hexagon:2611.00556v1

Counterexamples to the Mathieu Property

From Explicit Quotient Constructions

to a General Classification

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17 November 2026

Abstract

We construct regular functions P, Q on each of $\mathrm{SL}_2/2O$ and $\mathrm{SL}_2/2I$ for which every positive Haar moment of P vanishes, whereas $\mathsf{h}(QP^m)$ is nonzero for infinitely many m. The same elementary construction also supplies a radial witness on the defining module of SU(2). These three examples give the following classification: for a complex homogeneous space G/H, with G and H having reductive identity components, the kernel of normalized Reynolds integration is Mathieu–Zhao exactly when $[G^0, G^0] \subseteq H$. We give the reduction to the examples, including disconnected acting groups. The counterexamples are proved directly from Haar integration, polynomial identities and Dirichlet's theorem. For either exceptional example and each positive degree, one explicit fixed multiplier, chosen independently of the complex coefficients, gives a transcendental ordinary Cauchy transform for every polynomial recomposition of that degree. Over an arbitrary coefficient algebra, we also determine the pure and safe elements of the polynomial subalgebra generated by the example.

1 The problem and the results

Let A be a commutative complex algebra and let $\mathsf{h} : A \to \mathbb{C}$ be a linear functional with $\mathsf{h}(1) = 1$. A function $P \in A$ is called *pure* if

$$\mathsf{h}(P^m) = 0 \qquad (m \geq 1).$$

We call P *safe* if, for each fixed $Q \in A$, the moments $\mathsf{h}(QP^m)$ vanish for all sufficiently large m. The cutoff is allowed to depend on P and Q. Thus $\ker \mathsf{h}$ is a Mathieu–Zhao subspace precisely when every pure function is safe. This definition is equivalent to the version using eventual vanishing of the pure moments: take a sufficiently large power of P and then treat the finitely many residue classes of exponents separately.

For example, let $A = \mathbb{C}[t, t^{-1}]$ and let $\mathsf{h} = \mathrm{CT}$ extract the constant term. The function $P = t$ is pure and safe: for $Q = \sum_j q_j t^j$, one has $\mathrm{CT}(Qt^m) = q_{-m}$, which is zero once m exceeds the finite range of negative exponents of Q. The question is whether this eventual vanishing persists beyond tori.

Mathieu's Main Conjecture asks whether every pure K-finite function on a compact connected Lie group K is safe [11, pp. 264–265]. Here K-finite means that the translates span a finite-dimensional vector space. Long gave an explicit counterexample on SU(2) [8, Theorem 4.2], then classified the compact connected group case: the Mathieu property holds exactly for tori [9, Theorem 1.1]. Regular functions on a complex reductive group restrict to K-finite functions on a compact form. Thus the case $H = \{1\}$ of the classification below is already covered by Long's compact-group theorem.

We are interested in $A = \mathbb{C}[G/H]$, where G is a complex linear algebraic group, $H \subseteq G$ is a closed algebraic subgroup, and both identity components G^0 and H^0 are reductive. The algebraic Matsushima criterion gives affineness of G^0/H^0 [3, Proposition 2.2]. Each component of G/H is a finite quotient of a conjugate of this affine variety, so G/H is affine too. Its normalized Reynolds functional is the unique G-invariant functional with value one on the constant function one. When G is connected, it is integration over a compact form, with total mass one.

For connected G, the space G/H is *spherical* if a Borel subgroup has an open orbit. For reductive H in this setting, Long proves that the Mathieu property and equality of the pure set with the nullcone are both equivalent to $[G, G] \subseteq H$ [10, Theorem 1.1]. We extend the Mathieu-property classification to arbitrary reductive homogeneous spaces and disconnected acting groups.

Theorem 1.1. *Let G and H be as above. Then*

$$\ker \mathsf{h} \subseteq \mathbb{C}[G/H] \text{ is Mathieu–Zhao} \quad \Longleftrightarrow \quad [G^0, G^0] \subseteq H.$$

If the containment fails, there are fixed regular functions P, Q on G/H such that P is pure and $\mathsf{h}(QP^m) \neq 0$ for infinitely many m.

The important word in the second assertion is *fixed*. A different multiplier at each order would not disprove the Mathieu property. Our examples use one polynomial Q, independent of m.

A counterexample on $\mathrm{SU}(2)$ does not automatically descend to SL_2/Γ, with $\Gamma \subseteq \mathrm{SU}(2)$ finite: both functions must be invariant under right multiplication by Γ. Averaging P enforces invariance, but powers of that average contain mixed products of translates of P, whose moments are not controlled by $\mathsf{h}(P^m) = 0$. Our binary-form construction retains all-power vanishing while supplying the required right invariance. The exceptional quotients below have dimension three, while a Borel subgroup of SL_2 has dimension two; they supply the nonspherical cases left by the rank-one reduction.

Here are the two exceptional examples explicitly. Write

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \qquad xw - yz = 1,$$

and take one of the following binary forms:

$$\begin{aligned} f_8(X, Y) &= X^8 + 14X^4Y^4 + Y^8, \\ f_{12}(X, Y) &= XY(X^{10} + 11X^5Y^5 - Y^{10}). \end{aligned} \tag{1.1}$$

The groups $2O$ and $2I$ are the binary octahedral and binary icosahedral subgroups of $\mathrm{SU}(2)$. Their standard embeddings fix f_8 and f_{12}, respectively. An elementary verification is given in Appendix A.

For $n = 8$ or 12, put

$$\begin{aligned} b = \frac{n}{2}, \qquad a = b - 1, \qquad U = f_n(x, y), \qquad F = f_n(z, w), \\ R = \frac{f_n(z + Fx, w + Fy)}{F}, \qquad P = U^a R^b, \qquad Q = F^a. \end{aligned} \tag{1.2}$$

Although R is written as a quotient, it is a polynomial with integer coefficients. We will prove this before using it.

The parameters used throughout the construction are collected below. The first row also gives the radial example in Section 5. Here $c = b(n - 1)$ and $p = cm - a$ is the prime index for the exceptional examples; $N(T)$ denotes the normalizer of the diagonal torus.

Binary form	n	a	b	c	Quotient
X^2Y^2	4	1	2	6	$\mathrm{SL}_2/N(T)$
f_8	8	3	4	28	$\mathrm{SL}_2/2O$
f_{12}	12	5	6	66	$\mathrm{SL}_2/2I$

Theorem 1.2. *The functions in* (1.2) *are regular invariants on* $\mathrm{SL}_2/2O$ *when* $n = 8$*, and on* $\mathrm{SL}_2/2I$ *when* $n = 12$*. In both cases*

$$\mathsf{h}(P^m) = 0 \qquad (m \geq 1).$$

For the same fixed Q*, one has*

$$\begin{aligned} v_p\big(\mathsf{h}(QP^m)\big) &= -1 \quad \text{if } n = 8 \text{ and } p = 28m - 3 \text{ is prime,} \\ v_p\big(\mathsf{h}(QP^m)\big) &= -1 \quad \text{if } n = 12 \text{ and } p = 66m - 5 \text{ is prime.} \end{aligned} \tag{1.3}$$

Consequently both exceptional quotients fail the Mathieu property.

Here v_p is the usual valuation on rational numbers, normalized by $v_p(p) = 1$. The moments in this theorem are rational. Dirichlet's theorem [15, Theorem 18.1] gives infinitely many primes in the progressions $25 \pmod{28}$ and $61 \pmod{66}$, so (1.3) proves nonvanishing at arbitrarily large orders.

The proof has two independent parts. Torus weights and integration by parts give purity. For the marked moments, a congruence reduces Haar integration to one coefficient of a binary form modulo p. Two explicit Klein identities give a map from the curve defined by the binary form to a nonsingular cubic curve. A differential identity and compatibility of the Cartier operator with pullback transfer the coefficient calculation to that cubic. There it reduces to a binomial coefficient that is nonzero modulo p.

For the classification, equivariant pullback carries a bad pair to any homogeneous space mapping onto one of these examples. A simple factor of rank at least two instead produces a two-dimensional SL_2-summand in a representation, to which the radial example applies. In the remaining case, inactive higher-rank factors can be removed, leaving only rank-one factors. A proper stabilizer projection gives a map onto one of the three quotients in the table; otherwise a diagonal coupling again produces such a summand. If $[G, G] \subseteq H$ for connected G, the quotient is a torus. For disconnected G, a finite-trace argument separates the moment conditions on its torus components.

Sections 2–4 prove Theorem 1.2. Section 5 gives the radial example, and Sections 6–7 prove Theorem 1.1. Section 7 also distinguishes purity from the invariant-theoretic nullcone. The final section treats polynomial recompositions, ordinary Cauchy transforms and coefficient algebras with nilpotents. We use standard structural facts about reductive groups, the classical finite subgroups of SL_2, the two branching rules stated in Lemma 6.2, and the Duistermaat–van der Kallen theorem in its explicit form (6.2). All special moment identities used in the proof are derived here.

2 Haar integration and a polynomial construction

On SU(2) the second row is determined by the first:

$$z = -\overline{y}, \qquad w = \overline{x}, \qquad |x|^2 + |y|^2 = 1. \tag{2.1}$$

Normalized Haar measure is normalized surface measure on this unit three-sphere. The variables $|x|^2$, $\arg x$ and $\arg y$ have product measure

$$\mathrm{d}t\, \frac{\mathrm{d}\theta}{2\pi}\, \frac{\mathrm{d}\phi}{2\pi}, \qquad 0 \leq t \leq 1.$$

One can see this directly by writing $x = \sqrt{t}\, e^{i\theta}$ and $y = \sqrt{1-t}\, e^{i\phi}$ in the surface element. In particular, for nonnegative integers i, k,

$$\mathsf{h}(x^i y^k z^k w^i) = (-1)^k \int_0^1 t^i (1-t)^k \, \mathrm{d}t = (-1)^k \frac{i!k!}{(i+k+1)!}. \tag{2.2}$$

Every monomial whose two phase exponents do not cancel has integral zero.

We shall also use left invariance. The compact diagonal subgroup gives weight $+1$ to x, y and weight -1 to z, w; every nonzero weight has zero integral. The complexified infinitesimal left action contains

$$D = z\partial_x + w\partial_y.$$

Thus $\mathsf{h}(DT) = 0$ for every regular polynomial T, and the product rule gives

$$\mathsf{h}(SD^jT) = (-1)^j \mathsf{h}((D^jS)T). \tag{2.3}$$

These identities involve polynomials on SL_2 only.

We first work with any homogeneous $f \in \mathbb{Z}[X, Y]$ of even degree $n = 2b \geq 4$. Set $a = b-1$ and define U, F, R, P, Q as in (1.2), with f in place of f_n. Taylor expansion in a new scalar t gives

$$f(z + tx, w + ty) = \sum_{k=0}^{n} H_k t^k, \qquad H_0 = F.$$

Substitution $t = F$ shows

$$R = 1 + \sum_{k=1}^{n} F^{k-1} H_k \in \mathbb{Z}[x, y, z, w]. \tag{2.4}$$

If a subgroup fixes f, its simultaneous right action on the two rows fixes all the polynomials in this formula. Hence the construction respects right invariants.

Proposition 2.1. *For every such binary form f, the polynomial P is pure.*

Proof. Fix $m \geq 1$, and write $A = am$ and $B = bm$. Define

$$H_k^{(B)} = [t^k] f(z + tx, w + ty)^B \qquad (0 \leq k \leq nB).$$

Expanding R^B gives

$$P^m = U^A \sum_{k=0}^{nB} F^{k-B} H_k^{(B)}. \tag{2.5}$$

Every summand is a polynomial, including those with $k < B$. Indeed, at most k of the B factors contributing to $H_k^{(B)}$ can use a positive power of t. The remaining factors each contribute F, so F^{B-k} divides $H_k^{(B)}$ over $\mathbb{Z}$.

The kth summand has left weight

$$nA - (n-2)k = (n-2)(B-k).$$

Only $k = B$ can contribute to its Haar integral. Homogeneity also gives

$$H_k^{(B)} = \frac{D^{nB-k}(U^B)}{(nB-k)!}. \tag{2.6}$$

To verify this identity, replace t by t^{-1} in $f(x+tz, y+tw)^B$ and multiply by t^{nB}. Consequently, with $j = (n-1)B$,

$$\mathsf{h}(P^m) = \frac{1}{j!} \mathsf{h}\big(U^A D^j (U^B)\big).$$

The degree of U^A in the top row is $nA = (n-2)B < j$. Equation (2.3) therefore makes the last expression zero. □

Proposition 2.2. *Put $c = b(n-1)$. For $m \geq 1$, set $A = am$, $B = bm$ and $s = b(m-1)$. Then*

$$\mathsf{h}(QP^m) = \mathsf{h}(T_m), \qquad T_m = \frac{U^A}{F}[t^s]f(z+tx, w+ty)^B. \tag{2.7}$$

The displayed T_m is an integer polynomial, homogeneous of degree

$$N = cm - b \tag{2.8}$$

in each row. In particular its Haar integral has denominator dividing $(cm-a)!$.

Proof. Multiplication by $Q = F^a$ changes the weight in (2.5) to $(n-2)(B-k-b)$. The zero weight is therefore $k = s = B - b$, and its power of F is -1. The same factor-counting argument shows that F^b divides the coefficient in (2.7), so division by F is harmless. The top degree is $nA + s$ and the bottom degree is $nB - s - n$; both equal $cm - b$. Formula (2.2) gives the denominator bound. □

3 A congruence for the marked moments

Lemma 3.1. *If $T \in \mathbb{Z}[x,y,z,w]$ is homogeneous of degree $p-1$ in each row, where p is an odd prime, then $p\mathsf{h}(T)$ is p-integral and*

$$p\mathsf{h}(T) \equiv [X^{p-1}Y^{p-1}]T(X,Y,X,Y) \pmod{p}. \tag{3.1}$$

Proof. It suffices to consider $x^i y^{p-1-i} z^k w^{p-1-k}$. Its integral vanishes unless $i + k = p - 1$. In that case (2.2) gives $(-1)^k i!k!/p!$. Wilson's theorem implies

$$i!(p-1-i)! \equiv (-1)^{i+1} \pmod{p},$$

so multiplication by p makes this integral congruent to one. The same condition $i + k = p - 1$ selects precisely the monomials on the right of (3.1). □

Choose an index m for which

$$p = cm - a \quad \text{is prime.} \tag{3.2}$$

Then $N = p - 1$ in Proposition 2.2. At two equal rows,

$$[t^s]f(X+tX, Y+tY)^B = \binom{nB}{s} f(X,Y)^B.$$

It follows that

$$p\mathsf{h}(QP^m) \equiv \binom{nB}{s}\mathcal{H}_p \pmod{p}, \qquad \mathcal{H}_p = [X^{p-1}Y^{p-1}]f(X,Y)^{(p-1)/b}. \tag{3.3}$$

We used $A + B - 1 = (p-1)/b$ here. Notice also that

$$nB = p + q, \qquad q = bm + a = s + n - 1, \qquad 0 \leq s \leq q < p.$$

Taking the coefficient of t^s in $(1+t)^{p+q} \equiv (1+t^p)(1+t)^q$ proves

$$\binom{nB}{s} \equiv \binom{q}{s} \not\equiv 0 \pmod{p}. \tag{3.4}$$

Thus the entire nonvanishing problem has become the calculation of one binary coefficient $\mathcal{H}_p$.

4 The two binary coefficients

For each form in (1.1), define the companion G by

$$\begin{aligned} G_8(X,Y) &= X^2Y^2(X^4-Y^4)^2, \\ G_{12}(X,Y) &= X^{20}+Y^{20}-228X^{15}Y^5+494X^{10}Y^{10}+228X^5Y^{15}. \end{aligned} \tag{4.1}$$

Let $v = \deg G$, so $v = 12$ or 20, and put

$$J = \frac{f_X G_Y - f_Y G_X}{nv}.$$

The polynomial identities we need are

$$\begin{aligned} J^2 &= f^3G/36 - 3G^3, \quad n = 8, \\ J^2 &= 12f^5 + G^3/144, \quad n = 12. \end{aligned} \tag{4.2}$$

These identities can be checked by substitution and multiplication; the definitions specify every coefficient. They are a convenient normalization of the classical Klein identities. Appendix C gives explicit verifications in scalar coordinates.

We use a coefficient-extraction rule for differentials in characteristic p. Its short algebraic proof is included in Appendix B. If L is a one-variable function field over $\mathbb{F}_p$ and t is a separating variable, every differential has a unique expression

$$\omega = \sum_{i=0}^{p-1} a_i^p t^i \, \mathrm{d}t.$$

Here "separating" means that L is finite and separable over $\mathbb{F}_p(t)$. The rule $\mathcal{C}(\omega) = a_{p-1}\,\mathrm{d}t$ is independent of the chosen separating variable and commutes with separable field extensions. It is usually called the Cartier operator. In particular,

$$\mathcal{C}\left(\frac{\sum_j c_j t^j}{Y^p}\,\mathrm{d}t\right) = \frac{1}{Y}\sum_{r\ge 0} c_{pr+p-1}t^r\,\mathrm{d}t \quad (c_j \in \mathbb{F}_p). \tag{4.3}$$

Work now over $\mathbb{F}_p$, with $p > 5$ and $p \equiv 1 \pmod b$, and form the field $L = \mathbb{F}_p(t, Y)$ defined by

$$Y^b = f(t, 1).$$

The polynomials $f(t,1)$ have simple roots for these primes: for f_8 the relevant quadratic discriminant is $14^2 - 4 = 192$, and for f_{12} it is $11^2 + 4 = 125$. Together with $p \nmid 4 \cdot 5$, these facts check squarefreeness directly. The polynomial $Y^b - f(t,1)$ is Eisenstein at any simple root of f after extending constants; hence it defines a field and t is separating.

In L put

$$g = \frac{G(t,1)}{Y^{v/2}}, \qquad j = \frac{J(t,1)}{Y^{(n+v-2)/2}}.$$

Equation (4.2) gives

$$\begin{aligned} j^2 &= g/36 - 3g^3, \quad n = 8, \\ j^2 &= 12 + g^3/144, \quad n = 12. \end{aligned} \tag{4.4}$$

These are nonsingular cubic equations for $p > 5$. More importantly, differentiation gives the exact identity

$$\frac{\mathrm{d}g}{j} = -v\frac{\mathrm{d}t}{Y}. \tag{4.5}$$

Indeed, Euler's homogeneous formula yields $J = f'G/n - fG'/v$ on the chart with second coordinate one, while $Y'/Y = f'/(bf)$. Substituting these into the derivative of g proves (4.5). Since $p > 5$ and $v \in \{12, 20\}$, the scalar $-v$ lies in $\mathbb{F}_p^\times$. Thus $\mathrm{d}g \neq 0$. Over the perfect field $\mathbb{F}_p$, this means that g is separating: $L/\mathbb{F}_p(g)$ is separable. The cubic function field $E = \mathbb{F}_p(g, j)$ defined by (4.4) is an intermediate field, so the inclusion $E \subseteq L$ is separable as well.

Homogeneity identifies the binary coefficient with the affine one:

$$\mathcal{H}_p = [t^{p-1}] f(t, 1)^{(p-1)/b},$$

because $f(X, Y)^{(p-1)/b}$ has total degree $2(p-1)$. Since $Y^{p-1} = f(t, 1)^{(p-1)/b}$, the extraction rule gives

$$\mathcal{C}\left(\frac{\mathrm{d}t}{Y}\right) = \mathcal{H}_p \frac{\mathrm{d}t}{Y}.$$

The numerator has degree at most $2(p-1)$, so only the coefficient of t^{p-1} survives. If the cubic on the right of (4.4) is denoted by $h(g)$, the same calculation gives

$$\mathcal{C}\left(\frac{\mathrm{d}g}{j}\right) = [g^{p-1}] h(g)^{(p-1)/2} \frac{\mathrm{d}g}{j}.$$

Applying $\mathcal{C}$ to (4.5) and using separable pullback compares these two eigenvalues. Cartier is $\mathbb{F}_p$-linear, so the common nonzero factor $-v \in \mathbb{F}_p^\times$ cancels. The two coefficients are therefore equal. They are

$$\begin{aligned} n = 8: &\quad \mathcal{H}_p = \binom{(p-1)/2}{(p-1)/4} (-1/12)^{(p-1)/4}, \\ n = 12: &\quad \mathcal{H}_p = \binom{(p-1)/2}{(p-1)/3} 12^{(p-1)/6} 144^{-(p-1)/3}. \end{aligned} \tag{4.6}$$

Every factor is nonzero in $\mathbb{F}_p$. For primes in (3.2), the required congruence $p \equiv 1 \pmod{b}$ is automatic. Equations (3.3) and (3.4) now show that $p\mathsf{h}(QP^m)$ is a p-adic unit. This proves (1.3). Proposition 2.1, Appendix A and Dirichlet's theorem complete the proof of Theorem 1.2.

5 A radial witness from the same construction

Long's compact-group argument already supplies a radial counterexample for invariant measures on $\mathbb{C}^2$ [9, Theorem 4.1]. We give an alternative pair and its closed marked-moment formula. Take $f_4(X, Y) = X^2Y^2$ in Section 2. The resulting pair has the particularly simple form

$$P_4 = x^2y^2(1 + xzw^2)^4(1 + yz^2w)^4, \qquad Q_4 = z^2w^2. \tag{5.1}$$

Indeed, the intermediate polynomial R is $(1 + xzw^2)^2(1 + yz^2w)^2$. Both functions in (5.1) are invariant under the right diagonal torus and under

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The latter transformation interchanges the two parenthesized factors. Thus they descend to $\mathrm{SL}_2/N(T)$, where $N(T)$ is the normalizer of the diagonal torus.

The first moment illustrates the cancellation. On SU(2), put $t = |x|^2$ and $u = \overline{xy}$. Then

$$P_4 = (xy)^2(1 - tu)^4(1 + (1 - t)u)^4, \qquad Q_4 = u^2.$$

Phase integration keeps the coefficient of u^2 for P_4 and the constant coefficient for Q_4P_4. Consequently,

$$\mathsf{h}(P_4) = \int_0^1 t^2(1-t)^2(6-28t+28t^2)\,\mathrm{d}t = 0, \qquad \mathsf{h}(Q_4P_4) = \int_0^1 t^2(1-t)^2\,\mathrm{d}t = \frac{1}{30}.$$

The following calculation extends this distinction to every order.

Lemma 5.1. *For every $m \geq 1$,*

$$\mathsf{h}(P_4^m) = 0, \qquad \mathsf{h}(Q_4P_4^m) = \beta_m = \frac{2m}{3m+1}\frac{(2m)!(4m)!}{(6m)!} > 0. \tag{5.2}$$

Proof. The phase calculation above also proves purity at every order, directly from the binomial theorem:

$$\begin{aligned}\mathsf{h}(P_4^m) &= \frac{1}{(6m+1)!}\sum_{r=0}^{2m}(-1)^r\binom{4m}{r}\binom{4m}{2m-r}(2m+r)!(4m-r)! \\ &= \frac{((4m)!)^2}{(6m+1)!(2m)!}\sum_{r=0}^{2m}(-1)^r\binom{2m}{r} = 0.\end{aligned}$$

For the marked moment, Proposition 2.2 gives

$$T_m = (xy)^{2m}(zw)^{-2}[t^{2m-2}](z+tx)^{4m}(w+ty)^{4m}.$$

Set $L = 2m-2$. Expanding this coefficient and using (2.2) gives

$$\beta_m = \frac{((4m)!)^2}{(6m-1)!L!}\sum_{r=0}^{L}\frac{(-1)^r\binom{L}{r}}{(4m-r)(4m-r-1)(2m+r+1)(2m+r+2)}. \tag{5.3}$$

There is no division by zw in the polynomial T_m; its displayed coefficient is divisible by $(zw)^4$.

Let S_m denote the sum in (5.3). Use $1/(k(k+1)) = \int_0^1 u^{k-1}(1-u)\,\mathrm{d}u$ for each pair of consecutive denominator factors and sum the binomial expansion. This gives

$$S_m = \int_0^1\int_0^1 s^{2m}t^{2m}(1-s)(1-t)(t-s)^L\,\mathrm{d}s\,\mathrm{d}t > 0.$$

The positivity is immediate because $L = 2m-2$ is even. For its value, use symmetry to integrate over $0 < s < t < 1$ and put $s = rt$. With

$$I_m = \int_0^1 r^{2m}(1-r)^L\,\mathrm{d}r = \frac{(2m)!L!}{(4m-1)!},$$

the t-integral and then the two r-integrals give

$$S_m = 2I_m\left(\frac{1}{6m(6m+1)} - \frac{2m+1}{4m(6m+1)(6m+2)}\right) = \frac{I_m}{6m(6m+2)}.$$

Substitution into (5.3) gives (5.2). □

The same pair works for much more than surface measure. For $v = (v_0, v_1) \in \mathbb{C}^2$, define

$$M(v) = \begin{pmatrix} v_0 & -\overline{v_1} \\ v_1 & \overline{v_0}\end{pmatrix}, \qquad \mathcal{P}(v) = P_4(M(v)), \quad \mathcal{Q}(v) = Q_4(M(v)).$$

These are polynomials in the real and imaginary coordinates of v. They are unchanged by $v \mapsto e^{i\theta}v$, because (5.1) is right-torus invariant.

Proposition 5.2 (Radial identity)**.** *For every compactly supported finite positive* SU(2)*-invariant measure μ on $\mathbb{C}^2$ and every $m \geq 1$,*

$$\int \mathcal{P}^m \, \mathrm{d}\mu = 0, \qquad \int \mathcal{Q}\mathcal{P}^m \, \mathrm{d}\mu = \beta_m \int \|v\|^{12m-4} \, \mathrm{d}\mu. \tag{5.4}$$

The second integral is positive if μ is not concentrated at zero.

Proof. The matrix identity $M(kv) = kM(v)$ holds for $k \in \mathrm{SU}(2)$. For $v \neq 0$, put $\rho = \|v\|$ and write $M(v) = \rho g$ with $g \in \mathrm{SU}(2)$. Using $t = |x|^2$ and $u = \overline{xy}$ for the unit-radius coordinates of g, we have

$$P_4(\rho g) = \rho^4 (xy)^2 (1 - \rho^4 tu)^4 (1 + \rho^4 (1-t)u)^4, \qquad Q_4(\rho g) = \rho^4 u^2.$$

Average the integrands first over k. Phase integration selects order $2m$ for the pure moment and $2m-2$ for the marked moment. The averages are therefore $\rho^{12m} \mathsf{h}(P_4^m) = 0$ and $\rho^{12m-4}\beta_m$, respectively. The formulas hold at $v = 0$ as well. Integrating these averaged identities against μ proves the result. □

6 Connected reductive groups

We now prove Theorem 1.1 for connected G. We use compact forms as in [2, Section 3] and the central-isogeny and complete-reducibility facts for reductive groups [4, Propositions 14.2 and 14.10, Sections 24.1 and 24.4]. The root-doublet and subdirect-product arguments follow Long [10, Proposition 3.1 and Lemmas 4.1, 4.2, 6.2]. We give the reductions here and use Proposition 5.2 for the moment input.

We use two elementary facts about normalized Reynolds functionals. If $q : X \to Y$ is a surjective equivariant morphism of homogeneous spaces, then

$$\mathsf{h}_X(q^* f) = \mathsf{h}_Y(f). \tag{6.1}$$

Both sides are normalized invariant functionals on $\mathbb{C}[Y]$, and such a functional is unique. The same observation applies when the acting group maps onto the acting group of Y. Also, a central isogeny $\widetilde{G} \to G$ identifies $\widetilde{G}/\pi^{-1}H$ with G/H and preserves normalized integration. Thus all witnesses constructed after this change are functions on the original homogeneous space.

We call a submodule a *defining doublet* if it is isomorphic to the standard two-dimensional SL_2-module.

Proposition 6.1 (A line and a doublet)**.** *Suppose an irreducible G-module V has a nonzero H-stable line $\mathbb{C}\xi$, and some homomorphism $\mathrm{SL}_2 \to G$ has a defining two-dimensional submodule $W \subseteq V$. Then $\mathbb{C}[G/H]$ contains fixed functions P, Q with*

$$\mathsf{h}(P^m) = 0, \qquad \mathsf{h}(QP^m) > 0 \qquad (m \geq 1).$$

Proof. Choose a compact form L of H and a compact form K of G containing L. This is possible also for disconnected H; see [2, Theorems 3.7–3.8]. By conjugating the SL_2 homomorphism and W together, arrange that SU(2) maps into K. Give V a K-invariant Hermitian form. Orthogonal projection Π_W commutes with SU(2), since both W and $W^\perp$ are stable. The map

$$\Phi : K \to W, \qquad \Phi(k) = \Pi_W(k\xi)$$

is SU(2)-equivariant. The pushforward of Haar measure is therefore a compact positive invariant measure on W. It is not concentrated at zero: continuity and full support of Haar

measure would otherwise give $K\xi \subseteq W^\perp$, whereas the span of $K\xi$ is all of the irreducible module V.

Identify W with $\mathbb{C}^2$ by an SU(2)-equivariant unitary isomorphism and apply Proposition 5.2 to this measure. The polynomials $P = \mathcal{P} \circ \Phi$ and $Q = \mathcal{Q} \circ \Phi$ are regular on G: their holomorphic factors are matrix coefficients. In an orthonormal basis, $\overline{\rho(k)} = (\rho(k)^{-1})^{\mathsf{T}}$ for $k \in K$, so the conjugate factors extend as matrix coefficients of the dual representation. Moreover $\Phi(kl) = \chi(l)\Phi(k)$ for the character χ of the H-stable line. On L that character is unitary. The phase invariance of $\mathcal{P}, \mathcal{Q}$ therefore makes P, Q right L-invariant. Zariski density of L in H makes their regular extensions right H-invariant. Thus they belong to $\mathbb{C}[G/H]$ and (5.4) proves the claim. □

Lemma 6.2. *Every nontrivial finite-dimensional rational representation of a connected simple complex group of rank at least two contains a defining doublet for some root* SL_2.

Proof. For type A_2, an irreducible module has highest weight (a, b), $a, b \geq 0$. The $\mathrm{GL}_3 \downarrow \mathrm{GL}_2$ interlacing rule says that the polynomial highest weight $(a + b, b, 0)$ contains every pair satisfying

$$a + b \geq \mu_1 \geq b \geq \mu_2 \geq 0.$$

If $b > 0$, use $(\mu_1, \mu_2) = (b, b - 1)$; otherwise use $(1, 0)$. In the latter case nontriviality gives $a \geq 1$. The difference is one, so the root SL_2 has a defining summand.

For B_2, write a highest weight as (a, b) with $a \geq b \geq 0$, both integral or both half-integral. The $\mathrm{Spin}_5 \downarrow \mathrm{Spin}_4$ rule contains all (c, d) of the same parity with

$$a \geq c \geq b \geq |d|.$$

The two SL_2 highest weights are $c - d, c + d$. Use $(c, d) = (b, b - 1)$ when $b > 0$, and $(1, 0)$ otherwise. The first choice is valid even when $b = \frac{1}{2}$, since $|b - 1| \leq b$; the second uses $a \geq 1$. These are the classical branching rules; see [12, Theorem 2.1 and equation (4.4)].

Every simple root system of rank at least two contains a root-generated subalgebra of type A_2 or B_2: take adjacent simple roots, using the long-root A_2 inside G_2 if necessary. A nontrivial representation of a simple Lie algebra is faithful, so its restriction to this subalgebra is nontrivial. A nontrivial irreducible constituent and the preceding branching arguments give the doublet. Root homomorphisms integrate it to the group, also after composing a simply connected cover. □

Call a normal simple factor *active* if it acts nontrivially on G/H. Since G/H is affine, an active factor acts nontrivially on some irreducible submodule $U \subseteq \mathbb{C}[G/H]$. Evaluation at eH is a nonzero H-fixed vector of U^*. It is nonzero because a G-stable space of functions vanishing at eH would vanish at every point. Thus Lemma 6.2 and Proposition 6.1 handle every active factor of rank at least two.

Lemma 6.3. *Let $G = Z \times \mathrm{SL}_2^r$, with Z a torus, and let H be reductive. If every projection $H \to \mathrm{SL}_2$ is surjective, then either $[G, G] \subseteq H$ or the hypothesis of Proposition 6.1 holds.*

Proof. The projections of H^0 are surjective as well, since their images have finite index in the connected target. The center of $\mathrm{Lie}(H^0)$ maps to zero in each $\mathfrak{sl}_2$. Hence its derived algebra projects onto every factor. Decompose that semisimple algebra into simple ideals. For each coordinate, exactly one ideal maps nontrivially: its image is an ideal of $\mathfrak{sl}_2$, and two nonzero such images could not commute. That ideal maps isomorphically onto $\mathfrak{sl}_2$. Every simple ideal appears in at least one coordinate, since it cannot map nontrivially to the torus.

Thus the coordinates partition into blocks on which the derived algebra is diagonal, up to inner automorphisms. After conjugating individual factors,

$$[H^0, H^0] = \prod_B \Delta_B \operatorname{SL}_2 .$$

The equality holds for groups because connected algebraic subgroups in characteristic zero are determined by their Lie algebras. The connected center of H^0 lies in Z. If every block is a singleton, $[G, G] \subseteq H$. Otherwise choose two indices i, j in one block. On the irreducible module $V = \mathbb{C}_i^2 \otimes \mathbb{C}_j^2$, that block acts diagonally and the connected center acts trivially. Hence the entire fixed subspace is

$$V^{H^0} = \mathbb{C}\big(e_{i,1} \otimes e_{j,2} - e_{i,2} \otimes e_{j,1}\big),$$

where the bases correspond under the chosen diagonal identifications. This is the alternating summand in $\mathbb{C}^2 \otimes \mathbb{C}^2 = \operatorname{Sym}^2 \mathbb{C}^2 \oplus \bigwedge^2 \mathbb{C}^2$. Normality of H^0 in H makes this whole fixed subspace, and thus its unique line, H-stable. The ith factor has the defining submodule $\mathbb{C}_i^2 \otimes \mathbb{C}e_1$ in V. □

Proof of Theorem 1.1 for connected G. If $[G, G] \subseteq H$, then G/H is a torus and its normalized functional is constant term: H is the inverse image of a closed subgroup of the torus $G/[G, G]$, hence is normal with torus quotient. We use the Duistermaat–van der Kallen theorem [5, pp. 222–224] in the following form:

$$\operatorname{CT}(f^m) = 0 \ (m \geq 1), \ f \neq 0 \quad \Longrightarrow \quad 0 \notin \operatorname{conv}(\operatorname{Supp} f). \tag{6.2}$$

A strictly positive linear functional on the finite support of f eventually dominates the weights of any fixed multiplier q. Therefore $\operatorname{CT}(qf^m) = 0$ for all large m.

Now assume $[G, G] \not\subseteq H$. Pass to a finite central cover $Z \times \prod S_i \to G$. If an active factor has rank at least two, the preceding argument supplies the required pair. Otherwise quotient out the higher-rank factors contained in the stabilizer. They are normal and act trivially, so this leaves an isomorphic homogeneous space with acting group $Z \times \operatorname{SL}_2^r$.

Consider the image H_i of the stabilizer in one SL_2 factor. This image is closed and reductive. If it is proper and positive-dimensional, its identity component is a torus; the full image normalizes that torus and so lies in a conjugate of $N(T)$. If finite, the classical finite-subgroup classification [6, p. 215] puts it in one of the following groups:

$$N(T), \qquad 2O, \qquad 2I.$$

Here cyclic and binary dihedral groups lie in $N(T)$, while $2T$ lies in $2O$. In each case projection gives a genuine surjection of the original homogeneous space onto the corresponding quotient of SL_2. Pull back the pair from Lemma 5.1 or Theorem 1.2, using (6.1). Finally, if every projection is surjective, apply Lemma 6.3 and Proposition 6.1. The alternative $[G, G] \subseteq H$ is excluded. This proves the connected classification and its fixed-multiplier assertion. □

7 Disconnected groups and moment nullcones

We first give the finite-trace fact needed for disconnected spaces. It does not rely on the signs of complex moments.

Lemma 7.1 (Finite-trace separation)**.** *Let $f_1, \dots, f_r$ be Laurent polynomials in finitely many variables over a characteristic-zero field. If*

$$\sum_{i=1}^{r} \operatorname{CT}(f_i^N) = 0 \qquad (N \geq 1),$$

then $\mathrm{CT}(f_i^m) = 0$ *for every* i *and* $m \geq 1$.

Proof. It is enough to consider a number field of coefficients. Indeed, if one selected moment were nonzero, specialize the finitely generated $\mathbb{Q}$-algebra of coefficients after inverting that moment. A closed point gives a number-field specialization preserving it and every assumed identity.

Fix $m \geq 1$ and $1 \leq k \leq r$. There are arbitrarily large primes p splitting completely in the coefficient field and avoiding its coefficient denominators. For completeness, this infinitude needs no density theorem. Take an integral primitive element of the Galois closure. Schur's elementary argument gives infinitely many prime divisors of values of its minimal polynomial. Outside its discriminant and index, such a divisor gives a degree-one prime in the Galois field; all residue degrees are equal, so the prime splits completely. Schur's argument itself is Euclid's: after translating a nonconstant integer polynomial to have nonzero constant term a, if all prime divisors of its nonzero integer values belonged to a finite set with product B, the polynomial $f(aBt)/a$ would be integral, nonconstant and congruent to one modulo B. A sufficiently large value has a prime divisor outside that set.

Choose such p larger than the absolute value of every exponent coordinate in every f_i^m, and put $N = m(1 + p + \cdots + p^{k-1})$. In the residue field,

$$f_i^N = \prod_{j=0}^{k-1} f_i(z^{p^j})^m.$$

An exponent equation $\alpha_0 + p\alpha_1 + \cdots + p^{k-1}\alpha_{k-1} = 0$ forces $\alpha_0 = 0$ by reduction modulo p and the exponent bound. Repeating gives every $\alpha_j = 0$. Hence, writing $a_i = \mathrm{CT}(f_i^m)$,

$$\mathrm{CT}(f_i^N) \equiv a_i^k \pmod{p}.$$

The assumed trace identity makes $\sum_i a_i^k$ divisible by infinitely many primes, so it is zero in the number field. We have proved this for $k = 1, \ldots, r$. Newton's identities now give $\prod_i (T - a_i) = T^r$, so every $a_i = 0$. This also contradicts the nonzero moment preserved in the initial specialization, if one existed. □

Let $K = G^0$ and choose $g_1, \ldots, g_r$ representing $G/(KH)$. The connected components of X are the affine, open-and-closed orbits

$$X_i = K/H_i, \qquad H_i = K \cap g_i H g_i^{-1}.$$

Consequently $\mathbb{C}[X] = \prod_i \mathbb{C}[X_i]$. Transitivity of G on these components and normalization imply

$$\mathsf{h}((f_i)) = \frac{1}{r} \sum_i \mathsf{h}_i(f_i). \tag{7.1}$$

If $[K, K] \not\subseteq H$, use the connected counterexample on $K/(K \cap H)$ and extend both functions by zero. This is a regular operation on an open-and-closed component. Equation (7.1) preserves its pure moments and its nonzero marked orders, up to $1/r$.

If $D = [K, K] \subseteq H$, then D lies in every H_i, because it is normal in G. Each component is a torus. Embed its character lattice $M_i = X^*(K/H_i)$ into $M = X^*(K/D)$. For a pure tuple (f_i), Lemma 7.1 gives purity on every component. Apply (6.2) and separate each component support. The maximum of the finitely many multiplier cutoffs proves safety in (7.1). This completes Theorem 1.1.

We can also distinguish the Mathieu property from the moment-nullcone statement. Let Z_X be the pure set and $\mathcal{N}_G$ the G-nullcone: the functions whose orbit closure contains zero in a finite-dimensional rational submodule containing the function.

Corollary 7.2. *One has $Z_X = \mathcal{N}_G$ if and only if $D \subseteq H$ and*

$$\theta : \bigoplus_i M_i \longrightarrow M, \qquad (\chi_i) \longmapsto \sum_i \chi_i \tag{7.2}$$

is injective. The lattices in (7.2) *are defined after imposing $D \subseteq H$.*

Proof. The G-nullcone equals the K-nullcone, since a G-orbit closure is a finite union of translates of the K-orbit closure, where $K = G^0$. For this connected group, the Hilbert–Mumford criterion gives a one-parameter subgroup with strictly positive weights on any unstable function [7, Introduction and Section 2]. These weights eventually dominate those of any fixed multiplier, so unstable functions are safe. The non-toral fixed bad pairs therefore lie outside the nullcone.

In the toral case, finite-trace separation and (6.2) give a separate positive functional on every component support. If θ is injective, these extend to one functional on $M \otimes \mathbb{Q}$, positive on the entire union of supports. A multiple lifts to a cocharacter of K, since the central torus maps onto K/D with finite kernel. Its inverse contracts the tuple to zero, with the translation action on functions. Conversely, a nonzero lattice relation $\sum_i \chi_i = 0$ gives a pure tuple with component χ_i when $\chi_i \neq 0$ and zero otherwise. To exclude zero from its orbit closure, let $I = \{i : \chi_i \neq 0\}$ and let $v_i \in \mathbb{C}[X]$ equal χ_i on X_i and zero elsewhere. The tuple is $v = \sum_{i \in I} v_i$ in the finite-dimensional rational K-module $W = \bigoplus_{i \in I} \mathbb{C} v_i$. If z_i are the coordinate functions on W, then $\prod_{i \in I} z_i \in \mathbb{C}[W]^K$ because $\sum_i \chi_i = 0$. This polynomial has value one at v and zero at the origin. Thus v is outside the nullcone. The product here is a polynomial on W, rather than a product of component-supported functions in $\mathbb{C}[X]$. Finally, all nullcone elements are pure because the moments are invariant homogeneous polynomials vanishing at zero. □

8 Recompositions and ordinary Cauchy transforms

Return to either exceptional pair P, Q in Theorem 1.2, with $\Gamma = 2O$ or $2I$, respectively. Write

$$(c, a) = (28, 3) \text{ or } (66, 5), \qquad b_j = \mathsf{h}(QP^j).$$

The facts needed in this section are

$$b_0 = 0, \qquad \operatorname{den}(b_j) \mid (cj - a)! \ (j \geq 1), \qquad v_p(b_j) = -1 \text{ when } p = cj - a \text{ is prime.} \tag{8.1}$$

The first is a weight calculation, the second is Proposition 2.2, and the third is Theorem 1.2. Here $\operatorname{den}(b_j)$ is the positive denominator of the reduced rational number b_j.

For regular S, V on SL_2/Γ, evaluate the Haar integral on the compact quotient $\mathrm{SU}(2)/\Gamma$. The ordinary marked Cauchy transform is the convergent germ

$$C_V^S(z) = \mathsf{h}\left(\frac{V}{z - S}\right) = \sum_{m \geq 0} \frac{\mathsf{h}(VS^m)}{z^{m+1}} \quad \text{for } |z| > \max |S|.$$

We use this ordinary series throughout this section.

Theorem 8.1. *For each $d \geq 1$, choose an integer e satisfying*

$$0 \leq e < d, \qquad \gcd(cd, ce - a) = 1, \tag{8.2}$$

and set $Q_d = QP^e$. Such an e always exists. For every complex polynomial H of degree d, the same multiplier Q_d satisfies

$$\mathsf{h}\big(Q_d H(P)^m\big) \neq 0 \quad \textit{for infinitely many } m, \qquad C_{Q_d}^{H(P)}(z) \textit{ is transcendental over } \mathbb{C}(z).$$

Furthermore

$$\mathsf{h}\big(H(P)^m\big) = H(0)^m, \qquad C_1^{H(P)}(z) = \frac{1}{z - H(0)}. \tag{8.3}$$

In particular, every nonzero polynomial recomposition with $H(0) = 0$ is pure and unsafe.

The multiplier depends on the degree, not on the coefficients of H. For example, the smallest possible e in (8.2) gives an unambiguous choice.

We first record two elementary algebraic facts used to allow arbitrary complex coefficients.

Lemma 8.2. *Let L be a finitely generated field over $\mathbb{Q}$, and let $S \subset L^*$ be finite. For all but finitely many rational primes p, there is a valuation w on L extending v_p, normalized by $w(p) = 1$, for which every member of S is a unit.*

Proof. Choose a transcendence basis t with L finite over $\mathbb{Q}(t)$. On $\mathbb{Q}(t)$ use the p-adic Gauss valuation: on a polynomial it is the minimum coefficient valuation, and on a quotient it is the difference. Choose monic equations over $\mathbb{Q}(t)$ for each $\alpha \in S$ and its inverse. All coefficients of these finitely many equations are integral for all but finitely many Gauss valuations. Extend the valuation to L; existence follows from the valuation-ring domination lemma [14, Lemma 10.50.2]. The monic equations force both $w(\alpha) \geq 0$ and $w(\alpha^{-1}) \geq 0$, so $w(\alpha) = 0$. Values in an algebraic extension are rational multiples of $w(p)$: in an algebraic equation the minimum valuation must occur at least twice, which gives an integer multiple of the element's value in the original value group. This justifies the normalization. □

Lemma 8.3 (Finite-generation form of Eisenstein's argument)**.** *If L has characteristic zero and $A(t) \in L[[t]]$ is algebraic over $L(t)$, all coefficients of A belong to one finitely generated $\mathbb{Z}$-subalgebra of L.*

Proof. Choose $F(t, Y) \in L[t, Y]$ with $F(t, A) = 0$ and $F_Y(t, A) \neq 0$; characteristic zero guarantees such a relation. Put $r = \operatorname{ord}_t F_Y(t, A)$, choose $N > r$, and write $A = a + t^N B$, where $a = A \bmod t^N$. Taylor expansion gives

$$G(t, Y) = t^{-N-r} F(t, a + t^N Y) \in L[t, Y].$$

For the constant term, divisibility follows by comparing with $F(t, A) = 0$; all terms of order at least two have t-order at least $2N > N + r$. The coefficient of Y has a nonzero constant term u, and all higher powers of Y have zero constant term. Thus $G(0, Y)$ is linear. Successive coefficient comparison in $G(t, B) = 0$ uses only addition, multiplication and division by u. It puts all coefficients of B in the ring generated by the coefficients of G and u^{-1}. Adjoining the finitely many coefficients of a proves the lemma. This is the part of Eisenstein's theorem needed here; see [13] for its classical and more general forms. □

Proof of Theorem 8.1. Since $\gcd(c, a) = 1$, the choice in (8.2) is possible. For every prime $\ell \mid d$ with $\ell \nmid c$, choose $e \not\equiv a/c \pmod{\ell}$. A prime dividing c cannot divide a, so imposes no condition. The Chinese remainder theorem combines the choices modulo $\operatorname{rad}(d)$, and a representative lies below d.

Write $H(T) = \sum_{j=0}^{d} h_j T^j$, and let L be the field generated by its coefficients over $\mathbb{Q}$. For a sufficiently large prime

$$p = c(dm + e) - a, \tag{8.4}$$

Lemma 8.2 gives a w making every nonzero h_j a unit. Expansion of $H(T)^m$ yields

$$\mathsf{h}\big(Q_d H(P)^m\big) = h_d^m b_{dm+e} + \sum_{j<dm} [T^j] H(T)^m \, b_{j+e}.$$

The first summand has valuation -1 by (8.1). Every other summand is integral: its coefficients are integral, and $c(j+e)-a<p$ makes its factorial denominator prime to p. The term b_0 is zero. A unique term of negative valuation cannot cancel. The progression (8.4) is coprime by (8.2); Dirichlet's theorem supplies infinitely many such primes after any finite exclusions. This proves the fixed-multiplier nonvanishing assertion.

Suppose the ordinary series were algebraic over $\mathbb{C}(z)$. It would be algebraic over $L(z)$ as well. To see this descent, a polynomial relation with fixed degree bounds is a linear system over L in finitely many unknown coefficients. Extending scalars does not change its rank. Even though the system has infinitely many rows, its row space has a finite basis, so a nonzero complex solution gives a nonzero L solution.

Apply Lemma 8.3 to $A(t)=\sum_{m\ge 0}\mathsf{h}(Q_dH(P)^m)t^m$. There is a finite set $S_A\subset L^*$ whose $\mathbb{Z}$-algebra contains every coefficient of A. For a sufficiently large prime in (8.4), apply Lemma 8.2 simultaneously to S_A and the nonzero coefficients of H. It makes every coefficient of A integral, whereas the preceding expansion gives valuation -1 for its mth coefficient. This contradiction proves transcendence. Finally, all positive powers of P integrate to zero, so expanding $H(P)^m$ leaves only $H(0)^m$. Summing its geometric series proves (8.3). □

Thus the unit Cauchy transform is rational while the marked transform is transcendental for the specified fixed polynomial multiplier.

Coefficient algebras

There is also an exact statement with nilpotents retained. Let R be any commutative unital $\mathbb{C}$-algebra, and extend h R-linearly to $\mathsf{h}_R : R\otimes_{\mathbb{C}}\mathbb{C}[\mathrm{SL}_2/\Gamma]\to R$. An element is *eventually pure* if its positive moments vanish for all sufficiently large orders. Safety in the following statement is against every multiplier in this entire algebra.

Corollary 8.4. *For $H(T)=\sum_{j=0}^{d}c_jT^j\in R[T]$,*

$$\begin{aligned} H(P)\ \textit{is pure} &\iff c_0=0,\\ H(P)\ \textit{is eventually pure} &\iff c_0\ \textit{is nilpotent},\\ H(P)\ \textit{is safe} &\iff \textit{every } c_j\ \textit{is nilpotent} \iff H(P)\ \textit{is nilpotent}. \end{aligned}$$

Proof. The formula $\mathsf{h}_R(H(P)^m)=c_0^m$ proves the first two assertions. If every coefficient is nilpotent, their finitely generated ideal is nilpotent. Explicitly, $c_j^{r_j}=0$ makes its $1+\sum_j(r_j-1)$ power zero. Thus $H(P)$ is nilpotent and safe.

Conversely, a nonnilpotent coefficient survives in a prime quotient of R. Pass to the fraction field of that quotient. If the image of H has positive degree, the valuation argument in Theorem 8.1 applies to the finitely generated subfield over $\mathbb{Q}$ generated by its coefficients. It gives a fixed multiplier QP^e with infinitely many nonzero moments. Those moments were therefore nonzero in R. If the image of H is a nonzero constant, the multiplier one already detects failure. Hence safety forces every coefficient to be nilpotent. Since any nilpotent function is safe, the final equivalence follows too. □

A Invariance of the exceptional binary forms

We give a verification using the root sets; it also fixes the embeddings of the binary groups. Identify $\mathbb{P}^1$ with the unit sphere by

$$q\longmapsto\frac{(2\Re q,2\Im q,|q|^2-1)}{1+|q|^2}.$$

The projective action of SU(2) is the rotation action.

For f_8, the finite roots satisfy $q^4 = -7 \pm 4\sqrt{3}$. They have arguments $(2k+1)\pi/4$ and squared moduli $2 \pm \sqrt{3}$. Their sphere coordinates are exactly $(\pm 1, \pm 1, \pm 1)/\sqrt{3}$, the eight vertices of a cube. Its rotation group is S_4, and its inverse image in SU(2) is $2O$. The root set makes the line $\mathbb{C}f_8$ stable. Its scalar character factors through S_4, because the central element $-I$ acts trivially in even degree. The matrix $\operatorname{diag}(e^{\pi i/4}, e^{-\pi i/4})$ fixes f_8 directly and projects to a quarter-turn, a four-cycle on the body diagonals. Every character of S_4 has the same value $\epsilon \in \{1, -1\}$ on its transpositions and value $\epsilon^3 = \epsilon$ on a four-cycle. Thus this character is trivial and f_8 itself is invariant.

For f_{12}, let $\varphi = (1 + \sqrt{5})/2$. Besides 0 and ∞, its roots are

$$q = \varphi^{-1} e^{2\pi i k/5}, \qquad q = \varphi e^{(2k+1)\pi i/5} \quad (0 \le k < 5).$$

Their sphere images are two poles and two regular pentagons at heights $-1/\sqrt{5}$ and $1/\sqrt{5}$, of radius $2/\sqrt{5}$, offset by $\pi/5$. These are the vertices of a regular icosahedron. Its rotation group is A_5 and its inverse image is $2I$. Again the line spanned by the binary form is stable, and its scalar character factors through A_5. A three-cycle is conjugate to its inverse in A_5, so its character value has both square and cube equal to one. It is therefore one. Three-cycles generate A_5, proving invariance.

B The differential extraction rule

Here we prove the coordinate independence used in Section 4. Let L be a one-variable function field over $\mathbb{F}_p$, and choose a separating element t. Then $L = L^p(t)$ and $1, t, \dots, t^{p-1}$ is a basis over L^p. Indeed, $L/L^p(t)$ is purely inseparable but is also separable, as a subextension of the separable extension $L/\mathbb{F}_p(t)$. It is therefore trivial; and $t \notin L^p$, since $\mathrm{d}t \neq 0$.

Every differential is consequently $\sum_{i=0}^{p-1} a_i^p t^i \,\mathrm{d}t$. Modulo exact differentials, only the last term remains: differentiating $\sum b_i^p t^i$ shows that the exact differentials have the other $p-1$ basis terms. Thus $\Omega_L / \mathrm{d}L$ is one-dimensional over L^p, with basis $[t^{p-1}\,\mathrm{d}t]$.

For $u \in L$ put $\gamma(u) = [u^{p-1}\,\mathrm{d}u]$ in this quotient. It satisfies

$$\gamma(uv) = u^p \gamma(v) + v^p \gamma(u), \qquad \gamma(u+v) = \gamma(u) + \gamma(v).$$

The product formula is literal. For additivity, differentiate the integer polynomial $((U+V)^p - U^p - V^p)/p$ and reduce modulo p; its differential is exactly the difference of the two sides before taking classes. These rules also imply $\gamma(u^p) = 0$.

Writing $u = \sum a_i^p t^i$ now gives

$$[u^{p-1}\,\mathrm{d}u] = \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^p [t^{p-1}\,\mathrm{d}t].$$

If u is another separating coordinate, the class of a differential written as $a^p u^{p-1}\,\mathrm{d}u$ is therefore $(a\,\mathrm{d}u/\,\mathrm{d}t)^p [t^{p-1}\,\mathrm{d}t]$. The extracted differential is $a\,\mathrm{d}u$ in either coordinate. This proves that the rule defining $\mathcal{C}$ is well defined. For a separable extension of function fields, the same t is separating in both and the same expansion works in both, proving compatibility with pullback. Formula (4.3) follows by grouping powers whose exponents are congruent to $p-1$. See also [1, Section 2.2] for the usual Cartier–Manin terminology.

C Checking the polynomial identities

Here are explicit forms of the algebra in (4.2), so that the proof does not require a table of invariant-theoretic normalizations. For $n = 8$, put

$$B = XY(X^4 - Y^4), \qquad T = X^{12} - 33X^8Y^4 - 33X^4Y^8 + Y^{12}.$$

Differentiation gives $G = B^2$ and $J = BT/6$. To check $T^2 = f_8^3 - 108B^4$, set $s = X^4 + Y^4$ and $u = X^4Y^4$. Then $f_8 = s^2 + 12u$, $T = s(s^2 - 36u)$ and $B^4 = u(s^2 - 4u)^2$. The identity follows by expanding these three short expressions.

For $n = 12$, differentiation gives $12J = -T$, where

$$T = X^{30} + Y^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}).$$

With $t = X^5/Y^5$, the required identity becomes the following one-variable polynomial identity:

$$\begin{aligned}(t^6 + 522t^5 - 10005t^4 - 10005t^2 - 522t + 1)^2 \\ = (t^4 - 228t^3 + 494t^2 + 228t + 1)^3 + 1728t(t^2 + 11t - 1)^5.\end{aligned}$$

Multiplying the displayed polynomials verifies it. Homogenization then gives $T^2 = G^3 + 1728f_{12}^5$, hence $J^2 = G^3/144 + 12f_{12}^5$ as asserted.

AI assistance and review. ChatGPT (OpenAI) assisted with mathematical arguments, literature checks, exposition, proof review and verification code.

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