Robust Strictly Positive Real Synthesis for Polynomial Polytopes

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Abstract

We determine when Hurwitz stability of a polynomial polytope guarantees a common numerator of prescribed degree. The denominators are real and monic of degree n, and their convex hull has at most m vertices. For numerator degrees $n-1$ and n, the guarantee holds exactly when $n \le 2$ or $m \le 2$. For degree $n+1$ there is one additional case: every Hurwitz-stable cubic triangle admits a common quartic numerator. We prove this existence result by classifying the possible contacts of a nonnegative frequency pairing. In every order $n \ge 4$ we construct a Hurwitz-stable triangle excluding all numerators of degree at most $n+1$.

Keywords: robust positive realness, polynomial polytope, Hurwitz stability, common numerator, degree constraint

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1 Introduction

A robust positive-real design uses one numerator for every admissible denominator and every frequency. For a polynomial segment, Hurwitz stability guarantees such a numerator [1, 2]. For a polynomial polytope, the edge theorem reduces Hurwitz stability to stability on its exposed edges [3]. Common-numerator synthesis asks whether the feasible numerator sets for those edges have a common intersection. A triangle is the first case in which this compatibility becomes an additional requirement.

Anderson et al. [4] studied the characterization and construction of robust strictly positive real transfer functions. Wang and Yu [5] developed a geometric approach through feasible regions in numerator coefficient space, and Henrion [6] gave a linear matrix inequality test for a specified polynomial polytope. Numerator degree already matters in Example 4 of Wang and Yu [7]: their cubic triangle excludes a quadratic numerator and admits a cubic one.

The geometry of the denominator family also matters. For interval polynomials, whose coefficients vary independently, Kharitonov's theorem reduces Hurwitz stability to four distinguished polynomials [8]. In our companion work, every Hurwitz-stable sixth-order interval family admits a common numerator of degree six [9]. Here we classify the guarantees for general polynomial polytopes using the denominator order n and a bound m on the number of vertices. We consider the three numerator degrees $n-1$, n, and $n+1$.

The transition occurs between orders three and four. A Hurwitz-stable cubic triangle can exclude every numerator of degree at most three, yet always admits a quartic numerator. We prove existence by deforming a single polynomial into the triangle and showing that a nonzero limiting numerator can be perturbed to restore strict positivity. At order four, six frequency inequalities certify the failure of quintic synthesis on an explicit triangle. We then strengthen this certificate to exclude nonzero nonnegative pairings and carry the obstruction to every higher order by a coefficient-limit argument.

2 Problem formulation and classification

A real polynomial is *Hurwitz stable* if all its complex roots have strictly negative real part. A polynomial family is Hurwitz stable if every member is Hurwitz stable.

Let n be the denominator degree and m an upper bound on the number of vertices, both positive integers. For an integer r with $1 \le r \le m$, let $a_1, \dots, a_r \in \mathbb{R}[s]$ be monic, meaning that their leading coefficient is one, and have degree n. Their convex hull is the polynomial polytope

$$\mathcal{P} = \Big\{ \sum_{j=1}^{r} \lambda_j a_j : \lambda_j \ge 0,\ \sum_{j=1}^{r} \lambda_j = 1 \Big\},$$

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where $\lambda_1, \dots, \lambda_r$ are the convex weights. Assume that $\mathcal{P}$ is Hurwitz stable. Given a nonnegative integer k, we seek a *common numerator* $B \in \mathbb{R}[s]$ of degree k such that

$$\operatorname{Re} \frac{B(\mathrm{i}\omega)}{a(\mathrm{i}\omega)} > 0 \qquad (a \in \mathcal{P},\ \omega \in \mathbb{R}), \tag{2.1}$$

where ω is the angular frequency and a ranges over the denominator family. Condition (2.1) requires positivity at each finite frequency; for $k = n + 1$ the numerator degree exceeds the denominator degree, so the quotient is improper. Write $U(n, m, k)$ for the assertion that every such Hurwitz-stable polytope admits a common numerator of degree k.

Theorem 2.1. For positive integers n, m, each of $U(n, m, n-1)$ and $U(n, m, n)$ holds if and only if $n \le 2$ or $m \le 2$. Furthermore,

$$U(n, m, n+1) \quad \Longleftrightarrow \quad n \le 2 \quad \text{or} \quad m \le 2 \quad \text{or} \quad (n, m) = (3, 3).$$

Table 1: Universal existence at each exact numerator degree.

Denominator order	Vertex bound	$n-1$	n	$n+1$
$1, 2$	any m	yes	yes	yes
$n \ge 3$	$1, 2$	yes	yes	yes
3	3	no	no	yes
3	$m \ge 4$	no	no	no
$n \ge 4$	$m \ge 3$	no	no	no

Put $t = \omega^2 \ge 0$, the squared frequency, and define the real pairing

$$P_{B,a}(t) = \operatorname{Re}\big(B(\mathrm{i}\sqrt{t})\overline{a(\mathrm{i}\sqrt{t})}\big).$$

The real part in (2.1) is $P_{B,a}(\omega^2)/|a(\mathrm{i}\omega)|^2$. Since the denominator is positive and $P_{B,a}$ is linear in a, condition (2.1) is equivalent to $P_{B,a_j}(t) > 0$ at every vertex and all $t \ge 0$.

2.1 Low orders and segments

For a linear denominator $a(s) = s + \alpha$, Hurwitz stability gives $\alpha > 0$, and the numerator 1 has positive pairing α. For quadratic vertices $a_j(s) = s^2 + \alpha_j s + \beta_j$, the real coefficients satisfy $\alpha_j, \beta_j > 0$. Choose a real number x with $0 < x < \min_{1 \le j \le r} \alpha_j$. The numerator $s + x$ has pairing $(\alpha_j - x)t + \beta_j x > 0$ at every vertex. On a Hurwitz-stable segment of arbitrary positive order the segment construction gives a strictly proper common numerator of degree $n - 1$ [1]. For a single denominator a one can take a', since the real part of a'/a at $\mathrm{i}\omega$ is a sum of the positive quantities $-\operatorname{Re}\zeta/|\mathrm{i}\omega - \zeta|^2$ over the roots ζ with multiplicity.

These witnesses also give the two higher degrees. Add εs^n to the degree-$n-1$ solution, where $\varepsilon > 0$ is a real perturbation parameter. At every vertex the real part of s^n/a_j tends to one on the imaginary axis. A common tail therefore admits any positive ε, and the strict margin on the complementary compact interval admits a sufficiently small one. The resulting numerator b has degree n and its quotient has real part bounded below by a common positive constant. If α_j denotes the coefficient of s^{n-1} in a_j, the real part of $(\mathrm{i}\omega)^{n+1}/a_j(\mathrm{i}\omega)$ tends to $-\alpha_j$ as $|\omega| \to \infty$. It is therefore bounded on the imaginary axis, so a small nonzero multiple of s^{n+1} added to b gives the required exact degree $n + 1$.

3 The cubic transition

3.1 A triangle requiring degree four

Write the three cubic vertices as $a_j(s) = s^3 + \alpha_j s^2 + \beta_j s + \gamma_j$, $j = 1, 2, 3$, where $\alpha_j, \beta_j, \gamma_j$ are real coefficients. For a numerator $B(s) = ds^4 + xs^3 + ys^2 + zs + w$ with real coefficients d, x, y, z, w, put $G_j(t) = P_{B,a_j}(t)$. Direct expansion gives

$$\begin{aligned} G_j(t) = {} & (x - \alpha_j d)t^3 + (\gamma_j d + \alpha_j y - \beta_j x - z)t^2 \\ & + (\beta_j z - \gamma_j y - \alpha_j w)t + \gamma_j w. \end{aligned} \tag{3.1}$$

A short example shows why the fourth-degree coefficient matters. Choose $(\alpha_1, \beta_1, \gamma_1) = (3, 1, 2)$, $(\alpha_2, \beta_2, \gamma_2) = (4, 4, 15)$, and $(\alpha_3, \beta_3, \gamma_3) = (1, 3, 2)$. A convex combination $a = \sum_{j=1}^{3} \lambda_j a_j = s^3 + \alpha s^2 + \beta s + \gamma$ has Hurwitz determinant

$$\alpha\beta - \gamma = 3(\lambda_2 - \tfrac{1}{2})^2 + \tfrac{1}{4} + 4\lambda_1\lambda_3 > 0.$$

All three coefficients are positive, so the full triangle is Hurwitz. Substitution in (3.1) gives

$$6G_1(2) + 8G_2(3) + 3G_3(4) + 3G_1(0) = 24d.$$

If $\deg B \le 3$, the right side vanishes. Strict positivity would make the left side positive. This establishes both negative entries for the cubic triangle in Table 1.

3.2 Existence of a quartic numerator

Theorem 3.1. Every Hurwitz-stable cubic triangle admits one real quartic numerator satisfying (2.1).

In this subsection the vertex indices i, j range over $\{1, 2, 3\}$. To prove the theorem, introduce the compact frequency coordinate $u = t/(1+t)$, which maps $t \ge 0$ to $0 \le u < 1$. The function $g_j(u) = G_j(t)/(1+t)^3$ extends polynomially to $[0, 1]$, with $g_j(1) = x - \alpha_j d$. Call a coefficient vector weakly feasible if all three g_j are nonnegative on $[0, 1]$, and strictly feasible if they are positive there. A contact is a zero of one of these functions; the endpoints $u = 0$ and $u = 1$ represent zero and infinite frequency, respectively.

Lemma 3.2 (Perturbing the contacts)**.** If a nonzero coefficient vector (w, z, y, x, d) is weakly feasible, there exists a strictly feasible vector.

Proof. A perturbation direction positive at every contact remains positive in a neighborhood of the contact set. On the compact complement, the original pairings have a positive minimum and the direction is bounded. A sufficiently small positive step therefore makes all three pairings strictly positive on $[0, 1]$.

First suppose $G_j \equiv 0$. Since $\gamma_j > 0$ and $\alpha_j\beta_j > \gamma_j$, comparison of the four coefficients in (3.1) gives $B = dsa_j$ with $d \neq 0$. Use a_j as the perturbation direction. At a positive finite contact for vertex i the quotient a_j/a_i is real and nonzero. A negative real value would put zero on the segment between the two denominator values, contrary to stability of the closed edge. Thus the direction has positive pairing there. At zero frequency both constant coefficients are positive, and at infinity its compact pairing equals one.

Assume now that every G_j is a nonzero polynomial. Each has degree at most three and is nonnegative on the half-axis. A positive zero has even multiplicity, so each vertex contributes at most one positive contact. The contact set is therefore finite.

Suppose no direction is positive at all contacts. Evaluation at each contact, including infinity, defines a linear functional on the five coefficients. The convex hull of these contact rows contains zero, since otherwise a separating functional would supply a positive direction. Hence the rows admit a nontrivial nonnegative linear dependence.

Merge equal finite nodes in this dependence. At each node t write $\kappa_t > 0$ for the sum of its dependence weights and $(\alpha_t, \beta_t, \gamma_t)$ for the coefficients of the denominator averaged with those weights. Absorb the positive compactifying factors into these weights. Let $\mu_i \ge 0$ be the weight at the infinity contact of vertex i, taking $\mu_i = 0$ when that contact is absent. Put $\rho = \sum_{i=1}^{3} \mu_i$ and $M = \sum_{i=1}^{3} \mu_i\alpha_i$. Define

$$U_t = \kappa_t(\gamma_t - \alpha_t t), \qquad V_t = \kappa_t t(\beta_t - t).$$

Comparing the coefficients of w, y, d, z, x, respectively, gives

$$\sum U_t = \sum tU_t = 0, \qquad \sum t^2U_t = M, \qquad \sum V_t = 0, \qquad \sum tV_t = \rho. \tag{3.2}$$

Each sum is over distinct finite nodes. At zero $U_0 > 0$ and $V_0 = 0$ if this node is present. At any positive node $U_t \ge 0$ implies $V_t > 0$, because the merged denominator is in the Hurwitz-stable hull and $t \le \gamma_t/\alpha_t < \beta_t$. At least one finite node is present by (3.2) and positivity of the α_i. If $\rho = 0$, the first three moments of the array (U_t) vanish. On at most three nodes, Lagrange interpolation would give $U_t = 0$ at every node. The zero node would then be impossible, and all V_t would be positive. This contradicts their zero sum. The same argument using two moments shows that in all cases at least three finite nodes are needed.

We distinguish three configurations. In each one the contact at a positive node gives the identity

$$(dt^2 - yt + w)U_t + (z - xt)V_t = 0. \tag{3.3}$$

Case 1: $\rho = 0$. The four nodes must be zero and three distinct positive contacts. Every U_t is nonzero, since deleting a zero coordinate would leave three vanishing moments on at most three nodes. On the positive nodes the vectors (V_t) and (tU_t) both have zeroth and first moments zero. This kernel on three nodes is one-dimensional, so $V_t = AtU_t$ for some real scalar A. The zero contact gives $w = 0$. Equation (3.3) at the positive nodes yields $d = Ax$ and $y = Az$. Consequently

$$G_i(t) = t(z - xt)\big[(A\alpha_i - 1)t + \beta_i - A\gamma_i\big]. \tag{3.4}$$

The bracket cannot vanish identically because $\alpha_i\beta_i > \gamma_i$. At a positive contact the factor $z - xt$ must vanish, or the nonzero affine bracket would have to supply a double root. Two distinct contacts therefore give $z = x = 0$, and all five coordinates vanish.

Case 2: $\rho > 0$ *and* $w > 0$. There are exactly three positive nodes, one per vertex. Now $M > 0$ and every U_t is nonzero by the two zero moments and the nonzero second moment. Set $A = \rho/M$. The vectors $(V_t - AtU_t)$ and (U_t) both have zeroth and first moments zero, hence $V_t = (At + D)U_t$ for some real scalar D. The polynomial $dt^2 - yt + w + (z - xt)(At + D)$ vanishes at three distinct points and has degree at most two. Comparing coefficients gives the factorization below, which defines the quadratic polynomials Q_i:

$$d = Ax, \quad y = Az - Dx, \quad w = -Dz, \qquad G_i(t) = (z - xt)Q_i(t),$$
$$Q_i(t) = (A\alpha_i - 1)t^2 + (D\alpha_i - A\gamma_i + \beta_i)t - D\gamma_i.$$

No Q_i vanishes identically, by its constant term and the positive Hurwitz determinant. If $x \neq 0$, choose a vertex index $h \in \{1, 2, 3\}$ with $\mu_h > 0$. Its infinity contact gives $x - \alpha_h d = 0$, so Q_h is at most affine. At its positive contact $z - xt$ must vanish. Every other G_i now has this positive root as well as its own distinct positive contact, which is impossible. If $x = 0$, each $G_i = zQ_i$ is at most quadratic, with positive constant term and a positive double root. Its quadratic coefficient $z(A\alpha_i - 1)$ is therefore strictly positive. The sum of these coefficients with weights μ_i equals $z(AM - \rho) = 0$, again a contradiction.

Case 3: $\rho > 0$ *and* $w = 0$. At an infinity contact the corresponding G_i is at most quadratic with zero constant term. Such a nonzero nonnegative polynomial cannot also have a positive double root. There are at most two positive nodes, so the finite support consists of zero and exactly two positive nodes. The first two zero moments and $U_0 > 0$ again force every U_t to be nonzero. On the two positive nodes both (V_t) and (tU_t) sum to zero. Thus $V_t = AtU_t$ for a real scalar A, and (3.3) gives $d = Ax$, $y = Az$ at the two distinct positive nodes. Formula (3.4) forces the whole vector to vanish. Since weak feasibility gives $w \geq 0$, the cases exhaust the possible dependences. □

Proof of Theorem 3.1. For a real deformation parameter $\tau \in [0, 1]$, define $a_i^\tau = (1 - \tau)a_1 + \tau a_i$. Every intermediate triangle lies in the original hull. Let $T \subseteq [0, 1]$ consist of the parameters admitting a strictly feasible vector on the compact frequency interval. At $\tau = 0$ the choice $B = (s+1)a_1$ has positive finite pairing $|a_1(\mathrm{i}\omega)|^2$ and compact value one at infinity. Thus $0 \in T$. A strict witness persists locally by compactness, so T is relatively open. Along a sequence of parameters in T converging to τ, choose a witness at each parameter and normalize its coefficient vector (w, z, y, x, d) to Euclidean norm one by positive scaling. A convergent subsequence gives a nonzero limit whose pairings are nonnegative by continuity. Lemma 3.2 puts τ in T, so T is also closed. Connectedness gives $T = [0, 1]$.

Finally add a sufficiently small multiple of s^4. Compact strictness persists and the new leading coefficient can be chosen nonzero. The vertex reduction proves (2.1). □

4 Two boundary obstructions

Compactness also turns boundary obstructions into Hurwitz-stable counterexamples. Suppose a sequence of Hurwitz-stable families of fixed order converges coefficientwise to a boundary family. Common numerators of degree at most k, represented by their $k + 1$ real coefficients and normalized to Euclidean norm one in $\mathbb{R}^{k+1}$, have a subsequence converging to a nonzero polynomial B. Continuity gives nonnegative pairings of B with every boundary vertex. Thus a boundary family admitting no such B forces failure of synthesis in a nearby Hurwitz-stable family.

4.1 Four cubic vertices

For positive real parameters σ, c, put $p_{\sigma,c}(s) = (s + \sigma)(s^2 + c)$. Choose the four pairs $(\sigma, c) = (2, 1), (2, 2), (1, 3), (1, 4)$ and translate the corresponding polynomials to $p_{\sigma,c}(s + \varepsilon)$, where $\varepsilon > 0$ is real. Let $\lambda_1, \ldots, \lambda_4$ be their convex weights in this order, and define

$$u = \lambda_1 + \lambda_2, \qquad L = \lambda_1 + 2\lambda_2, \qquad H = 3\lambda_3 + 4\lambda_4.$$

Before translation their convex combination is $s^3 + \alpha s^2 + \beta s + \gamma$, where $\alpha = 1 + u$, $\beta = L + H$, and $\gamma = 2L + H$. They are positive, and

$$\alpha\beta - \gamma = uH - (1 - u)L \geq u(1 - u),$$

since $H \geq 3(1 - u)$ and $L \leq 2u$. Translation preserves positive coefficients and increases the Hurwitz determinant by $(2\alpha^2 + 2\beta)\varepsilon + 8\alpha\varepsilon^2 + 8\varepsilon^3$. The translated hull is therefore Hurwitz stable.

Write $B(s) = \sum_{j=0}^{4} b_j s^j$, where $b_0, \dots, b_4$ are real coefficients. Its pairing with the untranslated polynomial $p_{\sigma,c}$ equals $(c-t)R_\sigma(t)$, where the auxiliary polynomial R_σ is

$$R_\sigma(t) = \sigma b_0 + (b_1 - \sigma b_2)t + (\sigma b_4 - b_3)t^2.$$

Nonnegative pairings at all four vertices force R_2 to vanish on $(1,2)$ and R_1 on $(3,4)$. The two coefficient identities $R_1 = R_2 = 0$ force all five b_j to be zero. By the normalization argument, some positive translate excludes every common numerator of degree at most four. This proves the sharp four-vertex threshold for quartic synthesis at order three.

4.2 A quartic triangle

For the high-order construction, define three monic quartics a_i, $i = 1, 2, 3$, by the following matrix: row i lists the coefficients of $s^3, s^2, s, 1$ in a_i, respectively.

$$\begin{pmatrix} 2/7 & 37/7 & 1 & 25/7 \\ 11/7 & 32/7 & 4 & 36/7 \\ 857/21 & 1130/21 & 709/3 & 576/7 \end{pmatrix}. \tag{4.1}$$

Let $q = s^4 + s^3 + 7s^2 + 2s + 7$. Its Routh first column is $1, 1, 5, 3/5, 7$. Direct multiplication gives

$$P_{q,a_1} = (t-1)^2(t-5)^2, \quad P_{q,a_2} = (t-2)^2(t-3)^2, \quad P_{q,a_3} = (t-4)^2(t-6)^2.$$

For each squared-frequency node $t = j$, $j = 1, \dots, 6$, define a vertex index ν_j and a real scale factor κ_j by

$$(\nu_j) = (1, 2, 2, 3, 1, 3), \qquad (\kappa_j) = (5/7, -2/7, 1/7, -307/21, 1/7, -179/21).$$

Substitution verifies $a_{\nu_j}(\mathrm{i}\sqrt{j}) = \mathrm{i}\sqrt{j}\,\kappa_j q(\mathrm{i}\sqrt{j})$, so no vertex vanishes at its contacts. A convex combination with two or more positive weights has strictly positive pairing with q, since the corresponding zero sets are disjoint. Thus every member a of the hull has no imaginary-axis root. Along the segment from q to a, the pairing with q is strictly positive up to the final endpoint. The right-half-plane root count is constant along this monic path, proving that a is Hurwitz. The two differences of the first two coefficient coordinates have determinant $274/3$, so the three vertices are affinely independent.

For any real polynomial B of degree at most five, write $\Im(B(\mathrm{i}\sqrt{t})\overline{q(\mathrm{i}\sqrt{t})}) = \sqrt{t}\,H_B(t)$. Expansion gives a polynomial H_B of degree at most four. Therefore $P_{B,a_{\nu_j}}(j) = j\kappa_j H_B(j)$. For $Z(t) = \prod_{j=1}^{6}(t-j)$, comparison of the leading coefficients in the Lagrange formula gives $\sum_j H_B(j)/Z'(j) = 0$. With the weights $w_1, \dots, w_6$ given below, this yields

$$\sum_{j=1}^{6} w_j P_{B,a_{\nu_j}}(j) = 0, \qquad (w_j) = \left(\frac{1}{600}, \frac{1}{96}, \frac{1}{36}, \frac{1}{4912}, \frac{1}{120}, \frac{1}{42960}\right). \tag{4.2}$$

Indeed $w_j = -1/(7j\kappa_j Z'(j))$. All six weights are positive, so (4.2) rules out strict positivity. The polynomial q still has nonnegative pairings; the next step perturbs the vertices to remove this remaining feasible direction.

5 Three vertices in every order from four onward

Theorem 5.1. For every $n \geq 4$ there exists a Hurwitz-stable triangle of monic degree-n polynomials for which no common numerator has degree at most $n+1$.

Lemma 5.2 (A seed with no nonzero nonnegative pairing)**.** There are three monic quartics b_1, b_2, b_3 with Hurwitz-stable hull such that the only polynomial of degree at most five with nonnegative pairings against all b_i is zero. For the boundary triple sb_i, the same conclusion holds up to degree six provided the numerator has nonnegative constant coefficient.

Proof. For a real polynomial B of degree at most five and $j = 1, \dots, 6$, the map $B \mapsto P_{B,a_{\nu_j}}(j)$ is a linear functional on the six coefficients of B. We call its coefficient vector a row. The six rows in (4.2) have joint kernel spanned by q. Indeed, if all six pairings vanish, H_B has six distinct roots and vanishes identically. Thus $B(s)q(-s) = B(-s)q(s)$. Since q and its reflection have disjoint root sets, q divides B. The quotient has degree at most one and is even, hence is constant. Add a seventh row at $t = 7$ with vertex a_1. Its value at q is 144, so the seven rows have rank six. Their sole dependence, up to scale, has weights $(w_1, \dots, w_6, 0)$.

Introduce a real perturbation parameter $\eta \geq 0$ and replace each a_i by $a_i - \eta q$, initially retaining its unnormalized leading coefficient. Set $w_1(\eta) = w_1$ and determine $w_2(\eta), \dots, w_7(\eta)$ by requiring the weighted row sum to vanish; at $\eta = 0$ the full weight vector is $(w_1, \dots, w_6, 0)$. Rows two through seven are independent

at $\eta = 0$ because the sole dependence has nonzero first coefficient. They remain independent for all sufficiently small η. The first six weights remain positive by continuity. Evaluating the dependence on q gives

$$w_7(\eta) = \frac{\eta \sum_{j=1}^{6} w_j(\eta)|q(\mathrm{i}\sqrt{j})|^2}{144 - \eta|q(\mathrm{i}\sqrt{7})|^2} > 0.$$

Put $b_i = (a_i - \eta q)/(1-\eta)$ for such a positive $\eta < 1$. Their hull remains Hurwitz stable by compactness and openness of Hurwitz stability, and normalization scales every row by the same positive number. The positive dependence and rank six imply the first assertion.

Let $E_i(t)$ and $O_i(t)$ be the real polynomials in the squared frequency t determined by $b_i(s) = E_i(-s^2) + sO_i(-s^2)$, so that $b_i(\mathrm{i}\sqrt{t}) = E_i(t) + \mathrm{i}\sqrt{t}\,O_i(t)$. Extend the vertex indices by setting $\nu_7 = 1$. At $\eta = 0$, substitution of the six weights gives $\sum_{j=1}^{6} w_j O_{\nu_j}(j) = 1/720$. Decrease η if necessary so that the seven perturbed weights give $S = \sum_{j=1}^{7} w_j(\eta) O_{\nu_j}(j) > 0$. For a numerator $C \in \mathbb{R}[s]$ of degree at most six, put $c = C(0)$ and write $C = sD + c$ with $D \in \mathbb{R}[s]$ of degree at most five. When $c \geq 0$, direct multiplication gives

$$P_{C,sb_i}(t) = t\big(P_{D,b_i}(t) - cO_i(t)\big).$$

If these pairings are nonnegative, divide by t at the seven nodes and sum with the positive weights. The result is $0 \leq -cS$, so $c = 0$. The pairings for D are nonnegative on the positive half-axis and by continuity at zero. The first assertion now gives $D = 0$. □

Proof of Theorem 5.1. Let $e \in \{0, 1\}$ be the parity of $n - 4$, and put $\ell = (n - 4 - e)/2$, a nonnegative integer. Choose distinct positive real numbers $T_1, \ldots, T_\ell$ sufficiently small that $b_1(\mathrm{i}\sqrt{T_j})$ and $b_2(\mathrm{i}\sqrt{T_j})$ are independent as real two-dimensional vectors for $j = 1, \ldots, \ell$. This is possible because at the unperturbed vertices the coefficient of ω in $\Im(a_1(\mathrm{i}\omega)\overline{a_2(\mathrm{i}\omega)})$ is $-64/7$, and this coefficient stays nonzero for sufficiently small η. For a real shift $\varepsilon > 0$ and $i = 1, 2, 3$, define new vertices

$$A_{i,\varepsilon}(s) = (s + \varepsilon)^e \prod_{j=1}^{\ell} \big((s + \varepsilon)^2 + T_j\big) b_i(s).$$

For $\varepsilon > 0$ every convex combination is a product of the same Hurwitz factor and a Hurwitz-stable convex combination of the b_i. The new family is monic of degree n and remains a triangle by injectivity of multiplication.

If all positive ε admitted a common numerator of degree at most $n + 1$, take $\varepsilon \downarrow 0$ and normalize as above. The resulting nonzero polynomial B has degree at most $n + 1$. Its constant term is nonnegative because each strict witness has positive constant term by its zero-frequency inequality. Put $h(s) = \prod_{j=1}^{\ell}(s^2 + T_j)$. Then $P_{B,s^e h b_i}(t) \geq 0$ for all $t \geq 0$. At $t = T_j$, the real factor $h(\mathrm{i}\sqrt{t})$ changes sign. Continuity forces

$$\operatorname{Re}\big(B(\mathrm{i}\sqrt{T_j})\overline{(\mathrm{i}\sqrt{T_j})^e b_i(\mathrm{i}\sqrt{T_j})}\big) = 0 \qquad (i = 1, 2, 3).$$

Independence of the first two values gives $B(\mathrm{i}\sqrt{T_j}) = 0$. Reality supplies the conjugate root, and the distinct quadratic factors together divide B. Let $C \in \mathbb{R}[s]$ be the quotient defined by $B = hC$. Since $h(0) > 0$, we have $C(0) \geq 0$ and $\deg C \leq 5 + e$. Cancel $|h(\mathrm{i}\sqrt{t})|^2$ away from its zeros and fill those finitely many points by continuity. The residual pairings $P_{C,s^e b_i}$ are all nonnegative. Each choice of e contradicts Lemma 5.2, since $C \neq 0$. □

Completion of the proof of Theorem 2.1. The low-order and segment arguments establish all positive entries except the cubic triangle, which is Theorem 3.1. The integer cubic example excludes degrees two and three on three vertices. The four translated cubics exclude degree four on four vertices. Theorem 5.1 supplies a three-vertex obstruction for each higher order. Each counterexample applies to every larger vertex bound because the bound counts at most m vertices. □

6 Conclusion

Allowing one extra numerator degree resolves synthesis for every Hurwitz-stable cubic triangle. From order four onward, three vertices can exclude all degrees up to $n+1$. Together with the sixth-order interval result [9], this classification shows how the geometry of coefficient uncertainty affects common-numerator existence. A next question is whether some degree bound depending only on n and m guarantees synthesis.

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The proofs have been formalized in Lean 4. The proof sources are available in the private repository at https://github.com/dgjszy/polytope-spr-proof.

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