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Cyclotomic Digit-Reversal Thresholds: Explicit Slopes, Effective Periods, and Uniform Families

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Abstract

For a base-B digit polynomial P, let $K_t(B)$ be the first length at which the represented integer and its reversal are divisible by the full integer $\Phi_t(B)$, although $\Phi_t(X)$ does not divide $P(X)$. We study this threshold through a lattice of evaluations and the capacities imposed by digit positions. For every fixed $t \geq 3$, we determine explicit linear slopes and construct an eventual affine description on residue classes, with computable periods, cutoffs and compressed minimum witnesses. No total-unimodularity assumption is needed. For every prime $p \geq 5$, exact formulas in stated base ranges give $K_{3p}(10) = 17p - 1$ and $K_{6p}(10) = 18p + 1$. Ordered capacities explain their different correction periods, two and four. A deductive scalar classification also describes the minimum algebraic values for $6p$ and the action of reversal on them. Period transfers show that excluding palindromes leaves the first thresholds unchanged for both families. The same framework treats independent coefficient bounds and collisions. Exact results for orders $105, 210$ require finite certificates beyond total unimodularity. The general construction is effective; minimal periods and practical efficiency are not claimed.

Keywords: cyclotomic polynomials; digit reversal; congruence lattices; zonotopes; affine monoids; quasi-polynomials.

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1 The problem and the results

To a word of k digits in base $B \geq 2$ associate

$$P(X) = \sum_{i=0}^{k-1} d_i X^i, \quad 0 \leq d_i \leq B-1, \qquad N = P(B), \quad N^{\mathrm{rev}} = B^{k-1} P(B^{-1}). \tag{1.1}$$

The coefficient d_0 is the least significant digit; displayed words are read from the most significant digit. For $t \geq 3$, the numerical modulus is always the full integer $Q = \Phi_t(B)$.

Definition 1.1. A *false positive* is a word for which Q divides both N and N^{rev}, but $\Phi_t(X)$ does not divide $P(X)$. Let $K_t(B)$ be the first length of a false positive.

Palindromes and leading zeros are allowed. We prove that $K_t(B)$ exists and that minimum words have nonzero end digits. Thus the maximum safe length is $K_t(B) - 1$.

Let $\mathcal{D}_{B,k}$ be the coefficient box in (1.1), and set

$$\mathcal{E}_{B,t}(P) = (P(B), P(B^{-1})) \pmod{Q}.$$

The inverse of B modulo Q exists by Lemma 2.1. Our question is exactness of the zero fibre under coefficient constraints:

$$\mathcal{D}_{B,k} \cap \ker \mathcal{E}_{B,t} = \mathcal{D}_{B,k} \cap (\Phi_t) \quad \Longleftrightarrow \quad k < K_t(B). \tag{1.2}$$

Padding by zeros gives monotonicity. Here *local* would mean within the coefficient box, not a p-adic localization. This is a test for polynomial divisibility, not unique coefficient reconstruction. For example, for $t = 4$, $B = 2$, the polynomials $1 + X^4 + X^8$ and $X^2 + X^6$ both evaluate to $(3, 3)$ modulo 5, whereas their remainders modulo $X^2 + 1$ are 3 and -2, already at length $9 < K_4(2) = 17$.

1.1 Two uniform families and the governing mechanism

A concrete consequence of our results is

$$\boxed{K_{3p}(10) = 17p - 1, \qquad K_{6p}(10) = 18p + 1 \quad (p \geq 5 \text{ prime}).}$$

For example, $K_{15}(10) = 84$ and $K_{30}(10) = 91$. The first identity says that a decimal word of at most 83 positions passing both divisibility tests modulo $\Phi_{15}(10)$ must have Φ_{15} as a polynomial factor, whereas a word of length 84 need not. The analogous safe length for order 30 is 90.

Theorem 8.1 gives, for every prime $p \geq 5$,

$$K_{3p}(B) = \begin{cases} 3pB/2 + 2p - 1, & B \geq 9 \text{ even}, \\ 3p(B+1)/2 + 1, & B \geq 9 \text{ odd}. \end{cases}$$

For $B = 4m + r$, Theorem 9.1 gives

$$K_{6p}(B) = 6pm + (3p + d_p,\ 5p - e_p,\ 6p + 1,\ 8p + 1)_r,$$

where $d_p = 2, 4$ according as $p \equiv 5, 1 \pmod 6$, and $e_p = 2$ for $p = 5, 7$ and $e_p = 4$ otherwise. The sufficient base bounds in the four branches are $16, 13, 10, 19$. The eventual correction period is exactly four, compared with two for $3p$. The base cutoffs are sufficient; they are not asserted to be minimal.

The proof follows

$$\text{kernel lattice} \longrightarrow \text{coefficient height} \longrightarrow \text{ordered capacities} \longrightarrow \text{length}.$$

The lattice identifies all possible false positives. Its first zonotope shell controls coefficient height. Digit positions impose unequal capacities in the last period, and those ordered capacities determine the intercept of $K_t(B)$. A complete short-shell classification and a graph on three rows make the two families accessible. For $3p$, a two-state local obstruction has a deductive

proof. For $6p$, a four-position gap lemma reduces the parameters to a single scalar, and equality components and short boundary calculations classify every support. The complete calculation is included in Appendix F.

The same analysis determines

$$h_{3p}(B) = \left\lceil \frac{B^2+1}{2} \right\rceil \quad (B \geq 9), \qquad h_{6p}(B) = \left\lceil \frac{B^2+B+1}{4} \right\rceil \quad (B \geq 10),$$

where $h_t(B)$ is the minimum full-period coefficient height. For $6p$, the first normalized algebraic values are unique on three branches in specified ranges, while their number can grow exponentially with p on the fourth. This counts values, not digit words. One witness can nevertheless be lifted in $O(p)$ arithmetic operations; expanding its digits still requires time proportional to the output. Section 10 describes reversal on the Boolean family of minimum values and gives exact fixed-value and orbit counts. A period-transfer lemma also shows that the first thresholds for $3p$ and $6p$ are unchanged when words are required to be non-palindromic. The resulting word counts are lower bounds, distinguished from the exact counts of values.

1.2 Explicit slopes and effectivity at every fixed order

Put $w = \omega(t)$, counting the prime 2, $m_t = 2^{w-1}$, and $\delta_t(B) = \gcd(\Phi_t(B), B^2-1)$. Proposition 2.5 shows that $\delta_t(B)$ is either 1 or a prime. Define

$$\mu_t(B) = \begin{cases} 1/m_t, & \delta_t(B) = 1, \\ (p-1)/(m_t p), & \delta_t(B) = p > 1. \end{cases} \tag{1.3}$$

Theorem 1.2 (Explicit slopes)**.** *For every fixed $t \geq 3$,*

$$K_t(B) = t\mu_t(B)B + O_t(1).$$

More precisely, with $D = B-1$, $E = 2^w - 1$, and $U_t(B) = \lfloor \mu_t(B)(B+1)^2 + E \rfloor$,

$$t\big(\lceil \mu_t(B)D \rceil - 1\big) + 1 \;\leq K_t(B) \leq\; t \left\lceil \frac{U_t(B)}{D} \right\rceil. \tag{1.4}$$

The error is bounded over all bases at fixed order. When $\delta_t(B)$ changes, no single limit of $K_t(B)/B$ is asserted.

Let R_t be the integer root matrix in the tensor basis constructed in §4. Let L_t be the least common multiple of the absolute values of its nonzero square minors, including 1. Set

$$\rho_t = \begin{cases} 2, & t = 2^a,\ a \geq 2, \\ p, & t = p^a,\ p \text{ odd prime}, \\ 2p, & t = 2p^a,\ p \text{ odd prime}, \\ 2^{w-1}, & \text{otherwise.} \end{cases} \tag{1.5}$$

Theorem 1.3 (Effective finite description)**.** *One can compute a cutoff C_t and rational numbers $b_{t,j}$, $0 \leq j < P_t = \rho_t L_t$, such that*

$$K_t(B) = t\mu_t(B)B + b_{t,B \bmod P_t} \qquad (B \geq C_t). \tag{1.6}$$

The initial values $2 \leq B < C_t$ are also computable by a finite procedure. The minimum eventual period of the coefficient functions divides P_t. Total unimodularity is not assumed.

Here *effective* means an exact terminating procedure; its bounds may be very large. We do not claim to have executed it for every order.

Corollary 1.4 (Compressed witnesses)**.** *After possibly refining the period and increasing the cutoff, one can choose a minimum false positive in each base such that, within each subsequence of positions modulo t, the number of full digits $B-1$ and the possible partial digit are affine functions of the base.*

The constructive proof is in §5.4. Blocks are contiguous within the residue subsequences, not necessarily in the word.

1.3 Exact cases and bounded coefficients

Theorem 1.5 (Three elementary orders)**.** *For every $B \geq 2$,*

$$K_4(B) = \begin{cases} 17, & B = 2, \\ 4B+5, & B \geq 4 \text{ even}, \\ 2B+4, & B \text{ odd}, \end{cases} \tag{1.7}$$

$$K_{10}(B) = \begin{cases} 14, & B = 4, \\ 4B-4, & B \equiv -1 \pmod 5,\ B \geq 9, \\ 5B+3, & B \text{ even},\ B \not\equiv -1 \pmod 5, \\ 5B, & B \text{ odd},\ B \not\equiv -1 \pmod 5, \end{cases} \tag{1.8}$$

$$K_{12}(B) = \begin{cases} 6B+2, & B \text{ even}, \\ 6B, & B \text{ odd}. \end{cases} \tag{1.9}$$

Appendices A–E give complete deductive proofs, including small bases. The companion provides additional regression checks. Section 6 treats all $t = 2^a$, $a \geq 3$.

Orders $15, 30$ are corollaries of the uniform families. Orders $105, 210$ in §11 show how to retain integer feasibility beyond total unimodularity. Their proofs require the specified finite certificates. Doubling changes the zonotope, the sign of a reciprocal term, and the order of capacities; a comparison of convex bodies alone does not determine the intercepts.

Section 4.4 allows $0 \leq d_i \leq H$ independently of the base while keeping the full modulus $\Phi_t(B)$. It gives

$$K_{t,H}(B) = t\mu_t(B)B^2/H + O_t(1 + B/H)$$

uniformly for $H \geq 1$ at fixed order. Collisions between distinct remainders are exactly false positives with coefficients in $[-H, H]$, with scale $t\mu_t(B)B^2/(2H)$. These are consequences of the same coefficient-box problem, without a measured factorization speedup.

1.4 Prior work and reading guide

Mossinghoff [8, §2.1–2.2, Algorithms 1–2] uses evaluations at 2 and -2 to filter restricted-coefficient multiples before checking polynomial divisibility. This is an antecedent of the evaluation method, but a different test. Abbott–Mexis [1, §4.1.2–4.1.3, Algorithm 3] use reciprocal evaluations in searching for cyclotomic factors. We study the first false-positive length for the stated alphabet. Their full algorithm contains further tests, whose correctness and cost are not the question here.

Bogart–Goodrick–Woods [2, Theorem 1.15 and Corollary 1.16] provide a general framework for quasi-polynomial behaviour and witness selection in parametric Presburger arithmetic. Our proof in §5 gives, for this object, a sufficient period expressed through minors and an explicit finite construction. It does not need their theorem as a premise. We do not claim the general phenomenon as new: an eventually quasi-polynomial function of growth $O(B)$ is already eventually affine on residue classes.

The tensor basis and total unimodularity in the regime with at most two odd primes belong to the cyclotomic-polytope setting of Beck–Hoşten [3]. Root dependencies also appear in the cyclotomic and simplicial matroids of Martin–Reiner [4]. Our body is a sum of segments, not the convex hull of the roots. De Carli–Laporta [5] study polynomial divisibility through coefficients, a predicate distinct from the two numerical congruences. The minimum-degree problems for nonnegative multiples [6, Theorem 1] and for $0, 1$ multiples with at least three terms and minimal vanishing sums [7, §1–2] also fix different objects.

The general proof proceeds from the lattice and first gauge to slopes and integer-monoid periodicity. The short shell then supports the uniform $3p/6p$ results, the reversal and transfer corollaries, and the certified cases. The elementary appendices can be read independently; Appendix F completes the scalar $6p$ proof.

2 Reciprocal evaluation and the lattice of false positives

Put $d = \varphi(t)$, $Q = \Phi_t(B)$, and choose a primitive t-th root ζ. Write $\mathcal{A}_t = \mathbb{Z}[\zeta] \simeq \mathbb{Z}[X]/(\Phi_t(X))$. We use the classical facts that cyclotomic polynomials are monic and irreducible, have constant term 1 for $t > 1$, and are reciprocal for $t > 2$; see the opening conventions of [7, §1]. The norm of a nonzero element is the determinant of its multiplication map in the integral basis $1, \zeta, \dots, \zeta^{d-1}$.

Lemma 2.1 (Reciprocal evaluation and end digits)**.** *The base B is invertible modulo Q, and*

$$Q \mid N, N^{\mathrm{rev}} \quad \Longleftrightarrow \quad P(B) \equiv P(B^{-1}) \equiv 0 \pmod{Q}. \tag{2.1}$$

A false positive can be extended to any greater length by adding zeros at the most significant end. Every minimum false positive has nonzero end digits. To prove $K_t(B) = K$, it is therefore enough to exclude the whole box at $K - 1$ and construct a false positive at K.

Proof. Since $\Phi_t(0) = 1$, we have $\gcd(B, Q) = 1$. Equation (1.1) gives (2.1) after cancelling the unit B^{k-1}. Reciprocity ensures that both evaluations vanish on the divisor and descend to $\mathcal{A}_t$. Adding a most significant zero leaves N unchanged and multiplies N^{rev} by B. If the least significant digit is zero, write $P = XP_1$: removing X multiplies the algebraic value by ζ^{-1} and cancels only units in the modular evaluations. A most significant zero can be removed directly. Either removal would preserve a shorter false positive. □

Proposition 2.2 (Image of reciprocal evaluation)**.** *For $\delta = \gcd(Q, B^2 - 1)$, the additive map*

$$E : \mathcal{A}_t \longrightarrow (\mathbb{Z}/Q\mathbb{Z})^2, \qquad f(\zeta) \longmapsto (f(B), f(B^{-1}))$$

has image exactly $\{(u, v) : u \equiv v \pmod{\delta}\}$. In particular,

$$[\mathcal{A}_t : \ker E] = Q^2/\delta. \tag{2.2}$$

Proof. Every difference $f(B) - f(B^{-1})$ is a multiple of $B - B^{-1}$ modulo Q. The ideal generated by this element is generated by δ, since multiplication by B turns it into $B^2 - 1$. Conversely, if $\delta \mid u - v$, choose b with $b(B - B^{-1}) \equiv u - v \pmod{Q}$ and put $a = u - bB$. Then $a + bX$ realizes (u, v). There are $Q(Q/\delta)$ such pairs. □

Proposition 2.3 (Intersection and product)**.** *Let*

$$I = (B - \zeta)\mathcal{A}_t, \qquad J = (B - \zeta^{-1})\mathcal{A}_t, \qquad \Lambda = I \cap J,$$

and

$$H = (B - \zeta)(B - \zeta^{-1}) = B^2 + 1 - B(\zeta + \zeta^{-1}). \tag{2.3}$$

Then

$$\Lambda = \ker E, \qquad H\mathcal{A}_t \subseteq \Lambda, \qquad [\mathcal{A}_t : H\mathcal{A}_t] = Q^2, \qquad [\Lambda : H\mathcal{A}_t] = \delta. \tag{2.4}$$

For every complex embedding σ, $\sigma(H) = B^2 + 1 - 2B\cos\theta \geq (B-1)^2 > 0$.

Proof. The quotient by I is $\mathbb{Z}/Q\mathbb{Z}$, with $\zeta \mapsto B$. In the quotient by J we have $B\zeta = 1$; reciprocity again gives $\mathbb{Z}/Q\mathbb{Z}$, now with $\zeta \mapsto B^{-1}$. Hence $\Lambda = \ker E$. The product $IJ = H\mathcal{A}_t$ is contained in the intersection. Both $B - \zeta$ and $B - \zeta^{-1}$ have norm Q, so multiplication by H has determinant Q^2. The index formula and (2.2) give the last index. The final inequality follows from $\cos\theta \leq 1$. □

Remark 2.4 (Retaining every coset)**.** Every $\alpha \in \Lambda$ has the form $H\beta$ with $\beta \in \mathcal{A}_t$ if and only if $\delta = 1$. For $\delta > 1$, that parametrization covers only the sublattice $H\mathcal{A}_t$. Section B.3 explicitly constructs the missing cosets at order ten.

Proposition 2.5 (Classification of $\delta_t(B)$)**.** *For $t \geq 3$,*

$$\delta_t(B) = \begin{cases} p, & t = p^a,\ p \text{ odd prime},\ a \geq 1,\ B \equiv 1 \pmod p, \\ p, & t = 2p^a,\ p \text{ odd prime},\ a \geq 1,\ B \equiv -1 \pmod p, \\ 2, & t = 2^a,\ a \geq 2,\ B \text{ odd}, \\ 1, & \text{otherwise}. \end{cases} \tag{2.5}$$

Proof. Evaluating $(X^n - 1)/(X - 1) = \prod_{e|n, e>1} \Phi_e(X)$ at 1 gives $\prod_{e|n,e>1} \Phi_e(1) = n$. Induction yields $\Phi_n(1) = p$ for $n = p^a$, and 1 otherwise: when n has at least two prime factors, the proper prime-power factors already supply the whole product n. The identities $\Phi_{2m}(X) = \Phi_m(-X)$ for odd $m > 1$, and evenness of the polynomial when $4 \mid n$, give, for $n \geq 3$,

$$\Phi_n(-1) = \begin{cases} p, & n = 2p^a,\ p \text{ odd}, \\ 2, & n = 2^a,\ a \geq 2, \\ 1, & \text{otherwise}. \end{cases}$$

For odd n one can also use $\Phi_n(-1) = \Phi_{2n}(1) = 1$. An odd prime dividing $B^2 - 1$ has $B \equiv \pm 1$ and must divide one of these two values. Only the first two cases of (2.5) can occur.

To exclude higher powers, in the first case put $C = B^{p^{a-1}} = 1 + ph$. Modulo p^2,

$$\Phi_{p^a}(B) = \sum_{j=0}^{p-1} C^j \equiv p + ph \sum_{j=0}^{p-1} j \equiv p \pmod{p^2}.$$

Replace B by $-B$ for the second case. For the prime 2, an even base contributes no factor to $B^2 - 1$. For odd B, $\Phi_t(B) \equiv \Phi_t(1) \pmod 2$, so t must be a power of 2. If $t = 2^a$, $a \geq 2$, then $\Phi_t(B) = B^{2^{a-1}} + 1 \equiv 2 \pmod 8$. The factor is exactly 2. □

3 Exact boxes and digit realization

Lemma 3.1 (Full-period sums)**.** *Write $k = tL + a$, $0 \leq a < t$, and $D = B - 1$. The realizable sums $s_r = \sum_{i \equiv r \pmod t} d_i$ are exactly*

$$0 \leq s_r \leq D(L + \mathbf{1}_{r<a}), \qquad s_r \in \mathbb{Z}, \qquad 0 \leq r < t.$$

Both modular evaluations and $P(\zeta)$ are preserved on replacing P by $\sum_{r<t} s_r X^r$.

Proof. Since Φ_t divides $X^t - 1$, all three substitutions have period t. Every integer sum within capacity can be distributed over the positions $r, r+t, \dots$ by first placing full digits D, then at most one remainder, and then zeros. The residue classes of positions are disjoint, so these choices are independent. □

Lemma 3.2 (Alternating folding and realization)**.** *For even t, put $m = t/2$. The word* (1.1) *determines*

$$c_r = \sum_{j \geq 0} (-1)^j d_{r+jm}, \qquad 0 \leq r < m. \tag{3.1}$$

With $N_r = \max\{0, \lfloor (k-1-r)/m \rfloor + 1\}$, the realizable vectors form exactly the product of the integer intervals

$$\ell_r = -(B-1)\lfloor N_r/2 \rfloor \leq c_r \leq u_r = (B-1)\lceil N_r/2 \rceil. \tag{3.2}$$

For $C(X) = \sum_{r=0}^{m-1} c_r X^r$, both vanishing in $\mathcal{A}_t$ and the congruences (2.1) *are unchanged on passing from P to C.*

Proof. We have $\zeta^m = -1$ and, since Φ_t divides $X^m + 1$, $B^m \equiv (B^{-1})^m \equiv -1 \pmod{Q}$. These identities prove invariance. To realize ℓ_r, place 0 at positive positions and $B-1$ at negative ones. Increasing a positive digit or decreasing a negative digit gives any increment in $\{0, \dots, B-1\}$. With N_r positions, every integer increment from 0 to $N_r(B-1)$ is realizable by filling positions successively to capacity $B-1$. The residue classes are disjoint, so their choices are independent. □

For the three elementary orders, monic division gives

t	$C(\zeta) = 0$
4	$(c_0, c_1) = (0, 0)$
10	$c = (a, -a, a, -a, a)$
12	$c = (a, b, -a, -b, a, b)$

$(a, b \in \mathbb{Z})$. (3.3)

Indeed $\deg C \leq 4$ for $t = 10$, so the quotient is constant; for $t = 12$ it has degree at most one.

4 The zonotope, its first gauge, and slopes

Write $A = \mathbb{Z}[\zeta]$, equivalently $\mathcal{A}_t$, and $\Gamma = H_B^{-1}\Lambda$, where $H_B = B^2 + 1 - B(\zeta + \zeta^{-1})$ is the element H in (2.3). The lattice Γ depends on the base only through the following arithmetic branch.

Lemma 4.1 (The normalized lattice)**.** *If $\delta = 1$, then $\Gamma = A$. On the exceptional branches,*

$$\Gamma = \begin{cases} (1-\zeta)^{-1}A, & t = p^a,\ B \equiv 1 \pmod{p}, \\ (1+\zeta)^{-1}A, & t = 2p^a,\ p \text{ odd},\ B \equiv -1 \pmod{p}. \end{cases}$$

The first case includes $p = 2$, $a \geq 2$.

Proof. Proposition 2.3 gives $[\Lambda : H_B A] = \delta$. On an exceptional branch choose $e = 1$ or -1 and put $u = 1 - e\zeta$. Since $|\,\mathrm{N}(u)| = \Phi_t(e) = p$, the quotient A/uA has p elements. In it $\zeta = e$ and the characteristic is p. Thus u divides both $B - \zeta$ and $B - \zeta^{-1}$, giving $(H_B/u)A \subseteq I \cap J = \Lambda$. Its index over $H_B A$ is p, the same as that of Λ, so equality holds. For $\delta = 1$ the index comparison suffices. □

Define

$$\pi_t : \mathbb{R}^t \longrightarrow A \otimes \mathbb{R}, \qquad x \longmapsto \sum_{r=0}^{t-1} x_r \zeta^r, \qquad Z_t = \pi_t([0,1]^t).$$

Since $\pi_t(\mathbf{1}) = 0$, the body Z_t equals $\pi_t([-1/2, 1/2]^t)$ and is centrally symmetric and full-dimensional. Its gauge is $\|\beta\|_{Z_t} = \min\{\lambda \geq 0 : \beta \in \lambda Z_t\}$.

4.1 The integer root matrix

Let $t = \prod_{i=1}^{w} p_i^{a_i}$ and $n_i = p_i^{a_i - 1}$. CRT idempotents allow roots η_i of order $p_i^{a_i}$ with $\zeta = \prod_i \eta_i$. For each factor use the basis

$$\eta_i^{k+jn_i}, \qquad 0 \le k < n_i, \quad 0 \le j < p_i - 1.$$

The power with $j = p_i - 1$ is minus the sum of the others in its block. Products of these bases form an integral basis of A: the tensor product of the rings, presented by their monic relations, is free of rank $\prod_i \varphi(p_i^{a_i}) = \varphi(t)$ and surjects onto A because it contains ζ. The kernel of this surjection between free modules of equal rank is zero.

In this basis the matrix R_t of π_t has entries $0, 1, -1$. Each row has exactly 2^w nonzero entries, half positive and half negative. Columns indexed by the basis monomials form an identity matrix. This is the classical tensor model in [3, §2]; these are the two properties needed here.

Lemma 4.2 (First gauge)**.** *On every arithmetic branch,*

$$\min_{0 \ne \beta \in \Gamma} \|\beta\|_{Z_t} = \mu_t(B).$$

Proof. The integral lattice. If $\beta = R_t x$ with $0 \le x_r \le \lambda$, each coordinate of β has absolute value at most $m_t \lambda$. A nonzero integer vector has a coordinate of absolute value at least 1, so $\lambda \ge 1/m_t$.

To attain the bound, choose one block in each factor. Give coefficient $1/2$ to position $j = 0$, coefficient $-1/2$ to its other $p_i - 1$ positions, and zero to the other blocks. The algebraic value is a basis monomial. The tensor product has coefficients between -2^{-w} and 2^{-w} and represents a nonzero monomial. Adding a constant to every coordinate produces coefficients in $[0, 2^{1-w}]$ without changing the value.

A ramified prime power. For $t = p^a$ and $n = p^{a-1}$,

$$\frac{1}{1-\zeta} = \frac{1}{p} \sum_{k=0}^{n-1} \sum_{j=0}^{p-2} (p - 1 - j) \zeta^{k+jn}. \tag{4.1}$$

Evaluate $(\Phi_t(X) - p)/(X - 1)$ at ζ and divide by p. In each block, adjoining the anchored zero coordinate gives the values $(p-1)/p, (p-2)/p, \dots, 0$. The quotient Γ/A has order p. The class in (4.1) is nonzero and therefore generates it. Every nonintegral class is a multiple ℓ, $1 \le \ell < p$, of this class plus an element of A. Multiplication by ℓ permutes the residues modulo p. Consequently, in every nonintegral class the fractional parts of the p coefficients in each block run through $(1/p)\mathbb{Z}/\mathbb{Z}$. Adding integers and a common constant cannot reduce their range below $(p-1)/p$: p representatives with distinct equally spaced fractional parts cannot fit in a shorter interval. The real kernel of each block consists precisely of constant translations. Formula (4.1) attains that range in every block simultaneously. The nonzero integral coset has minimum 1 and does not improve the bound.

Twice an odd prime power. If $t = 2N$ with $N = p^a$ odd, then $\eta = -\zeta$ has order N and $\mathbb{Z}[\eta] = \mathbb{Z}[\zeta]$. The $2N$ roots are the N roots in η and their negatives, so $Z_{2N} = 2Z_N$. The exceptional lattice is $(1 - \eta)^{-1} A$, whose first gauge is $(p-1)/(2p)$. These exhaust Proposition 2.5. □

4.2 From gauge to integer digits

Proof of Theorem 1.2. Put $D = B - 1$, $\mu = \mu_t(B)$, and $E = 2^w - 1$. On the full period let S be the cyclic shift and set

$$\widehat{H}_B = (B^2 + 1)I - B(S + S^{-1}).$$

Its eigenvalues $B^2+1-2B\cos(2\pi r/t)$ are positive, so it is invertible, and $\pi_t(\widehat{H}_B y) = H_B \pi_t(y)$.

A false positive of length k gives integer sums s_r with $0 \le s_r \le D\lceil k/t \rceil$. For $y = \widehat{H}_B^{-1} s$, the maximum principle gives

$$0 \le y_r \le \frac{\lceil k/t \rceil}{D}. \tag{4.2}$$

A negative minimum would produce a negative coordinate of $\widehat{H}_B y$, whereas at a maximum M the corresponding coordinate of s is at least $D^2 M$. Thus $\beta = \pi_t(y) = H_B^{-1}\pi_t(s) \in \Gamma \setminus \{0\}$ has gauge at most $\lceil k/t \rceil / D$. Lemma 4.2, with the integrality of $\lceil k/t \rceil$, gives the lower bound in (1.4).

For the upper bound, choose β attaining the first gauge and a centred representative x with $|x_r| \le \mu/2$. Then $h = \widehat{H}_B x$ satisfies $|h_r| \le \mu(B+1)^2/2$, and $\alpha = R_t h = H_B \beta$ is a nonzero integral point of Λ. The vector h need not be integral.

Round its nonpivot coordinates to nearest integers, then determine the pivot coordinates by $R_t c = \alpha$. Since the pivot columns form an identity matrix, all resulting coordinates are integers. Each row has one pivot and E other entries of absolute value 1, and no other pivot. Therefore $|c_r - h_r| \le E/2$ at the pivots as well. Translate by the integer minimum, setting $s = c - \min(c)\mathbf{1}$. Then $R_t s = \alpha$ and

$$0 \le s_r \le \lfloor \mu(B+1)^2 + E \rfloor = U_t(B).$$

Lemma 3.1 realizes these sums in $t\lceil U_t(B)/D \rceil$ digit positions. Their algebraic value is nonzero and lies in Λ, giving a false positive. Finally $(B+1)^2/(B-1) = B + 3 + 4/(B-1)$ proves the $O_t(1)$ error. □

4.3 Height thresholds and exactness in boxes

For coefficients collected over a full period, define

$$h_t(B) = \min\{h \in \mathbb{Z}_{\ge 0} : R_t([0,h]^t \cap \mathbb{Z}^t) \cap (\Lambda \setminus \{0\}) \ne \varnothing\}.$$

This is the minimum range of an integer vector c with $R_t c \in \Lambda \setminus \{0\}$: the identity $R_t \mathbf{1} = 0$ allows its minimum to be translated to zero. The preceding construction proves finiteness. Exactness of the zero fibre in the uniform box holds precisely for integer heights $h < h_t(B)$.

Corollary 4.3 (Height and complete periods)**.** *For every $t \ge 3$ and $B \ge 2$,*

$$\left\lceil \mu_t(B)(B-1)^2 \right\rceil \le h_t(B) \le \left\lfloor \mu_t(B)(B+1)^2 + 2^{\omega(t)} - 1 \right\rfloor. \tag{4.3}$$

With $L_* = \lceil h_t(B)/(B-1) \rceil$,

$$t(L_*-1)+1 \le K_t(B) \le tL_*. \tag{4.4}$$

In particular, $h_t(B) = \mu_t(B)B^2 + O_t(B)$ at fixed order.

Proof. If $s \in [0,h]^t \cap \mathbb{Z}^t$ has nonzero image in Λ, put $y = \widehat{H}_B^{-1} s$. The same maximum principle gives $0 \le y_r \le h/(B-1)^2$. Hence $H_B^{-1} R_t s \in \Gamma \setminus \{0\}$ has gauge at most $h/(B-1)^2$ and at least $\mu_t(B)$, proving the lower bound. The upper bound is precisely the range obtained by rounding above.

At length tL, Lemma 3.1 realizes exactly $[0, L(B-1)]^t \cap \mathbb{Z}^t$. It contains a false positive if and only if $L(B-1) \ge h_t(B)$. Thus tL_* is the first failing full period. The preceding one is safe, and the monotonicity in Lemma 2.1 gives (4.4). For $L_* = 1$, the lower bound simply states $K_t(B) \ge 1$. □

Height has quadratic scale, whereas each period adds capacity $B-1$ in each residue class. The final partial period can precede the first failing complete period by at most $t-1$ positions. This separates the geometric scale from the ordered-capacity correction without asserting an exact general formula for h_t.

4.4 Height independent of the base

For an integer $H \geq 1$, define $K_{t,H}(B)$ by replacing the digit restriction $0 \leq d_i \leq B-1$ with $0 \leq d_i \leq H$. The base remains $B \geq 2$ and the modulus remains $\Phi_t(B)$. If $H > B-1$, these are bounded coefficients, not canonical base-B digits; no carrying is performed.

Corollary 4.4 (An independent coefficient bound)**.** *Set $L_H = \lceil h_t(B)/H \rceil$. Then*

$$t(L_H - 1) + 1 \leq K_{t,H}(B) \leq tL_H. \tag{4.5}$$

At fixed t, uniformly for $H \geq 1$,

$$K_{t,H}(B) = \frac{t\mu_t(B)}{H} B^2 + O_t(1 + B/H).$$

A sufficient condition for exactness at length k is

$$H\lceil k/t \rceil < \mu_t(B)(B-1)^2.$$

Proof. At a full period tL, the folded sums realize every integer point of $[0, HL]^t$. Thus tL_H is the first failing full period and its predecessor is safe. Monotonicity under padding gives (4.5). For $\mu = \mu_t(B)$ and $E = 2^{\omega(t)} - 1$, (4.3) implies

$$|h_t(B) - \mu B^2| \leq 2\mu B + \mu + E.$$

Also $|K_{t,H}(B) - th_t(B)/H| \leq t$. The error relative to $t\mu B^2/H$ is therefore at most $t + (t/H)(2\mu B + \mu + E)$, uniformly in H. The sufficient condition follows from the height lower bound and the largest folded capacity. □

No joint quasi-polynomiality in B, H follows from this corollary. The excluded case $H = 0$ has no false positives.

4.5 Collisions between classes and signed coefficients

Let $J_{t,H}(B)$ be the first length at which two words with coefficients in $[0, H] \cap \mathbb{Z}$ have the same reciprocal evaluations but different remainders modulo $\Phi_t(X)$. Let $K^{\pm}_{t,H}(B)$ be the first false-positive length with integer coefficients in $[-H, H]$. The two polynomials are compared at the same length bound, padding with leading zeros if necessary.

Proposition 4.5 (Fibres, signs, and complete periods)**.** *For every $t \geq 3$, $B \geq 2$, and $H \geq 1$,*

$$K^{\pm}_{t,H}(B) = J_{t,H}(B). \tag{4.6}$$

A collision exists at length tL if and only if $2HL \geq h_t(B)$. With $M_H = \lceil h_t(B)/(2H) \rceil$,

$$t(M_H - 1) + 1 \leq J_{t,H}(B) \leq tM_H, \qquad J_{t,H}(B) = \frac{t\mu_t(B)}{2H} B^2 + O_t(1 + B/H).$$

The condition $2H\lceil k/t \rceil < \mu_t(B)(B-1)^2$ suffices to exclude signed false positives.

Proof. The difference of a collision has coefficients in $[-H, H]$, zero reciprocal evaluations, and a nonzero remainder. Conversely, every such polynomial can be written coefficientwise as $F^+ - F^-$, where both $F^{\pm}$ have coefficients in $[0, H]$. This proves (4.6) at every length.

At a full period, differences of folded sums fill exactly $[-HL, HL]^t \cap \mathbb{Z}^t$. Translating by $HL\mathbf{1}$ gives $[0, 2HL]^t \cap \mathbb{Z}^t$ without changing either the remainder or the two evaluations, since Φ_t divides $\sum_{r=0}^{t-1} X^r$. The characterization through $h_t(B)$ and the padding bounds follow. The asymptotic estimate is the preceding argument with $2H$ in place of H. At arbitrary lengths, signed folded coordinates belong to $[-H\lceil k/t \rceil, H\lceil k/t \rceil]$; the same translation and the height lower bound give the sufficient condition. □

There is no resulting identity $J_{t,H} = K_{t,H}/2$ at arbitrary lengths. The example $t = 4, B = 2, H = 1$ in §1 illustrates the difference between zero-fibre exactness and recovering the remainder from every fibre. Proposition 4.5 links them by an exact change of coefficient alphabet. In particular, (8.2) gives an exact criterion at complete periods for $t = 3p$, $B \geq 9$:

$$\text{signed false positive at length } 3pL \quad \Longleftrightarrow \quad 2HL \geq \left\lceil (B^2+1)/2 \right\rceil .$$

For $6p$ and $B \geq 10$, Theorem 9.6 gives the analogous criterion with $6pL$ and $\lceil (B^2+B+1)/4 \rceil$.

5 An effective period without total unimodularity

We prove Theorem 1.3 while retaining integer feasibility. The dimension depends on the fixed order t, not on B.

5.1 A finite list of elements and lengths

Let $d = \varphi(t)$, $m = 2^{w-1}$, $E = 2^w - 1$, and

$$\Delta_t = \max\{|\det R_t[I,J]| : \det R_t[I,J] \neq 0\}, \qquad L_t = \operatorname{lcm}\{|\det R_t[I,J]| : \det R_t[I,J] \neq 0\},$$

including the empty minor 1. Fix a residue j modulo $\rho = \rho_t$, so that the lattice Γ_j and first gauge $\mu = \mu_t(j)$ are constant. The choice (1.5) ensures that $\mu\rho$ is integral and that the coordinate denominators of Γ_j divide ρ.

The upper bound in Theorem 1.2 and (4.2) give, for a minimum false positive,

$$\|\beta\|_{Z_t} \leq \mu + \frac{4\mu+1}{D} + \frac{4\mu+E}{D^2}, \qquad D = B-1. \tag{5.1}$$

Indeed $\lceil K_t(B)/t \rceil \leq \lceil U_t(B)/D \rceil$. Choose an integer D_0 satisfying

$$D_0 \geq 4\mu + E, \qquad D_0 \geq \frac{2(4\mu+1)}{\mu}, \qquad D_0^2 \geq \frac{2(4\mu+E)}{\mu}. \tag{5.2}$$

For $D \geq D_0$ the gauge is at most 2μ, and each CRT coordinate has absolute value at most $2m\mu \leq 2$. Thus the finite list

$$\mathcal{S}_j = (\Gamma_j \cap [-2,2]^d) \setminus \{0\} \tag{5.3}$$

suffices. Enumerate the grid with denominator δ_j and test lattice membership: on a ramified branch require $(1-e\zeta)\beta \in A$. The full grid is not substituted for Γ_j.

Writing $B = \rho n + j$, the minimum length has the form

$$k = t\mu\rho n + c, \quad \lceil t\mu j - t(\mu+1) + 1 \rceil \leq c \leq \lfloor t\mu j + t(3\mu+2) \rfloor . \tag{5.4}$$

The lower bound follows from (1.4). For the upper bound use $U_t(B)/D \leq \mu B + 3\mu + (4\mu+E)/D$ and $D \geq 4\mu + E$. This offset interval is finite and independent of n. With $a = c \bmod t$ chosen in $\{0, \ldots, t-1\}$, we have $L = \mu\rho n + \lfloor c/t \rfloor$.

5.2 The fibre as a monoid

For $\beta \in \mathcal{S}_j$ and an offset c, define

$$\alpha(n) = H_{\rho n+j}\beta, \qquad v_r^{\mathrm{cap}}(n) = (\rho n + j - 1)(\mu\rho n + \lfloor c/t \rfloor + \mathbf{1}_{r<a}).$$

These are integer-coefficient polynomials of degree at most two. For α, use $\rho\beta, \rho(\zeta+\zeta^{-1})\beta \in A$ and $H_j\beta \in A$. For the latter inclusion, choose a valid base $B \equiv j \pmod{\rho}$ and subtract from $H_B\beta \in A$ the integral terms involving $B - j$.

Lemma 3.1 expresses feasibility as

$$Y(n) = \begin{pmatrix} \alpha(n) \\ v^{\mathrm{cap}}(n) \end{pmatrix} \in M_t \mathbb{N}^{2t}, \qquad M_t = \begin{pmatrix} R_t & 0 \\ I_t & I_t \end{pmatrix}. \tag{5.5}$$

The first variables are digit sums and the others are unused capacities. Here $\mathbb{N} = \{0, 1, \ldots\}$.

The absolute values of the nonzero minors of M_t are those of R_t: expand first along the second-block columns, each with a single entry 1, and then along remaining lower rows, each with at most one entry 1. The determinant is zero or, up to sign, a minor of R_t. The reverse inclusion is immediate. Moreover, every column of M_t has height 1 under the functional summing the last t coordinates.

Lemma 5.1 (Finite monoid decomposition)**.** *Let M be an integer matrix with N columns, all of height 1 under a linear functional. Let Δ be the maximum absolute value of its minors, including 1. Then*

$$M\mathbb{N}^N = \bigcup_{I,v} (Mv + M_I \mathbb{N}^I), \tag{5.6}$$

where I runs through the linearly independent column sets, including the empty set, and $v_i = 0$ on I, $0 \le v_i < \Delta$ outside I.

Proof. For $y = Mz$, all nonnegative integral representations have the same coordinate sum, the height of y, so there are finitely many. Choose a lexicographically minimum representation. The columns with $z_i \ge \Delta$ are independent. Otherwise a minimal dependence involves $r + 1$ columns of rank r. Choose r independent rows. The signed minors obtained by deleting one column give a kernel vector. The kernels of the submatrix and of the full columns coincide, since both have dimension one. Dividing by the greatest common divisor produces an integral circuit q, supported there, with $|q_i| \le \Delta$. Choose its sign so that its first nonzero coordinate is positive. Then $M(z - q) = Mz$, its first affected coordinate decreases, and earlier coordinates are unchanged. If $q_i > 0$, then $z_i \ge \Delta \ge q_i$; if $q_i \le 0$, that coordinate does not decrease. All coordinates remain nonnegative, contradicting lexicographic minimality. The set I of large coordinates and the vector of the remaining coordinates give (5.6). The opposite inclusion is immediate. □

Decomposition into translates of monoids with independent generators is classical; see Chistikov–Haase [9, Proposition 5]. The proof above gives a minor-based bound under the height-one hypothesis needed for M_t. It assumes neither normality nor absence of integer holes, and is not a new general theory of semilinear sets.

5.3 Congruences, intercepts, and cutoffs

In a component (I, v), choose rows J such that $M_t[J, I]$ is square and invertible, and put $d_I = |\det M_t[J, I]|$. Cramer's rule gives the only possible component coordinates:

$$z_i(n) = N_i(n)/d_I,$$

where the N_i are integer polynomials of degree at most two, absorbing the determinant's sign into the numerators. The exact conditions are

$$M_I N(n) = d_I(Y(n) - M_t v), \tag{5.7}$$

$$N_i(n) \ge 0, \tag{5.8}$$

$$N_i(n) \equiv 0 \pmod{d_I}. \tag{5.9}$$

For $I = \varnothing$, take $d_I = 1$ and require only $Y = M_t v$.

Every nonzero polynomial residual in (5.7) has finitely many roots. Every nonzero polynomial in (5.8) eventually has the sign of its leading coefficient. Conditions (5.9) have period d_I in n, and d_I divides L_t.

For $0 \le h < L_t$, let $\mathcal{F}_{j,h}$ consist of those offsets in (5.4) for which some $\beta \in \mathcal{S}_j$ and (I, v) have identically zero span residuals, numerators that are zero or have positive leading coefficient, and $d_I \mid N_i(h)$ for all i. This is a finite list of integer tests.

To compute a common cutoff, collect all nonzero span and sign polynomials from every component. For $p(n) = a_s n^s + \cdots + a_0$, $s \ge 1$, any number strictly greater than

$$1 + \max_{i<s} |a_i/a_s|$$

exceeds the modulus of every root: beyond it a geometric-series bound on the lower terms is strictly smaller than $|a_s n^s|$. Choose an integer N exceeding all these bounds, imposing $B - 1 \ge D_0$, and making lengths positive and capacities nonnegative. Constant and zero polynomials are treated separately.

For $n \ge N$, every component is stable on each residue class modulo L_t. A nonidentically zero span residual excludes the component beyond N. If all span residuals vanish, a numerator with negative leading coefficient excludes it. Otherwise only congruences remain, and $N_i(n + d_I) \equiv N_i(n) \pmod{d_I}$ for an integer polynomial. Since $d_I \mid L_t$, feasibility depends only on the residue. There is a common cutoff because the candidates, offsets, and components are finite in number. The upper bound in Theorem 1.2 guarantees a solution at every base, and the finite reduction guarantees its presence in the list. Therefore $\mathcal{F}_{j,h} \neq \varnothing$ and

$$K_t(\rho n + j) = t\mu_j \rho n + \min \mathcal{F}_{j,n \bmod L_t}. \tag{5.10}$$

On the class $B = j + \rho h \pmod{\rho L_t}$, this gives

$$b_{t,j+\rho h} = \min \mathcal{F}_{j,h} - t\mu_j j.$$

For example, $C_t = \rho(N + 1)$ is a sufficient cutoff. This proves the eventual assertion of Theorem 1.3. For each smaller base, the explicit upper bound (1.4) allows a finite enumeration of the boxes, using integer congruence and polynomial-division tests up to the first false positive. The same bound guarantees termination.

Remark 5.2. Hadamard's inequality gives $\Delta_t \le 2^{w\varphi(t)/2}$, since each row has Euclidean norm $2^{w/2}$. Hence

$$P_t \mid \rho_t \operatorname{lcm}(1, 2, \ldots, \lfloor 2^{w\varphi(t)/2} \rfloor).$$

This is a large explicit bound, not a practical complexity bound. The minimum period of the coefficient functions divides every period of those functions, proving the final assertion of the theorem.

5.4 Constructive witness selection

Proof of Corollary 1.4. Fix a class $B = B_0 + \rho_t L_t n$. Beyond the cutoff, the minimum offset is constant. Order the finite list (β, I, v) and choose the first admissible component realizing that offset. Span, signs, and congruences are stable on the class, so the same choice works for every sufficiently large base in it.

The first t coordinates of the representation are digit sums $s_r(B)$, rational polynomials of degree at most two, integral on the base class and between 0 and $(B - 1)N_r(B)$. Write $s_r(B) = aB^2 + bB + c$. Exact division gives

$$\frac{s_r(B)}{B-1} = aB + (a + b) + \frac{a + b + c}{B - 1}. \tag{5.11}$$

Refine the classes by the denominators of all coefficients. The fractional part of the affine term is then constant. If it is nonzero, the reciprocal term, once small enough, does not change the floor. If it is zero, the floor decreases by 1 when $a+b+c<0$ and is unchanged when $a+b+c\geq 0$, after requiring $|a+b+c|/(B-1)<1$. Thus $n_r(B)=\lfloor s_r(B)/(B-1)\rfloor$ is affine on the class.

The remainder $e_r(B)=s_r(B)-(B-1)n_r(B)$ is affine because the quadratic term cancels, and $0\leq e_r(B)<B-1$. At positions congruent to r, place $n_r(B)$ full digits, then $e_r(B)$ if positive, then zeros. Capacity guarantees enough space: if $n_r=N_r$, then $e_r=0$. The sums remain s_r, preserving the two divisibilities and the nonzero remainder. The length is the minimum in (5.10); Lemma 2.1 also ensures nonzero end digits. Every refinement and cutoff used here follows from finitely many rational comparisons. □

This selects one witness; it does not classify all minimum words. The eventual quasi-polynomial selection framework of [2] remains an antecedent and an alternative route.

6 An exact infinite family: powers of two

Theorem 6.1. *For $a\geq 3$, $t=2^a$, and every $B\geq 2$,*

$$K_{2^a}(B)=\begin{cases}2^aB+2, & B \text{ even},\\ 2^{a-1}(B+2), & B \text{ odd}.\end{cases}$$

The minimum period of the coefficient functions is 2.

Proof. Put $m=t/2\geq 4$, $A_0=B^2+1$, and $D=B-1$. Since $\Phi_t=X^m+1$, the alternating folding in (3.1) is already an integral basis with no further relations. Extend β antiperiodically by $\beta_{r+m}=-\beta_r$. Multiplication by H_B becomes

$$c_r=A_0\beta_r-B(\beta_{r-1}+\beta_{r+1}). \tag{6.1}$$

Lemma 4.1 gives $\beta\in\mathbb{Z}^m$ for even B and

$$\beta\in\mathbb{Z}^m\ \cup\ (\mathbb{Z}+1/2)^m$$

for odd B. Indeed $(1-\zeta)^{-1}=(1+\zeta+\cdots+\zeta^{m-1})/2$, and the lattice quotient has order 2; mixed fractional parts are not allowed.

A plateau boundary. If $M=\max|\beta_r|>0$, the antiperiodic extension has a maximal run of value M with a boundary. Its jump to the outside neighbour is at least 1 in either coset, and the other neighbour is at most M. Equation (6.1) yields

$$\max_r|c_r|\geq D^2M+B. \tag{6.2}$$

Indices outside the period correspond to indices inside it with opposite sign.

Even bases. At length $tB+1$ the intervals are $[-BD,(B+1)D]$ at coordinate 0 and $[-BD,BD]$ elsewhere. If $M\geq 2$, (6.2) exceeds $(B+1)D$, since $2D^2+B-(B+1)D=B^2-3B+3>0$. Thus $M=1$. A one-position plateau of value 1 has image at least $A_0>(B+1)D$, as both neighbours are at most zero. A longer plateau has two ends distinct modulo m, both with image of absolute value at least $D^2+B=BD+1$. Antiperiodicity bounds its length by m, but only coordinate 0 allows absolute value exceeding BD. This excludes the boundary $tB+1$.

At length $tB+2$, choose $\beta=e_0+e_1$, with image

$$c=(A_0-B,A_0-B,-B,0,\ldots,0,B).$$

For $m = 4$ the four displayed coordinates are distinct. The first two have upper bound $(B+1)D$, the others BD, and all lower bounds are $-BD$. The identities $A_0 - B = BD + 1$ and $B \le BD$ verify every constraint. This is a nonzero element of Λ, realized by Lemma 3.2.

Odd bases. Here $B \ge 3$. At length $m(B+2)-1$, every lower bound is $-(A_0/2-1)$; the first $m-1$ upper bounds are $U = A_0/2 + B - 2$, and the last is $A_0/2 - 1$. In the integral coset, (6.2) gives at least $A_0 - B > U$, with difference $(B^2 - 4B + 5)/2 > 0$. In the half-integral coset, $M \ge 3/2$ gives $3D^2/2 + B - U = B^2 - 3B + 3 > 0$. Only coordinates $\pm 1/2$ remain.

If a negative sign occurs, there is a negative coordinate adjacent to a positive one: use an internal boundary for a mixed sequence and the antiperiodic boundary for the all-negative sequence. There $c_r \le -A_0/2$, below the lower bound. Hence all coordinates are $1/2$, but then $c_{m-1} = A_0/2$, above its upper bound. The preceding boundary is excluded.

At length $m(B+2)$, choose $\beta = (1/2, \dots, 1/2)$. Then $c_0 = c_{m-1} = A_0/2$ and $c_r = D^2/2$ for $1 \le r \le m-2$. These are integers and satisfy the box, whose upper bound is now U in every coordinate. This gives a false positive. The minimum period is 2 since the two slopes differ. □

The case $a = 2$ is (1.7), not a substitution in this formula. The restriction $m \ge 4$ prevents overlap of the endpoints in the even-base construction.

Remark 6.2 (Units do not suffice at every order)**.** The even-base construction uses $\beta = 1 + \zeta$, whose norm is $\Phi_{2^a}(-1) = 2$. It is not a unit. Its occurrence among minimum witnesses rules out a universal classification of all minimum multipliers as units.

7 The first tensor shell and doubling

The first gauge determines the slopes. Exact formulas also require a description of its minimizers and a gap to other lattice points.

Theorem 7.1 (Signed rectangles and a gap)**.** *Let $p_i \ge 2$ and*

$$R = \bigotimes_{i=1}^{w} [I_{p_i - 1} \mid -\mathbf{1}], \qquad Z = R[0,1]^{\prod_i p_i}, \quad \mu = 2^{1-w}, \quad \nu = \frac{2}{2^w - 1}.$$

The nonzero integer points of gauge less than ν are precisely

$$b = \varepsilon \mathbf{1}_{S_1 \times \cdots \times S_w}, \qquad \varepsilon \in \{1, -1\}, \quad \varnothing \ne S_i \subseteq \{0, \dots, p_i - 2\}.$$

All have gauge μ. Every other nonzero integer point has gauge at least ν, without a claim that ν is the actual second value.

Proof. Let ρ_a be the rows of R. For a zero-sum vector v, set $h(v) = \sum_r \max(v_r, 0)$, so that $v \cdot x \le \lambda h(v)$ whenever $0 \le x_r \le \lambda$. If a, b differ in e coordinates, their rows share 2^{w-e} entries with the same sign. Hence

$$h(\rho_a - \rho_b) = 2^w - 2^{w-e} \le 2^w - 1 \quad (a \ne b). \tag{7.1}$$

If c is obtained by replacing one coordinate of a by that of b, then also

$$h(\rho_a + \rho_b - \rho_c) \le 2^w - 1. \tag{7.2}$$

When that coordinate changes, write $\rho_a = U \otimes (e_j - e_*)$, $\rho_b = V \otimes (e_k - e_*)$, and $\rho_c = U \otimes (e_k - e_*)$. The three sections are $U, V - U, -V$; their total ℓ^1 norm is $2^{w+1} - 2^{w-e+1}$. If the coordinate is unchanged, the combination is simply ρ_b.

Now let $0 < \|b\|_Z < \nu$. The single-row bound gives $|b_a| < 2$, so every integral coordinate is $-1, 0, 1$. Opposite signs would contradict (7.1), since $2 \le (2^w - 1)\|b\|_Z < 2$. After reversing sign, all nonzero values are 1. If $b_a = b_b = 1$ but a recombination c has value zero, (7.2) gives the same contradiction. Thus the support is closed under one-coordinate recombination. Replacing coordinates one at a time shows that it is the product of its nonempty projections.

Conversely, let f_i equal 1 on S_i and -1 elsewhere, including the anchored index $p_i - 1$. Then $[I \mid -\mathbf{1}]f_i = 2\mathbf{1}_{S_i}$, and

$$x = \frac{\mathbf{1} + \varepsilon \bigotimes_i f_i}{2^w}$$

has coordinates 0 or μ and satisfies $Rx = b$. The lower bound μ follows from a nonzero integral coordinate and the row bound $2^{w-1}\lambda$. □

For $t = \prod p_i^{a_i}$, the CRT matrix is, up to permutations, a direct sum of $h = \prod p_i^{a_i - 1}$ copies of this model. The body is a product, and its gauge is the maximum of the block gauges. On the *integral* branch, each block of a minimum point is zero or a signed rectangle, with at least one nonzero block. Their number is

$$\left(1 + 2\prod_i (2^{p_i - 1} - 1)\right)^h - 1.$$

This statement does not silently include ramified cosets.

Proposition 7.2 (Doubling an odd order)**.** *For odd $m > 1$, choose $\xi = -\zeta_m$ of order $2m$. The two rings coincide, and in the same integral basis*

$$R_{2m}[:, r] = (-1)^r R_m[:, r \bmod m], \qquad Z_{2m} = 2Z_m.$$

The two matrices have the same absolute values of nonzero minors. If T represents multiplication by $\zeta_m + \zeta_m^{-1}$, the order-$2m$ operator is $(B^2 + 1)I + BT$.

Proof. The reordered columns are $[R_m, -R_m]$. Since Z_m is centrally symmetric, their zonotope is $Z_m + (-Z_m) = 2Z_m$. A minor using two copies of one column is zero; the other minors differ from those of R_m only in sign and order. Every minor of R_m occurs. Finally, $(B - \xi)(B - \xi^{-1}) = (B + \zeta_m)(B + \zeta_m^{-1})$. □

Capacity transport is a separate matter. For $N_r = \max\{0, 1 + \lfloor (k - 1 - r)/m \rfloor\}$, the coordinate

$$z_r = \sum_{q \ge 0,\ r + qm < k} (-1)^{r+q} d_{r+qm}$$

runs through exactly the integer interval

$$(-1)^r [-(B - 1)\lfloor N_r/2 \rfloor,\ (B - 1)\lceil N_r/2 \rceil], \tag{7.3}$$

with the endpoints reordered. The proof is that of Lemma 3.2. Both the ordered capacities and the sign of the operator's linear term change. Doubling the body does not determine the intercepts.

Corollary 7.3. *At order 105 the first shell has exactly $2(2^2 - 1)(2^4 - 1)(2^6 - 1) = 5670$ points of gauge $1/4$; every other nonzero integer point has gauge at least $2/7$. At order 210 the same points have gauge $1/8$, and every other nonzero integer point has gauge at least $1/7$.*

Proof. Apply Theorem 7.1 to $3, 5, 7$ and Proposition 7.2. At both orders $\delta = 1$ in every base. For 210 the gap $1/7$ is stronger than $2/15$, which the bound with four factors alone would give. □

Remark 7.4 (A real fibre can contain no integer point)**.** In R_{105}, rows $(0,0,0),(0,1,1),(1,0,1)$ and CRT columns $(0,4,6),(2,0,6),(2,4,1)$ give the minor

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det = -2.$$

Allow these three columns and the 45 pivots of the other rows with capacity 1, and capacity zero elsewhere. Place $1/2$ on the three columns and 0 or $1/2$ on the remaining pivots to make the right-hand side integral. In the three chosen rows it is $(1,1,1)$. An integer realization would require $x_1 + x_2 = x_1 + x_3 = x_2 + x_3 = 1$, impossible by parity. This is a specially constructed box, not a digit-capacity box at the first threshold. It motivates integer lifts without refuting any threshold formula.

8 Uniform thresholds of order $3p$

Fix a prime $p \geq 5$ and put $t = 3p$. Proposition 2.5 gives $\delta_t(B) = 1$ in every base, so $\Lambda = H_B \mathcal{A}_t$. We obtain a cutoff independent of p without enumerating the first shell.

Theorem 8.1 (Order-$3p$ thresholds)**.** *For every prime $p \geq 5$ and every integer base $B \geq 9$,*

$$K_{3p}(B) = \begin{cases} 3pB/2 + 2p - 1, & B \text{ even}, \\ 3p(B+1)/2 + 1, & B \text{ odd}. \end{cases} \tag{8.1}$$

Moreover,

$$h_{3p}(B) = \left\lceil \frac{B^2+1}{2} \right\rceil. \tag{8.2}$$

In particular $K_{3p}(10) = 17p - 1$. The cutoff 9 is sufficient, not asserted to be minimal.

8.1 Arrays, potentials, and cycles

Choose explicitly

$$\eta_3 = \zeta^{p(p^{-1} \bmod 3)}, \qquad \eta_p = \zeta^{3(3^{-1} \bmod p)}, \qquad \zeta = \eta_3 \eta_p.$$

A full representative is a $3 \times p$ array: cell (i,j) represents $\eta_3^i \eta_p^j$ and occupies the position $r(i,j) \in [0, 3p-1]$ with $r \equiv i \pmod 3$, $r \equiv j \pmod p$. Deleting the last row and column, the tensor coordinates of an array F are $F_{ij} - F_{i,p-1} - F_{2,j} + F_{2,p-1}$. Its kernel therefore consists exactly of arrays $u_i - w_j$.

For an integer target A and integer capacities $c_{ij} \geq 0$, a lift to the box is equivalent to potentials with

$$0 \leq A_{ij} + u_i - w_j \leq c_{ij}.$$

These are difference constraints on the bidirected graph $K_{3,p}$: $w_j - u_i \leq A_{ij}$ and $u_i - w_j \leq c_{ij} - A_{ij}$. Absence of negative-weight cycles is necessary and sufficient. For sufficiency, add a source with zero-weight edges and take shortest distances: every constraint holds and the distances are integers. A negative closed walk contains a negative simple cycle. Simple cycles here have length 2, 4, or 6.

To an orientation of a $2h$-cycle associate its alternating functional ℓ, with h positive and h negative cells. Its condition is

$$\ell(A) \leq \sum_{\text{positive cells}} c_{ij}. \tag{8.3}$$

Both orientations must be considered; length 2 reduces to $c_{ij} \geq 0$. This directly proves the integer-lift property used here. The graph and matroid context is Martin–Reiner [4, Theorem 1 and §3]; the more general TU result of Beck–Hoşten [3, Theorem 13] is not needed for this deduction.

The following elimination also applies to the signed boxes used later.

Lemma 8.2 (Linear-cost lifting with three rows)**.** *For integer $a \times p$ arrays T, l, h, $l \leq h$, the system*

$$l_{ij} \leq T_{ij} + u_i - w_j \leq h_{ij}$$

is feasible if and only if the a-variable difference system

$$u_i - u_b \leq D_{bi}, \qquad D_{bi} = \min_j (h_{ij} - l_{bj} - T_{ij} + T_{bj}) \tag{8.4}$$

is feasible. Integral data admit an integral lift. It can be constructed, or infeasibility detected, in $O(a^2 p + a^3)$ arithmetic operations, hence $O(p)$ when $a = 3$.

Proof. For each column j, the admissible interval for w_j is

$$\max_i (T_{ij} + u_i - h_{ij}) \ \leq w_j \leq \ \min_b (T_{bj} + u_b - l_{bj}).$$

It is nonempty exactly when every lower endpoint is at most every upper endpoint. Rewriting those inequalities and taking the minimum over j gives (8.4). Computing its a^2 weights takes $O(a^2 p)$ operations. Shortest paths on the fixed a-vertex graph, with an added zero source, detect negative cycles or give integral potentials in $O(a^3)$ operations. Then set $w_j = \max_i (T_{ij} + u_i - h_{ij})$ and reconstruct the array. This last step costs $O(ap)$. □

This is classical difference-constraint elimination specialized to the three-row model. It is an arithmetic operation count, not a bit bound or an empirical speed comparison.

Lemma 8.3 (The three-row short shell)**.** *Every $\beta \in \mathcal{A}_{3p} \setminus \{0\}$ with $\|\beta\|_{Z_{3p}} < 2/3$ has gauge $1/2$ and a unique expression*

$$\beta = \eta_3^i \sum_{j \in S} \eta_p^j, \qquad i \in \{0, 1, 2\}, \quad \varnothing \neq S \subsetneq \mathbb{Z}/p\mathbb{Z}. \tag{8.5}$$

There are $3(2^p - 2)$ such elements.

Proof. For two prime factors, §7 says that vectors below gauge $2/3$ are precisely single-sign rectangles in the $2 \times (p-1)$ basis. The three nonempty row masks are third roots, possibly with reversed sign. Absorb the sign by complementing S, using $\sum_{j=0}^{p-1} \eta_p^j = 0$, to obtain (8.5). Sums over distinct nonempty proper subsets are distinct: a difference of degree at most $p-1$ can be a multiple of Φ_p only if every coefficient agrees; the two nonzero constant possibilities would require the full and empty sets. The three row factors are distinct, as their tensor masks show. The gauge and count follow from the same classification. □

Write $f_j = \mathbf{1}_{j \in S}$. An array for $H_B \beta$ is

$$A_{qj} = (B^2 + 1)\mathbf{1}_{q=i} f_j - B(\mathbf{1}_{q=i+1} f_{j-1} + \mathbf{1}_{q=i-1} f_{j+1}). \tag{8.6}$$

Indices are cyclic. For a cycle of half-length h put $z = \ell(\beta)$ and $v = \ell((\zeta + \zeta^{-1})\beta)$. Then $\ell(A) = (B^2+1)z - Bv$, $|z| \leq h/2$, and $|v| \leq h$: root multiplication preserves Z_{3p} and these elements have gauge $1/2$. A critical cycle with $z/h = 1/2$ must have length 4, because for a 6-cycle z is integral and $z \leq 1$. It compares row i with $i + \sigma$, $\sigma = \pm 1$, and a column $j \in S$ with $k \notin S$. Its positive cells are $(i, j), (i + \sigma, k)$, and

$$z = 1, \qquad v = f_{k-\sigma} - f_{j-\sigma} \in \{-1, 0, 1\}. \tag{8.7}$$

For every nonempty proper S, one such cycle has $v \leq 0$. Otherwise, fixing σ, translation would exchange S and its complement, forcing $2|S| = p$.

8.2 Exact height and odd bases

Lemma 8.4 (Strict positive-inverse bound)**.** *Let $M = (B^2+1)I - B(S+S^{-1})$ act on cyclic coefficient space, where S shifts by one position. If $R_t s = H_B\beta$ and $0 \le s \le u\mathbf{1}$, then $\|\beta\|_{Z_t} \le u/(B-1)^2$. If any coordinate of s is less than u, the inequality is strict.*

Proof. The convergent series

$$M^{-1} = \frac{1}{B^2+1}\sum_{n\ge 0}\left(\frac{B}{B^2+1}(S+S^{-1})\right)^n$$

has nonnegative entries and row sum $(B-1)^{-2}$. Every entry is strictly positive: a walk between any two cycle positions contributes positively to a power. Thus $M^{-1}s \in [0, u/(B-1)^2]^t$. If any coordinate of s is strictly below u, every upper inequality for $M^{-1}s$ is strict. Its maximum is strictly below the bound because there are finitely many coordinates. Finally $R_t M^{-1} s = \beta$. No positive margin uniform in t is required. □

Proof of (8.2). Put $h_0 = \lceil (B^2+1)/2 \rceil$. For the upper bound choose $\beta = 1$, or $i = 0, S = \{0\}$. Critical cycles have $v = 0$ or 1 and target at most B^2+1. Other 4-cycles have $z \le 0$ and target at most $2B$. The 6-cycle targets are at most B^2+1+3B. Uniform capacity h_0 satisfies all these inequalities for $B \ge 6$; for the last, $B^2 - 6B + 1 \ge 0$ suffices. Integral potentials give a lift.

For the lower bound, a false positive of integer height $u \le h_0$ would give, by Lemma 8.4, a nonzero element with

$$\|\beta\|_{Z_{3p}} \le \frac{h_0}{(B-1)^2} \le \frac{B^2+2}{2(B-1)^2} < \frac{2}{3} \qquad (B \ge 9).$$

The last inequality is $B^2 - 8B - 2 > 0$. Lemma 8.3 is exhaustive. The critical cycle with $v \le 0$ requires $2u \ge B^2+1$, so $u \ge h_0$. □

Proof of (8.1) *for odd B.* At length $3p(B+1)/2$, every capacity is $(B^2-1)/2 = h_{3p}(B) - 1$, so the full period is safe. At the next position choose $\beta = 1$ again. The capacity of $(0,0)$ increases by $B-1$. Every critical cycle has this positive cell, giving capacity B^2+B-2 against target at most B^2+1. Other 4-cycles have capacity at least B^2-1 and target at most $2B$; 6-cycles have capacity at least $3(B^2-1)/2$ and target at most B^2+1+3B. These comparisons hold for $B \ge 7$. Integral potentials and digit realization give a false positive, since $H_B \ne 0$. Padding completes the proof. □

8.3 Even bases: local obstruction and prefix cost

At length $k = 3pB/2 + a$, $0 \le a < 3p$, put $M_{qj}(a) = \mathbf{1}_{r(q,j)<a}$. Capacities are $(B-1)(B/2 + M_{qj})$. For even $B \ge 4$, a critical-cycle constraint is equivalent to

$$f_{k-\sigma} - f_{j-\sigma} + M_{i,j} + M_{i+\sigma,k} \ge 2 \quad (j \in S,\ k \notin S,\ \sigma = \pm 1). \tag{8.8}$$

Indeed the original inequality is $1 + B(1-v) \le (B-1)(M_{i,j} + M_{i+\sigma,k})$. A marker sum of 0 fails; a sum of 1 requires $v = 1$; a sum of 2 requires $v \ge 0$. Noncritical 4-cycles hold for $B \ge 3$ since $2B \le B(B-1)$. The 6-cycles hold for even $B \ge 10$ because

$$\tfrac{3}{2}B(B-1) - (B^2+1+3B) = \tfrac{1}{2}(B^2 - 9B - 2) > 0.$$

Lemma 8.5 (Two local systems)**.** *A nonconstant cyclic binary sequence f satisfies* (8.8) *if and only if it satisfies one of the following systems.*

A. *No adjacent* 11 *occurs;* $f_j = 1$ *requires* $M_{i,j} = 1$*; a pair* $f_j = f_{j+1} = 0$ *requires* $M_{i-1,j} = M_{i+1,j+1} = 1$.

B. *No adjacent* 00 *occurs;* $f_j = 0$ *requires* $M_{i-1,j} = M_{i+1,j} = 1$*; a pair* $f_j = f_{j+1} = 1$ *requires* $M_{i,j} = M_{i,j+1} = 1$.

Proof. A critical cycle with $v = -1$ is impossible even with two markers. Such a cycle exists exactly when both 11 and 00 occur. Since p is odd, a cyclic sequence cannot avoid both, so an admissible sequence belongs to exactly one system. In A, 00 occurs. In each direction choose an unselected k with unselected predecessor: then $v = 0$ forces the main marker of every selected j and the lateral markers precisely at the 00 pairs. Other critical pairs have $v = 1$ and are satisfied by the main marker. In B, 11 occurs. In each direction choose a selected j with selected predecessor: then $v = 0$ forces both lateral markers of every unselected k and the main markers at the ends of 11. Other constraints have $v = 1$ and are satisfied by the lateral marker. These arguments also prove sufficiency. □

Lemma 8.6 (Exact prefix cost)**.** *The minimum* a *admitting a nonconstant solution to* (8.8) *for some row* i *is* $2p - 1$.

Proof. Write $p = 3q + r$, $r \in \{1, 2\}$. At $a = 2p - 1$, choose system B and

r	i	columns with $f_j = 0$
1	0	$1, 4, \ldots, p-3$
2	1	$0, 3, \ldots, p-2$

.

Zeros are isolated, including across the cyclic join. Their two lateral cells occupy $j, j+p < 2p-1$. Every selected main cell occupies its column's first or second occurrence, also below $2p - 1$. Thus B holds.

For the lower bound it suffices to exclude $a = 2p - 2$. The marker vector in column j is

$$m_j = \begin{cases} \mathbf{1}_{\{j \bmod 3,(j+r) \bmod 3\}}, & 0 \le j < p-2, \\ \mathbf{1}_{\{j \bmod 3\}}, & j = p-2, p-1. \end{cases} \tag{8.9}$$

For each row i and system A or B, define a Boolean matrix $E(m, n)$ with states $x, y \in \{0, 1\}$. Entry (x, y) is 1 exactly when the rules of Lemma 8.5 hold at both ends and for the pair. For A, state 1 requires the row-i marker, 11 is forbidden, and 00 requires $m_{i-1}n_{i+1} = 1$. For B, state 0 requires the two lateral markers, 00 is forbidden, and 11 requires $m_i n_i = 1$. A cyclic product has a diagonal entry 1 exactly when an admissible cyclic sequence exists.

In the two-marker portion, the sequence has period 3. Let E_0, E_1, E_2 be its transitions and $C = E_0E_1E_2$. This portion has $p - 3 = 3(q - 1) + r$ edges. With Boolean products, put

$$D = E_0 \cdots E_{r-1}E(m_{p-3}, m_{p-2})E(m_{p-2}, m_{p-1})E(m_{p-1}, m_0).$$

The full product is $C^{q-1}D$. Substituting (8.9) into the four possible transitions gives the complete table:

Row i	System	D, $r = 1$	D, $r = 2$	C if $D \neq 0$
0	A	0	N	N
0	B	0	0	–
1	A	0	0	–
1	B	0	0	–
2	A	0	0	–
2	B	0	0	–

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the sole nonzero case, $r = 2, i = 0$, system A, explicitly

$$E_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = D = N.$$

The other eleven cases follow from the same rules on $\{0,1\}^2$ and give $D = 0$, independently of q. The exceptional product is N^q, $q \geq 1$: N has zero diagonal and $N^2 = 0$. This includes $p = 5$, where $q = 1$. In every case no admissible cyclic sequence exists. □

The prefix lemma alone requires only that p be odd and coprime to 3; primality remains necessary for the cyclotomic model and shell.

Proof of (8.1) *for even* B. At the preceding proposed length, $a = 2p - 2$, every capacity is at most $(B-1)(B/2+1)$. For $B \geq 12$, a false positive would give

$$\|\beta\|_{Z_{3p}} \leq \frac{B+2}{2(B-1)} < \frac{2}{3}.$$

The shell is exhaustive, and Lemma 8.6 excludes every candidate.

For $B = 10$, the preceding length is $17p - 2$. Capacities are 54 on $2p - 2$ cells and 45 on the other $p + 2$ cells, so every admissible vector has a coordinate below 54. The strict form of Lemma 8.4 gives $\|\beta\|_{Z_{3p}} < 54/81 = 2/3$. The same prefix exclusion therefore applies in this base as well.

At $a = 2p - 1$, the lemma's explicit support satisfies all critical cycles. The preceding estimates cover every other cycle for even $B \geq 10$; at $B = 10$ they include $20 \leq 90$ and $131 \leq 135$. Integral potentials give admissible sums and hence digits. The value $H_B\beta$ is nonzero. Padding proves minimality. □

8.4 Intercepts and witness construction

Relative to $\lfloor 3pB/2 \rfloor$, the intercepts are $2p - 1$ in even bases and $(3p + 3)/2$ in odd bases; they agree only at $p = 5$. The minimum coefficient period in the rational affine form is 2: period 1 would force constant increments $3p/2$, incompatible with integer values.

Formula (8.1) computes the threshold directly. The explicit row and support avoid scanning the $3(2^p - 2)$ candidates. Lemma 8.2 constructs potentials in $O(p)$ arithmetic operations. The weights have absolute value $O(B^2)$; this is not a unit-cost bit-complexity estimate. Realize the $3p$ sums by division with remainder by $B - 1$, as in Lemma 3.1. The description is compressed; expanding every digit still costs time proportional to $\Theta(pB)$.

9 Uniform thresholds of order $6p$

We retain the definitions of $K_t(B)$, full-period height and digit length from the preceding sections. This family shares the three-row geometry of $3p$, but opposite roots change both the operator and the ordered capacity boxes. The critical-prefix proof below is scalar and deductive; its complete boundary calculations are included in Appendix F.

Write $B = 4m + r$, $0 \leq r < 4$. For primes $p \geq 5$, set

$$d_p = \begin{cases} 2, & p \equiv 5 \pmod 6, \\ 4, & p \equiv 1 \pmod 6, \end{cases} \qquad e_p = \begin{cases} 2, & p = 5, 7, \\ 4, & p \geq 11. \end{cases}$$

Define the four offsets

$$(c_{p,0}, c_{p,1}, c_{p,2}, c_{p,3}) = (3p + d_p,\ 5p - e_p,\ 6p + 1,\ 8p + 1). \tag{9.1}$$

Theorem 9.1 (Uniform order-$6p$ threshold)**.** *For every prime $p \geq 5$,*

$$\boxed{K_{6p}(4m+r) = 6pm + c_{p,r}}$$

whenever the base satisfies the corresponding bound:

r	0	1	2	3
B *at least*	16	13	10	19.

In particular all four formulas hold for every $B \geq 16$. The minimal eventual period of the coefficient functions is 4. No minimal base cutoff is asserted.

Relative to the common slope, the same formula is

$$K_{6p}(B) = \frac{3p}{2}B + \begin{cases} 3p + d_p, & B \equiv 0 \pmod 4, \\ 7p/2 - e_p, & B \equiv 1 \pmod 4, \\ 3p + 1, & B \equiv 2 \pmod 4, \\ 7p/2 + 1, & B \equiv 3 \pmod 4. \end{cases}$$

The specialization $p = 5$ gives the order-30 formula in Corollary 11.2; the theorem treats the entire prime family.

Corollary 9.2 (Base ten)**.** *For every prime $p \geq 5$, $K_{6p}(10) = 18p + 1$. Thus the lengths for orders $30, 42, 66, 78$ are respectively $91, 127, 199, 235$.*

9.1 Opposite roots, signed boxes and short elements

Put $n = 3p$, $t = 2n$, take a primitive t-th root ξ, and put $\zeta = -\xi$, of order n. Let $A = \mathbb{Z}[\xi] = \mathbb{Z}[\zeta]$. With

$$\mathcal{H}_B = (B - \xi)(B - \xi^{-1}) = B^2 + 1 + B(\zeta + \zeta^{-1}),$$

the lattice of the double-evaluation kernel is $\Lambda = \mathcal{H}_B A$. This follows from Propositions 2.2, 2.3, and 2.5: here $\Phi_{6p}(1) = \Phi_{6p}(-1) = 1$, so $\delta = 1$. The reversal's power of B does not change the kernel.

Choose η_3, η_p with $\zeta = \eta_3\eta_p$. Explicit choices are $\eta_3 = \zeta^{p(p^{-1} \bmod 3)}$ and $\eta_p = \zeta^{3(3^{-1} \bmod p)}$. The root at digit position s is

$$\xi^s = (-1)^s \eta_3^{s \bmod 3} \eta_p^{s \bmod p}.$$

Represent elements of A by full $3 \times p$ arrays. Equal algebraic values differ by a row potential plus a column potential; in the anchored basis their coordinates are $F_{ij} - F_{i,p-1} - F_{2,j} + F_{2,p-1}$.

At length $k = tL + a$, each residue modulo t has capacity $(B-1)(L + \mathbf{1}_{s<a})$. Every array cell has one even digit position, contributing the positive root, and one odd position, contributing its negative. Define markers

$$P_{ij} = \mathbf{1}_{\text{even position for } (i,j)<a}, \qquad N_{ij} = \mathbf{1}_{\text{odd position for } (i,j)<a}.$$

The signed coefficient in that cell fills exactly the integer interval

$$-(B-1)(L + N_{ij}) \ \leq y_{ij} \leq \ (B-1)(L + P_{ij}). \tag{9.2}$$

To realize a positive signed coefficient, use its positive position; to realize a negative one, use its negative position. Folded sums then split greedily into actual digits in $[0, B-1]$.

For a target array T, the lift has form $T_{ij} + u_i - w_j$. The interval inequalities are difference constraints on the bidirected graph $K_{3,p}$. They are feasible exactly when every directed cycle has nonnegative weight, and integral data yield integral shortest-path potentials. Simple cycles have length 2, 4 or 6. An oriented $2h$-cycle with alternating functional u requires

$$u(T) \le \sum_{u_{ij}=1} (B-1)(L+P_{ij}) + \sum_{u_{ij}=-1} (B-1)(L+N_{ij}). \tag{9.3}$$

Length 2 states only that each interval is nonempty. The graph model has the classical context of Martin–Reiner, Theorem 1 and Section 3; the above difference-constraint argument supplies the needed integral lift directly.

Let Z_s be the image of $[0,1]^s$ under its root columns. Pairing opposite roots gives

$$Z_{6p} = R_{3p}[-1,1]^{3p} = 2Z_{3p},$$

because the sum of the $3p$ roots is zero. By Lemma 8.3 and Proposition 7.2, the complete short-shell description is

$$0 \ne \beta \in A, \quad \|\beta\|_{Z_{6p}} < \frac{1}{3} \quad \Longrightarrow \quad \beta = \eta_3^i \sum_{j\in S} \eta_p^j, \tag{9.4}$$

uniquely for $i \in \{0,1,2\}$ and $\varnothing \ne S \subsetneq \mathbb{Z}/p\mathbb{Z}$. There are $3(2^p-2)$ such elements and their Z_{6p} gauge is $1/4$. The proof of Theorem 7.1 uses the tensor coordinates to exclude opposite signs and nonrectangular supports below gauge $2/3$ in Z_{3p}. Primality is needed for this cyclotomic model; the binary lemma below alone works for odd p coprime to 3.

For the inverse estimate we apply Lemma 8.4 in the original $6p$ coefficient space. Although the operator on the $3 \times p$ array has a plus sign, the full cyclic coefficient operator is $(B^2+1)I - B(S+S^{-1})$ for the root ξ. For a nonzero $\alpha = R_t s = \mathcal{H}_B \beta$ with $0 \le s \le U\mathbf{1}$, we have $s \ne U\mathbf{1}$ because $R_t\mathbf{1} = 0$. Consequently

$$\|\beta\|_{Z_t} < U/(B-1)^2. \tag{9.5}$$

9.2 Four-cycle conditions become local binary rules

For (9.4), write $f_j = \mathbf{1}_{j\in S}$ and $A_B = B^2+1$. A target array for $\mathcal{H}_B\beta$ is

$$T_{qj} = A_B \mathbf{1}_{q=i} f_j + B(\mathbf{1}_{q=i+1} f_{j-1} + \mathbf{1}_{q=i-1} f_{j+1}). \tag{9.6}$$

All row and column indices are cyclic. If $z = u(\beta)$ and $v = u((\zeta+\zeta^{-1})\beta)$, then $u(T) = A_B z + Bv$. For a cycle with h positive cells, $|z| \le h/2$ and $|v| \le h$, using the gauge in Z_{3p} and invariance under root multiplication.

The critical cycles have length 4, $z = 1$, and compare the selected main cell (i,j) with an unselected column k in row $i+\sigma$, $\sigma = \pm 1$. The positive cells are $(i,j), (i+\sigma,k)$ and the negative cells are $(i,k), (i+\sigma,j)$. Then

$$v = f_{k-\sigma} - f_{j-\sigma} \in \{-1,0,1\}, \quad M_u = P_{i,j} + N_{i+\sigma,j} + P_{i+\sigma,k} + N_{i,k}.$$

Write the offset as $c = 6p\ell + a$, so $L = m + \ell$, and put $h = r + 2 - 4\ell$. The critical condition is

$$\boxed{M_u - v \ge h.} \tag{9.7}$$

Indeed the exact original inequality is

$$B^2 + 1 + Bv \le (B-1)(4L + M_u).$$

Putting $g = M_u - r + 4\ell$ gives $B(v - g + 1) + (g + 1) \le 0$. For integer $v \in \{-1, 0, 1\}$ and $B \ge 4$, this holds precisely when $g \ge v + 2$, which is (9.7). The constant term excludes the apparent equality $g = v + 1$.

For each direction define the marker-only quantities

$$a_j^\sigma = P_{i,j} + N_{i+\sigma,j}, \qquad b_j^\sigma = P_{i+\sigma,j} + N_{i,j}.$$

All selected/unselected pairs satisfy (9.7) if and only if there are integers $q_-, q_+ \in \{0, 1, 2, 3\}$ such that

$$f_j = 1 \Rightarrow a_j^\sigma + f_{j-\sigma} \ge q_\sigma, \qquad f_j = 0 \Rightarrow b_j^\sigma - f_{j-\sigma} \ge h - q_\sigma. \tag{9.8}$$

For necessity take the selected minimum in each direction; it lies between zero and three. For sufficiency add the two inequalities for each selected/unselected pair. Thus the local formulation retains all global pair constraints, not just constraints between adjacent columns.

9.3 A gap lemma removes the off-diagonal parameters

Lemma 9.3 (Four ordered positions)**.** *Fix a row i and adjacent columns $j, j+1$ cyclically. Each of the two four-marker sums*

$$b_j^- + a_{j+1}^+, \qquad a_j^- + b_{j+1}^+$$

uses four distinct positions on the $6p$-cycle with cyclic gaps $p, 2p-1, p, 2p+1$, up to reversal and cyclic order. Hence an initial prefix of length $a < 3p$ contains at most two of them; if $a < 4p$, it contains at most three.

Proof. For the first sum, start at the odd position s of cell (i, j). The even position of $(i-1, j)$ differs by p or $-p$ modulo $6p$, according to p mod 3. The other two positions are obtained from these by adding $3p + 1$. Thus the four positions are a translate of $\{0, \epsilon p, 3p+1, 3p+1+\epsilon p\}$, $\epsilon = \pm 1$. The same construction applies to the other sum; reversing the shift replaces $3p+1$ by $3p-1$. Sorting either set gives the displayed gaps. The shortest arc containing all four has span $4p - 1$; the shortest arc containing any three has span $3p - 1$, because the largest sum of two consecutive gaps is $3p + 1$. A prefix of length a has span $a - 1$. □

Lemma 9.4 (Equal channel parameters)**.** *At either offset $c_{p,r} - 1$ or $c_{p,r}$, every nonconstant support satisfying* (9.7) *admits* (9.8) *with $q_- = q_+ = q$. The only values that need consideration are*

r	0	1	2	3
q	1, 2	2	0, 1	0, 1, 2.

Proof. In each direction a nonconstant cyclic binary word has a selected position with unselected predecessor and an unselected position with selected predecessor. Therefore its selected minimum is at most 2, and its unselected minimum at most 1. The selected minimum can serve as q_σ, so $\max(0, h-1) \le q_\sigma \le 2$. When $r = 2$, there is at most one marker in total, improving the upper bound to 1.

For a transition $f_j = 0, f_{j+1} = 1$, the two endpoint inequalities give

$$b_j^- \ge h - q_- + 1, \qquad a_{j+1}^+ \ge q_+.$$

If $q_+ > q_-$, their sum is at least $h + 2$. This is impossible: the four-marker sum is at most $h + 1$. For $r = 0$ this follows from $a \le 3p + d_p < 4p$ and Lemma 9.3; for $r = 1$ it is the trivial bound four; for $r = 2$ at most one marker is present; for $r = 3$, $a \le 2p + 1 < 3p$ and the lemma gives at most two. Similarly, if $q_- > q_+$, every transition $1 \to 0$ is impossible, using $a_j^- + b_{j+1}^+$. A nonconstant cyclic word needs both transition types. Thus the parameters coincide, with the stated ranges. □

On the branch $r = 1$, the common parameter is forced to be 2.

9.4 Scalar rules and their six-step bulk patterns

We use six relations between consecutive bits x, y:

symbol	E	A	Z	O	U	D
meaning	$x = y$	any x, y	$x = y = 0$	$x = y = 1$	$x \le y$	$x \ge y$.

These are ordinary scalar constraints, not matrices or software states. For the common parameter q, the allowed pair $x = f_j, y = f_{j+1}$ is given by four tests:

$$\begin{array}{c|l} 00 & b_j^- \ge h - q, \quad b_{j+1}^+ \ge h - q \\ 01 & b_j^- \ge h - q + 1, \quad a_{j+1}^+ \ge q \\ 10 & a_j^- \ge q, \quad b_{j+1}^+ \ge h - q + 1 \\ 11 & a_j^- \ge q - 1, \quad a_{j+1}^+ \ge q - 1. \end{array} \tag{9.9}$$

All tables below are obtained by these four substitutions. Their entries involve only equality, a fixed bit, or a one-way inequality, so their consequences can be read directly.

For completeness, the marker rule requires no CRT search. In column j, visit number k, $0 \le k < 6$, marks row $(j + kp) \bmod 3$, positively if $j + k$ is even and negatively if it is odd. A column with b visits uses $k = 0, \dots, b - 1$. At the two boundaries, with $\varepsilon = 0$ for upper and 1 for lower, the visit counts are

$$\begin{array}{c|l} r = 0 & 3 + \mathbf{1}_{j<d_p-\varepsilon} \\ r = 1 & 5 - \mathbf{1}_{j\ge p-e_p-\varepsilon} \\ r = 2 & \mathbf{1}_{\varepsilon=0,\ j=0} \\ r = 3 & 2 + \mathbf{1}_{\varepsilon=0,\ j=0}. \end{array} \tag{9.10}$$

Both endpoints of a bulk edge have the same visit count. Put $\theta = j + 2i + \delta \pmod 6$, where $\delta = 0$ for $p \equiv 5$; for $p \equiv 1$, take $\delta = 2$ in the three-visit bulk and $\delta = 1$ in the two-visit bulk. Formula (9.9) gives

visits	$q\backslash\theta$	0	1	2	3	4	5
3	1	O	O	E	E	E	E
3	2	E	E	E	Z	Z	E
2	0	O	O	E	A	E	O
2	1	E	E	E	E	E	E
2	2	A	E	Z	Z	Z	E.

(9.11)

For example, with three visits, $p \equiv 5$, $i = j = 0$, the two marker vectors are $(1, 1, 0; 0, 0, 1)$ and $(1, 0, 0; 0, 1, 1)$; at $h = 2, q = 1$, only 11 passes (9.9), giving O. Incrementing j or changing i gives the displayed phases by the visit rule. With five visits and $q = 2, h = 3$, the rule is simply

$$\text{A} \quad \text{if } j \equiv i + p + 1 \pmod 3, \qquad \text{E} \quad \text{otherwise.} \tag{9.12}$$

Indeed the only absent visit is $k = 5$, whose row is $j - p \pmod 3$ and whose sign is opposite to the parity of j. Substituting this one absent marker in (9.9) gives (9.12). Thus its free edges separate constant triples.

9.5 The exact critical prefix

Lemma 9.5 (Critical-prefix classification)**.** *For odd $p \ge 5$ coprime to 3, let $c_{p,r}$ be* (9.1)*, with $e_p = 2$ for $p = 5, 7$ and $e_p = 4$ otherwise. The least offset $c \ge 0$ admitting a nonconstant support satisfying all critical conditions is $c_{p,r}$. At that offset the support is unique on residues $0, 2, 3$. On residue 1, all supports are the independent-triple families described in Appendix F.4, with the two explicit exceptions at $p = 5, 7$.*

Proof. Only $c_{p,r}-1$ and $c_{p,r}$ need be considered, since the physical capacities are monotone in c. Lemma 9.4 reduces all possible parameters to the listed common values q. Substituting the visit counts (9.10) in the four tests (9.9) gives the bulk patterns (9.11)–(9.12). Appendix F gives the complete boundary relations for every row and every remaining q, and contracts their equalities and fixed bits. At the lower boundary these force a constant word or a contradiction. At the upper boundary they give exactly the supports stated here. These are finite scalar substitutions and their uniform consequences for the arbitrary bulk length; no matrix certificate is a premise. □

Explicit upper supports are also available:

r	$p \equiv 5 \pmod 6$	$p \equiv 1 \pmod 6$
0	$i=2,\ S=(\mathbb{Z}/p\mathbb{Z})\setminus\{0,1\}$	$i=0,\ S=(\mathbb{Z}/p\mathbb{Z})\setminus\{1,2\}$
1, $p\geq 11$	$i=1,\ S=\{2,3,4\}$	$i=0,\ S=(\mathbb{Z}/p\mathbb{Z})\setminus\{3,4,5\}$
2	$i=0,\ S=\{0\}$	$i=0,\ S=\{0\}$
3	$i=2,\ S=(\mathbb{Z}/p\mathbb{Z})\setminus\{0\}$	$i=1,\ S=(\mathbb{Z}/p\mathbb{Z})\setminus\{0\}$

For $r=1$, use $i=2, S=\{3,4\}$ at $p=5$ and $i=1, S=\{1,2,3\}$ at $p=7$. Substitution in the local rules verifies these templates; the $r=1$ family is described more fully in Section 9.8.

9.6 Completing the cyclotomic proof and the base bounds

Proof of Theorem 9.1. Consider first the preceding length $k_- = 6pm + c_{p,r} - 1$. A nonzero false positive would give $\beta \in A \setminus \{0\}$. The largest folded capacity U satisfies

$$\frac{U}{(B-1)^2} = \begin{cases} (B+4)/(4(B-1)), & r=0, \\ (B+3)/(4(B-1)), & r=1, \\ (B+2)/(4(B-1)), & r=2, \\ (B+5)/(4(B-1)), & r=3. \end{cases}$$

In the $r=2$ case the preceding offset is exactly $6p$, so all capacities equal $(B-1)(m+1)$; no extra prefix capacity is present. These four quantities are at most $1/3$ at, respectively, $B \geq 16, 13, 10, 19$. Lemma 8.4 makes the actual gauge strictly smaller than $1/3$, even at equality. The complete short-shell classification (9.4) applies. Lemma 9.5 excludes every such element at the preceding offset, proving the lower bound.

At the proposed upper length, select an accepting support. All critical four-cycles hold by (9.7). Other four-cycles have $z \leq 0$, so their target is at most $2B$. Six-cycles have integer $z \leq 1$ and target at most B^2+1+3B.

At this length put $d = r - 4\ell$, with values $0, 1, -2, -1$ on the four branches. The capacity of a four-cycle is at least $(B-1)(B-d)$; that of a six-cycle is at least $\frac{3}{2}(B-1)(B-d)$. The difference between this last quantity and its target bound is

r	$2(\frac{3}{2}(B-1)(B-d) - (B^2+1+3B))$
0	B^2-9B-2
1	$B^2-12B+1$
2	B^2-3B-8
3	B^2-6B-5.

All are positive in the stated branch domains. The four-cycle comparison with $2B$ is also immediate there. Thus every cycle constraint holds, the difference graph gives an integer lift, and (9.2) realizes it with actual digits. Its algebraic value $\mathcal{H}_B\beta$ is nonzero and belongs to Λ. This is a false positive at the proposed length.

Padding proves minimality against all shorter lengths. Period 4 is exhibited by the formula. Period 2 is impossible because the intercepts on residues 0 and 2 differ by $d_p - 1 \neq 0$. The minimal eventual coefficient period divides 4, so it is exactly 4. □

Lemma 8.2 constructs an integral lift in $O(p)$ arithmetic operations after choosing the support. The weights have size $O(B^2)$; no bit-complexity or empirical speedup is claimed. Greedy digit realization costs time proportional to the output length. The diagnostic code retains a different graph-based lift for comparison.

9.7 Exact full-period height

Let $h_t(B)$ be the least integer h for which folded coefficients in $[0, h]^t$ meet $\Lambda \setminus \{0\}$.

Theorem 9.6. *For every prime $p \geq 5$ and every $B \geq 10$,*

$$\boxed{h_{6p}(B) = \left\lceil (B^2 + B + 1)/4 \right\rceil .}$$

Proof. Write h_0 for the proposed value. For the upper bound choose $\beta = 1$. Its critical cycles have $v = 0$ or 1, so their targets are at most $B^2 + B + 1 \leq 4h_0$. Noncritical four-cycles are bounded by $2B$, and six-cycles by $B^2 + 1 + 3B \leq 6h_0$ for $B \geq 3$, since $B^2 - 3B + 1 > 0$. The signed intervals $[-h_0, h_0]$ admit an integer lift and hence folded coefficients in $[0, h_0]^{6p}$.

For the lower bound suppose a false positive has folded height $u \leq h_0 - 1$. For $B \geq 11$,

$$\frac{u}{(B-1)^2} \leq \frac{B^2 + B}{4(B-1)^2} \leq \frac{1}{3},$$

the last comparison being $B^2 - 11B + 4 \geq 0$. For $B = 10$, $u \leq 27$ and $27/81 = 1/3$. The strict inverse bound puts β in the short shell in either case. Every nonconstant cyclic binary sequence has a selected position with unselected predecessor and an unselected position with selected predecessor. Choosing those two columns gives a critical cycle with $v = 1$. Its capacity is $4u$, so $4u \geq B^2 + B + 1$, contradicting $u \leq h_0 - 1$. □

For independent coefficient height $H \geq 1$, the earlier complete-period bridge immediately yields

$$6p\left(\lceil h_{6p}(B)/H \rceil - 1\right) + 1 \leq K_{6p,H}(B) \leq 6p \lceil h_{6p}(B)/H \rceil .$$

This is a bound for arbitrary H, not an exact formula for every partial period.

9.8 Exponential multiplicity of the first algebraic values

Here we count distinct normalized values $\beta = \mathcal{H}_B^{-1} P(\xi)$ at the first threshold, not digit words or orbits under rotations.

Theorem 9.7. *Let $B \equiv 1 \pmod 4$, $B \geq 13$. For $p = 5, 7$ there is one such value. For primes $p \geq 11$ the number is*

$$\boxed{2^{\lambda_p} - 1}, \qquad \lambda_p = \begin{cases} (p-8)/3, & p \equiv 5 \pmod 6, \\ (p-10)/3, & p \equiv 1 \pmod 6. \end{cases}$$

More explicitly:

- *If $p \equiv 5 \pmod 6$, $i = 1$ and S is a nonempty union of the disjoint triples $\{2 + 3j, 3 + 3j, 4 + 3j\}$, $0 \leq j < \lambda_p$.*
- *If $p \equiv 1 \pmod 6$, $i = 0$ and the complement of S is a nonempty union of $\{3 + 3j, 4 + 3j, 5 + 3j\}$, $0 \leq j < \lambda_p$.*

Proof. At the upper length on this branch the maximum capacity is the same as at the preceding length. Thus the strict shell reduction remains exhaustive for every minimum word. Lemma 9.5 and Appendix F.4 classify every admissible support: the free triples are independent, and the constant choice is excluded. For $p = 5, 7$, the appendix gives the single exceptional support. Each nonconstant support satisfies all critical constraints, and the noncritical bounds in Theorem 9.1 supply its integral lift. The short-shell parametrization is injective, so no normalized values are counted twice. □

For example the numbers at $p = 11, 13, 17, 19, 23, 29$ are $1, 1, 7, 7, 31, 127$. The exponential number of values is compatible with constructing one value directly from a single triple. This does not enumerate all minimum digit words.

Corollary 9.8 (Uniqueness on the other three branches)**.** *For every prime $p \geq 5$, there is exactly one normalized algebraic value β at the first threshold on each of the following domains:*

$B \bmod 4$	0	2	3
B *at least*	16	22	19.

Together with Theorem 9.7, this classifies the number of normalized minimum values in the four stated ranges. It does not classify the minimum digit words.

Proof. Lemma 9.5 gives precisely one nonconstant support on each of these three branches. Its scalar classification is completed in Appendices F.1, F.2 and F.3.

It remains to make the short shell exhaustive *at* K, rather than only at $K - 1$. The maximal capacity divided by $(B - 1)^2$ is respectively

$$\frac{B+4}{4(B-1)}, \qquad \frac{B+6}{4(B-1)}, \qquad \frac{B+5}{4(B-1)}$$

on residues $0, 2, 3$. These ratios are at most $1/3$ in the stated domains. The strict inverse bound places every nonzero normalized value in the short shell. Its parametrization is injective, and the upper construction supplies a lift of the single support. □

For $B = 10, 14, 18$, the order-$6p$ threshold remains covered by Theorem 9.1, but the preceding corollary makes no uniqueness claim. The capacity increase at K prevents reuse of the $K - 1$ shell bound without further work.

9.9 Computational cross-checks

The companion retains the independent matrix reconstruction (F6) and integer path-count identities (F7) as regression checks on the prefix and multiplicity formulas. They are not premises of the scalar proof above. Component F8 checks the scalar boundary classification directly against selected/unselected inequalities on a finite panel, and checks the digit-transfer and reversal consequences below. All such panels are finite diagnostics; the uniform conclusions use the written gap, boundary and cyclotomic arguments. The certificate dependencies for orders $105, 210$ are separate and remain as stated in Section 12.

10 Reversal and non-palindromic minimum words

We distinguish a normalized algebraic value from a digit word realizing it. Reversal acts on both, but its fixed points need not correspond: a fixed value can have non-palindromic digit realizations.

10.1 Boolean coordinates for the minimum values

Suppose $p \geq 11$ is prime and $B \equiv 1 \pmod 4$, $B \geq 13$. Let $\lambda = \lambda_p$ be as in Theorem 9.7, and put

$$\gamma_j = \begin{cases} \eta_3 \eta_p^{2+3j}(1 + \eta_p + \eta_p^2), & p \equiv 5 \pmod 6, \\ -\eta_p^{3+3j}(1 + \eta_p + \eta_p^2), & p \equiv 1 \pmod 6, \end{cases} \qquad 0 \leq j < \lambda.$$

The normalized values at the first threshold are exactly

$$\mathcal{V}_p = \left\{ \sum_{j=0}^{\lambda-1} \epsilon_j \gamma_j : \epsilon_j \in \{0, 1\}, \quad \epsilon \neq 0 \right\}. \tag{10.1}$$

Indeed this is the support classification, using $\sum_{j=0}^{p-1} \eta_p^j = 0$ on the complement branch. The γ_j are linearly independent over $\mathbb{Q}$: after removing the common nonzero factor, a relation would be a polynomial supported on disjoint triples, of degree less than $p - 1$, vanishing at η_p. Its coefficients must vanish. They are consequently independent over $\mathbb{R}$ in $A \otimes_{\mathbb{Z}} \mathbb{R}$. Thus (10.1) is a Boolean parallelotope with the zero vertex removed. This assertion concerns the real scalar extension of the lattice, not one complex embedding. The enumerator by selected support size on the first branch, or omitted support size on the second, is $(1 + z^3)^\lambda - 1$.

Theorem 10.1 (Reversal of minimum values)**.** *At length $K_{6p}(B)$ in the preceding domain, reversal acts on* (10.1) *by*

$$(\epsilon_0, \ldots, \epsilon_{\lambda-1}) \longmapsto (\epsilon_{\lambda-1}, \ldots, \epsilon_0).$$

The numbers of fixed normalized values and reversal orbits of values are

$$F_p = 2^{\lceil \lambda/2 \rceil} - 1, \qquad O_p = \frac{2^\lambda + 2^{\lceil \lambda/2 \rceil} - 2}{2}.$$

For $p = 5, 7$ the unique normalized value is fixed, with one orbit.

Proof. For $P^*(X) = X^{K-1} P(X^{-1})$, reality of $\mathcal{H}_B$ gives

$$J(\beta) := \mathcal{H}_B^{-1} P^*(\xi) = \xi^{K-1} \overline{\beta}.$$

Here $K - 1 \equiv 5p - 5 \pmod{6p}$, an even exponent. On the active row, conjugation and this root factor preserve that row and send column j to $-5 - j \pmod p$. In either residue class the triple numbered j is therefore sent to the triple numbered $\lambda - 1 - j$. The common minus sign on the complement branch is unchanged. Independence of the generators gives the asserted coordinate action. A reversal-fixed binary vector has $\lceil \lambda/2 \rceil$ free bits; exclude the zero vector. Every other value has an orbit of size two, so the total is $(|\mathcal{V}_p| + F_p)/2$. At $p = 5, 7$ reversal preserves the singleton of minimum values. □

Here λ is odd. The enumerator of the fixed choices, by number of chosen triples, is $(1 + z)(1 + z^2)^{(\lambda-1)/2} - 1$. These are value counts, not counts of palindromic words.

10.2 A transfer within a residue class

Call $\beta \in \mathbb{Z}[\xi]$ *primitive* if $\beta \notin n\mathbb{Z}[\xi]$ for every integer $n > 1$. Let $K_t^{\mathrm{np}}(B)$ denote the first false-positive length restricted to non-palindromic words, with the same full modulus $\Phi_t(B)$ and digit alphabet as $K_t(B)$.

Lemma 10.2 (Period transfer)**.** *Let $t \geq 3$, $B \geq 3$, $\Phi_t(1) = 1$, and $K = K_t(B) \geq 2t$. Put $A = \mathbb{Z}[\xi]$ and $\mathcal{H}_B = (B - \xi)(B - \xi^{-1})$. If a word at length K realizes $P(\xi) = \mathcal{H}_B \beta$ with $\beta \in A$ primitive, then the same value has a non-palindromic realization at length K.*

Proof. Write $N = B - 1 \geq 2$. From $\Phi_t(X) = 1 + (X-1)R(X)$ we obtain $(1-\xi)R(\xi) = 1$. Both $1-\xi$ and $1-\xi^{-1}$ are units in A. Modulo NA, the element $\mathcal{H}_B$ is their product, hence a unit. If every digit of P were 0 or N, then $\mathcal{H}_B\beta = P(\xi) \in NA$ would imply $\beta \in NA$, contrary to primitivity. Thus some digit d_a satisfies $0 < d_a < N$.

Since $K \geq 2t$, another position $b \neq a$ satisfies $b \equiv a \pmod{t}$. If $d_b < N$, decrease d_a and increase d_b by one; if $d_b = N$, do the opposite. Both moves respect $[0, N]$. The change is $\pm(X^b - X^a)$, a multiple of $\Phi_t(X)$, so it preserves the algebraic value and its nonzero remainder. It preserves both evaluations modulo the full integer $\Phi_t(B)$: for the reciprocal evaluation use $\gcd(B, \Phi_t(B)) = 1$. The new endpoint digits cannot vanish, by minimality of K and Lemma 2.1.

If P is already non-palindromic there is nothing further to prove. Otherwise the change cannot be invariant under $a \mapsto K - 1 - a$. Its two nonzero coefficients have opposite signs, so invariance would require two distinct fixed positions of this reflection, which has at most one. The modified word is non-palindromic. □

Corollary 10.3. *For every prime $p \geq 5$,*

$$K_{3p}^{\mathrm{np}}(B) = K_{3p}(B) \quad (B \geq 9), \qquad K_{6p}^{\mathrm{np}}(B) = K_{6p}(B)$$

in all four base domains of Theorem 9.1. In particular,

$$K_{3p}^{\mathrm{np}}(10) = 17p - 1, \qquad K_{6p}^{\mathrm{np}}(10) = 18p + 1.$$

Proof. For either order, $\Phi_t(1) = 1$. The constructions at the threshold use $\beta = \eta_3^i \sum_{j \in S} \eta_p^j$ with S nonempty and proper. These elements are primitive. In the integral product basis $\{1, \eta_3\}\{\eta_p^j : 0 \leq j \leq p-2\}$, the coefficients of $\sum_{j \in S} \eta_p^j$ are $\mathbf{1}_S(j) - \mathbf{1}_S(p-1) \in \{-1, 0, 1\}$, and at least one is ± 1. The products form a basis: they generate $\mathbb{Z}[\eta_3, \eta_p] = \mathbb{Z}[\eta_{3p}]$ and their number equals its rank $2(p-1)$. Multiplication by the unit η_3^i preserves primitivity. On the odd-base $3p$ branch the construction uses $\beta = 1$.

The threshold formulas give $K_{3p}(B) \geq 15p + 1 > 2(3p)$ and $K_{6p}(B) \geq 18p + 1 > 2(6p)$ in the respective domains. Lemma 10.2 supplies a non-palindromic word at the unrestricted first threshold. The opposite inequality follows by restriction of the set of words. □

For bases $10, 14, 18$ in the $6p$ family this uses the constructed short value only. It does not extend the uniqueness domain of Corollary 9.8.

Corollary 10.4 (Words above primitive values)**.** *Under the hypotheses of Lemma 10.2, each primitive normalized value at length K has at least two distinct non-palindromic realizations. An ensemble of V distinct such values therefore supplies at least $2V$ non-palindromic words and at least V reversal orbits of words. In particular, for $B \equiv 1 \pmod 4$, $B \geq 13$, $p \geq 11$ prime, there are at least $2(2^{\lambda_p} - 1)$ non-palindromic words at $K_{6p}(B)$, and at least $2^{\lambda_p} - 1$ word orbits.*

Proof. Reversal induces $J(\beta) = \xi^{K-1}\overline{\beta}$. If $J(\beta) \neq \beta$, every word in the fibre is non-palindromic, and the nonzero period transfer supplies a second distinct word in that fibre. If $J(\beta) = \beta$, take a non-palindromic word from the lemma; its reverse is a second distinct word in the same fibre. Different values have disjoint fibres. Every non-palindromic reversal orbit has two words, which proves the lower bound on the number of word orbits. Apply Theorem 9.7 for the last assertion. □

For $p = 5, 7$, and on each other constructed branch of $3p$ or $6p$, a single primitive value already supplies two non-palindromic words. For example, $p = 17, B = 13$ gives $K = 387$ and $\lambda = 3$: there are exactly seven normalized values, three fixed values and five value orbits, while there are at least fourteen non-palindromic words and seven word orbits. No exact count of all minimum words is asserted.

11 Recipes and certified formulas

This section combines the reductions proved above with finite exact identities. Section 12 identifies the required files and checkers. The solvers used to discover the certificates are not premises.

11.1 The totally unimodular regime

By Beck–Hoşten [3, Theorem 13 and the decomposition in §2], the CRT root matrix R_t is totally unimodular when t has at most two distinct odd prime factors. We use this property of the matrix, rather than a property of a different convex body.

Corollary 11.1. *Under this hypothesis, the least eventual coefficient period of $K_t(B)$ divides ρ_t.*

Proof. Every nonzero minor has absolute value 1, so $L_t = 1$ in Theorem 1.3. □

There is a smaller recipe for computing the intercepts. Let r_i be the columns of R_t. Let $\mathcal{N}_t$ contain both primitive normals to each span of $d-1$ independent columns, with repetitions removed. These are the facet directions of the zonotope. Total unimodularity and the cofactor construction give $u \cdot r_i \in \{0, 1, -1\}$. Put

$$h(u) = \sum_{i<t} \max(u \cdot r_i, 0), \qquad h_a(u) = \sum_{i<a} \max(u \cdot r_i, 0).$$

Since $\sum r_i = 0$, we have $0 < h(u) \le t/2$. For $L > 0$, the integer fibre at length $tL + a$ is nonempty precisely when

$$u \cdot H_B\beta \le (B-1)(Lh(u) + h_a(u)) \quad \text{for every } u \in \mathcal{N}_t. \tag{11.1}$$

Indeed these inequalities describe the sum of segments of capacities $(B-1)(L + \mathbf{1}_{i<a})$. A nonempty real fibre has a vertex. Its active constraints consist of rows of R_t and coordinate bounds with integer endpoints. Total unimodularity and Cramer's rule make that vertex integral. This implication is unavailable for a general root matrix.

Fix a branch Γ, δ, μ, and put

$$h_{\max} = \max h(u), \qquad g = \frac{1}{\operatorname{den}(\mu)\delta h_{\max}}, \qquad C_{\mathrm{red}} = 2 + \left\lceil \frac{8\mu + 1}{g} \right\rceil.$$

The gauge of $\beta \in \Gamma$ is the maximum of $u \cdot \beta / h(u)$. If it exceeds μ, it exceeds it by at least g. Total unimodularity removes the rounding error from the upper bound in (1.4), giving

$$K_t(B) \le t \left\lceil \frac{\mu(B+1)^2}{B-1} \right\rceil.$$

The maximum principle bounds a first failure by $\|\beta\|_{Z_t} \le \mu + (8\mu+1)/(B-1)$. Thus $B \ge C_{\mathrm{red}}$ reduces it to the first shell, which is exactly

$$\mathcal{S} = \{\beta \in \Gamma \cap [-1,1]^d : \beta \ne 0,\ u \cdot \beta \le \mu h(u) \text{ for all } u \in \mathcal{N}_t\}.$$

Let T represent multiplication by $\zeta + \zeta^{-1}$. For fixed $\beta \in \mathcal{S}$ and a, put

$$z_u = u \cdot \beta, \quad v_u = u \cdot T\beta, \quad r_u = \frac{z_u}{h(u)}, \quad c_u = \frac{z_u - v_u - h_a(u)}{h(u)}, \quad d_u = \frac{2z_u - v_u}{h(u)}.$$

Division by $B-1$ gives

$$\frac{u \cdot H_B\beta/(B-1) - h_a(u)}{h(u)} = r_u B + c_u + \frac{d_u}{B-1}. \tag{11.2}$$

The largest slope is μ. Among the normals with $r_u = \mu$, choose (c_u, d_u) lexicographically largest, denoted $(c_{\beta,a}, d_{\beta,a})$. Beyond a computable cutoff the least L for this pair is

$$\left\lceil \mu B + c_{\beta,a} + \frac{d_{\beta,a}}{B-1} \right\rceil.$$

For $B \equiv j \pmod{\rho}$ its eventual value is

$$\mu(B-j) + \lceil \mu j + c_{\beta,a} \rceil + \mathbf{1}_{\{\mu j + c_{\beta,a} \in \mathbb{Z},\ d_{\beta,a} > 0\}}.$$

The reciprocal term therefore matters when the affine part is integral. The intercept recipe is

$$b_{t,j} = \min_{\beta \in \mathcal{S},\ 0 \le a < t} \left\{ t\left(\lceil \mu j + c_{\beta,a} \rceil + \mathbf{1}_{\{\mu j + c_{\beta,a} \in \mathbb{Z},\ d_{\beta,a} > 0\}} - \mu j \right) + a \right\}. \tag{11.3}$$

For an effective cutoff, require for each winning pair (c, d)

$$\begin{aligned} (\mu - r_u)B &> |c - c_u| + |d - d_u| && (r_u < \mu), \\ (B-1)(c - c_u) &> |d - d_u| && (r_u = \mu,\ c_u < c), \\ \mu B &> |c| + |d|. \end{aligned}$$

If $c_u = c$, the inequality $d \ge d_u$ resolves the tie. Also require $|d|/(B-1) < \varepsilon_j$, where ε_j is the distance of the nonintegral number $\mu j + c$ to the nearest adjacent integer, or 1 if it is integral. Taking the maximum of these rational bounds and C_{red} proves the recipe without fitting sampled bases.

11.2 Orders fifteen and thirty

Corollary 11.2. *The formulas of Theorems 8.1 and 9.1 at $p = 5$ give the exact thresholds for order 15 when $B \ge 9$, and for order 30 when $B \equiv j \pmod 4$ and $B \ge (16, 13, 10, 19)_j$. The least eventual coefficient periods are 2 and 4, respectively. In particular $K_{15}(10) = 84$ and $K_{30}(10) = 91$.*

Proof. Apply Theorems 8.1 and 9.1 with $p = 5$, including their period assertions. □

The facet recipe provides a redundant check for these two orders. Its computational details are recorded in `docs/COMPUTATIONS.md` in the companion; its sufficient cutoffs do not restrict this corollary.

11.3 Orders one hundred five and two hundred ten

Theorem 11.3 (Thresholds beyond total unimodularity)**.** *For $B = 4n + j \ge 44$, $0 \le j < 4$,*

$$K_{105}(B) = 105n + (74, 86, 106, 127)_j. \tag{11.4}$$

For $B = 8n + j \ge 65$, $0 \le j < 8$,

$$K_{210}(B) = 210n + (71, 91, 127, 144, 161, 186, 197, 211)_j. \tag{11.5}$$

The least eventual coefficient periods are 4 and 8. The cutoffs are sufficient; minimality of the cutoffs is not claimed.

Use the $2 \times 4 \times 6$ basis of monomials $\eta_3^i \eta_5^j \eta_7^k$, where $\eta_3 = \zeta_{105}^{70}$, $\eta_5 = \zeta_{105}^{21}$ and $\eta_7 = \zeta_{105}^{15}$. Put $R = R_{105}$, and let T represent $\zeta_{105} + \zeta_{105}^{-1}$. For order 210 take $\xi = -\zeta_{105}$. The targets are, respectively,

$$((B^2+1)I - BT)\beta, \qquad ((B^2+1)I + BT)\beta.$$

Reduction and certification. For order 105, a failure of length at most $105n + c_j - 1$ has gauge at most

$$\frac{n+q_j}{4n+j-1}, \qquad (q_0,q_1,q_2,q_3) = (1,1,1,2).$$

The inequality $7(n+q_j) < 2(4n+j-1)$ holds for $n \geq (10,8,6,11)_j$. Corollary 7.3 therefore reduces every earlier failure to one of the 5670 signed rectangles. For order 210 the same argument gives

$$\|\beta\|_{Z_{210}} \leq \frac{n+1}{8n+j-1} < \frac{1}{7} \quad \Longleftrightarrow \quad n \geq 9-j.$$

This list is again exhaustive. The reduction excludes length $K-1$; it does not classify the fibre at K.

Put $(t,b,\epsilon) = (105,4,-1)$ or $(210,8,1)$, and $B = bn+j$. For any integer normal u, define

$$h = \sum_r \max((R_t^T u)_r, 0), \quad h_a = \sum_{r<a} \max((R_t^T u)_r, 0), \quad z = u \cdot \beta, \quad v = u \cdot T\beta.$$

Write $c_j - 1 = tL_0 + a_0$ and $e = L_0 h + h_{a_0}$. Every point in the preceding box must satisfy $u \cdot ((B^2+1)\beta + \epsilon BT\beta) \leq (B-1)((n+L_0)h + h_{a_0})$. The violation margin is

$$\begin{aligned} &(b^2 z - bh)n^2 \\ &\quad + (2bjz + \epsilon bv - be - (j-1)h)n \\ &\quad + (j^2+1)z + \epsilon jv - (j-1)e. \end{aligned} \tag{11.6}$$

This supporting inequality is valid for every u; a facet or an exhaustive facet list is unnecessary.

For each of the 5670 candidates in each branch, the order-105 certificate selects one of 699 integer normals; the order-210 certificate uses 478. In all 22 680 and 45 360 rows, respectively, the quadratic coefficient in (11.6) vanishes, the linear coefficient is nonnegative, and the constant is positive. The checkers reconstruct the candidates, actions and margins as integer identities. They exclude every candidate for all $n \geq 0$, hence on the domains of the exhaustive reduction.

For example, at order 105, $j=2$, $\beta=1$, and $u = e_{(0,0,0)}$, the target is $B^2+B+1 = 16n^2+20n+7$ and the capacity of the four positive entries is $4(4n+1)(n+1) = 16n^2+20n+4$: the margin is 3. At order 210, $j=7$, $\beta=1$, and $u = e_{(0,0,0)} - e_{(0,0,1)}$, we have $z=1$, $v=0$, $h=8$, and margin 2. The first coordinate alone would give $-8n-5$ and fail to separate. These examples do not replace checking the whole list.

For the upper bound at order 105 use the positive rectangles

j	S_3	S_5	S_7	sufficient n
0	$\{0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	3
1	$\{0,1\}$	$\{0,2,3\}$	$\{0,1\}$	2
2	$\{0\}$	$\{0\}$	$\{0\}$	1
3	$\{0\}$	$\{0,1\}$	$\{0\}$	1

For each branch the file gives 105 integer polynomial sums $s_r(n)$. Coefficient comparison verifies $Rs(n) = ((4n+j)^2+1)\beta - (4n+j)T\beta$. After substituting $n = N_j + m$, all coefficients of s_r and $\mathrm{cap}_r - s_r$ are nonnegative for the N_j in the table. These are admissible sums for every $n \geq N_j$.

For order 210 the certificate gives 105 integer folded polynomials $z_r(n)$ in the intervals (7.3):

j	β in ξ	sufficient n
0	ξ^{35}	3
1	ξ^{45}	3
2	ξ^{63}	3
3	ξ^{63}	2
4	ξ^{80}	2
5	ξ^{102}	1
6	$\xi^{83}+\xi^{113}$	0
7	1	0

The checker verifies $Rz(n) = ((8n+j)^2+1)\beta + (8n+j)T\beta$ and, after the indicated shifts, coefficientwise nonnegativity of $z_r - \ell_r$ and $u_r - z_r$. To recover the full period put $d'_r = (-1)^r z_r$, $s_r = \max(d'_r, 0)$ and $s_{r+105} = \max(-d'_r, 0)$. The folded bounds ensure the separate capacities. Lemma 3.1 realizes actual digits. In both orders $\beta \neq 0$ and $H_B\beta \in \Lambda \setminus \{0\}$.

The final domains are

t	least n in each branch	first B in each branch
105	$(10, 8, 6, 11)$	$(40, 33, 26, 47)$
210	$(9, 8, 7, 6, 5, 4, 3, 2)$	$(72, 65, 58, 51, 44, 37, 30, 23)$.

The last uncovered bases are 43 and 64, giving uniform cutoffs 44 and 65.

The common slope 105/4 forces $4 \mid M$ for any eventual coefficient period M, since $K(B+M) - K(B)$ is integral. The order-105 formula has period 4. For order 210, $K(8n+4) - K(8n) = 90 \neq 105$, excluding period 4 and every $M \equiv 4 \pmod 8$. Its least period is 8. □

11.4 Two distinctions between geometry and capacity

For $B \geq 65$, the formulas give

$$K_{210}(B) - K_{105}(B) = (-3, 5, 21, 17, -18, -5, -14, -21)_{B \bmod 8}. \tag{11.7}$$

Doubling can advance or delay the first failure even though the absolute minors and the first-gauge candidates are unchanged.

Let $K_{\mathrm{rad}}(B)$ be the first failure length at order 210 when $\beta = \xi^r$, $0 \leq r < 210$, is required.

Proposition 11.4 (Loss from restricting to roots)**.** *For $B = 8n+6$, $n \geq 0$, we have $K_{\mathrm{rad}}(B) = 210n + 204$. For $n \geq 3$ this exceeds $K_{210}(B)$ by 7. The optimal multiplier $\xi^{83} + \xi^{113}$ is nevertheless a unit.*

Proof. The certificate `root_residue6.json` excludes all 210 roots at length $210n + 203$, using 202 margins equal to 2, five equal to $8n+7$, and three equal to 1. This list is exhaustive by definition, without a shell reduction. A polynomial folded lift for $\beta = \xi^{114}$ attains length $210n + 204$ for every $n \geq 0$; its checker verifies 144 scalar identities and 210 box constraints. Realization follows the preceding proof. The gap 7 follows from branch $j = 6$ for $n \geq 3$. Finally, with $\eta = \xi^{30}$ of order 7, $(1+\eta)(-\eta - \eta^3 - \eta^5) = 1$, whence

$$(\xi^{83} + \xi^{113})(-\xi^7 - \xi^{67} - \xi^{157}) = 1.$$

The binomial is not a root of unity: under $\xi \mapsto e^{2\pi i/210}$ its modulus is $2\cos(\pi/7) > 1$. The difference concerns representation in the box, not a new ideal index. □

11.5 No real–integer gap at the certified thresholds

Define $K_t^{\mathrm{fib},\mathbb{R}}(B)$ by allowing real folded sums within the digit capacities, while requiring their image to remain a nonzero point of the same discrete lattice Λ. The lattice variable is not relaxed.

Corollary 11.5.

$$K_{105}^{\mathrm{fib},\mathbb{R}}(B) = K_{105}(B) \quad (B \geq 44), \qquad K_{210}^{\mathrm{fib},\mathbb{R}}(B) = K_{210}(B) \quad (B \geq 65).$$

Proof. The maximum-principle reduction also holds for real sums, while $\beta = H_B^{-1} R_t s$ remains in the discrete lattice. Thus the shell to exclude at length $K-1$ is unchanged. All lower-bound separators are supporting inequalities valid on the whole real box. No real lift exists at $K-1$, whereas the upper certificates give integer lifts at K. □

This does not make all fibres integral. The non-TU example in Section 7 uses a different box and does not imply a gap at these thresholds.

12 Reproducibility, dependencies and scope

12.1 Computational dependencies

Companion version 1.1.0 supplies the article's sources, certificates and verifiers in the project repository. Run `python -B scripts/verify.py` from its root; repeat under `python -O -B` to check independence from assertions. All arithmetic is exact. Unit tests mutate certificates and exercise boundary cases. `MANIFEST_SHA256.txt` fixes the distributed files; hash agreement alone is not a mathematical check.

The finite components have distinct roles:

ID	component	role in the argument
F1	`period_recipe.json`	Redundant facet recipes for 15 and 30.
F2	`threshold_105.json`	Essential separators and polynomial lifts for order 105.
F3	`threshold_210.json`	Essential separators and polynomial lifts for order 210.
F4	`root_residue6.json`	Essential root-restricted comparison at order 210.
F5	`core/three_p.py`	Diagnostic replay of the deductive $3p$ proof and digit witnesses.
F6	`core/six_p/`	Regression checks of transition, boundary and multiplicity formulas.
F7	`core/six_p_uniqueness.py`	Regression checks of the bounded uniqueness classification.
F8	`core/scalar_transfer.py`	Diagnostic scalar comparisons, reversal and period-transfer witnesses.

F2/F3 check the lower-bound separators and polynomial lifts specified in Section 11. Their proofs also require the exhaustive shell reduction, not merely valid separators for selected candidates. F4 checks all 210 roots and the matching upper witness. Panel definitions, execution counts and toolchain information are in `docs/COMPUTATIONS.md` and `REPRODUCTION_RECORD.md`.

For both $3p$ and $6p$, the prefix classification, shell reduction and noncritical bounds are proved in the text. The scalar $6p$ boundary proof does not import the F6/F7 certificates. Their replay is diagnostic, as are the finite F8 panels; none proves a conclusion for all primes or bases by sampling. F2–F4 retain their essential roles for the separately stated results.

The witness audit reconstructs actual digit words, verifies the two evaluations modulo the complete cyclotomic value, and independently checks a nonzero polynomial remainder. A folded vector is not itself a digit word. Lemma 8.2 supplies an $O(p)$ arithmetic construction of three-row potentials; explicit expansion into a word still takes time proportional to its length.

12.2 Scope and further questions

The general theorems hold for each fixed t. Their effective monoid construction is finite but is not advertised as a practical algorithm when t varies. The companion does not implement that construction for arbitrary t; its fixed-order certificates use the smaller geometric reductions. The period $\rho_t L_t$ is sufficient, and need not be the least period. The exact families and the non-TU cases illustrate stronger period information.

The first failure threshold, the first shell, the set of normalized algebraic multipliers, the folded vectors and the digit words are different objects. A classification of one does not automatically classify the others. In particular the multiplicity and uniqueness statements for $6p$ count normalized algebraic values; the uniqueness domain for $B \equiv 2 \pmod 4$ starts at 22, even though its threshold formula already starts at 10. No uniqueness assertion is made there for bases 10, 14 or 18. The non-palindromic threshold corollary does apply at those bases: it needs one primitive value, not exhaustion of the minimum shell. Fixed-value counts under reversal do not count palindromic words; the word counts in Corollary 10.4 are lower bounds. Nor is a full minimum-fibre classification claimed at orders 105 and 210.

Natural remaining questions include the smallest valid cutoffs, exact formulas for further prime products, and the relation between minor denominators and least coefficient periods. These are extensions of the same lattice–height–capacity problem, rather than prerequisites for the results proved here. The alphabet-height and collision interpretations likewise concern the same coefficient map. Passing from the full modulus to a proper divisor changes the problem: in base 4 the word `11` has value 5, a divisor of $\Phi_{10}(4) = 205$, whereas the full-modulus threshold is 14.

The eventual quasi-polynomial behavior and general witness selection have antecedents in Bogart–Goodrick–Woods [2]; the effective semilinear decomposition uses Chistikov–Haase [9]. The cyclotomic lattice, matroid and total-unimodularity inputs are attributed in the relevant proofs. The contribution developed here is the explicit slope and threshold analysis with its geometric reductions, uniform formulas, and fully specified finite components. The literature comparison is targeted and does not certify priority.

Use of computational and AI assistance. OpenAI Codex and, through ChatGPT in a web browser, GPT-6 Pro and Astra contributed substantially to mathematical exploration, proof development and checking, literature comparison, software and exact computations, critical review, drafting and editing. The author remains responsible. The programs check the finite components described above; these checks are not formal verification or independent human peer review.

A Order four at arbitrary height

Proposition A.1. *For $B \geq 2$, $H \geq 1$, and $q = B^2 + 1$,*

$$K_{4,H}(B) = \begin{cases} 4\lceil q/H \rceil - 3, & B \text{ even}, \\ \min\{4\lceil q/H \rceil - 3,\ 4\lceil q/(2H) \rceil - 2\}, & B \text{ odd}. \end{cases}$$

Proof. With $\zeta = i$, alternating folding has coordinates c_0, c_1. Polynomial divisibility means

$c_0 = c_1 = 0$. Since $B^{-1} \equiv -B \pmod q$, the evaluation tests are

$$c_0 + Bc_1 \equiv c_0 - Bc_1 \equiv 0 \pmod q. \tag{A.1}$$

To first attain absolute value $T > 0$ in a class requires $2h - 1$ positions of that class, where $h = \lceil T/H \rceil$: positive positions are more numerous. Class 0 reaches this capacity at length $4h - 3$; both classes reach it at length $4h - 2$. Every intermediate integer value is attainable.

If B is even, q is odd. Adding and subtracting (A.1) gives $q \mid c_0, c_1$. The first nonzero point can be $(q, 0)$, giving $4\lceil q/H \rceil - 3$. If B is odd, the lattice consists of $(q/2)(a, b)$ with a, b of equal parity. The even–even branch requires magnitude at least q in one coordinate; the odd–odd branch requires at least $q/2$ in both. The points $(q, 0)$ and $(q/2, q/2)$ attain the two first times. Taking their minimum is essential: if $H \geq q$, the constant polynomial q is already a failure of length 1. □

Proof of (1.7). Set $H = B - 1$. For $B = 2$, $\lceil q/(B-1) \rceil = 5$, giving 17. For even $B \geq 4$, $q/(B-1) = B+1+2/(B-1)$ has ceiling $B+2$, giving $4B+5$. For odd $B \geq 3$, $\lceil q/(2(B-1)) \rceil = (B+3)/2$ gives $2B + 4$ on the diagonal branch; the axial branch gives the larger $4B + 5$. □

B Order ten: a pentagon and fractional classes

B.1 An elementary separation criterion

Lemma B.1 (Translated intervals)**.** *Let $[L_i, U_i]$ be coordinate intervals and h_i a fixed representative. If every representative of the same element is $h_i + s$ for a common translation, any pair i, j with*

$$g = (L_i - h_i) - (U_j - h_j) > 0 \tag{B.1}$$

excludes feasibility even for real s.

Proof. Feasibility requires $s \geq L_i - h_i$ and $s \leq U_j - h_j$. □

The following arguments select separators structurally. No exclusion table is a premise, and n always denotes a nonnegative integer.

B.2 The ordinary branch: three support types

Let $Y = -\zeta$ be a primitive fifth root and $F(Y) = 1 + Y + \cdots + Y^4$. Set $z_r = (-1)^r c_r$. The algebraic element vanishes exactly when z is constant, and five-coordinate representatives differ by multiples of $\mathbf{1} = (1, 1, 1, 1, 1)$.

Lemma B.2 (Pentagonal oscillation)**.** *Multiplication by H on Y-coordinates is*

$$(\mathcal{H}_B v)_r = (B^2 + 1)v_r + B(v_{r-1} + v_{r+1}), \qquad r \pmod 5. \tag{B.2}$$

For every $v \in \mathbb{R}^5$,

$$\operatorname{osc}(\mathcal{H}_B v) \geq (B - 1)^2 \operatorname{osc}(v). \tag{B.3}$$

Proof. Substitute $\zeta = -Y$ in (2.3). Between a maximum and a minimum coordinate of v, the diagonal contributes $(B^2 + 1) \operatorname{osc}(v)$. The two neighbours subtract at most $2B \operatorname{osc}(v)$. □

Lemma B.3 (Three supports in marked boxes)**.** *For every $B \geq 2$, the transformed box of length $5B - 1$ for odd B, or $5B + 2$ for even B, contains no vector $\mathcal{H}_B v + s\mathbf{1}$ with real s and nonconstant $v \in \{0, 1\}^5$. Three support types suffice in the original coordinates of the box.*

Proof. Put $A = B^2 + 1$, $D = B - 1$. The transformed intervals are

$$L_r = -T - D\mathbf{1}_{E_L}(r), \qquad U_r = T + D\mathbf{1}_{E_U}(r), \tag{B.4}$$

with the following data, obtained from (3.2):

Base	T	E_U	E_L
odd	$D^2/2$	$\{0,2\}$	$\{1,3\}$
even	$BD/2$	$\{0\}$	$\{1\}$

Thus

$$U_j - L_i = 2T + D(\mathbf{1}_{E_U}(j) + \mathbf{1}_{E_L}(i)). \tag{B.5}$$

A high coordinate j and low coordinate i of $h = \mathcal{H}_B v$ give margin $g = h_j - h_i - (U_j - L_i)$ in Lemma B.1. Choose the smaller of the support and its complement, denoted S. Do not rotate the box. For $v = \mathbf{1}_S$, with p arbitrary modulo 5,

Type	S	high indices	low indices	difference
P1	$\{p\}$	$\{p\}$	$\{p+2, p+3\}$	A
P2	$\{p, p+1\}$	S	$\{p+3\}$	$A+B$
P3	$\{p, p+2\}$	S	$\{p+3, p+4\}$	$A-B$

For P3 the support entries equal A, their common neighbour equals $2B$, and the low entries equal B. We have $A > 2B$. For complements, $\mathcal{H}_B(\mathbf{1} - v) = (B+1)^2\mathbf{1} - \mathcal{H}_B v$ exchanges highs and lows without changing the difference. It does not assert that the box is invariant under complementation.

For P1 and odd B, capacity is at most $D^2 + 2D = B^2 - 1$, giving $g \geq 2$. For even B, each chosen high–low pair has cyclic distance two, also after complementation. It cannot be $(0, 1)$, the only pair activating both markers. Capacity is at most $BD + D = B^2 - 1$, again giving 2.

For P2, global capacity is at most $B^2 - 1$ for odd B and $BD + 2D = B^2 + B - 2$ for even B. The difference $A + B$ gives margins at least $B + 2$ and 3, in every position and for complements.

For P3 both index sets have size two. For even B, choose high $j \neq 0$ and low $i \neq 1$. Neither marker contributes, and $A - B = BD + 1$ gives margin 1. For odd B, it suffices to choose $j \notin \{0, 2\}$ or $i \notin \{1, 3\}$; capacity is then at most $D^2 + D = BD$. If no high index lies outside $\{0, 2\}$, the high set equals $\{0, 2\}$. It cannot arise from a complement, whose two high indices are adjacent. Hence $S = \{0, 2\}$, the low set is $\{3, 4\}$, and $i = 4$ works.

All margins are positive. The classification is exhaustive: a smaller support has one or two elements, and a pair is adjacent or nonadjacent. Each type has five positions and their complements; this count is not used in proving the inequalities. □

Proposition B.4 (Ordinary order-ten boundary)**.** *Suppose $B \not\equiv 4 \pmod 5$. No failure exists at length $5B - 1$ for odd $B \geq 5$, or at length $5B + 2$ for even $B \geq 6$.*

Proof. Proposition 2.5 gives $\delta = 1$, so $\alpha \in \Lambda$ has form $H\beta$ with $\beta \in \mathcal{A}_t$. Choose an integer representative v of β with $\min v = 0$. A failure means that v is nonconstant. The class digit counts are $(B, B, B, B, B - 1)$ at the odd boundary and $(B + 1, B + 1, B, B, B)$ at the even boundary. Equation (3.2) and the alternating signs give

$$\operatorname{osc}(z) \leq \begin{cases} (B+1)(B-1), & B \text{ odd}, \\ (B+2)(B-1), & B \text{ even}. \end{cases}$$

Lemma B.2 bounds $\operatorname{osc}(v)$ by $(B + 1)/(B - 1) < 2$ or $(B + 2)/(B - 1) < 2$ on the stated domains. Integrality forces a nonconstant binary vector, excluded by Lemma B.3. The box argument also holds at exceptional bases, but the reduction to integral classes needs $\delta = 1$. □

Bases 2 and 3 are handled in Proposition E.3.

B.3 The exceptional branch

Lemma B.5 (The fractional classes)**.** *If $B \equiv -1 \pmod 5$, put*

$$V(Y) = \sum_{r=0}^{4} rY^r, \qquad \beta_* = \frac{V(Y)}{5} = \frac{1}{Y-1}.$$

Then

$$H^{-1}\Lambda = \mathcal{A}_t + \mathbb{Z}\beta_*. \tag{B.6}$$

Proof. The identity $(Y-1)V = 4(Y^5-1)+5-F$ becomes $(Y-1)V = 5$ modulo F. The coordinates $(0, 1/5, 2/5, 3/5, 4/5)$ cannot become integral by a common translation, so β_* has order 5 modulo $\mathcal{A}_t$. With $S = (B+1)^2/5 \in \mathbb{Z}$,

$$\mathcal{H}_B(0, 1/5, 2/5, 3/5, 4/5) = (B, S, 2S, 3S, 4S - B).$$

The corresponding X-polynomial has coefficients $c_* = (B, -S, 2S, -3S, 4S-B)$ and satisfies

$$\frac{C_*(B)}{\Phi_{10}(B)} = \frac{4B(B+1)}{5}, \qquad \frac{B^4 C_*(B^{-1})}{\Phi_{10}(B)} = \frac{4(B+1)}{5}. \tag{B.7}$$

These quotients are integral, so $H\beta_* \in \Lambda$. The subgroup $\mathcal{A}_t + \mathbb{Z}\beta_*$ has index 5 over $\mathcal{A}_t$, equal to $[H^{-1}\Lambda : \mathcal{A}_t]$ by Proposition 2.3. The established inclusion is therefore equality. □

Lemma B.6 (A single endpoint identity)**.** *For $j \in \{1,2,3,4\}$ and $a \in \mathbb{Z}/5\mathbb{Z}$, put*

$$w_r = ((r-a)j \bmod 5), \qquad v_r = w_r/5, \tag{B.8}$$

using residues in $\{0,1,2,3,4\}$. Let b be the unique index with $w_b = 4$. For $h = \mathcal{H}_B v$,

$$h_a = B, \quad h_b = \frac{4(B^2+1)+3B}{5}, \quad h_b - h_a = \frac{4B^2 - 2B + 4}{5}. \tag{B.9}$$

Proof. The neighbours of a have residues $j, 5-j$, summing to 5. Those of b have residues $4-j, j-1$, summing to 3. Substitute in (B.2). The exact values suffice; a, b need not be extrema of h. □

Proposition B.7 (Exceptional boundary without a table)**.** *For $B = 5n+9$, $n \geq 0$, no failure exists at length $4B-5$.*

Proof. Put $D = B-1$, $T = (2n+3)D$. Class 0 has $4n+7$ digits and the others $4n+6$. After the sign change the box is

$$L_r = -T, \qquad U_r = T + D\mathbf{1}_{\{0\}}(r). \tag{B.10}$$

Write $\alpha = H\beta$ using (B.6) and choose a rational representative v. Lemma B.2 gives

$$\operatorname{osc}(v) \leq \frac{2T+D}{D^2} = \frac{4B-1}{5(B-1)} < 1.$$

In the integral class, normalization to minimum zero therefore gives zero. In each other class the fractional parts, up to a common constant, are $jr/5$ modulo 1 for $j = 1,2,3,4$, and are distinct. Subtracting the minimum at index a forces precisely (B.8): adding a positive or negative integer to any normalized coordinate violates the range. This proves completeness, not merely feasibility of twenty examples.

The entries h_r are integral, since $w_{r-1}+w_{r+1} \equiv 2w_r \pmod 5$ and $B \equiv -1 \pmod 5$. Every representative is $h + s\mathbf{1}$. The endpoint identity gives

$$\begin{aligned} h_b - h_a - (U_b - L_a) &= 8n + 14 - (5n+8)\mathbf{1}_{\{0\}}(b) \\ &= \begin{cases} 8n+14, & b \neq 0, \\ 3n+6, & b = 0. \end{cases} \end{aligned} \tag{B.11}$$

Both are positive, so Lemma B.1 excludes every step and position without rotating the box. □

For each j exactly one position has $b = 0$, explaining the multiplicities 16 and 4 of the two margins; these counts are not proof premises.

B.4 The entire first exceptional fibre

Theorem B.8 (One nonzero class at the threshold)**.** *Let $B = 5n+9$, $n \geq 0$, and $K = 4B-4$. The folded failure vectors at length K are precisely*

$$c = c^{(0)} + \ell(1,-1,1,-1,1), \qquad \ell \in \mathbb{Z}, \quad 0 \leq \ell \leq 2n+2, \tag{B.12}$$

where

$$c^{(0)} = (10n^2+34n+30,\ 10n^2+36n+32,\ -5n^2-21n-21,\ n+1,\ 5n^2+19n+19).$$

Thus there are $2n+3 = (2B-3)/5$ folded vectors, all representing the same nonzero class modulo Φ_{10}. This does not count digit words.

Proof. At length K, only the lower marker at coordinate 1 is added:

$$L_r = -T - D\mathbf{1}_{\{1\}}(r), \qquad U_r = T + D\mathbf{1}_{\{0\}}(r). \tag{B.13}$$

The representative of β has oscillation at most

$$\frac{2T+2D}{D^2} = \frac{4n+8}{5n+8} \leq 1.$$

For $n \geq 1$, no nonzero integral class remains. For $n = 0$ ($B = 9$), it would have a nonconstant binary representative; the box at $K = 32$ lies in the box at $5B-1 = 44$, so Lemma B.3 excludes it without needing $\delta = 1$. Distinct fractional parts cannot differ by exactly 1. Thus (B.8) remains exhaustive even at $B = 9$.

The endpoint margin is $8n + 14 - D(\mathbf{1}_{\{0\}}(b) + \mathbf{1}_{\{1\}}(a))$. With zero or one marker it is positive, forcing $a = 1$, $b = 0$. Then $(b-a)j \equiv 4 \pmod 5$ forces $j = 1$. The unique normalized representative is $v = (4,0,1,2,3)/5$. With $S = (B+1)^2/5 = 5(n+2)^2$,

$$h = \mathcal{H}_B v = (4S - B,\ B,\ S,\ 2S,\ 3S).$$

Since $S - B - D = 5n^2 + 10n + 3 > 0$, coordinate 1 imposes the greatest lower bound for s, and coordinate 0 the least upper bound: for $r \neq 1$, $h_r \geq B + D$, and for $r \neq 0$, $h_r \leq h_0 - D$. Hence feasibility is exactly

$$s_- = -T - D - B \leq s \leq s_+ = T + D - 4S + B, \qquad s_+ - s_- = 2n+2. \tag{B.14}$$

Both endpoints and s are integral. Set $s = s_- + \ell$ and return to $c_r = (-1)^r(h_r + s)$ to obtain (B.12). The element $H\beta$ is nonzero since H is invertible and v nonconstant, and it lies in Λ by Lemma B.5. Lemma 3.2 realizes every vector with digits. Proposition B.7 and padding establish minimality.

The difference between two folded polynomials is $(\ell - \ell')\Phi_{10}(X)$, so all have the same nonzero remainder:

$$\begin{aligned}(5n^2+15n+11) + (15n^2+55n+51)X \\ - (10n^2+40n+40)X^2 + (5n^2+20n+20)X^3.\end{aligned} \tag{B.15}$$

□

The cyclotomic vectors are $z = s\mathbf{1}$ with $-T \le s \le T$, giving $2T+1$ vectors at both $K-1$ and K. At K precisely the $2n+3$ failures above are added. This does not count words realizing a vector.

B.5 Base four without box enumeration

Proposition B.9 (Base four and folded uniqueness)**.** *We have $K_{10}(4) = 14$. The unique noncyclotomic, doubly divisible folded vector at length 14 is $(6,1,1,6,0)$. The 49 000-state enumeration at length 13 is only a check.*

Proof. Here $D = 3$, and the transformed boxes are

k	L	U
13	$(-3,-6,-3,-3,-3)$	$(6,3,6,3,3)$
14	$(-3,-6,-3,-6,-3)$	$(6,3,6,3,3)$.

Both have global oscillation at most 12, so $\operatorname{osc}(\mathcal{H}_4 v) \ge 9\operatorname{osc}(v)$ gives $\operatorname{osc}(v) \le 4/3$. The normalized integral class is binary and is excluded by Lemma B.3, since both boxes lie in the box at 22.

For a fractional class normalize the minimum to zero at a and put $w = 5v$. Its five residues occur once each as in (B.8), with $0 \le w_r \le 6$. There are only two types: the ramp with values $0,1,2,3,4$, or the ramp with its value 1 replaced by 6. This reduction does not enumerate 7^5 tuples. For the second type let d be the position of 6. If $j = 1$ or 4, its neighbours are 0 and 2, while the neighbours of a sum to 10. Thus $h_d = 22$, $h_a = 8$, giving difference $14 > 12$. If $j = 2$ or 3, the respective neighbour sums are 7 and 5, giving $h_d = 26$, $h_a = 4$, and difference $22 > 12$. This type is excluded.

For the ramp, Lemma B.6 gives $h_a = 4$, $h_b = 16$. A difference of 12 requires $L_a = -6$, $U_b = 6$, forcing $s = -10$. At length 13 this leaves $a = 1$, $b \in \{0,2\}$; at 14, $a = 3$ is added. Three of the four possible pairs fail directly:

(a,b)	j	coordinate of $h - 10\mathbf{1}$ outside the box
$(1,0)$	1	$z_4 = 5 > 3$
$(1,2)$	4	$z_3 = 5 > 3$
$(3,2)$	1	$z_1 = 5 > 3$

The remaining pair $(3,0)$ forces $j = 2$ and gives

$$w = (4,1,3,0,2), \quad h = (16,9,11,4,10), \quad z = (6,-1,1,-6,0).$$

It lies in the box at 14 but not at 13. The folded vector is $c = (6,1,1,6,0)$. For $C(X) = 6 + X + X^2 + 6X^3$,

$$C(4) = 410 = 2 \cdot 205, \qquad 4^4 C(1/4) = 1640 = 8 \cdot 205, \qquad 205 = \Phi_{10}(4).$$

The remainder is nonzero since $0 \ne C$ has degree less than 4. Digit realization gives length 14 and padding excludes all smaller lengths. The same classification proves uniqueness. □

The exceptional family for $B \geq 9$ cannot be blindly extended: at the hypothetical boundary $4B - 5 = 11$ the oscillation bound is 1, not strictly less than 1; at lengths 13 and 14 it is $4/3$, requiring the second ramp type to be excluded separately. The accepted ramp has step 2, not step 1. No argument assuming $n \geq 0$ is used at $n = -1$.

C Order twelve: two coupled triangles

Proposition 2.5 gives $\delta = 1$ in every base. Thus $\alpha \in \Lambda$ has form $H\beta$, $\beta \in \mathcal{A}_t$. For an integer representative $u \in \mathbb{Z}^6$ put

$$y = (u_0, -u_2, u_4), \qquad z = (u_1, -u_3, u_5).$$

By (3.3), the represented element vanishes exactly when both triangles are constant. Independent integer translations normalize $\min y = \min z = 0$.

Lemma C.1 (Triangular multiplication and oscillation)**.** *Multiplication by H becomes, with indices modulo three,*

$$y'_r = (B^2 + 1)y_r + B(z_{r-1} - z_r), \tag{C.1}$$
$$z'_r = (B^2 + 1)z_r + B(y_{r+1} - y_r). \tag{C.2}$$

For $M = \max\{\operatorname{osc}(y), \operatorname{osc}(z)\}$,

$$\max\{\operatorname{osc}(y'), \operatorname{osc}(z')\} \geq (B - 1)^2 M. \tag{C.3}$$

Proof. Multiply $\sum u_r X^r$ by $B^2 + 1 - B(X + X^{-1})$, using $X^6 = -1$. The first original coordinate is $(B^2 + 1)u_0 - Bu_1 + Bu_5 = y'_0$. Also $y'_2 = (B^2 + 1)u_4 - Bu_3 - Bu_5$ and $z'_2 = (B^2 + 1)u_5 + Bu_0 - Bu_4$. The same index shifts give the remaining coordinates, including the endpoint signs. In the triangle attaining M, choose a maximum and a minimum. The diagonal difference is $(B^2 + 1)M$ and the coupling subtracts at most $2BM$. □

Lemma C.2 (Four binary regimes and markers)**.** *Let $y, z \in \{0, 1\}^3$ have minimum zero and not both vanish. For every $B \geq 2$, their images in* (C.1)–(C.2)*, even after independent real translations, lie outside the transformed box at length $6B - 1$ for odd B, or $6B + 1$ for even B. Four oscillation regimes suffice.*

Proof. Put $A = B^2 + 1$, $D = B - 1$ and use (B.4) separately in the two triangles. The exact boxes are

Base	T	E_U^y	E_L^y	E_U^z	E_L^z
odd	$D^2/2$	$\{0, 2\}$	$\{1\}$	$\{0\}$	$\{1\}$
even	$BD/2$	$\{0\}$	$\varnothing$	$\varnothing$	$\varnothing$

Both permitted oscillations are at most $B^2 - 1$. For even B the second triangle has the stronger bound BD. If one triangle vanishes and the other is nonconstant, the latter's image has oscillation $A > B^2 - 1$, with margin 2.

Otherwise, modulo their separate constants,

$$y = \varepsilon e_p, \quad z = \eta e_q, \qquad \varepsilon, \eta \in \{1, -1\}, \quad p, q \in \mathbb{Z}/3\mathbb{Z}.$$

A singleton has positive sign and its complement negative sign. Adding a constant to y adds A times that constant to y' and does not change z'; likewise with y, z exchanged. For $d = q - p$ and $\sigma = \varepsilon\eta$,

$$y' = A\varepsilon e_p + B\eta(e_{q+1} - e_q), \qquad z' = A\eta e_q + B\varepsilon(e_{p-1} - e_p).$$

Since $A > 2B$, both oscillations are

d	$\sigma = 1$	$\sigma = -1$
0	$A - B$	$A + 2B$
1	$A + B$	$A + B$
2	$A + 2B$	$A - B$

This follows by substitution in three basis vectors with arbitrary p, not by enumerating 36 pairs; simultaneous sign reversal preserves oscillation. The regimes $A + B, A + 2B$ exceed $B^2 - 1$ by $B + 2, 2B + 2$.

In the remaining regime $A - B = BD + 1$, maximum and minimum indices are

Type	y	z	(j_y, i_y)	(j_z, i_z)
I+	e_p	e_p	$(p, p-1)$	$(p, p+1)$
I−	$-e_p$	$-e_p$	$(p-1, p)$	$(p+1, p)$
II+	e_p	$-e_{p-1}$	$(p, p+1)$	$(p+1, p-1)$
II−	$-e_p$	e_{p-1}	$(p+1, p)$	$(p-1, p+1)$

Omitted constants affect neither differences nor extremal indices. For even B, use the second triangle, whose capacity is $2T$: the margin is 1. For odd B, in types I+, II+, II− the minimum indices differ, so at least one is not the sole lower marker 1. Choose that triangle; even with a marked maximum, capacity is $2T + D = BD$, leaving margin 1. In type I− both minima equal p. The same argument works if $p \neq 1$. If $p = 1$, the second triangle has maximum at 2, outside $E_U^z = \{0\}$, so again at most one marker contributes.

This exhausts all vectors. As a count check, there are 12 cases with one active triangle. Of the other 36, each regime $A - B, A + B, A + 2B$ occurs 12 times, from three choices of p, two simultaneous signs and the table rows. □

Proposition C.3 (Order-twelve boundary)**.** *For $B \geq 4$ no failure exists at length $6B - 1$ for odd B, or $6B + 1$ for even B.*

Proof. The six digit counts are $(B, B, B, B, B, B - 1)$ and $(B + 1, B, B, B, B, B)$, respectively. In coordinates $(c_0, -c_2, c_4)$ and $(c_1, -c_3, c_5)$, (3.2) bounds each oscillation by B^2-1. Lemma C.1 gives $M \leq (B + 1)/(B - 1) < 2$. The normalized integer triangles are therefore binary, neither is $(1, 1, 1)$, and a failure cannot have both zero. Lemma C.2 excludes them. □

The $(2^3 - 1)^2 - 1 = 48$ pairs per parity are a redundant check. Bases 2 and 3 require the inverse estimates in Proposition E.3; a binary lemma alone does not justify omitting the height reduction.

D Explicit families at the threshold

Put $D = B - 1$, $A = B^2 + 1$, $m = t/2$, and use the original folded coordinates c_r of (3.1). A single level T describes four ordinary families, including bases 2 and 3, without electronic tables.

Proposition D.1 (Four ordinary families and the exceptional branch)**.** *For odd $B \geq 3$ put $T = D^2/2$. The following vectors are realizable by failures at the stated lengths:*

$$c^{10}_{\text{odd}} = (T + 1, T + D, T - D, T + D, T + 1), \qquad K = 5B, \tag{D.1}$$

$$c^{12}_{\text{odd}} = (T + 1, T - D, T + D, T + D, T - D, T + 1), \qquad K = 6B. \tag{D.2}$$

For even $B \geq 2$ put $T = BD/2$. The same holds for

$$c^{10}_{\text{even}} = (T + 1, T - B, T + 1, T, -T), \qquad K = 5B + 3, \tag{D.3}$$

$$c^{12}_{\text{even}} = (T + 1, T + 1, T - B, T, -T, B - T), \qquad K = 6B + 2. \tag{D.4}$$

For $B = 5n + 9$, $n \geq 0$, the vector $c^{(0)}$ of Theorem B.8 gives a failure at length $4B - 4$. The ordinary order-ten families still give failures in this branch, but are not minimal.

Proof. The numbers T are integral on their respective parity domains. For odd B, every class has B digits and interval $[-T, T + D]$. Since $D \geq 2$, $T - D = D(D - 2)/2 \geq 0$ and $T + 1 \leq T + D$, so all coordinates are admissible. For even B, the first three classes at order 10 and first two at order 12 have $B + 1$ digits and interval $[-T, T + D]$. The others have B digits and interval $[-T, T]$. Here $T + 1 \leq T + D$ and $B \leq BD = 2T$, implying $-T \leq T - B \leq T$ and $-T \leq B - T \leq T$, including $B = 2$.

For double divisibility and nonvanishing, in the pentagon

$$\mathcal{H}_B(1, 0, 1, 0, 1) = (A + B, 2B, A, 2B, A + B),$$
$$\mathcal{H}_B(1, 0, 1, 0, 0) = (A, 2B, A, B, B).$$

Adding constants $T - B^2 - B$ and $T - B^2$ respectively gives the transformed vectors of (D.1) and (D.3). Constants represent zero and the prototypes are nonconstant.

At order 12 use the two triangles of Section C. For $y = z = (1, 0, 1)$ the images are $(A, B, A - B)$ and $(A - B, B, A)$. Adding $T - B^2$ to both yields (D.2). For $y = z = (1, 0, 0)$ the images are $(A - B, B, 0)$ and $(A - B, 0, B)$; adding $-T$ gives (D.4). These prototypes are again nonzero modulo constants.

Each constructed class is therefore H times a nonzero integral class. Proposition 2.3 gives $H\mathcal{A}_t \subseteq \Lambda$, and H is invertible in the cyclotomic field. Thus both evaluations vanish while the class does not. Lemma 3.2 realizes digits and preserves these properties. The exceptional branch follows from Theorem B.8. □

These formulas agree with the earlier parametrizations $B = 2n + 5$, $2n + 6$ or $2n + 4$, while also covering their omitted small bases. No certificate assuming $n \geq 0$ is evaluated at negative n. Bases 2 and 3 give the four vectors of Proposition E.4; the exceptional base 4 is treated in Proposition B.9.

Corollary D.2 (One coordinate exceeds its bound by one)**.** *Each ordinary folded vector above violates the box at $K - 1$ in exactly one coordinate r_*, exceeding its upper bound by 1. Passing to K increases just that bound by $B - 1$.*

Order	*Parity*	$K - 1$	r_*	*upper bound at* $K - 1$
10	*odd*	$5B - 1$	*4*	$(B - 1)^2/2$
10	*even*	$5B + 2$	*2*	$B(B - 1)/2$
12	*odd*	$6B - 1$	*5*	$(B - 1)^2/2$
12	*even*	$6B + 1$	*1*	$B(B - 1)/2$

Proof. The new digit has index $K - 1$ in the indicated class, whose digit count changes from even to odd. It adds a positive folded position. Equation (3.2) increases only that upper bound by $B - 1$. In every row the displayed family has $c_{r_*} = T + 1$. All other unchanged bounds hold by Proposition D.1; the newly opened coordinate has slack $B - 2 \geq 0$. □

This explains contact for these four witnesses, not a classification of all minimal witnesses. A one-unit excess alone would not exclude other representatives or lattice points: minimality still needs the structural lower bounds and Section E. At exceptional bases the order-ten ordinary families give upper bounds but not the minimum; use Theorem B.8 and Proposition B.9. Once minimality is established, the digit realizations at the true thresholds have nonzero endpoints.

E Four residual boundaries without enumeration

The direct oscillation bounds omit bases 2 and 3. Inverse identities recover the lost precision and settle these boundaries without enumerating boxes.

E.1 Inverse identities and optimal constants

Lemma E.1 (Inversion on differences)**.** *Let $B \geq 2$ and $A = B^2 + 1$. In the pentagon, with $h = \mathcal{H}_B v$ and $q_{10} = \Phi_{10}(B)$, indices are modulo five and*

$$q_{10}(v_r - v_{r+1}) = A(h_r - h_{r+1}) + B(h_{r+2} - h_{r+4}), \tag{E.1}$$
$$q_{10}(v_r - v_{r+2}) = (A - B)(h_r - h_{r+2}) + B(h_{r+3} - h_{r+4}). \tag{E.2}$$

For the triangles, put $a = y'$, $b = z'$ in (C.1)–(C.2) *and $q_{12} = \Phi_{12}(B)$. With indices modulo three,*

$$q_{12}(y_r - y_{r+1}) = A(a_r - a_{r+1}) + B(2b_r - b_{r-1} - b_{r+1}), \tag{E.3}$$
$$q_{12}(z_r - z_{r+1}) = A(b_r - b_{r+1}) + B(a_r + a_{r-1} - 2a_{r+1}). \tag{E.4}$$

These identities are invariant under adding a constant to h, or independent constants to a, b.

Proof. Substitute the multiplication formulas of Lemmas B.2 and C.1. In the first identity the coefficients of v_r, v_{r+1} are $A^2 - AB - B^2 = q_{10}$ and $-q_{10}$, with the other three zero. The second has $q_{10}, -q_{10}$ at positions $r, r+2$. In the triangular identities the other triangle cancels, leaving $A^2 - 3B^2 = q_{12}$. Coefficients in each block sum to zero, proving invariance under constants. □

Proposition E.2 (Exact inverse oscillation norms)**.** *For real vectors in these coordinates,*

$$\operatorname{osc}(v) \leq \frac{B^2 + B + 1}{\Phi_{10}(B)} \operatorname{osc}(h), \tag{E.5}$$
$$\operatorname{osc}(y) \leq \frac{A \operatorname{osc}(a) + 2B \operatorname{osc}(b)}{\Phi_{12}(B)}, \qquad \operatorname{osc}(z) \leq \frac{A \operatorname{osc}(b) + 2B \operatorname{osc}(a)}{\Phi_{12}(B)}. \tag{E.6}$$

In particular, for $M = \max\{\operatorname{osc}(y), \operatorname{osc}(z)\}$ and $M' = \max\{\operatorname{osc}(a), \operatorname{osc}(b)\}$,

$$M \leq \frac{(B+1)^2}{\Phi_{12}(B)} M'. \tag{E.7}$$

The constants in (E.5) *and* (E.7) *are optimal on the respective real spaces modulo constants.*

Proof. Pentagon vertices have distance one or two. Equation (E.1) bounds a difference by $(A + B)\operatorname{osc}(h)/q_{10}$; (E.2) gives the smaller $A \operatorname{osc}(h)/q_{10}$, since $A - B > 0$. Every distinct triangular pair is consecutive in one orientation, and $|2b_r - b_{r-1} - b_{r+1}| \leq 2 \operatorname{osc}(b)$. This gives the mixed inequalities and then (E.7).

For optimality both real multiplication matrices are invertible: their diagonal $A > 2B$ exceeds the sum of absolute off-diagonal entries. A maximum absolute coordinate of a kernel vector would give $A|v_r| \leq 2B|v_r|$, forcing zero. We may therefore prescribe the image. In the pentagon take $h = (1, 0, 1, 0, 0)$: at $r = 0$ the identity gives $v_0 - v_1 = (A + B)/q_{10}$ and $\operatorname{osc}(h) = 1$. In the triangles take $a = b = (1, 0, 0)$: then $y_0 - y_1 = (A + 2B)/q_{12}$ and $M' = 1$. The upper bounds already proved show equality in both norms. □

The improved direct constants satisfy

$$\frac{\Phi_{10}(B)}{B^2+B+1} = (B-1)^2 + \frac{B^2}{B^2+B+1},$$
$$\frac{\Phi_{12}(B)}{(B+1)^2} = (B-1)^2 + \frac{B^2}{(B+1)^2}.$$

The positive terms matter at the integral boundary. Optimality concerns arbitrary real vectors, not the digit box or only the integer lattice; the boundary arguments still use integer differences and, in one case, markers.

E.2 Four arithmetic exclusions

Proposition E.3 (Residual boundaries)**.** *There are no failures in the boxes*

$$(t, B, k) = (10, 2, 12),\ (10, 3, 14),\ (12, 2, 13),\ (12, 3, 17).$$

Only inverse identities and the structural binary lemmas are needed.

Proof. In all four cases $\delta = 1$, so a lattice element is $H\beta$, $\beta \in \mathcal{A}_t$. Choose integer representatives with minimum zero in each block. Their images differ from transformed digit vectors only by block constants. The inverse identities therefore apply, with integer differences on the left. Oscillation less than 2 reduces to binary representatives, excluded by Lemmas B.3 and C.2, unless they represent zero.

For order 10, base 3, the transformed box at $k = 14$ is

$$[-2, 4] \times [-4, 2] \times [-2, 4] \times [-4, 2] \times [-2, 2].$$

Its oscillation is at most 8. Since $q_{10} = 61$ and $A + B = 13$, (E.5) gives $\operatorname{osc}(v) \le 104/61 < 2$.

For order 12, base 2, $k = 13$, the box is

$$a \in [-1, 2] \times [-1, 1]^2, \qquad b \in [-1, 1]^3.$$

Thus $\operatorname{osc}(a) \le 3$, $\operatorname{osc}(b) \le 2$, and with $A = 5$, $q_{12} = 13$,

$$\operatorname{osc}(y) \le \frac{5 \cdot 3 + 4 \cdot 2}{13} = \frac{23}{13} < 2, \qquad \operatorname{osc}(z) \le \frac{5 \cdot 2 + 4 \cdot 3}{13} = \frac{22}{13} < 2.$$

For base 3 and $k = 17$, both transformed oscillations are at most 8. Now $q_{12} = 73$, $A+2B = 16$, so $M \le 128/73 < 2$.

For order 10, base 2, $k = 12$, the transformed box is

$$z \in [-1, 2] \times [-2, 1] \times [-1, 1]^3. \tag{E.8}$$

Relative to $[-1, 1]^5$, only the upper bound of coordinate 0 and the lower bound of coordinate 1 are extended by 1. Hence

$$\max \sum_i \lambda_i z_i = \sum_i |\lambda_i| + \max(\lambda_0, 0) + \max(-\lambda_1, 0). \tag{E.9}$$

An oriented adjacent difference in (E.1) has coefficients $5, -5, 2, 0, -2$ cyclically, or their negatives, with absolute sum 14. The two marker increments total 10 only when $+5$ is at 0 and -5 at 1, namely for $v_0 - v_1$. Otherwise they total at most $5 + 2 = 7$. Since $q_{10} = 11$, every other oriented adjacent difference is at most $21/11 < 2$, while $v_0 - v_1 \le 24/11 < 3$. For a nonadjacent pair the coefficients are $3, 0, -3, 2, -2$ or their negatives: absolute sum 10, with marker increments at most $3 + 3$. Such differences are at most $16/11 < 2$.

All integer differences are therefore at most 1, except possibly $v_0 - v_1 = 2$. Without this exception the binary lemma applies. With it, minimum-zero normalization forces all remaining values to 1:

$$v = (2, 0, 1, 1, 1), \qquad \mathcal{H}_2 v = (12, 6, 7, 9, 11).$$

The image difference between coordinates 0 and 1 is 6, but (E.8) allows only $2 - (-2) = 4$. No common translation repairs it. □

E.3 Safe counts, witnesses and assembly

Proposition E.4 (Five boundaries and witnesses)**.** *In each row below, the doubly divisible integer folded vectors at length $K - 1$ are exactly the cyclotomic vectors, with the stated counts. The displayed word at length K is a failure.*

t	B	K	*states at* $K-1$	*doubly divisible*	*word at* K
10	*2*	*13*	*432*	*3*	*1011001001101*
10	*3*	*15*	*12 005*	*5*	*120210020022222*
10	*4*	*14*	*49 000*	*7*	*30033022033333*
12	*2*	*14*	*972*	*9*	*11010100101011*
12	*3*	*18*	*84 035*	*25*	*102201020020222222*

Proof. Propositions E.3 and B.9 exclude all failures in the preceding boxes. For order 10 the cyclotomic vectors are $(a, -a, a, -a, a)$ with $-(B-1) \le a \le B-1$, giving 3,5,7. For order 12 they are $(a, b, -a, -b, a, b)$ with independent $a, b \in [-(B-1), B-1]$, giving 9,25. Multiplying the integer interval lengths in (3.2) gives the state counts.

The words have respective folded vectors

$$(2, -1, 2, 1, -1), \quad (3, 4, 0, 4, 3), \quad (6, 1, 1, 6, 0),$$

$$(2, 2, -1, 1, -1, 1), \qquad (3, 0, 4, 4, 0, 3).$$

They lie in the boxes at K, and direct folding gives an explicit realization. For the folded polynomial C, all necessary arithmetic identities are below; remainders use increasing coefficients in $1, X, X^2, X^3$.

t	B	q	$C(B)/q$	$B^{m-1}C(1/B)/q$	remainder
10	2	11	0	3	$(3, -2, 3, 0)$
10	3	61	6	6	$(0, 7, -3, 7)$
10	4	205	2	8	$(6, 1, 1, 6)$
12	2	13	2	7	$(3, 1, -2, 2)$
12	3	73	12	12	$(3, -3, 4, 7)$

Integer substitution and monic division verify the table without search. The quotients are integral and the remainders nonzero. Lemma 3.2 transfers both properties to the words. The zero folded value $C(2) = 0$ in the first row does not mean that the polynomial or its displayed remainder vanishes. □

Assembly of Theorem 1.5. Appendix A proves (1.7). Propositions B.4, B.7 and C.3 exclude the boundaries $K - 1$ on their domains. Proposition B.9 treats base 4 and Proposition E.3 covers the four remaining cases. No table of exclusions or box enumeration is needed. Proposition D.1 gives failures at K for all ordinary bases, including 2 and 3. Theorem B.8 and Proposition B.9 cover the exceptional branch. The numerical words above are examples rather than substitutes for the proved families. Lemma 2.1 extends exclusion at the boundary to every smaller length. These are therefore first failure lengths. □

Corollary E.5 (Decimal specialization)**.** $\Phi_{10}(10) = 9091$, $\Phi_{12}(10) = 9901$, $K_{10}(10) = 53$, *and* $K_{12}(10) = 62$. *The safe bounds are 52 and 61 digits, with the full modulus and with palindromes allowed.*

Proof. Substitute $B = 10$ in (1.8)–(1.9). □

F Scalar boundary classification for the order-$6p$ proof

This appendix completes Lemma 9.5. The six relation symbols and their four defining tests are in Section 9.4. The tables enumerate all rows and the common parameters allowed by Lemma 9.4. They are evaluations of the displayed marker rule, not imported computational certificates.

F.1 Residue zero: a short boundary calculation

Let $d = d_p - \varepsilon$. The columns $d, \dots, p-1$ have three visits. By (9.11), their internal edges are equalities, sometimes with the common value fixed. Write their common bit as x. There are at most four exceptional initial bits.

The next table lists the boundary relations on $(f_0, f_1), \dots, (f_{d-1}, x)$, followed after the semicolon by the wrap relation (x, f_0). It follows directly from (9.10) and (9.9); it is valid for all p in the indicated residue, including $5, 7$.

p mod 6	i	q	lower boundary	upper boundary
5	0	1	O; U	E O; U
5	0	2	U; Z	A E; Z
5	1	1	E; O	E D; O
5	1	2	D; E	A Z; E
5	2	1	E; D	E U; D
5	2	2	Z; E	Z E; E
1	0	1	A E D; E	A E A E; E
1	0	2	E Z Z; U	E Z E D; U
1	1	1	A E O; E	A E O O; E
1	1	2	E A E; Z	E A E U; Z
1	2	1	O E U; O	O E A E; O
1	2	2	E A E; D	E A E Z; D

Only four bulk fixed-bit facts are needed in reading the table:

p mod 6	i	q	edge j	consequence
5	2	1	2	$x = 1$
5	1	2	1 (lower)	$x = 0$
1	0	1	4	$x = 1$
1	2	2	3 (lower)	$x = 0$.

Each is a bulk edge even at the smallest admissible prime; its symbol is O or Z by (9.11).

For $p \equiv 5, q = 1$, the first two row choices force every bit to 1. The third has $x = 1$: the lower boundary forces $f_0 = x$, whereas the upper permits $f_0 = f_1 \le x$. Its only nonconstant word therefore has zeros exactly at $0, 1$. For $p \equiv 1, q = 1$, the second and third rows force all bits to 1. In row zero, $x = 1$; the lower boundary forces $f_1 = f_2 \ge x$, while the upper leaves $f_1 = f_2$ free and fixes $f_0 = f_3 = x$. Its only nonconstant word has zeros exactly at $1, 2$.

For $q = 2$, all rows force the zero word. For example, the upper boundary in residue one, row zero, fixes $f_0 = f_1 = f_2 = f_3 = 0$ and requires $f_3 \ge x$, so $x = 0$. Row one fixes $x = f_0 = f_1 = 0$ and $f_2 = f_3 \le x$; row two fixes $f_2 = f_3 = x = 0$ and $x \ge f_0 = f_1$. The lower

cases follow from the displayed fixed bits and equalities. In residue five the Z or wrap relation, together with the indicated bulk fixed bit in the one necessary case, gives the same conclusion.

Thus no lower word is nonconstant, and precisely the support stated in Lemma 9.5 survives at the upper boundary.

F.2 Residue two: one extra position

The preceding offset is exactly $6p$: all markers vanish and $h = 0$. For $q = 0$ or 1, every edge is E, excluding a nonconstant word. At the upper offset only $P_{0,0}$ is 1. Every edge except the two touching column zero is E. Their symbols are

q	i	(f_0, f_1)	(f_{p-1}, f_0)
0	0	E	E
0	1	U	E
0	2	E	D
1	0	D	U
1	1, 2	E	E.

A cycle of equalities with at most one one-way inequality is constant. Only $q = 1, i = 0$ has two relaxed edges. Writing $x = f_1 = \cdots = f_{p-1}$, both require $x \le f_0$. Its sole nonconstant solution is $f_0 = 1, x = 0$, giving $S = \{0\}$.

F.3 Residue three: fixed equality components

Here $h = 1$. The lower prefix has two visits per column; the upper adds a third visit at column zero only. On the interior edges $j = 1, \ldots, p-2$, the two-visit patterns in (9.11) apply.

For $q = 1$, the interior is entirely E and the two boundary edges are equalities except possibly one U or D. This follows by inserting the third visit in (9.9); it changes at most one of the two boundary relations for this value of q. Thus the word is constant at both boundaries.

For $q = 0, 2$, contract the interior E edges. Every resulting component either has its bit fixed by O (for $q = 0$) or Z (for $q = 2$), or touches an endpoint f_1 or f_{p-1}. To see this directly, in (9.11) successive A edges are six positions apart, and the intervening equality block contains a fixed-bit edge. Only the two truncated end blocks can lack one. Let $x = f_1, y = f_{p-1}, z = f_0$. The endpoint conditions and the two boundary relations are listed below. An asterisk leaves the endpoint free before boundary relations are imposed.

			interior		lower		upper	
p mod 6	i	q	x	y	(z, x)	(y, z)	(z, x)	(y, z)
5	0	0	1	$*$	O	O	O	E
5	1	0	$*$	1	E	O	E	O
5	2	0	1	1	E	D	U	D
5	0	2	0	0	A	Z	A	Z
5	1	2	0	0	Z	D	Z	A
5	2	2	$*$	$*$	Z	Z	Z	Z
1	0	0	$*$	1	O	O	O	O
1	1	0	1	1	A	U	A	A
1	2	0	1	$*$	O	O	O	O
1	0	2	0	$*$	E	U	E	U
1	1	2	0	0	Z	Z	Z	Z
1	2	2	$*$	0	E	Z	D	E

For instance, the third row at the upper boundary says $x = y = 1$ and $z \le x$, $y \ge z$, leaving z free. The ninth row has the same freedom. Every other $q = 0$ row forces $x = y = z = 1$. Every

$q = 2$ row forces $x = y = z = 0$; in the last upper row, $y = z = 0$ and $z \geq x$ force $x = 0$ as well. The fixed interior components then fix all remaining bits.

All entries are consequences of the two-visit bulk pattern and the two endpoint substitutions; each endpoint is either separated from the fixed interior by a A edge or connected to a fixed-bit edge after at most two interior edges. There are already three interior edges at $p = 5$, so these endpoint tests apply at $p = 5, 7$ as well. We obtain only constant words at the lower boundary, and at the upper boundary the sole nonconstant word is $f_0 = 0$, all other bits 1, in row $i = p \bmod 3$.

F.4 Residue one: free triples and a boundary contradiction

The common parameter is $q = 2$, $h = 3$. First suppose $p \geq 11$, so the upper prefix has five visits in the first $p-4$ columns and four visits in the last four; the lower prefix has five four-visit columns at its end.

Away from those last columns, (9.12) gives constant triples. The complete exceptional tail is described by the six relations on edges $p-6, p-5, \ldots, p-1$, with the last edge wrapping to column zero:

$p \bmod 6$	i	lower tail	upper tail
5	0	E E O E E Z	E U O E E Z
5	1	O E E Z E E	E E E Z E E
5	2	D Z E E O D	A Z E E O D
1	0	Z E E O E U	E D E O E U
1	1	E O E E Z E	E O E E Z E
1	2	U E Z E E O	A E Z E E O

Every lower row contains O E E Z or Z E E O as a consecutive subword. It forces the same bit to be both 1 and 0. The four inactive upper rows contain the same contradiction.

In the active upper row for $p \equiv 5$, the tail forces zeros outside the free interval $[2, p-7]$: its Z propagates through the tail to column zero, and the first bulk equality fixes column one as well. The free edges are $j \equiv 1 \pmod 3$, yielding the triples starting at $2, 5, \ldots, p-9$. In the active upper row for $p \equiv 1$, its O, D and wrap U force ones outside $[3, p-8]$. The bulk free edges are $j \equiv 2 \pmod 3$, yielding the triples starting at $3, 6, \ldots, p-10$.

In either case all free triples can be chosen independently. The one constant word is discarded. This proves the exact supports and $2^{\lambda_p} - 1$ count directly.

For $p = 5, 7$, $e_p = 2$. To leave no unreported small case, here are the full cyclic relation words, starting at edge zero:

p	i	lower word	upper word
5	0	A O E E Z	A E E E Z
5	1	E D Z E E	E A Z E E
5	2	E E E O D	E E U O D
7	0	E E A E O E U	E E A E O E U
7	1	A E E U E Z E	A E E A E Z E
7	2	E A E Z E E O	E A E E D E O

Contracting the equalities gives only constants or inconsistency at the lower boundary. At the upper boundary the only additional words are 00011 in row 2 for $p = 5$, and 0111000 in row 1 for $p = 7$; these specify supports $\{3, 4\}$ and $\{1, 2, 3\}$. For example, the first exceptional upper word has $f_0 = f_1 = f_2$, $f_3 = f_4 = 1$, and only the inequalities $f_2 \leq f_3$, $f_4 \geq f_0$. The other has two free edges around the equal triple $f_1 = f_2 = f_3$, with every other bit fixed to zero.

References

[1] J. Abbott, N. Mexis, *Cyclotomic Factors and LRS-Degeneracy*, arXiv:2403.08751v5 (19 June 2026). Specific reference: §4.1.2–4.1.3 and Algorithm 3.

[2] T. Bogart, J. Goodrick, K. Woods, *Parametric Presburger arithmetic: logic, combinatorics, and quasi-polynomial behavior*, Discrete Analysis 2017:4. doi:10.19086/da.1254. Version consulted: arXiv:1608.08520v2, Theorem 1.15 and Corollary 1.16.

[3] M. Beck, S. Hoşten, *Cyclotomic Polytopes and Growth Series of Cyclotomic Lattices*, Mathematical Research Letters 13 (2006), 607–622. Version consulted: arXiv:math/0508136v4, Propositions 4–6 and Theorems 7, 13.

[4] J. L. Martin, V. Reiner, *Cyclotomic and simplicial matroids*, Israel Journal of Mathematics 150 (2005), 229–240. Version consulted: arXiv:math/0402206v1, Theorem 1, Corollary 2, Remark 6, and §2–3.

[5] L. De Carli, M. Laporta, *Divisibility criteria and coefficient formulas for cyclotomic polynomials*, The Ramanujan Journal 70, article 44 (2026). doi:10.1007/s11139-026-01421-6. Comparison based on: arXiv:2508.02722v1, Theorem 1.1.

[6] J. P. Steinberger, *The lowest-degree polynomial with nonnegative coefficients divisible by the n-th cyclotomic polynomial*, The Electronic Journal of Combinatorics 19(4), P1 (2012). doi:10.37236/2755. The 18-page journal-distributed text, Theorem 1 and §2 (Lemma 1 and Proposition 1). The PDF retains provisional headers; bibliographic data follow the journal record.

[7] A. Satyanarayana Reddy, *The lowest degree $0,1$-polynomial divisible by cyclotomic polynomial*, arXiv:1106.1271v2 (2011). §1–2: $0,1$ multiples with at least three terms and minimal vanishing sums.

[8] M. J. Mossinghoff, *Polynomials with Restricted Coefficients and Prescribed Noncyclotomic Factors*, LMS Journal of Computation and Mathematics 6 (2003), 314–325. doi:10.1112/S1461157000000474. §2.1–2.2, Algorithms 1–2.

[9] D. Chistikov, C. Haase, *The Taming of the Semi-Linear Set*, ICALP 2016, LIPIcs 55, article 128. doi:10.4230/LIPIcs.ICALP.2016.128. Publisher version corrected on 26 October 2016, pp. 128:1–128:14, Proposition 5, pp. 128:5–128:6.