses and notation of Theorem 6.6, assume $0 < r < n$ and define*

$$S = (6k + 14)J - 2I - g_k(A).$$

Then S is a signed adjacency matrix:

$$S_{uu} = 0, \qquad S_{uv} \in \{-1, 0, 1\} \quad (u \ne v).$$

It has constant signed row sum $(n - r - 4)/2$, and

$$S + 2I \succeq 0, \qquad E_{-2}(S) = E_{-2}(A).$$

Proof. Write $x = r/n \in (0,1)$, $\rho = (1+x)/2$, and $\mathcal{E} = \mathcal{E}_k$. Since

$$\mathcal{S}_k = I - \mathcal{E} + \rho J \succeq 0,$$

its 2×2 principal minors give $-1 \le \mathcal{E}_{uv} \le 2+x$. The entries of $\mathcal{E}$ are nonnegative integers, so $\mathcal{E}_{uv} \in \{0,1,2\}$ for $u \ne v$, and $S = J - I - \mathcal{E}$ has the asserted entries (so $S = A(X)$, with X as in the proof of Theorem 6.6 in Appendix D, when $\mathcal{E}_k$ is a 0/1 matrix). The row sum follows from $\varepsilon = (n+r+2)/2$. Moreover,

$$S + 2I = \mathcal{S}_k + \frac{n-r}{2n} J \succeq 0.$$

To identify the shared eigenspace, use the fact that A and S commute. A nonprincipal adjacency eigenvector with eigenvalue θ is an eigenvector of S with eigenvalue $-2 - g_k(\theta)$. Inside the open shifted WOW window, g_k vanishes only at $\theta = -2$, since its other zeros are the excluded endpoints. The eigenvalue of S on the constant vector is $(n-r-4)/2 > -2$, since $r < n$. Thus the two (-2)-eigenspaces coincide. □

Remark F.8 (Signed-root Gram formulation). Under the hypotheses of Theorem F.5, put

$$T = 50J - g_6(A) - 2I.$$

Then

$$T\mathbf{1} = 2\mathbf{1}, \qquad T + 2I \succeq 0, \qquad 25\mathcal{S}_6 = 25(T+2I) - 2J.$$

This is the specialization of Proposition F.7: the excess parameter is $r = 42$, so the signed row sum is 2. By Proposition F.7, $T_{uu} = 0$ and $T_{uv} \in \{-1,0,1\}$ for $u \ne v$. Explicitly, $T_{uv} = -h_{uv}$ on edges, $T_{uw} = 31 - R_{uvw}$ at distance two, and $T_{uz} = 50 - q_{uz}$ at distance three, where $q_{uz} = 6(A^3)_{uz} + (A^4)_{uz} \in \{49,50,51\}$. Consequently $T + 2I$ is the Gram matrix of 50 vectors of norm $\sqrt{2}$, with inner products in $\{-1,0,1\}$ between distinct vectors. Its row sum is 4. The kernel of $T + 2I$ is $E_{-2}(A)$, so Proposition F.3 gives

$$\operatorname{rank}(T + 2I) \ge 30.$$

Thus the unresolved degree-six boundary can be viewed as a root-type integral Gram problem constrained by the graph's distance classes and local cycle data.

F.5. **Why the signed complement is disconnected.** Connectedness of a signed graph means connectedness after forgetting the edge signs. We show that connectedness here would force a root representation incompatible with the parity of the number of negative edges.

Theorem F.9 (Disconnected signed complement)**.** *Under the hypotheses of Theorem F.5, the edge-signed graph with adjacency matrix T from Remark F.8 is disconnected.*

Proof. Let N_i denote the number of i-cycles in G. In terms of the degree-six nonbacktracking polynomials,

$$\begin{aligned}(g_6+2)^2 = {} & 28144F_0 + 18220F_1 + 8838F_2 + 3576F_3 + 1233F_4 \\ & + 352F_5 + 78F_6 + 12F_7 + F_8.\end{aligned}$$

Girth at least five gives $\operatorname{tr} F_i(A) = 0$ for $1 \le i \le 4$. For lengths five and six, every closed nonbacktracking walk is a directed cycle. At lengths seven and eight there are also the walks obtained by attaching a one-edge tail to a directed five- or six-cycle; a tail at a cycle vertex avoids the two cycle edges there, so each directed five- or six-cycle carries $5\cdot 4$ or $6\cdot 4$ such walks. Hence

$$\begin{aligned}\operatorname{tr} F_5(A) &= 10N_5, & \operatorname{tr} F_6(A) &= 12N_6, \\ \operatorname{tr} F_7(A) &= 14N_7 + 40N_5, & \operatorname{tr} F_8(A) &= 16N_8 + 48N_6.\end{aligned}$$

These walk counts give the full trace of the squared polynomial. On the nonprincipal space, the signed matrix and the polynomial differ only by a sign, so their squares have the same trace there. We therefore subtract the principal adjacency contribution $(g_6(6)+2)^2 = 2498^2$ and add the square of the signed matrix's eigenvalue 2 on the constant vector, giving

$$\operatorname{tr} T^2 = 8(500N_5 + 123N_6 + 21N_7 + 2N_8 - 604100).$$

If P_+ and P_- are the numbers of positive and negative signed edges, then the signed row sum and the zero diagonal give

$$P_+ - P_- = 50, \qquad \operatorname{tr} T^2 = 2(P_+ + P_-) = 100 + 4P_-.$$

The polynomial expression for the trace is a multiple of eight. Comparing it with the signed-edge count shows that P_- is odd.

We show that connectedness is incompatible with this parity. Suppose T is connected. Proposition F.3 gives $\operatorname{rank}(T + 2I) \ge 30$. The representation theorem for connected edge-signed graphs with smallest

eigenvalue at least -2 [24, Theorem 2] (see also [17]) represents $T + 2I$ in a root system of type D_ℓ or E_8. The exceptional root system has rank eight. The rank bound therefore rules out E_8, leaving

$$B^{\mathsf{T}}B = T + 2I$$

for an integral matrix B whose columns b_u are roots $\pm e_i \pm e_j$. Put $s = B\mathbf{1}$. Since $(T + 2I)\mathbf{1} = 4\mathbf{1}$,

$$b_u \cdot s = 4 \quad \text{for every } u, \qquad \|s\|^2 = 200.$$

Changing signs of coordinate axes if necessary, assume $s_i \geq 0$. Form a multigraph with the used coordinates as vertices and one edge for each root column, joining its two nonzero coordinates. This is the coordinate-support multigraph. It is connected: otherwise the columns of B would split into two families with zero inner products between them, and T would be disconnected. If v coordinates are used, then

$$30 \leq v \leq 51 :$$

the lower bound is the Gram rank, and a connected support multigraph with fifty root-edges has at most fifty-one vertices.

Call the nonnegative coordinate values the levels. Each root joins two coordinates whose levels satisfy $\pm s_i \pm s_j = 4$. Along such an edge, the levels either differ by four or sum to four. Starting from one coordinate and following the connected support multigraph, all levels must therefore lie in one of

$$0, 4, 8, \ldots; \qquad 2, 6, 10, \ldots; \qquad 1, 3, 5, \ldots.$$

The first family would make $\|s\|^2$ divisible by 16, contrary to $200 \equiv 8 \pmod{16}$.

In the second family, a level at least 10, together with the other $v - 1 \geq 29$ levels, would contribute at least $10^2 + 29 \cdot 2^2 > 200$. Thus only levels 2 and 6 occur. If their multiplicities are ν_2, ν_6, then

$$\nu_2 + 9\nu_6 = 50, \qquad v = \nu_2 + \nu_6 = 50 - 8\nu_6 \geq 30,$$

so

$$(\nu_2, \nu_6) \in \{(50, 0), (41, 1), (32, 2)\}.$$

Every root is now either $e_i + e_j$ with $s_i = s_j = 2$, or $e_i - e_j$ with $s_i = 6$ and $s_j = 2$, after ordering the two coordinates. On a fixed support the sign pattern is unique, and duplicate columns would have inner product 2. Distinct roots in this family therefore share at most one coordinate. At a level-two coordinate, let p_i, m_i count positive and negative incidences. We have $p_i - m_i = 2$. Each level-six coordinate has six incidences, all paired with negative incidences at level-two coordinates. A negative Gram product occurs when two roots meet at a level-two coordinate with opposite signs. The shared coordinate is unique, so each such product is counted once in

$$P_- = \sum_{i : s_i = 2} p_i m_i.$$

Consequently

$$P_- \equiv \sum_{i : s_i = 2} p_i m_i \equiv \sum_{i : s_i = 2} m_i = 6\nu_6 \equiv 0 \pmod{2},$$

contradicting the parity already proved.

For the odd levels, let a_t, c_t count coordinates at levels $4t + 1, 4t + 3$, respectively. These form two chains, one in each odd residue class modulo four. Sum the signed incidences along each chain. A difference root has one positive and one negative coordinate, so its contributions cancel within that chain. The only roots joining the chains are sum roots $e_i + e_j$ between levels 1 and 3, and they contribute equally to the two sums. Hence

$$\sum_{t \geq 0} (4t + 1)a_t = \sum_{t \geq 0} (4t + 3)c_t.$$

To see the cancellation in the squared levels, write

$$(4t + 1)^2 - 3 = 16t(t + 1) - 2(4t + 1),$$
$$(4t + 3)^2 - 3 = 16t(t + 1) + 2(4t + 3).$$

After multiplying by a_t and c_t and summing, the linear terms cancel by the signed-incidence identity above. Since $\|s\|^2 = 200$, this gives

$$\frac{200 - 3v}{32} = \sum_{t \geq 1} \binom{t + 1}{2} (a_t + c_t).$$

The right-hand side is an integer. Hence $v \equiv 24 \pmod{32}$, which is impossible for $30 \leq v \leq 51$. All three level families are excluded, proving that T is disconnected. □

F.6. **Six-cycle bounds from shifted spectral moments.** We give the determinant calculation for the second pair of six-cycle bounds in Theorem F.5. Work under its hypotheses. The earlier part of its proof has established diameter three and $N_5 = 360 + m/5$, where m counts the high edges. Let $\theta_1, \dots, \theta_{49}$ be the nonprincipal adjacency eigenvalues, and put

$$y_i = \theta_i + 1, \qquad s_j = \sum_{i=1}^{49} y_i^j \quad (0 \le j \le 6).$$

Theorem 3.1 gives $|y_i| < \sqrt{10}$.

Girth and regularity give

$$\operatorname{tr} A = \operatorname{tr} A^3 = 0, \qquad \operatorname{tr} A^2 = 300, \qquad \operatorname{tr} A^4 = 3300.$$

For the degree-six nonbacktracking polynomials from Proposition 6.2, the recurrence gives

$$F_5(x) = x^5 - 21x^3 + 85x, \qquad F_6(x) = x^6 - 26x^4 + 165x^2 - 150.$$

A closed nonbacktracking walk of length five or six is a cycle: a repeated vertex before the final return would leave a cycle of length at least five and require at least two further steps. Each cycle is counted once for every starting vertex and orientation. Thus $\operatorname{tr} F_5(A) = 10N_5$ and $\operatorname{tr} F_6(A) = 12N_6$, yielding

$$(\operatorname{tr} A^j)_{j=0}^6 = (50, 0, 300, 0, 3300, 3600 + 2m, 43800 + 12N_6).$$

The principal adjacency eigenvalue is six. Binomial expansion therefore gives

$$s_j = \sum_{r=0}^{j} \binom{j}{r} \operatorname{tr} A^r - 7^j,$$

and hence

$$(s_0, \dots, s_6) = (49, 43, 301, 607, 2749, 6343 + 2m, 1801 + 12m + 12N_6).$$

For the lower bound, take the vectors $u_i = (1, y_i, y_i^2, y_i^3)^{\mathsf{T}}$. Their Gram sum is

$$\mathsf{M} = \sum_{i=1}^{49} u_i u_i^{\mathsf{T}} = (s_{a+b})_{a,b=0}^3 = \begin{pmatrix} 49 & 43 & 301 & 607 \\ 43 & 301 & 607 & 2749 \\ 301 & 607 & 2749 & 6343 + 2m \\ 607 & 2749 & 6343 + 2m & 1801 + 12m + 12N_6 \end{pmatrix} \succeq 0.$$

For the upper bound, put $v_i = (1, y_i, y_i^2)^{\mathsf{T}}$ and weight each rank-one matrix by $10 - y_i^2 > 0$. This gives the following weighted moment matrix, also called the localizing matrix:

$$\mathsf{L} = \sum_{i=1}^{49} (10 - y_i^2) v_i v_i^{\mathsf{T}} = (10 s_{a+b} - s_{a+b+2})_{a,b=0}^2 = \begin{pmatrix} 189 & -177 & 261 \\ -177 & 261 & -273 - 2m \\ 261 & -273 - 2m & 25689 - 12m - 12N_6 \end{pmatrix} \succ 0.$$

To show that this weighted moment matrix is positive definite, note that diameter three implies that I, A, A^2, A^3 are linearly independent: evaluate a linear relation successively at vertex pairs of distances three, two, one, and zero. Since A is symmetric, it has at least four distinct eigenvalues. Connectedness makes the principal eigenvalue simple, so at least three distinct values occur among the y_i. A nonzero polynomial of degree at most two cannot vanish at all three. Together with $10 - y_i^2 > 0$, this shows that the quadratic form of L is positive on every nonzero vector. For the four-by-four matrix M, the test polynomials can have degree three. Three distinct values do not rule out a nonzero polynomial vanishing at all of them, so the same argument gives only semidefiniteness.

Expanding the determinants gives

$$\det \mathsf{M} = 1200(58500N_6 - 43m^2 + 70200m - 119632500),$$
$$\det \mathsf{L} = 108(4220000 - 2200m - 7m^2 - 2000N_6).$$

Thus $\det \mathsf{M} \ge 0$ and $\det \mathsf{L} > 0$. Dividing by the positive constants and isolating N_6 gives

$$\frac{43m^2 - 70200m + 119632500}{58500} \le N_6 < \frac{4220000 - 2200m - 7m^2}{2000}.$$

Appendix G. Further deletions inside a neighborhood

The deletion sets in this appendix lie in the neighborhood of a vertex, so Proposition 8.4 applies to them. Neither result is needed for the deletion bounds of Section 8.

Proposition G.1 (Six-neighbor punctures)**.** *Let M be the Hoffman–Singleton graph. For every $w \in V(M)$ and every $S \subseteq N_M(w)$ with $|S| = 6$, the graph $M - S$ is a strict counterexample to WOW-284.*

Proof. Put $H = M - S$. Each vertex in $\Gamma_2(w)$ has a unique neighbor in $N_M(w)$, which we call its parent; the vertices adjacent to a given parent in the second sphere are its children. Partition $V(H)$ into $\{w\}, \mathcal{A}, \mathcal{B}, C$, where $\mathcal{A} = N_M(w) \setminus S$, $\mathcal{B}$ consists of children of the deleted parents, and C consists of children of the surviving parents. Their sizes are

$$1, \qquad 1, \qquad 36, \qquad 6.$$

The Moore property gives no edge between children of the same parent and exactly one neighbor in every other child class. Indeed, if x has parent p and $q \in N_M(w) \setminus \{p\}$, the unique common neighbor of the nonadjacent pair x, q belongs to $\Gamma_2(w)$ and has parent q.

The distance sum from a vertex to each part depends only on its own part. For example, a vertex of $\mathcal{B}$ has five siblings at distance three, five other vertices of $\mathcal{B}$ at distance one, and twenty-five at distance two. Its distance sum to $\mathcal{B}$ is therefore $5 \cdot 3 + 5 \cdot 1 + 25 \cdot 2 = 70$. Thus the partition is equitable for the distance matrix. On vectors constant on each part, the distance operator acts by the following row-sum quotient, in the order $\{w\}, \mathcal{A}, \mathcal{B}, C$:

$$Q = \begin{pmatrix} 0 & 1 & 108 & 12 \\ 1 & 0 & 72 & 6 \\ 3 & 2 & 70 & 11 \\ 2 & 1 & 66 & 10 \end{pmatrix}.$$

The distances in the original Moore graph, restricted to the surviving vertices, are given by $D_0 = 2(J - I) - A(H)$. Proposition 8.4 changes only the distances from w to $\mathcal{B}$ and between two children of the same deleted parent, adding one in each case.

Let U be the four-dimensional space of vectors constant on each part. Its orthogonal complement $U^\perp$ consists of vectors whose coordinates sum to zero separately on each part. Both D_0 and the distance-increase matrix E_S preserve U. Their symmetry also makes $U^\perp$ invariant. We can therefore check the shifted distance matrix on these two spaces separately.

We first bound the correction on $U^\perp$. The w-coordinate is zero because its part has only one vertex. For $x \in U^\perp$,

$$x^{\mathsf{T}} E_S x = \sum_{z \in S} \left(\sum_{y \in N_M(z) \setminus \{w\}} x_y \right)^2 - \sum_{y \in \mathcal{B}} x_y^2 \geq -\|x\|^2.$$

The distance matrix of the original Moore graph has least eigenvalue -4. Taking its principal submatrix on the surviving vertices gives $D_0 \succeq -4I$, and the correction estimate above gives $D(H)|_{U^\perp} \succeq -5I$. Theorem 8.1 gives $\delta^*(H) = 43/7 > 5$.

It remains to check the restriction to U. In the basis of cell indicators, the quadratic form of $7D(H) + 43I$ has symmetric matrix

$$T = \operatorname{diag}(1, 1, 36, 6)(7Q + 43I) = \begin{pmatrix} 43 & 7 & 756 & 84 \\ 7 & 43 & 504 & 42 \\ 756 & 504 & 19188 & 2772 \\ 84 & 42 & 2772 & 678 \end{pmatrix}.$$

Its leading principal minors are

$$43, \qquad 1800, \qquad 4374000, \qquad 226834992.$$

All four are positive, so Sylvester's criterion gives $T \succ 0$. The shifted distance matrix is therefore positive definite on both U and $U^\perp$, proving $\Phi(H) > 0$. □

These successful six-vertex deletions do not increase the universal radius: the deletion in Theorem 8.6 still fails.

For a hypothetical degree-57 Moore graph, the neighborhood calculation reaches the endpoint of the small-puncture theorem.

Corollary G.2 (Degree-57 neighborhood punctures)**.** *If M is a degree-57 Moore graph of diameter two, then for every $w \in V(M)$ and every $S \subseteq N_M(w)$ with $|S| \leq 56$,*

$$\Phi(M - S) > \frac{685}{57}.$$

In particular, every such induced graph is a strict counterexample to WOW-284.

Proof. Theorem 2.2 gives the original Moore graph a score of 48, which handles $S = \varnothing$. For nonempty S, put $s = |S|$. Theorem 8.1 applies through $s = 56$: the survivor graph is connected and has minimum dual degree $57 - s/57$. Its order is at least 3194, and it inherits girth at least five. Proposition 8.4 gives

$$\lambda_{\min}(E_S) = \frac{55 - \sqrt{55^2 + 224s}}{2} > -35,$$

since

$$55^2 + 224s \le 15569 < 15625 = 125^2.$$

The loss of minimum dual degree is exactly $s/57$, so Proposition 8.2 yields

$$\Phi(M - S) \ge 48 - \frac{s}{57} + \lambda_{\min}(E_S) > 48 - \frac{56}{57} - 35 = \frac{685}{57} > 0.$$

□

Appendix H. Small orders, deletions of the Hoffman–Singleton graph, and an irregular identity

This appendix completes Section 9 and uses its labels *proved*, *computer-assisted*, *Lean-checked* and *conjectured*, and its exact decision procedure.

Throughout, M is the coordinate Hoffman–Singleton graph of Proposition 2.3, and R, H_{38}, H_{39} and X_{42} are the graphs defined before Theorem 4.2. For a graph G, $\Delta_G = \operatorname{diag}(d(v))$ is the diagonal degree matrix.

H.1. **An irregular form of the diameter-three identity.**

Theorem H.1 (Irregular diameter-three identity; proved, identities Lean-checked)**.** *Let G be a finite connected graph of girth at least five and diameter at most three, with adjacency matrix A and distance matrix D. Then*

$$D = 3J - 3I - 2A - A^2 + \Delta_G, \qquad D + tI = 3J + \Delta_G + (t-2)I - (A+I)^2 \quad (t \in \mathbb{R}).$$

Consequently $\Phi(G) > 0$ if and only if $3J + \Delta_G + (\delta^*(G) - 2)I - (A+I)^2$ is positive definite.

Proof. Since G has no triangle, adjacent vertices have no common neighbor; since it has no four-cycle, distinct vertices have at most one. Hence $(A^2)_{uu} = d(u)$, and for $u \ne v$ the entry $(A^2)_{uv}$ is 1 if $d_G(u, v) = 2$ and 0 otherwise. Thus $A_2 = A^2 - \Delta_G$ is the distance-two indicator, and the diameter bound gives $A_3 = J - I - A - A_2$ for the distance-three indicator. Substituting into $D = A + 2A_2 + 3A_3$ gives the first identity, and expanding $(A+I)^2$ gives the second. The last assertion is $\lambda_{\min}(D) > -\delta^*(G)$, that is, positive definiteness of $D + \delta^*(G)I$. □

For a k-regular graph of diameter three, $\Delta_G = kI$ and the first identity is the one in Theorem 3.1. The two identities and, for every real t, the equivalence of positive definiteness of the two sides of the second are Lean-checked (Appendix J); the passage to $\Phi(G) > 0$ through the least distance eigenvalue is the written argument above.

Corollary H.2 (Deletions from a Moore graph; proved)**.** *Let M' be a Moore graph of diameter two and let $S \subseteq V(M')$ be such that $H = M' - S$ is connected of diameter at most three. Let C_S be the $0/1$ matrix on $V(H)$ whose (x, y) entry is 1 exactly when x, y are distinct and nonadjacent and their unique common neighbor in M' lies in S. Then*

$$D(H) = 2(J - I) - A(H) + C_S,$$

so $\Phi(H) > 0$ if and only if $2J - A(H) + C_S + (\delta^*(H) - 2)I$ is positive definite.

Proof. Nonadjacent vertices of M' have exactly one common neighbor. In H, two vertices are at distance one if adjacent, at distance two if their common neighbor survives, and otherwise at distance at least three, hence exactly three by the diameter hypothesis. □

With B as in Theorem 8.1, $C_S = BB^{\mathsf{T}} - \operatorname{diag}(BB^{\mathsf{T}})$; that theorem is the case $|S| \le k - 1$, where the diameter hypothesis holds automatically. Every strict induced subgraph of M found in Section 9, and the equality graph E_{43} of Proposition H.3, has diameter at most three and is an instance of Corollary H.2; connected induced subgraphs of larger diameter already occur for $|S| = 7$. The corollary is a reformulation and not a classification: for $|S| = 6, 7, 8$ in Proposition 9.4, the number of edges inside S, the largest number of deleted neighbors of a surviving vertex, and the presence of a closed neighborhood in S do not separate strict from non-strict deletions.

H.2. **The search on at most nineteen vertices.** We describe the computation behind Proposition 9.3. The graphs were generated by `geng` from nauty [21] with the options `-c -t -f` (connected, triangle-free, four-cycle-free), one graph per isomorphism class. For $n \le 14$, every generated graph was decided. For $15 \le n \le 19$, the maximum degree was capped by Theorem 9.2 (`-D4` for $n \le 18$ and `-D5` for $n = 19$), and only graphs with $\delta^* > 2 + \sqrt{2}$ were kept, the test being done in integer arithmetic; both restrictions are necessary for strictness. Graphs are decided as described at the start of Section 9.

n	graphs generated	tested exactly	strict
3–14	329,766	329,766	0
15	1,417,899	0	0
16	11,152,778	1	0
17	98,858,189	19	0
18	977,178,397	221	0
19	25,975,541,173	7,459	0

For $15 \le n \le 19$, the column "tested exactly" counts the graphs that pass the dual-degree test; a discarded graph has $\delta^* < 2 + \sqrt{2}$ and is therefore not strict, by Lemma 9.1 and the proof of Theorem 9.2. Exactly one graph is not certified by an integer vector: the 3-regular graph of girth five on ten vertices, the Petersen graph, for which $\Phi = 0$ by Theorem 2.2. The proposition is a statement about orders at most 19; orders 20 to 37 are not covered by it.

H.3. **The orbit enumeration.** We describe the enumeration behind Proposition 9.4. The orbits were enumerated by orderly generation. All 252,000 automorphisms of M were found by an exhaustive backtracking search over adjacency-preserving bijections. Number the vertices by $P_{i,j} \mapsto 5i + j$ and $Q_{k,\ell} \mapsto 25 + 5k + \ell$ ($0 \le i, j, k, \ell \le 4$), and take S as the representative of its orbit when $\sum_{z \in S} 2^{49-z}$ is maximal over the orbit. If S is a representative and $y = \max S$, then $T = S \setminus \{y\}$ is one too. Otherwise $g(T)$ beats T for some automorphism g; let p be the first vertex number at which they differ, so $p \in g(T) \setminus T$. If $p \ge y$, then $g(T) \supseteq T$, so $g(T) = T$; hence $p < y$, and $g(S) = g(T) \cup \{g(y)\}$ beats S at $g(y)$ or at p, whichever comes first. Representatives of size s therefore arise from representatives of size $s - 1$ by adding a larger vertex. At every size the number of representatives equals the number of orbits obtained from the cycle index of the automorphism group. Each connected representative is decided as in Proposition 9.3; the equality case is decided by the explicit certificate of Proposition H.3. The proposition concerns induced subgraphs of M of order at least 36 only.

H.4. **An equality case of order** 43**.**

Proposition H.3 (computer-assisted)**.** *Let* $E_{43} = M - \{P_{0,0}, P_{0,2}, P_{1,0}, P_{1,2}, P_{2,1}, P_{4,1}, Q_{1,4}\}$*; the deleted set is independent. Then* E_{43} *has order* 43*,* 126 *edges, degree counts* $5^{21}\,6^{7}\,7^{15}$*, girth five and diameter three,* $\delta^*(E_{43}) = 6$*, and*

$$\lambda_{\min}(D(E_{43})) = -6 \quad \textit{with multiplicity } 2, \qquad \Phi(E_{43}) = 0.$$

The certificate is explicit. A principal submatrix of $D + 6I$ of order 41 has an exact rational LDL^{T} factorization with all pivots positive. For each of the two remaining indices r, a rational vector equal to e_r plus a vector supported on those 41 indices lies in the kernel of $D + 6I$. Hence $D + 6I$ is congruent to the direct sum of a positive definite matrix of order 41 and the zero matrix of order 2, so it is positive semidefinite of rank 41. This gives $\lambda_{\min}(D(E_{43})) = -6$ with multiplicity 2, and $\Phi(E_{43}) = 6 - 6 = 0$. This is an irregular case of equality in WOW-284; Jørgensen's equality example after Theorem 3.3 is regular. It is the only orbit with $\Phi = 0$ in Proposition 9.4. This proposition is not formalized in Lean.

H.5. **Two-vertex deletions of** R **and** X_{42}**.**

Proposition H.4 (computer-assisted)**.** *Let* $G \in \{R, X_{42}\}$ *and let* u, v *be distinct vertices of* G*. Then* $G - \{u, v\}$ *is a strict counterexample if and only if* $u \sim v$*. Every edge deletion has* $\delta^* = 17/3$*. For the edge* $P_{0,2}P_{0,3}$ *of* X_{42}*, that is, for* X_{40}*,*

$$\det(xI - D(X_{40})) = x^4(x+5)^{11}(x^2-4x+2)(x^2+6x+2)^4(x^3-83x^2+390x-414)(x^3+2x^2-20x-4)^4,$$

and the least root is $-3 - \sqrt{7}$*, with multiplicity four, so* $\Phi(X_{40}) = 17/3 - 3 - \sqrt{7} = \Phi(H_{38})$.

For R, the edge half is part of Theorem 4.2. The computation decides all 780 pairs of R (120 edges) and all 861 pairs of X_{42} (126 edges) exactly, and the least root of the displayed polynomial is located by exact real-root isolation. The value $17/3$ is elementary: after the endpoints of an edge uv of a k-regular graph of girth five are deleted, the vertices of $N(u) \cup N(v)$ keep $d^* = k$, and every other vertex x has $d^*(x) \ge k - 2/k$, with equality exactly when x has a neighbor in $N(u) \setminus \{v\}$ and one in $N(v) \setminus \{u\}$.

Conjecture H.5 (conjectured)**.** *Let G be a connected 6-regular graph of girth five and diameter three with $\Phi(G) > 0$. Then, for distinct vertices u, v, the graph $G - \{u, v\}$ is a strict counterexample if and only if $u \sim v$.*

The conjecture is supported only by R and X_{42}, the two known graphs satisfying its hypotheses. Both are induced subgraphs of M, and the least root $-3 - \sqrt{7}$ of the edge deletions of R (the polynomial P_{38} of Appendix B) and of X_{40} carries the degree of M, as does $-2 - \sqrt{7}$ for an edge deletion of M itself (Theorem A.2). The pattern may therefore be specific to the Hoffman–Singleton graph rather than to degree six.

Appendix I. Spectral exclusions for two construction families

We consider two possible sources of new examples: a prime-field construction and certain perfect-matching deletions from Hoffman–Singleton. The first gives no strict counterexample of diameter three, and none of the deletions in the second family is strict. We give the definitions and proofs together so these exclusions can be read independently of the main results.

I.1. **The prime-field construction in diameter three.** For an odd prime $q \geq 7$ and $1 \leq m \leq q$, define $G(q, m)$ on vertices $P_{i,j}, Q_{k,\ell}$, with $0 \leq i, k < m$ and $j, \ell \in \mathbb{F}_q$, by

$$P_{i,j} \sim P_{i,j\pm 1},$$
$$Q_{k,\ell} \sim Q_{k,\ell\pm 2},$$
$$P_{i,j} \sim Q_{k,ik+j}.$$

This is a balanced specialization of known finite-field girth-five constructions [25]. It is $(m + 2)$-regular. A coordinate common-neighbor calculation shows that it has no triangle or 4-cycle for $q \geq 7$: for a cross pair, the possible same-side common neighbors require residues in the disjoint sets $\{\pm 1\}$ and $\{\pm 2\}$.

Theorem I.1. *If $G(q, m)$ has diameter three, then it is not a strict counterexample to WOW-284.*

Proof. The degree restriction first excludes $m \leq 3$ by Theorem 7.1. For the remaining degrees, we first consider vectors that assign the same value to all vertices of a given type with a fixed first coordinate. Taking neighbor sums leaves these vectors constant on the same sets. The restricted adjacency matrix has eigenvalues $m + 2$, $2 - m$, and 2 with multiplicity $2m - 2$; the shifted WOW window leaves only $m \in \{4, 5, 6\}$ ($\theta = 2$ needs $3 < \sqrt{2m + 2}$, i.e. $m \geq 4$; $\theta = 2 - m$ needs $(m - 3)^2 < 2m + 2$, i.e. $m \leq 6$).

We decompose vectors into Fourier modes in the second coordinate. A mode assigns values proportional to successive powers of a root of unity as that coordinate varies. The vectors just tested form the zero-frequency block. A nonzero mode supplies the remaining restriction and excludes the last three cases. Let $\omega = e^{2\pi \mathrm{i}/q}$. On the nonzero character $t = 1$, the adjacency block has form

$$\begin{pmatrix} aI & M \\ M^* & bI \end{pmatrix}, \qquad a = 2\cos\frac{2\pi}{q}, \quad b = 2\cos\frac{4\pi}{q}, \quad M_{ik} = \omega^{ik}.$$

The star in the block matrix denotes conjugate transpose. Every vector in this character space has coordinate sum zero, since $\sum_{j\in\mathbb{F}_q} \omega^j = 0$. Its adjacency eigenvalues are therefore nonprincipal. If σ is a singular value of M, the associated two-dimensional invariant subspace carries $\left(\begin{smallmatrix} a & \sigma \\ \sigma & b \end{smallmatrix}\right)$, whose larger eigenvalue $\frac{a+b}{2} + \sqrt{(\frac{a-b}{2})^2 + \sigma^2}$ is at least $\frac{a+b}{2} + \sigma$. We need only the largest singular value, rather than the full Fourier spectrum. The squared Frobenius norm is the sum of the squared singular values. Since $\|M\|_F^2 = m^2$, one singular value satisfies $\sigma^2 \geq m$. Its two-dimensional block therefore supplies a nonprincipal eigenvalue at least

$$\sqrt{m} + \frac{a + b}{2} \geq \sqrt{m} + \cos(\pi/7) - \frac{1}{2}.$$

For $q \geq 7$, both $\cos(2\pi/q)$ and $\cos(4\pi/q)$ increase with q, so their sum is minimized at $q = 7$. The last inequality then uses $2\cos(2\pi/7) + 2\cos(4\pi/7) = 2\cos(\pi/7) - 1$. Moreover $\cos(\pi/7) > \sqrt{3}/2$, and $h(m) = \sqrt{2m + 2} - \sqrt{m}$ is increasing for $m \geq 1$ and satisfies

$$h(m) \leq h(6) = \sqrt{14} - \sqrt{6} < \frac{27}{20} < \frac{\sqrt{3} + 1}{2}.$$

Thus the displayed nonprincipal eigenvalue lies above $-1 + \sqrt{2m + 2}$ for $m = 4, 5, 6$. The common-neighbor case split, Fourier reduction, radical comparisons, and exact $q = 7$ controls are independently checked by exact computation. □

I.2. **Why layer-respecting edge deletions satisfy WOW-284.** For a permutation $\pi \in S_5$ of the five first-coordinate labels, delete the perfect matching

$$\mathcal{M}_\pi = \{\{P_{i,j}, Q_{\pi(i),i\pi(i)+j}\} : i, j \in \mathbb{F}_5\}$$

from the Hoffman–Singleton graph. We call the permutation affine when it multiplies a label by a nonzero residue and then adds a fixed residue, all modulo five.

Theorem I.2. *Each deletion produces a connected simple* 6*-regular graph of order* 50*, girth five, and diameter four. The* 120 *labeled graphs form exactly two isomorphism classes. The* 20 *affine permutations have*

$$\lambda_{\min}(D) = -13, \qquad \Phi = -7,$$

and the 100 *nonaffine permutations have*

$$\lambda_{\min}(D) = -6 - \sqrt{61}, \qquad \Phi = -\sqrt{61}.$$

Thus none of these graphs is a counterexample.

Proof. Every $P_{i,j}$ occurs once in $\mathcal{M}_\pi$; for a fixed $Q_{k,\ell}$, the unique incident matching edge is obtained from $i = \pi^{-1}(k)$ and $j = \ell - ik$. Thus $\mathcal{M}_\pi$ is a perfect matching, and its deletion leaves a simple 6-regular graph. Deleting edges cannot create a short cycle, while the same-layer pentagons remain; exact breadth-first search gives connectedness and diameter four. Non-strictness already follows from Theorem 5.3 since $k = 6 \le 9$; the exact least eigenvalues below sharpen it.

Explicit type-preserving and type-swapping coordinate automorphisms generate orbits of sizes 20 and 100. The representatives have different adjacency characteristic polynomials, so the orbits are distinct isomorphism classes. Exact distance characteristic polynomials and Sturm separators give the stated least roots. All 120 matchings, 400 coordinate maps, 48,000 matching images, orbit coverage, graph hypotheses, and root certificates are checked by exact computation. □

Appendix J. Computations, code, and the scope of Lean verification

The computer-assisted arguments verify the labeled finite graphs, check the complete enumeration of the stated finite orbit families, and evaluate the symbolic identities derived in the text. All signs and eigenvalue orderings used in the theorems are established by exact arithmetic. The computations use SymPy and NetworkX [26, 27]. For Jørgensen's graph after Theorem 3.3, an exact computation checks three local representations of the graph, its graph6 decoding, the characteristic polynomials and the root intervals. Wherever the text says that a step is checked by exact computation, or by computer, the step is reproduced by a script in the code described in the next subsection. The Lean results below provide a separate layer of formal verification.

J.1. **Code and data.** The source, graph data, exact certificates, verification scripts, and Lean development are available at `github.com/SamPetkov/wow284`. Release `v2.2.8` of that repository archives the graph encodings (`data/graphs/` and `data/cages55/`), the exact rational and polynomial certificates, the finite-orbit records, and the verification scripts (`scripts/`). The Lean development is in `lean/`; its toolchain and dependency revisions are pinned in `lean/lean-toolchain` and `lean/lake-manifest.json`. The order-fifty computations of Appendix F and those of Section 9 and Appendix H postdate release `v2.2.8`; they are reproduced by the scripts at commit `c546c8a` of the same repository, and the order-fifty profile scripts are also included as ancillary files (`anc/scripts/`) with this article. The scripts for Section 9 and Appendix H are `scripts/small_order_core.py` (graph constructions and exact decisions), `scripts/small_order_hs_orbits.py` (the orbit enumeration of Proposition 9.4), `scripts/small_order_deletions.py` (labeled deletion screens), `scripts/small_order_g6filter.c` and `scripts/small_order_certify_g6.py` (the dual-degree filter and the exact decisions for the `geng` output of Proposition 9.3), `scripts/recertify_order40.py` (the second check in Proposition 9.5), `scripts/verify_order43_equality.py` (the certificate of Proposition H.3) and `scripts/verify_edge_deletion_dichotomy.py` (Proposition H.4); `scripts/generate_lean40_hs_extra.py` generates the order-40 Lean certificates. The orbit counts, representative graphs, generation counts and exact pivots are in `data/hs_induced_classification/`, `data/small_order_exhaustive/`, `data/hs_order40_extra/` and `data/hs_two_vertex_deletions/` at the same commit. Analytic results not covered by these computations are proved in the text.

J.2. **Lean verification.** For the explicit 50-vertex graph, Lean 4.31 with Mathlib 4.31 [28, 29] checks structural and exact rational matrix certificates. The development pins the toolchain and dependency revisions (see the preceding subsection). It checks the coordinate graph, regularity, the exhaustive common-neighbor certificate, girth five, the adjacency-square identity, the algebraic formula for the case-defined $0/1/2$ matrix, and an exact rational diagonalization with multiplicities. The diameter-two certificate identifies that matrix with the actual distance matrix in the mathematical proof. Regularity and diagonalization then give

$$\delta^*(G) = 7, \qquad \lambda_{\min}(D(G)) = -4, \qquad \Phi(G) = 3.$$

These conclusions are assembled in the paper; they are not exported from Lean as a combined graph-distance theorem.

Lean also kernel-checks finite spectral certificates attached to the explicit constructions of orders 38, 39, 40, 42. At orders 38, 39, 42, they certify the dual-degree data and positive definiteness of the relevant shifted distance matrix. At order 40, they certify an invertible exact diagonalization, multiplicities, an attained minimum diagonal entry of -5, dual degree six, and gap one. These certificates concern the finite matrices supplied to Lean and do not provide end-to-end formalizations via `SimpleGraph.dist`. Certificates of the same kind are checked for the order-40 graphs X_{40}, Y_{40} and Z_{40} of Proposition 9.5, in the modules `Wow284.Induced40X`, `Wow284.Induced40Y` and `Wow284.Induced40Z`: the degree data, the exact minimum dual degree, girth five through the embedding in the coordinate graph, diameter three, the distance matrix, and positive definiteness of $3D + 17I$, respectively $7D + 39I$.

The module `Wow284.DiameterThreeMatrix` checks graph-level statements whose hypotheses use Mathlib's `SimpleGraph.Connected`, `SimpleGraph.egirth` and `SimpleGraph.diam`, and whose distance matrix has the entries `SimpleGraph.dist`. For a connected graph of girth at least five and diameter at most three, it proves the identities of Theorem H.1 and, for every real t, the equivalence of positive definiteness of the two sides of the second, as the declarations

`distMatrix_eq_of_diam_le_three`,
`distMatrix_add_smul_one_eq_of_diam_le_three`,
`posDef_shift_iff_of_diam_le_three`.

For a connected k-regular graph of girth at least five and diameter three, it proves the identities of Theorem 3.1, the transfer of a zero-sum adjacency eigenvector with eigenvalue θ to a distance eigenvector with eigenvalue $k - 2 - (\theta + 1)^2$, the principal distance eigenvalue $3n - k^2 - k - 3$, and, for every adjacency eigenvalue θ with a nonzero zero-sum eigenvector, the implication from positive definiteness of $D + kI$ to $|\theta + 1| < \sqrt{2k - 2}$. The passage from these identities to the least distance eigenvalue, and hence to Φ, is not formalized. Among the small-order results of Section 9 and Appendix H, only the identities and the positive-definiteness equivalence of Theorem H.1 and the certificates of Proposition 9.5 are Lean-checked; the final equivalence with $\Phi(G) > 0$ in Theorem H.1 is a written argument.

Separately, Lean formalizes the analytic optimization statement in Theorem 6.4. For every integer $k \geq 4$ and every admissible finitely supported expansion

$$f = \sum_i c_i F_i, \qquad c_0 > 0, \qquad c_i \geq 0 \quad (i \geq 5), \qquad f|_{I_k} \leq 0,$$

it proves

$$B_k c_0 \leq f(k), \qquad B_k = \frac{(k+2)(k^2+3)}{6}.$$

The formal development defines the coefficient family of the displayed quartic f_*, proves that it is admissible and attains equality, and proves that the equality cases are exactly the positive scalar multiples of f_*, both as polynomials and at coefficient level. The imported release development contains no `sorry` placeholders and is kernel-checked by Lean 4.31.

This LP formalization is deliberately graph-independent. It does not formalize the trace interpretation of the $F_i(A)$, the girth-five vanishing and nonnegativity statements, or the passage from the LP inequality to graph-order bounds. The integral-slack consequences, signed-complement theory, the high-edge profile filter in Corollary F.6, puncture spectra, and the general deletion theorems lie outside this Lean development. They are proved in the text using the analytic and, where explicitly identified, exact computer-assisted components. The paper makes no Lean claims beyond those stated in this appendix.

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