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DAUGAVET RENORMINGS VIA POOR SUBSPACES

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Abstract. Let Y be a separable Banach space with the Daugavet property containing a complemented copy of ℓ_1. We prove that $Y \oplus_1 E$ admits an equivalent norm with the Daugavet property for every separable Banach space E. We also construct a subspace of $C[0,1]$ isomorphic to the James space such that the corresponding quotient fails the Daugavet property.

1. Introduction

A Banach space X has the *Daugavet property*, written $X \in \mathrm{DPr}$, if every rank-one operator $T : X \to X$ satisfies the Daugavet equation

$$\|\operatorname{Id} + T\| = 1 + \|T\|.$$

Every separable Banach space containing an isomorphic copy of $C[0,1]$ admits an equivalent norm with the Daugavet property [3, Corollary 8.4.12]. We obtain such renormings for direct sums by taking quotients by poor subspaces. Recall that a closed subspace M of a Daugavet space X is called *poor* if X/R has the Daugavet property for every closed subspace $R \subset M$ [4]. Every closed subspace of a poor subspace is poor in the same ambient space. Our first main result is the following.

Theorem 1.1. *Let Y be a separable Banach space with the Daugavet property containing a complemented subspace isomorphic to ℓ_1. Then there exists a complemented poor subspace $M \subset Y$, isomorphic to ℓ_1, such that, for every separable Banach space E, there is a closed subspace $R \subset M$ satisfying*

$$Y/R \cong Y \oplus_1 E.$$

Consequently, $Y \oplus_1 E$ admits an equivalent norm with the Daugavet property for every separable Banach space E.

Theorem 5.1 shows that the existence of a complemented copy of ℓ_1 is also necessary for realizing all separable direct-sum extensions as quotients of Y.

Our second result concerns a limitation of quotient permanence. If X has the Daugavet property and $Z \subset X$ is reflexive, then X/Z has the Daugavet property [4, Corollary 5.9]. Quasilacunary Müntz spaces provide non-poor copies of c_0 in $C[0,1]$ [2, Corollary 3.3], showing that a subspace with separable dual need not be poor. We show that even separability of the bidual does not suffice: we construct a subspace $Z \subset C[0,1]$ isomorphic to the James space such that $C[0,1]/Z$ fails the Daugavet property. Thus Z is non-poor despite having separable bidual and being quasireflexive of order one.

Theorem 1.2. *For every $0 < \eta < 1$, there is a closed subspace $Y \subset C[0,1]$ isomorphic to the James space J such that $Q = C[0,1]/Y$ admits a rank-one projection P satisfying*

$$\|P\| = 1, \qquad \|\operatorname{Id}_Q - P\| \le 1 + \eta < 2.$$

In particular, Y is quasireflexive of order one and Q does not have the Daugavet property.

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2. Preliminaries

For $x^* \in S_{X^*}$ and $\varepsilon > 0$, put

$$\mathrm{Slice}(B_X, x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}.$$

We have the following standard characterization of the Daugavet property [5, Lemma 2.2]:

$$\begin{aligned} X \in \mathrm{DPr} \quad \iff \quad & \text{for every } x \in S_X,\ \varepsilon > 0, \text{ and every slice } S \subset B_X, \\ & \text{there is } u \in S \text{ such that } \|x + u\| > 2 - \varepsilon. \end{aligned} \tag{2.1}$$

The finite-dimensional version of this characterization will be used repeatedly.

Lemma 2.1. [5, Lemma 2.8] *Let X have the Daugavet property, let $F \subset X$ be finite-dimensional, and let $0 < \delta < 1$. Define*

$$\mathcal{U}(F, \delta) = \{u \in S_X : \|v + \lambda u\| \geq (1 - \delta)(\|v\| + |\lambda|) \text{ for all } v \in F,\ \lambda \in \mathbb{R}\}.$$

Every slice of B_X meets $\mathcal{U}(F, \delta)$. Moreover,

$$\overline{\mathrm{conv}}\,\mathcal{U}(F, \delta) = B_X.$$

We will also use the following elementary form of quotient universality of ℓ_1.

Lemma 2.2. [1, Theorem 5.9] *For every nonzero separable Banach space F there exists a bounded linear surjection $q : \ell_1 \to F$ such that*

$$\|f\| = \inf\{\|a\|_1 : a \in \ell_1,\ qa = f\} \qquad (f \in F). \tag{2.2}$$

In particular, $\|q\| = 1$.

3. Poor copies of ℓ_1

Every Banach space with the Daugavet property contains an isomorphic copy of ℓ_1 [5, Theorem 2.9]. The following theorem shows that, in the separable case, one can choose a copy that is poor in the ambient space.

Theorem 3.1. *Let X be a separable Banach space with the Daugavet property, and let $0 < \rho < 1$. There exists a sequence $(x_n) \subset S_X$ such that*

$$(1 - \rho) \sum_n |a_n| \leq \left\| \sum_n a_n x_n \right\| \leq \sum_n |a_n| \tag{3.1}$$

for every finitely supported scalar sequence (a_n), and

$$L = \overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\}$$

is poor in X. Moreover, at each stage n, the vector x_n can be chosen in any prescribed nonempty slice of B_X.

Proof. Choose a norm-dense sequence $(d_i) \subset B_X$. Let $(\delta_n) \subset (0, 1/4)$ satisfy

$$\sum_{n=1}^{\infty} \delta_n < \rho,$$

and set

$$c = \prod_{n=1}^{\infty} (1 - \delta_n) > 1 - \rho, \qquad b_n = \prod_{m=n}^{\infty} (1 - \delta_m).$$

Then $b_n \to 1$.

We construct $x_n \in S_X$ and finite subsets $W_{n,i} \subset S_X$, where $1 \leq i \leq n$. Suppose all choices before stage n have been made. Put

$$F_n = \mathrm{span}\left(\{d_1, \ldots, d_n\} \cup \{x_1, \ldots, x_{n-1}\} \cup \bigcup_{m<n} \bigcup_{i \leq m} W_{m,i} \right).$$

By Lemma 2.1, choose

$$x_n \in \mathcal{U}(F_n, \delta_n).$$

If a nonempty slice has been prescribed at this stage, choose x_n in that slice as well.

Define

$$D_n = \operatorname{span}\{d_1, \dots, d_n\}, \qquad G_n = D_n + \operatorname{span}\{x_1, \dots, x_n\}.$$

For every $i \le n$, Lemma 2.1 provides a finite set

$$W_{n,i} \subset \mathcal{U}(G_n, \delta_n)$$

and a vector $w_{n,i} \in \operatorname{conv} W_{n,i}$ such that

$$\|d_i - w_{n,i}\| < \delta_n. \tag{3.2}$$

This completes the induction.

Repeated use of the defining inequality for x_n gives

$$\left\| \sum_{n=1}^{N} a_n x_n \right\| \ge \left(\prod_{n=1}^{N} (1 - \delta_n) \right) \sum_{n=1}^{N} |a_n| \ge c \sum_{n=1}^{N} |a_n|.$$

The upper estimate in (3.1) follows from $\|x_n\| = 1$. Thus (x_n) is equivalent to the canonical basis of ℓ_1, and every $z \in L$ has a unique representation $z = \sum_{m=1}^{\infty} a_m x_m$ with $(a_m) \in \ell_1$.

We prove that for every n, every $i \le n$, and every $u \in W_{n,i}$,

$$\|a + z + u\| \ge b_n(1 + \|a + z\|) \qquad (a \in D_n,\ z \in L). \tag{3.3}$$

Fix such n, i, u, a, z, and write

$$z = z_{\le n} + z_{>n} = \sum_{m \le n} a_m x_m + \sum_{m > n} a_m x_m.$$

For $m > n$, the space F_m contains D_n, the vector u, and $x_1, \dots, x_{m-1}$. Iterating the defining inequality for x_m, first for finite tails and then passing to the limit, yields

$$\|a + z + u\| \ge \left(\prod_{m > n} (1 - \delta_m) \right) \left(\|a + z_{\le n} + u\| + \sum_{m > n} |a_m| \right).$$

Since $u \in \mathcal{U}(G_n, \delta_n)$ and $a + z_{\le n} \in G_n$, the right-hand side is at least

$$b_n \left(1 + \|a + z_{\le n}\| + \sum_{m > n} |a_m| \right) \ge b_n(1 + \|a + z\|).$$

This proves (3.3). Taking $a = 0$ and the infimum over $z \in L$ gives $\operatorname{dist}(u, L) \ge b_n > 0$.

Let $R \subset L$ be an arbitrary closed subspace, and let $Q_R : X \to X/R$ be the quotient map. We verify (2.1) in X/R. Fix

$$\xi \in S_{X/R}, \qquad \varphi \in S_{(X/R)^*}, \qquad \alpha > 0, \qquad 0 < \varepsilon < 1.$$

Choose $x \in X$ with $Q_R x = \xi$, and put $f = \varphi Q_R \in S_{X^*}$. Choose i such that

$$f(d_i) > 1 - \alpha/2.$$

Fix $0 < \eta < \varepsilon/4$. Since $\bigcup_n D_n$ is dense in X, choose n_0 and $a \in D_{n_0}$ with $\|x - a\| < \eta$. Now take $n \ge \max\{i, n_0\}$ so large that

$$\delta_n < \alpha/2, \qquad b_n > 1 - \varepsilon/4.$$

By (3.2),

$$f(w_{n,i}) > f(d_i) - \delta_n > 1 - \alpha.$$

Hence some $u \in W_{n,i}$ satisfies $f(u) > 1 - \alpha$. In particular,

$$Q_R u \in \operatorname{Slice}(B_{X/R}, \varphi, \alpha).$$

For every $r \in R$, estimate (3.3) gives

$$\begin{aligned}\|x + r + u\| &\geq \|a + r + u\| - \eta \\ &\geq b_n(1 + \|a + r\|) - \eta \\ &\geq b_n(1 + \|x + r\|) - (1 + b_n)\eta \\ &\geq 2b_n - (1 + b_n)\eta.\end{aligned}$$

Taking the infimum over $r \in R$, we obtain

$$\|\xi + Q_R u\| \geq 2b_n - (1 + b_n)\eta > 2 - \varepsilon.$$

Therefore $X/R \in \mathrm{DPr}$. Since R was arbitrary, L is poor in X. □

4. Complemented poor copies

We use the following standard factorization result for operators with values in ℓ_1.

Lemma 4.1. [7, Corollary 5.5] *Let W be a Banach space and let $S : W \to \ell_1$ be a bounded noncompact operator. Then there exist bounded operators $V : \ell_1 \to W$ and $U : \ell_1 \to \ell_1$ such that*

$$USV = \mathrm{Id}_{\ell_1}.$$

Theorem 4.2. *Let X be a separable Banach space with the Daugavet property. If X contains a complemented copy of ℓ_1, then it contains a complemented poor subspace isomorphic to ℓ_1.*

Proof. Since X contains a complemented subspace isomorphic to ℓ_1, there exist bounded operators

$$J_0 : \ell_1 \to X, \qquad T : X \to \ell_1, \qquad TJ_0 = \mathrm{Id}_{\ell_1}.$$

For every $k \in \mathbb{N}$,

$$\|T^* e_k^*\| \geq c := \|J_0\|^{-1} > 0.$$

Carry out the construction of Theorem 3.1, with the following additional slice condition. After choosing $x_1, \ldots, x_{n-1}$, select k_n so large that

$$|(Tx_m)_{k_n}| < c/4 \qquad (m < n).$$

This is possible because $Tx_m \in \ell_1$. The set

$$S_n = \{x \in B_X : (Tx)_{k_n} > c/2\}$$

is a nonempty slice. Choose $x_n \in S_n$ at stage n, as allowed by Theorem 3.1. Then

$$\|Tx_n - Tx_m\|_1 > c/4 \qquad (m < n).$$

The resulting subspace $L = \overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\}$ is poor in X. Since (Tx_n) is separated in norm, $T|_L : L \to \ell_1$ is noncompact. By Lemma 4.1, there exist bounded operators $V : \ell_1 \to L$ and $U : \ell_1 \to \ell_1$ with $UTV = \mathrm{Id}_{\ell_1}$. Hence V is bounded below, and $P = VUT$ is a bounded projection from X onto $M = V(\ell_1) \subset L$. Thus M is isomorphic to ℓ_1, complemented in X, and poor in X. □

5. Daugavet quotients and renormings

Proof of Theorem 1.1. By Theorem 4.2, choose a complemented poor subspace $M \subset Y$ isomorphic to ℓ_1. Let Z be a closed complement of M in Y, and fix an isomorphism $J : \ell_1 \to M$.

Let E be separable. By Lemma 2.2, choose a bounded surjection

$$q : \ell_1 \to \ell_1 \oplus_1 E$$

and set $R = J(\ker q)$. This is a closed subspace of M. The map

$$z + Ja \longmapsto (z, qa)$$

is a bounded surjection from Y onto $Z \oplus_1 (\ell_1 \oplus_1 E)$ with kernel R. Hence the open mapping theorem gives

$$Y/R \cong Z \oplus_1 \ell_1 \oplus_1 E \cong Y \oplus_1 E.$$

Since M is poor, Y/R has the Daugavet property. Transporting that norm through the isomorphism gives an equivalent Daugavet norm on $Y \oplus_1 E$. □

We now identify the exact hypothesis required by the universal quotient realization in Theorem 1.1.

Theorem 5.1. *Let Y be a separable Banach space with the Daugavet property. The following assertions are equivalent:*

(i) *Y contains a complemented subspace isomorphic to ℓ_1.*
(ii) *For every separable Banach space E there exists a closed subspace $R \subset Y$ such that $Y/R \in \mathrm{DPr}$ and $Y/R \cong Y \oplus_1 E$.*
(iii) *There exists a closed subspace $R \subset Y$ such that $Y/R \cong Y \oplus_1 \ell_1$.*

Proof. The implication (i)⇒(ii) follows from Theorem 1.1, and (ii)⇒(iii) is immediate.

Assume (iii). Composing the quotient map, the given isomorphism, and the projection onto ℓ_1 gives a bounded surjection $T : Y \to \ell_1$. The open mapping theorem provides vectors $y_n \in Y$ satisfying

$$Ty_n = e_n, \qquad \sup_n \|y_n\| < \infty.$$

Thus

$$J : \ell_1 \to Y, \qquad Ja = \sum_n a_n y_n,$$

is bounded and $TJ = \mathrm{Id}_{\ell_1}$. It follows that JT is a bounded projection onto $J(\ell_1) \cong \ell_1$. This proves (i). □

The complementability hypothesis in Theorem 1.1 is automatic for the following class of spaces.

Corollary 5.2. *Let Y be a closed subspace of $L_1[0,1]$ which admits an equivalent norm with the Daugavet property. Then $Y \oplus_1 E$ admits an equivalent Daugavet norm for every separable Banach space E.*

Proof. Theorem 3.1, applied to a Daugavet norm on Y, shows that Y contains ℓ_1. In particular, Y is nonreflexive. By the classical theorem of Kadec and Pełczyński, every nonreflexive closed subspace of $L_1[0,1]$ contains an ℓ_1-subspace complemented in $L_1[0,1]$; see [8]. Restricting the projection gives a complemented copy in Y. Complementability is unchanged by equivalent renorming, so Theorem 1.1 applies. □

6. A quotient with a quasireflexive kernel

Recall that a Banach space Y is *quasireflexive of order one* if its canonical image in Y^{**} has codimension one.

We use the following realization of the real James space:

$$J = \{a = (a_k) \in c_0 : \|a\|_J < \infty\},$$

$$\|a\|_J = \sup_{\substack{m \geq 2 \\ 1 \leq p_1 < \cdots < p_m}} \left(\sum_{j=1}^{m-1} |a_{p_{j+1}} - a_{p_j}|^2 \right)^{1/2}. \tag{6.1}$$

This is a separable Banach space with a shrinking canonical basis, and it is quasireflexive of order one; see [6, Section 5]. Consequently, J^* and J^{**} are separable. Put

$$s_n = (\underbrace{1, \ldots, 1}_{n \text{ terms}}, 0, 0, \ldots).$$

The definition of the norm gives

$$\|s_n\|_J = 1, \qquad \|a\|_\infty \le \|a\|_J \quad (a \in J). \tag{6.2}$$

Indeed, an increasing sequence of indices meets at most one jump of s_n; the second assertion follows by letting the later index tend to infinity.

Lemma 6.1. *The formula*

$$(Ra)(t) = a_1 t + \sum_{k=2}^{\infty} a_k (t^k - t^{k-1})$$

defines a bounded linear operator $R : c_0 \to C[0,1]$ with $\|R\| \le 2$. It satisfies

$$(Ra)(0) = 0, \qquad (Ra)(1) = a_1, \qquad (Rs_n)(t) = t^n.$$

Proof. For $M > N \ge 1$ and $0 \le t \le 1$,

$$\left| \sum_{k=N+1}^{M} a_k (t^k - t^{k-1}) \right| \le \sup_{k>N} |a_k| \, (t^N - t^M) \le \sup_{k>N} |a_k|.$$

Thus the series converges uniformly for every $a \in c_0$, and its sum is continuous. The same estimate, including the first term, yields $\|Ra\|_\infty \le 2\|a\|_\infty$. The endpoint identities follow by substitution, and $Rs_n = t^n$ follows by telescoping. □

Lemma 6.2. *For every $h \in B_{C[0,1]}$,*

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} |h(t) - h(1)t^n| \le 1.$$

Proof. Write $\alpha = h(1)$. Given $\varepsilon > 0$, choose $r \in (0,1)$ such that $|h(t) - \alpha| < \varepsilon$ for $r \le t \le 1$. On $[0,r]$, $|h(t) - \alpha t^n| \le 1 + r^n$, whereas on $[r,1]$,

$$|h(t) - \alpha t^n| \le |h(t) - \alpha| + |\alpha|(1 - t^n) \le \varepsilon + 1.$$

Let $n \to \infty$ and then $\varepsilon \downarrow 0$. □

Proof of Theorem 1.2. Fix $0 < \eta < 1$. By the Banach–Mazur embedding theorem, there is a linear isometry $H : J \to C[0,1]$. Define $F : J \to C[0,1]$ by

$$(Fa)(t) = \begin{cases} \eta(Ha)(3t), & 0 \le t \le \frac{1}{3}, \\ \eta \cdot (2 - 3t)(Ha)(1), & \frac{1}{3} \le t \le \frac{2}{3}, \\ (Ra)(3t-2), & \frac{2}{3} \le t \le 1. \end{cases} \tag{6.3}$$

Here R is restricted to J using (6.2). The formulas agree at the endpoints of the three intervals, so Fa is continuous. Moreover,

$$\eta\|a\|_J \le \|Fa\|_\infty \le 2\|a\|_J. \tag{6.4}$$

Hence F is an isomorphic embedding with closed range.

Set

$$J_0 = \{a \in J : a_1 = 0\}, \qquad Y = F(J_0).$$

The subspace Y is closed. The right shift $Ua = (0, a_1, a_2, \ldots)$ is a bijection from J onto J_0, and

$$\|a\|_J \le \|Ua\|_J \le \sqrt{2}\,\|a\|_J.$$

For the lower bound, shift each index in (6.1) by one. For the upper bound, the initial zero can contribute at most one additional term $|a_j|^2 \le \|a\|_J^2$. Thus $Y \cong J_0 \cong J$.

Let $q : C[0,1] \to Q = C[0,1]/Y$ be the quotient map and put $g_n = Fs_n$. By (6.2) and (6.3), we obtain

$$|g_n(t)| \le \eta \quad (0 \le t \le \tfrac{2}{3}), \qquad g_n(t) = (3t-2)^n \quad (\tfrac{2}{3} \le t \le 1). \tag{6.5}$$

In particular, $\|g_n\|_\infty = g_n(1) = 1$. Since $s_n - s_m \in J_0$, the class

$$z = qg_n$$

does not depend on n. Every element of Y vanishes at 1, because $(Fa)(1) = a_1$. Consequently,

$$\varphi(qf) = f(1)$$

defines a bounded linear functional on Q with $\|\varphi\| \le 1$. As $\varphi(z) = 1$ and $\|z\| \le \|g_n\|_\infty = 1$, we have $\|\varphi\| = \|z\| = 1$.

Define $P : Q \to Q$ by $Pu = \varphi(u)z$. Then $P^2 = P$ and $\|P\| = 1$. If $f \in B_{C[0,1]}$ and $\alpha = f(1)$, then, for every n,

$$(\mathrm{Id}_Q - P)qf = q(f - \alpha g_n). \tag{6.6}$$

On $[0, 2/3]$, (6.5) gives $|f(t) - \alpha g_n(t)| \le 1 + \eta$. On $[2/3, 1]$, Lemma 6.2, applied to $h(t) = f((t + 2)/3)$, gives

$$\limsup_{n\to\infty} \sup_{2/3\le t\le 1} |f(t) - \alpha g_n(t)| \le 1.$$

It follows that

$$\|(\mathrm{Id}_Q - P)qf\| \le \limsup_{n\to\infty} \|f - \alpha g_n\|_\infty \le 1 + \eta.$$

For $u \in B_Q$ and $\delta > 0$, choose a lift f with $qf = u$ and $\|f\|_\infty \le 1 + \delta$. By homogeneity,

$$\|(\mathrm{Id}_Q - P)u\| \le (1 + \eta)(1 + \delta).$$

Letting $\delta \downarrow 0$ proves $\|\mathrm{Id}_Q - P\| \le 1 + \eta$. Hence the rank-one operator $-P$ fails the Daugavet equation.

Finally, Y is quasireflexive of order one because $Y \cong J$. □

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