A Fifth-Order Hurwitz Interval Polynomial Family Without a Degree-Four Weak Strictly Positive Real Numerator

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Abstract

We construct an explicit fifth-order Hurwitz-stable interval polynomial family with five independent coefficient intervals of positive width that admits no common weak strictly positive real (WSPR) numerator of degree four. For every real numerator of degree at most four, one of six fixed denominator–frequency pairs has a nonpositive real part. Robust Hurwitz stability follows from three common separator points for the even and odd denominator polynomials. Nonexistence follows from an exact linear identity: a strictly positive integer combination of the six frequency pairing functionals is identically zero. Thus robust Hurwitz stability does not guarantee a degree-four common WSPR numerator for fifth-order interval polynomial families.

Keywords: strictly positive real synthesis; weak strictly positive real; interval polynomial family; Hurwitz stability; numerator degree

hexagon:2611.00557v1

1 Introduction

Strictly positive real (SPR) and weak strictly positive real (WSPR) transfer functions arise in passivity, adaptive control, and output-error system identification [2, 8]. For an uncertain denominator polynomial, robust synthesis seeks a single real numerator whose ratio with every admissible denominator has positive real part at all real frequencies. The degree of this common numerator determines whether the resulting transfer functions are strictly proper or have equal numerator and denominator degrees.

Kharitonov's theorem reduces Hurwitz stability of an interval polynomial family to four vertex polynomials [4]. For a prescribed numerator, positivity on the imaginary axis also reduces to four vertex combinations [1, 2]. Common-numerator synthesis requires these frequency inequalities to hold simultaneously for one coefficient vector. For Hurwitz-stable interval families of order $n \le 4$, the constructions in [5, 7] yield degree-$(n-1)$ common WSPR numerators. The WSPR-region approach in [8] uses such a numerator as the first step toward equal-degree SPR synthesis.

The sixth-order counterexample in [6] has a degree-six common SPR numerator and no degree-five common WSPR numerator. In this paper, we construct a fifth-order Hurwitz interval family for which no numerator of degree at most four satisfies strict frequency positivity. The family has rational interval endpoints. Three separator points establish its stability, and six evaluations at four Kharitonov vertices give a positive linear dependence that excludes every candidate numerator. These two certificates form the stability and nonexistence proofs below.

2 Problem Formulation and the Main Counterexample

A real polynomial is Hurwitz stable if all of its complex roots lie in the open left half-plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$. Consider the monic fifth-order interval polynomial family

$$\mathcal{K} = \left\{ a(s) = s^5 + \sum_{k=0}^{4} a_k s^k : \ell_k \le a_k \le u_k,\ 0 \le k \le 4 \right\}, \tag{2.1}$$

where the coefficients vary independently in the closed intervals $[\ell_k, u_k]$. The family is robustly Hurwitz stable if every $a \in \mathcal{K}$ is Hurwitz stable.

Following [8], for a Hurwitz-stable denominator a of degree five, b/a is weak strictly positive real (WSPR) of degree four if $b \in \mathbb{R}[s]$, $\deg b = 4$, and

$$\operatorname{Re} \frac{b(i\omega)}{a(i\omega)} > 0 \qquad (\forall\, \omega \in \mathbb{R}). \tag{2.2}$$

A common WSPR numerator for $\mathcal{K}$ is a single polynomial b satisfying (2.2) for every $a \in \mathcal{K}$. The synthesis question is whether robust Hurwitz stability guarantees such a numerator.

For $t = \omega^2 \geq 0$, write

$$a(i\omega) = E_a(t) + i\omega O_a(t), \qquad E_a(t) = a_4 t^2 - a_2 t + a_0, \quad O_a(t) = t^2 - a_3 t + a_1. \tag{2.3}$$

For a candidate $v(s) = \sum_{k=0}^{4} \beta_k s^k$, define

$$P_v(t) = \beta_4 t^2 - \beta_2 t + \beta_0, \qquad Q_v(t) = \beta_1 - \beta_3 t, \tag{2.4}$$

so that $v(i\omega) = P_v(t) + i\omega Q_v(t)$. Its frequency pairing polynomial with a is

$$\Phi(a, v; t) = P_v(t) E_a(t) + t Q_v(t) O_a(t). \tag{2.5}$$

Whenever $a(i\omega) \neq 0$,

$$\operatorname{Re} \frac{v(i\omega)}{a(i\omega)} = \frac{\Phi(a, v; \omega^2)}{|a(i\omega)|^2}. \tag{2.6}$$

The extreme even and odd denominator polynomials on $t \geq 0$ are

$$\begin{aligned} E_0(t) &= \ell_4 t^2 - u_2 t + \ell_0, \quad E_1(t) = u_4 t^2 - \ell_2 t + u_0, \\ O_0(t) &= t^2 - u_3 t + \ell_1, \qquad O_1(t) = t^2 - \ell_3 t + u_1. \end{aligned} \tag{2.7}$$

For $\varepsilon, \delta \in \{0, 1\}$, let $a^{(\varepsilon,\delta)}$ denote the Kharitonov vertex determined by

$$a^{(\varepsilon,\delta)}(i\omega) = E_\varepsilon(\omega^2) + i\omega O_\delta(\omega^2). \tag{2.8}$$

Each of these four polynomials belongs to $\mathcal{K}$.

Lemma 2.1. For every $a \in \mathcal{K}$ and $t \geq 0$,

$$E_0(t) \leq E_a(t) \leq E_1(t), \qquad O_0(t) \leq O_a(t) \leq O_1(t). \tag{2.9}$$

For fixed v and $t \geq 0$, the inequality $\Phi(a, v; t) > 0$ holds for all $a \in \mathcal{K}$ if and only if it holds at the four vertices $a^{(\varepsilon,\delta)}$.

Proof. The coefficient bounds give (2.9). The even and odd parts depend on disjoint sets of independent coefficients, so $(E_a(t), O_a(t))$ ranges over the rectangle $[E_0(t), E_1(t)] \times [O_0(t), O_1(t)]$. The pairing (2.5) is linear in these two coordinates and attains its minimum at a corner of this rectangle. □

We now choose the interval endpoints in Table 1.

Theorem 2.2. Let $\mathcal{K}$ be the fifth-order interval polynomial family defined by (2.1) and Table 1. Then:

(i) Every polynomial $a \in \mathcal{K}$ is Hurwitz stable.

(ii) For every real polynomial v of degree at most four, one of the six fixed pairs (a_j, t_j) in Table 2 satisfies

$$\operatorname{Re} \frac{v(i\sqrt{t_j})}{a_j(i\sqrt{t_j})} \leq 0. \tag{2.10}$$

Consequently, $\mathcal{K}$ admits no common WSPR numerator of degree four.

Table 1: Coefficient intervals of the fifth-order counterexample.

k	0	1	2	3	4
ℓ_k	1	5454/5	2734/3	104001/100	1
u_k	8002/9	200002/5	1000001/1000	108101/100	200/9

3 Robust Hurwitz Stability

For the intervals in Table 1, the endpoint polynomials (2.7) are

$$\begin{aligned} E_0(t) &= t^2 - \frac{1000001}{1000}t + 1, \qquad E_1(t) = \frac{200}{9}t^2 - \frac{2734}{3}t + \frac{8002}{9}, \\ O_0(t) &= t^2 - \frac{108101}{100}t + \frac{5454}{5}, \quad O_1(t) = t^2 - \frac{104001}{100}t + \frac{200002}{5}. \end{aligned} \tag{3.1}$$

Set

$$\alpha = \frac{201}{200}, \qquad \beta = \frac{8001}{200}, \qquad \gamma = \frac{200001}{200}. \tag{3.2}$$

Direct evaluation gives

$$\begin{aligned} E_1(\alpha) &= -\frac{7801}{1800}, \quad E_1(\beta) = -\frac{7801}{1800}, \qquad E_0(\gamma) = \frac{250001}{50000}, \\ O_0(\alpha) &= \frac{215799}{40000}, \quad O_1(\beta) = -\frac{192001}{40000}, \quad O_1(\gamma) = -\frac{192001}{40000}. \end{aligned} \tag{3.3}$$

By Lemma 2.1, every $a \in \mathcal{K}$ therefore satisfies

$$\begin{aligned} &E_a(\alpha) < 0, \quad E_a(\beta) < 0, \quad E_a(\gamma) > 0, \\ &O_a(\alpha) > 0, \quad O_a(\beta) < 0, \quad O_a(\gamma) < 0. \end{aligned} \tag{3.4}$$

3.1 Even–Odd Root Interlacing

Fix $a \in \mathcal{K}$. Since $a_0 \geq 1$ and $a_4 \geq 1$, the quadratic E_a is positive at zero and has positive leading coefficient. The sign conditions (3.4) give two distinct roots

$$e_1 \in (0, \alpha), \qquad e_2 \in (\beta, \gamma).$$

The monic quadratic O_a is positive at α, negative at β and γ, and tends to $+\infty$ as $t \to +\infty$. It therefore has two distinct roots

$$o_1 \in (\alpha, \beta), \qquad o_2 \in (\gamma, +\infty).$$

Thus

$$0 < e_1 < \alpha < o_1 < \beta < e_2 < \gamma < o_2, \tag{3.5}$$

and

$$O_a(t) = (t - o_1)(t - o_2), \qquad E_a(o_1) < 0 < E_a(o_2). \tag{3.6}$$

This is the fifth-order Hermite–Biehler interlacing pattern [3]. The following decomposition proves Hurwitz stability directly from these signs.

3.2 A Positive-Real Decomposition

Define

$$A = \frac{a_0}{o_1 o_2}, \qquad B = -\frac{E_a(o_1)}{o_1(o_2 - o_1)}, \qquad D = \frac{E_a(o_2)}{o_2(o_2 - o_1)}. \tag{3.7}$$

All three constants are positive by (3.5)–(3.6). Comparing coefficients gives

$$a_4 = A + B + D, \quad a_2 = A(o_1 + o_2) + Bo_2 + Do_1, \quad a_0 = Ao_1 o_2,$$

and hence

$$\frac{a(s)}{s(s^2 + o_1)(s^2 + o_2)} = 1 + \frac{A}{s} + B\frac{s}{s^2 + o_1} + D\frac{s}{s^2 + o_2}. \tag{3.8}$$

For $s = x + iy$ with $x > 0$ and any $r > 0$,

$$\operatorname{Re}\frac{1}{s} = \frac{x}{x^2 + y^2} > 0, \qquad \operatorname{Re}\frac{s}{s^2 + r} = \frac{x(x^2 + y^2 + r)}{|s^2 + r|^2} > 0. \tag{3.9}$$

The factors s, $s^2 + o_1$, and $s^2 + o_2$ have no zeros in the open right half-plane. Equations (3.8)–(3.9) show that the real part of the quotient in (3.8) exceeds 1 there, so a has no roots with positive real part.

If $a(i\omega) = 0$, then (2.3) gives $E_a(\omega^2) = 0$ and $\omega O_a(\omega^2) = 0$. The case $\omega = 0$ is excluded by $a_0 > 0$. Otherwise $\omega^2 = o_1$ or o_2, contradicting (3.6). Thus a has no imaginary-axis roots either. Every $a \in \mathcal{K}$ is Hurwitz stable, proving Theorem 2.2(i).

4 Nonexistence of Degree-Four WSPR Numerators

4.1 Six Frequency Pairing Functionals

Choose the six vertex–frequency pairs (a_j, t_j), $0 \le j \le 5$, in Table 2. The numbers t_j are squared frequencies; the corresponding real frequencies are $\omega_j = \sqrt{t_j}$. For each pair, define

$$f_j(v) = \Phi(a_j, v; t_j). \tag{4.1}$$

Writing $\mathbf{b} = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)^{\mathsf{T}}$, these are linear functionals $f_j(v) = r_j \mathbf{b}$, with row vectors

$$r_j = \big(E_{a_j}(t_j),\ t_j O_{a_j}(t_j),\ -t_j E_{a_j}(t_j),\ -t_j^2 O_{a_j}(t_j),\ t_j^2 E_{a_j}(t_j)\big). \tag{4.2}$$

Table 2: The six evaluation pairs and their exact even and odd values.

j	a_j	t_j	$E_{a_j}(t_j)$	$O_{a_j}(t_j)$
0	$a^{(0,0)}$	1/40	−119997/5000	8510203/8000
1	$a^{(1,0)}$	9/10	3911/45	118701/1000
2	$a^{(1,1)}$	42	16318/9	−95801/50
3	$a^{(0,1)}$	400	−1199997/5	−1080018/5
4	$a^{(0,1)}$	900	−899999/10	−430043/5
5	$a^{(0,1)}$	100000	9899999901	49480195002/5

4.2 An Exact Positive Linear Dependence

Proposition 4.1. Let $w_0, \dots, w_5$ be the positive integers in Appendix A. For every real polynomial v of degree at most four,

$$\sum_{j=0}^{5} w_j f_j(v) = 0. \tag{4.3}$$

Proof. Set $\mathsf{M} = (r_j)_{j=0}^{5} \in \mathbb{R}^{6\times 5}$ and $\mathbf{w} = (w_0, \dots, w_5)^{\mathsf{T}}$. Substitution of Table 2 and Appendix A into (4.2) gives the five exact identities

$$\begin{aligned} \sum_{j=0}^{5} w_j t_j^k E_{a_j}(t_j) &= 0 \quad (k = 0, 1, 2), \\ \sum_{j=0}^{5} w_j t_j^{k+1} O_{a_j}(t_j) &= 0 \quad (k = 0, 1). \end{aligned} \tag{4.4}$$

These identities state that $\mathsf{M}^{\mathsf{T}}\mathbf{w} = 0$. Therefore

$$\sum_{j=0}^{5} w_j f_j(v) = \mathbf{w}^{\mathsf{T}}\mathsf{M}\mathbf{b} = 0.$$

□

Proof of Theorem 2.2(ii). Let v be any real polynomial of degree at most four. Since all $w_j > 0$, the identity (4.3) implies $f_j(v) \le 0$ for at least one j. By part (i), a_j is Hurwitz stable, so $|a_j(i\sqrt{t_j})|^2 > 0$. Equation (2.6) now gives

$$\operatorname{Re} \frac{v(i\sqrt{t_j})}{a_j(i\sqrt{t_j})} = \frac{f_j(v)}{|a_j(i\sqrt{t_j})|^2} \le 0.$$

In particular, no degree-four polynomial satisfies (2.2) for every $a \in \mathcal{K}$. □

The same certificate describes the obstruction in coefficient space. Each necessary inequality $f_j(v) > 0$ defines an open half-space in $\mathbb{R}^5$. After normalization by $W = \sum_{j=0}^{5} w_j$, equation (4.4) gives

$$\sum_{j=0}^{5} \frac{w_j}{W} r_j = 0, \qquad \frac{w_j}{W} > 0, \qquad \sum_{j=0}^{5} \frac{w_j}{W} = 1.$$

Thus the origin is a convex combination of the six evaluation rows, and no coefficient vector can have a strictly positive scalar product with every row.

5 Conclusion

The interval family in Table 1 is robustly Hurwitz stable and has no degree-four common WSPR numerator. Its stability is certified by three separator points, and its nonexistence result by six finite frequency evaluations with positive integer weights. The construction shows that the degree-$(n-1)$ common WSPR property of low-order interval families already fails at order five.

Acknowledgements

GPT-6 Astra was used for calculations, proof development, and editorial assistance. AI-generated suggestions were checked against the mathematical arguments and the accompanying formal development; the formal claims are supported by Lean kernel checking. Responsibility for the final manuscript rests with the authors. The main counterexample theorem has been formalized in Lean 4, and the source code is available at `https://github.com/XPiPiT/n_random/tree/main/SPR/simplified`, in the file `n5-1.lean`.

A Exact Integer Weights

The weights in Proposition 4.1, in the order of Table 2, are

$$w_0 = 2001510813470376083043278683598175556546639440000000000,$$
$$w_1 = 5439492056541499374339578187742327107316660058850000000,$$
$$w_2 = 4707833972453221448396095842788111487554195655825000,$$
$$w_3 = 5439959166953135881632284160626887889264212200 9,$$
$$w_4 = 8967183883997763155493353400634431143590337809 76,$$
$$w_5 = 6658146969843145950313796630228477733 97.$$

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