An upper bound for three-sunflower-free families of four-element sets

Oleksiy Babanskyy

Abstract

Let $M(4,3)$ be the maximum size of a finite simple family of four-element sets containing no three distinct members whose pairwise intersections are equal. The empty intersection is included in the forbidden configurations. We give a computer-assisted exclusion of families of size 69, giving the bound $54 \le M(4,3) \le 68$. The argument combines an imported bound of 27 for intersecting families, structural analysis of point-links, exact trace constraints and finite exhaustions. The proof is organized at the cardinality 69. A cap on families containing a complete $R16$ point-link provides a shorter final contradiction. We make the imported results, computational lemma contracts and remaining verification limits explicit. The lower construction is classical. Neither exclusion of size 68 nor optimality of the lower construction is claimed.

Keywords: Sunflowers; four-uniform families; extremal set theory; exact certificates.

hexagon:2611.00552v1

1 The finite extremal problem

A three-sunflower is a triple of distinct sets A, B, C such that

$$A \cap B = A \cap C = B \cap C.$$

The common intersection is the kernel. Allowing the empty kernel means that three pairwise disjoint sets are also forbidden. Throughout, families are simple: a set occurs at most once.

The quantity studied here is

$$M(4,3) = \max\{|\mathcal{F}| : \mathcal{F} \text{ is a finite simple four-uniform family without a three-sunflower}\}.$$

Thus an upper bound 68 is equivalent to saying that every such family of *at least* 69 sets contains a three-sunflower. The convention that counts the first cardinality forcing a sunflower adds one to M.

Theorem 1.1 (Computer-assisted bound)**.** *With H27 and the classification and finite-exclusion lemmas stated in this manuscript, no simple four-uniform three-sunflower-free family of size* 69 *exists. Consequently,*

$$\boxed{54 \le M(4,3) \le 68}.$$

The general sunflower conjecture of Erdős and Rado [1] concerns exponential dependence on the uniformity for each fixed number of petals. A bound for this particular four-uniform case does not resolve that asymptotic conjecture. The distinction also prevents a finite improvement from being described as a solution of the entire Erdős problem.

The contemporary primary comparison is the preprint of Axante et al. [4], whose Theorem 6.9 gives $54 \le f(4,3) \le 83$, with f defined there as the maximum sunflower-free size. Their Proposition 6.7 proves that an intersecting four-uniform three-sunflower-free family has at most 27 members, and Proposition 6.8 gives equality. We use the upper bound as an imported theorem, denoted H27. It is not a claim that general three-uniform families have maximum 27.

The lower bound 54 is attributed to Abbott and Hanson [2] in the table of Axante et al. We reproduce the construction and its elementary verification in Section 10. No new lower bound or priority for that construction is asserted. The numerical comparison with 83 provides the comparison for this bound; it does not by itself establish that 68 is a new record. The historical upper bound 142 of Abbott, Hanson and Sauer [3] is also reported in that table; the numerical attribution here is through the checked preprint rather than a new reading of the original article.

The imported intersecting-family theorem and the exact finite lemmas are stated with their domains. The companion provides the complete named finite verification route; the written reductions establish why these finite domains cover every hypothetical family.

2 Notation and the proof architecture

We abbreviate a finite simple r-uniform three-sunflower-free family as SFr; the forbidden configurations include the empty kernel. Only points that occur in a member of $\mathcal{F}$ belong to its support. Write $d(x) = |\{A \in \mathcal{F} : x \in A\}|$ and $d(x,y) = |\{A \in \mathcal{F} : \{x,y\} \subseteq A\}|$. For $R \in \mathcal{F}$,

$$N_{\mathcal{F}}[R] = \{A \in \mathcal{F} : A \cap R \neq \varnothing\}, \qquad D_{\mathcal{F}}(R) = \{A \in \mathcal{F} : A \cap R = \varnothing\},$$

and $\mu(\mathcal{F}) = \min_{R \in \mathcal{F}} |N_{\mathcal{F}}[R]|$. The neighborhood is closed: it includes R itself. The point-link

$$L_x = \{A \setminus \{x\} : A \in \mathcal{F},\ x \in A\}$$

is a complete link of the ambient family. A point degree in L_x is the global codegree $d(x,y)$, not the global degree $d(y)$.

Lemma 2.1 (The neighborhood lower bound)**.** *If $\mathcal{F}$ is three-sunflower-free, then $D_{\mathcal{F}}(R)$ is intersecting for every $R \in \mathcal{F}$. Under H27,*

$$|N_{\mathcal{F}}[R]| \geq |\mathcal{F}| - 27.$$

In particular, a hypothetical family of size 69 *has $\mu \geq 42$.*

Proof. If two members of $D_{\mathcal{F}}(R)$ were disjoint, they and R would be a sunflower with empty kernel. H27 applies to this intersecting subfamily, which is still four-uniform, simple and sunflower-free. □

The disjointness graph has the *members* of $\mathcal{F}$ as vertices; two are adjacent exactly when they are disjoint. This graph is triangle-free. Its degree at R is $|\mathcal{F}| - |N_{\mathcal{F}}[R]|$. Homomorphism fibers in this graph consist of members of $\mathcal{F}$, whereas the degrees and codegrees above count points. Maintaining this distinction is essential to the incidence arguments.

Our imported graph theorem is due to Jin [5], in the exact form stated as Theorem A by Nikiforov [6]: for $1 \leq k \leq 9$, every triangle-free graph of order n with minimum degree strictly greater than $(k+1)n/(3k+2)$ is homomorphic to the Andrásfai graph A_k. Its cyclic definition is given in Section 7. We verify the strict inequality at $k = 3$ or $k = 4$ in each use; equality does not suffice.

The proof follows three reductions. First, universal link bounds and cardinality-69 trace constraints exclude maximum point degrees $20, 19, 18, 17$ and at most 15. In the remaining maximum-16 case, a triangle-free graph homomorphism and a near-transversal argument force every degree-16 point to have a complete $R16$ link. Finally, the local $R61$ lemma contradicts $|\mathcal{F}| = 69$. The first reduction and all finite coverings are recorded with their hypotheses; none is transferred from cardinality 70 without its own bridge.

3 Elementary link structure

3.1 Graphs without three-sunflowers

Lemma 3.1. *A simple graph contains no three-sunflower of edges if and only if its maximum degree and matching number are both at most two. It has at most six edges. Equality gives two vertex-disjoint triangles.*

Proof. The two possible kernels of a three-sunflower of distinct edges are a single vertex and the empty set. They respectively give a three-edge star and a three-edge matching. Every nontrivial component of a graph with maximum degree two is a path or a cycle. With matching number q, a path has at most $2q$ edges; an even cycle has $2q$ and an odd cycle $2q+1$. The total matching number is at most two, so the maximum six edges is attained precisely by two triangles. With five edges the possibilities are a five-cycle or a triangle and a two-edge path. □

3.2 The canonical ten- and eight-member families

We abbreviate $\{0,1,2\}$ as 012 when all labels are single digits. Put

$$\mathcal{P} = \{012, 015, 023, 034, 045, 124, 134, 135, 235, 245\}.$$

This intersecting family has ten triples, every point degree five and every pair codegree two. Every point-link is a five-cycle, so it is three-sunflower-free. Write P_9 and P_8 for subfamilies obtained by deleting respectively one and two blocks of $\mathcal{P}$.

On seven distinct points let

$$Q(h,k;T;a,b) = \{\{h\}\cup S, \{k\}\cup S : S \in \binom{T}{2}\} \cup \{\{h,k,a\},\{h,k,b\}\}, \qquad |T| = 3.$$

This is the eight-member family Q_8. The points h,k are its hubs, T its triad and a,b its leaves. The hubs have degree five, the triad points degree four and the leaves degree one. Pair codegrees are at most two, and every point-link has matching number at most two.

Proposition 3.2 (Intersecting triple classification)**.** *An intersecting three-uniform three-sunflower-free family has at most ten members. At ten it is $\mathcal{P}$; at nine it is P_9; at eight it is P_8 or Q_8, up to relabeling.*

Proof. Every point-link is a graph from Lemma 3.1, so its point degrees are at most six. A degree-six point has two triangles as its link. A triple avoiding it would have to cover both triangles with only three points, which is impossible; the whole family then has only six members. A degree-five point has link either a five-cycle or a triangle and a two-edge path. In the first case every triple avoiding the point is one of the five three-point vertex covers of the cycle, giving a subfamily of $\mathcal{P}$. In the second case there are at most three such triples, and at size eight all three occur, giving Q_8.

We justify the existence of a degree-five point at sizes at least eight. For an intersecting family $\mathcal{H}$ of m triples, every $A \in \mathcal{H}$ satisfies

$$\sum_{x\in A} d_{\mathcal{H}}(x) \geq m+2.$$

If all degrees are at most four this gives $m \leq 10$; at $m = 10$ all used points would have degree four, contrary to a degree sum of 30. At $m = 9$ all used points have degree three or four. If their number is n, and B counts pairs of points of codegree two, counting intersections of pairs

of triples gives $B = 45 - 6n$. The degree sum 27 and $B \geq 0$ force $n = 7, B = 3$. The triples would use $27 - B = 24$ distinct pairs on seven points, exceeding $\binom{7}{2} = 21$.

At $m = 8$ and maximum degree four all used degrees are at least two. Let a, b, c count degrees two, three, four. Then $a + b + c = n$ and $2a + 3b + 4c = 24$. Intersection counting gives $B = 20 - 3n + c$. For each point the number $r(x)$ of codegree-two neighbors is at least $d(x) - 2$: the line graph of its point-link has independence number at most two, whereas deleting at most one endpoint per edge gives an independent set of size at least $d(x) - r(x)$. Consequently $2B \geq b + 2c$, so $40 - 6n \geq b$ and $n \leq 6$. The degree sum forces $n = 6$ and all degrees four. Now $B = 8$, so the eight triples use $24 - B = 16$ distinct pairs on six points, exceeding 15. This final contradiction completes the classification without a support cutoff. □

3.3 Isolation and a separated neighborhood

Lemma 3.3 (Isolation)**.** *In a sunflower-free triple family containing P_{10}, P_9 or P_8, every additional triple meeting the six-point support is one of the deleted blocks. For $Q(h, k; T; a, b)$, every additional triple meeting the five-point core $\{h, k\} \cup T$ is $\{h, a, b\}$ or $\{k, a, b\}$.*

Proof. For P_{10} all pairs on its support already have codegree two. Meeting the support in one point gives a third disjoint edge in that point-link, using a matching of size two in a five-cycle. For P_9 or P_8, a new internal triple can use only pairs of the deleted blocks. The only triangles in the union of those one or two triangles are the deleted blocks themselves. A single-point intersection is still excluded by a matching of size two in a five-cycle with at most two edges removed. For a two-point intersection ij, choose an endpoint contained in just one deleted block if the pair has remaining codegree one. Its link is a four-edge path with the other endpoint at an end; after removing that endpoint a matching of size two remains. If the pair has remaining codegree zero, the two deleted blocks both contain it; removing their two incident edges again leaves a three-edge path with matching number two. These matchings create a forbidden singleton-kernel sunflower with the new triple.

All ten pairs of the Q_8 core have codegree two. A new triple cannot use two core points. A single triad point has a matching of size two within the core, which excludes it. For a single hub, its link consists of the triad triangle and the two edges joining the other hub to the leaves. Unless both leaves occur in the new triple, one of those two edges and an edge of the triangle avoid its remaining points and form a forbidden matching. This leaves only the two stated triples. □

Lemma 3.4 (B4 separation)**.** *Let A belong to a sunflower-free triple family $\mathcal{L}$. If at least eight members of $\mathcal{L}$ are disjoint from A, its closed neighborhood $N_{\mathcal{L}}[A]$ is intersecting and has at most ten members.*

Proof. The disjoint members form an intersecting family $\mathcal{H}$. Proposition 3.2 classifies it. For a P-type, isolation forces every member of the remaining part to avoid its support: deleted blocks cannot occur there, since they are also disjoint from A and would belong to $\mathcal{H}$. Two disjoint remaining members and a member of $\mathcal{H}$ would be an empty-kernel sunflower. For Q_8, the same argument uses the five-point core; the two exceptional triples of Lemma 3.3 are disjoint from A and cannot belong to the remaining part. A core-only block of Q_8 gives the required empty-kernel sunflower if two remaining members were disjoint. The intersecting bound of ten now applies. □

The Q_8 conclusion uses only core isolation. Its leaves may be shared by the remaining part; treating all seven support points as isolated would be incorrect.

Corollary 3.5. *The maximum size of a simple three-uniform family without a three-sunflower is* 20*. Therefore in a four-uniform such family,*

$$d(x) \le 20, \qquad d(x,y) \le 6, \qquad d(x,y,z) \le 2.$$

Proof. Relative to a triple A, an exact singleton class has at most three members and an exact pair class at most one. The residues are intersecting sunflower-free graphs or singleton families, respectively. Thus $|N_{\mathcal{L}}[A]| \le 13$. If $|\mathcal{L}| \ge 21$, at least eight members avoid A; B4 bounds the neighborhood by ten, and its disjoint intersecting part also has size at most ten, a contradiction. Two copies of $\mathcal{P}$ on disjoint supports give twenty members. The point, pair and triple links of a four-uniform family now give the three displayed caps. □

4 Exact traces and the near-transversal argument

For $R \in \mathcal{F}$, let s_x, p_{xy}, t_m count members with exact trace $\{x\}, \{x,y\}, R \setminus \{m\}$ on R. Put $S = \sum s_x$, $P = \sum p_{xy}$ and $T = \sum t_m$. Then

$$|N_{\mathcal{F}}[R]| = 1 + S + P + T, \qquad \sum_{x \in R} d(x) = 4 + S + 2P + 3T.$$

Exact residues are intersecting and sunflower-free: two disjoint residues with R give a sunflower with their exact trace as kernel. In particular $s_x \le 10, p_{xy} \le 3, t_m \le 1$. For $r_x = \sum_{y \ne x} p_{xy}$, the part of L_x meeting $R \setminus \{x\}$ has size

$$k_x = 1 + r_x + T - t_x, \qquad d(x) = s_x + k_x.$$

When $s_x \ge 8$, B4 makes this part intersecting. The trace constraints keep the actual external points shared by its members; separate integer histograms do not certify their compatibility.

Call a four-point set a *complete transversal* of a triple family if it meets every member. Relative to $A \in \mathcal{L}$, a *near-transversal* is a four-point set avoiding A and meeting every other member of $\mathcal{L}$.

Lemma 4.1 (A near-transversal is unique)**.** *Let $\mathcal{L}$ be an SF3 family with* 16 *members and, for some $A_0 \in \mathcal{L}$, a B4 separation $H = D_{\mathcal{L}}(A_0)$, $K = N_{\mathcal{L}}[A_0]$ into two intersecting parts of size eight. If it has a complete four-point transversal, each $A \in \mathcal{L}$ has at most one near-transversal.*

Proof. The parts are P_8 or Q_8. If a part is P_8, its support is isolated from the other part. If H is P_8, its deleted blocks are disjoint from A_0 and cannot belong to K. If K is P_8 and one member of H is a deleted block on its support, intersection of H forces all eight members of H to meet that support; isolation would restrict them to at most two deleted blocks, a contradiction. Meeting at least seven members requires at least two points from its support, since its maximum point degree is five. All four transversal points are therefore allocated two to each part. For two Q_8 parts, their five-point cores are disjoint; only leaves may be shared. Indeed, isolation forces K to avoid the core of H: both exceptional triples avoid A_0 and would belong to H. Conversely a member of H meeting the core of K would have to be one of its two exceptional triples; this is impossible since every member of H uses core points of H, which lie outside the entire support of K. Each core has six core-only triples, and each point occurs in at most four of them. Meeting at least five core-only triples in each part requires two points in each core. Again all four points are allocated.

A P_8 part with a two-point cover must have its two deleted blocks intersecting in two points. The 60 automorphisms of $\mathcal{P}$ act transitively on these 15 pairs of deleted blocks (out of 45). We may take $P^* = \mathcal{P} \setminus \{012, 015\}$, whose unique complete two-point cover is 34. For the canonical Q_8 with hubs $0, 1$, triad $2, 3, 4$ and leaves $5, 6$, the unique cover is 01. The two-point near-transversals are exactly the following:

P^* block	Near-cover	Q_8 block	Near-cover
023	45	023	14
034	none	024	13
045	23	034	12
124	35	123	04
134	none	124	03
135	24	134	02
235	none	015	none
245	none	016	none

For an omitted block in one part, the two points in the other part form its unique complete cover; the first pair is unique when it exists by the table. This proves the assertion with zero, one or two shared Q_8 leaves. The finite table and automorphism orbit are checked independently by `primary_check.py`; the reduction above, rather than a chosen support cutoff, makes those finite checks complete. □

5 The universal sixteen-link classification

For an SF3 family L, write $D_L(T) = \{U \in L : U \cap T = \varnothing\}$ and $\rho(L) = \max_{T \in L} |D_L(T)|$. In this paper, "R16" means $|L| = 16$ and $\rho(L) \le 7$, together with the SF3 condition. It is not an assertion that all SF3 families of size 16 are impossible.

Theorem 5.1 (Universal R16 classification)**.** *Every R16 is isomorphic to the following family. Take four distinct hubs $K = \{u, v, a, b\}$ and two disjoint three-element sets D, E, disjoint also from K. Then*

$$R = \{i \cup P : i \in \{u, v\},\ P \in \tbinom{D}{2}\} \cup \tbinom{K}{3} \cup \{i \cup Q : i \in \{a, b\},\ Q \in \tbinom{E}{2}\}.$$

Here and below, a point in a union denotes its singleton. The proof uses the explicitly certified finite computations described below.

Proof. We first classify the possible seven-member intersecting family that will occur as $D_L(T)$. Its support has at most seven points. Here is an arbitrary-support proof of this reduction. A degree-six point would force only six triples: a triple avoiding it cannot cover two disjoint triangles. For maximum degree at most four and no degree-one point, let b, c, d count points of degrees 2,3,4. Then $2b + 3c + 4d = 21$. If B counts used pairs of codegree two, counting intersections gives $B = b + 3c + 6d - 21$. For a point w, its SF2 link has matching number at most two; if $r(w)$ counts its codegree-two neighbours, $r(w) \ge d(w) - 2$. Hence $2B \ge c + 2d$, or $2d \ge 2b + c$. Together with $21 = 3n - b + d$, this implies $n \le 7$. Equality would force $b = d, c = 0$ and $n = 2d$, impossible for $n = 7$. Thus this case has support at most six.

If ℓuv contains a degree-one point ℓ, the other six triples meet $\{u, v\}$. Their distributions between only u, only v, and both are $(3, 3, 0)$ or $(3, 2, 1)$, after interchanging u, v. A three-edge intersecting SF2 family is a triangle; cross-intersection confines the two residual graphs to the same triad. The first case has support six; the second has support seven. Finally, for a degree-five point, its link is C_5 or $K_3 \sqcup P_3$, where P_3 has two edges. A triple avoiding the point must be a three-vertex cover. In the cycle case it lies on those five vertices; in the other case it contains two vertices of the triangle and the middle vertex of the path. This gives support six or the unique seven-point model, respectively.

The finite catalogue on six points is complete: every intersecting seven-triple family selects seven of the ten complementary pairs of triples, choosing one side from each. This gives $\binom{10}{7}2^7 = 15360$ choices. The checker reconstructs all 1860 SF3 labelled families, partitions them into the seven disjoint orbits below, and adds the analytic seven-point case. It also

checks every mixed trace of size 1 or 2 and its forbidden-pair witness, for 175 traces in total. Strings denote sets, so $012 = \{0, 1, 2\}$.

Type	Seven triples	Six-point orbit	Mixed traces allowed
H_0	$012, 034, 134, 035, 235, 145, 245$	120	$01, 02, 12$
H_1	$012, 034, 234, 135, 235, 145, 245$	360	$01, 02, 12$
H_2	$012, 134, 234, 135, 235, 145, 245$	60	$0, 01, 02, 12$
H_3	$013, 024, 134, 125, 235, 045, 345$	240	none
H_4	$023, 124, 134, 125, 235, 045, 345$	360	none
H_5	$014, 234, 025, 125, 035, 145, 345$	360	$01, 23$
H_6	$134, 234, 125, 035, 135, 045, 245$	360	$0, 12$
H_7	$023, 024, 034, 124, 134, 015, 016$	—	$5, 6, 23, 56$

For a forbidden trace X, the certificate exhibits two actual triples h_1, h_2 with $h_1 \cap h_2 = h_1 \cap X = h_2 \cap X$. Adding points outside the catalogue support does not change these intersections. Thus this is a trace obstruction on arbitrary support, rather than a cutoff on new point labels. Allowed traces are not assumed jointly realizable.

We next rule out $|L| = 16, \rho(L) \leq 6$. Every closed triple-neighbourhood has size at least ten. A degree-one point cannot have a degree-six neighbour, because every used pair in the link of such a neighbour has codegree two. Its other two neighbours therefore have degrees at most five, and its root neighbourhood has size at most nine. For a degree-two point x with a degree-six neighbour a, its triples are xab, xac, and the link of a forces abc. Hence $d(xb) = 1, d(ab) = d(ax) = 2, d(b) \leq 5$, giving $|N_L[xab]| \leq 9$. Without a degree-six neighbour, a root xyz could have neighbourhood ten only with $d(y) = d(z) = 5$ and all three used pairs of codegree one. But the link of y then has x as a leaf of its P_3 component and z as that component's middle vertex, forcing $d(yz) = 2$. The minimum used degree is consequently at least three.

Let I count intersecting pairs of triples. Then $I \geq 72$ and

$$I = \sum_w \left(\binom{d(w)}{2} - \frac{r(w)}{2} \right), \qquad \sum_w d(w) = 48.$$

At degrees 3,4,5,6 the minimum values of $r(w)$ are 1,2,4,6. For those degrees the summand is at most $19d(w)/6 - 7$, so $72 \leq 152 - 7n$ and $n \leq 11$. If the maximum degree is at most five, the stronger upper bound $(11d(w) - 23)/4$ gives $n \leq 10$; a degree-five point then exists, because otherwise $48 \leq 4n \leq 40$. The following three padded domains therefore cover every remaining family. Padding permits unused points and does not discard smaller supports.

Complete normalized point-star	Universe	Proof-tree nodes	Solutions
Two triangles	11	16357	0
C_5	10	5874	0
Triangle and two-edge path	10	3641	0

For the first domain, six triples through point 0 are fixed and ten of the 120 triples avoiding 0 are required. In the other domains five are fixed and eleven of the 84 triples avoiding 0 are required. The supplied proof trees cover every choice. Their set-based checker verifies each branch and each terminal capacity, degree, disjointness or colouring bound. In particular, all 99953 pairs in a colour class conflict with an actual triple of the fixed star. A branch is assigned by its last chosen candidate in the certified order; the omitted prefix has fewer colours than the number of triples still required. The three supplied complete trees have 25872 nodes in total and no open leaves. The files are `companion/terminal/r16/results/r16_mu10_certificate.json.gz` and `companion/terminal/r16/src/check_r16.py`. Thus $\rho(L) \leq 6$ is impossible.

Choose $A \in L$ with exactly seven disjoint triples. Put $H = D_L(A)$, $W = \operatorname{supp} H$ and $K' = L \setminus H$. The family H is intersecting and has one of the eight types above. No member of K' is entirely in W, since it would be disjoint from A and belong to H. The part K_0 disjoint from W is intersecting, and $|K_0| \le 7$ because each member of H avoids all of it. For H_3, H_4 there are no mixed members, giving $|K_0| = 9$. For H_0, H_1, H_5 there are at most three mixed members, each on a pair already used once. Hence there are at least two mixed members and at least six in K_0. If a mixed member is $P + r$, take a member of H avoiding P. Every member of K_0 must contain r to avoid three disjoint triples. Distinct external points for two mixed members would force a common pair in at least six members of K_0; a single common external point instead has degree at least eight. Both are impossible. For H_6, its block 125 is disjoint from all of K' except possibly the single member with trace 12, giving at least eight disjoint members. The same argument uses 023 and 23 for H_7.

Only H_2 remains. Its six-triple core can be renamed $\{u + P, v + P : P \in \binom{D}{2}\}$, and its seventh member is uvb. Pair capacities and the two triples uij, vik forbid any further triple meeting D. All remaining members have the forms uvw, $u + e$, $v + f$, or avoid $\{u, v\} \cup D$; call the last part K^*. The part K^* is intersecting and has size at most seven. There are either one or two members of the first form, by the capacity of uv. If uvb is the only such member, the other two forms can only be ubr, vbs, at most one each. Size 16 would require both and $|K^*| = 7$. For $r \neq s$, members of K^* avoiding b contain rs, at most two; members containing b have residual edges meeting r or s, each with at most one remaining incidence, at most two more. For $r = s$, the pair br is already saturated and K^* splits into the b and r stars. Their residual graphs are cross-intersecting. If the first has 0,1,2,3 edges, the second has at most 6,4,3,3, respectively. The two-edge case is a two-edge path and the three-edge case a triangle. Thus $|K^*| \le 6$ in every case, again impossible.

With two members uva, uvb, each extra edge through u or v must be ab, so each contributes at most one triple. At least one is necessary for size 16. Every member of K^* then meets $\{a, b\}$. In each hub-link the edge uv is already present, so residual edges form an intersecting SF2 family, at most three edges. Therefore $|K^*| \le 6$. Equality in $6 + 2 + 1 + 1 + 6 = 16$ forces both extra triples and two residual triangles, one through a and one through b. Cross-intersection forces the same triad E, disjoint from all existing core points. This is precisely R. □

Lemma 5.2 (Maximality and the high-link decompositions)**.** *No new triple, including one with new points, can be added to R. Consequently every SF3 family of 17 triples is $\mathcal{P} \sqcup I_7$ or $P_9 \sqcup I_8$, and every such family of 18 triples is $\mathcal{P} \sqcup I_8$ or $P_9 \sqcup P_9$, on disjoint supports. Here I_j denotes an intersecting SF3 family of size j, and P_9 is $\mathcal{P}$ with one block removed. The size 19 and 20 decompositions are respectively $\mathcal{P} \sqcup P_9$ and $\mathcal{P} \sqcup \mathcal{P}$, again on disjoint supports.*

Proof. Every hub has degree six and its complete SF2 link is already maximum. A new triple avoids all hubs. Used pairs within D and within E have codegree two, so it uses at most one point in each triad. A pair of D and a pair of E avoiding the new triple give two disjoint members of R, disjoint also from that triple. This is a forbidden empty sunflower. If a family of 17–20 triples had at most seven disjoint triples per root, every sixteen-member subfamily would be R and could not extend. Thus some root has at least eight disjoint triples. B4 separates the family into two intersecting parts of size at most ten. Their size pairs are 10+7 or 9+8 at size 17,10+8 or 9+9 at size 18,10+9 at size 19, and 10+10 at size 20. Isolation makes their supports disjoint: a missing block of a P9 belongs to the same disjointness class as that P9 and cannot be inserted in the other part. □

6 The direct reduction of a hypothetical family of size 69

This section gives the local finite exclusions at cardinality 69. The companion includes their full mathematical reductions, coefficient constructors, exact certificates and coverage checkers. The nine blocks are identified by their mathematical roles in `proof/DEPENDENCIES.md`. All file paths in this manuscript are relative to the companion repository root. All domains below use $\mu \geq 42$; no exclusion at a larger cardinality is used as a premise.

Proposition 6.1 (Bootstrap at 42)**.** *In an SF4 family with $\mu \geq 42$, a singleton of size 9 or 10 at an actual anchor requires at least seven other members through its centre. Every used point has degree at least four; co-occurring points have degree sum at least 20, and every member has degree sum at least 52. Moreover*

$$\frac{37}{170}|\mathcal{F}| \leq |\operatorname{supp}\mathcal{F}| \leq \frac{17}{50}|\mathcal{F}|.$$

Proof. For an ordered anchor $R = \{r_0, r_1, r_2, r_3\}$ let p_{ij} be its six exact pair traces, t_i its exact triple trace omitting r_i, and s_i its singleton traces. Then

$$k_i = 1 + \sum_{j \neq i} p_{ij} + \sum_h t_h - t_i, \quad d(r_i) = k_i + s_i, \quad |N[R]| = 1 + \sum p + \sum t + \sum s.$$

The intervals are $0 \leq p \leq 3$, $t \in \{0, 1\}$, $0 \leq s \leq 10$. The universal rooted signatures of the intersecting triple models and the high-link decompositions are imposed only in their valid cases. The full extension-point identifications, all five trace margins and the high-point bound are proved in `companion/upstream/MU42/TRACE_FOUNDATIONS.md`. The two complete row constructors yield 980624 degree-ordered rows and 1233 degree profiles. They imply minimum degree 2 and minimum co-occurring degree sum 20. Permutations order an actual anchor; they do not impose symmetry on its completion.

For an isolated P10 or P9 on six points, all internal quadruples are chosen, including absences. The respective catalogues have 292 and 373 members, covering all 2^{15} masks. At each of the six neighbour points, the actual external triple family must be shared by every central anchor that mentions it. These families are cross-intersecting, and a triple appears in at most two of them. If one has size 10 their total is at most 20; if one has size 9 and none has 10 it is at most 22, by the complete triple-transversal catalogues. A singleton 8 in a Q8 only guarantees its six core triples are external; it is not replaced by an external family of size 8. The 256 correlated lower-bound vectors in $\{0, 6, 9, 10\}^6$ cover every actual choice. The exact joint check eliminates all 16 P10 and 24 P9 scalar survivors with $k \leq 5$.

At $k = 6$, the full 13 P10 interior orbits are checked before the 79 P9 orbits. The P9 check may therefore use the already proved P10 bound 7. Each system uses the 41 nonempty traces of sizes 1–3 on the six points; empty traces and the number of exterior points are unrestricted. The same variables encode the same members in all central anchors. Ninety systems have an empty anchor domain. The two remaining P9 systems have exact integer certificate gaps -53 and -200, including all variable-box contributions. Thus $k \geq 7$ for both singleton sizes.

The only degree-two raw row is $p = (0, 0, 0, 3, 3, 3)$, $t = (1, 0, 0, 0)$, $s = (1, 10, 10, 10)$, $d = (2, 18, 18, 18)$. Thus a degree-two point x has two members xA, xB with disjoint triples A, B, whose six points all have degree 18. In every L_a, $a \in A$, the point x is a leaf, so the 18-link decomposition forces P10+Q8, with x a Q8 leaf and $A \setminus \{a\}$ its two hubs. The other leaf is the same point y for all a: otherwise the global triple A would have more than two completions. Comparing the actual members through a pair $a, b \in A$ forces the same Q8 triad C in all three links. Hence xA, yA and all nine members $E_A \cup E_C$, $E_A \in \binom{A}{2}, E_C \in \binom{C}{2}$, are present. Each opposite P10H_a avoids $A \cup C \cup \{x, y\}$. The point y is not in B: otherwise its degree 18 link would contain A with all three points of degree one in that link, which no 18-link model permits.

If $B \cap C = \varnothing$, then B avoids all of that core. For $T \in H_a$, a disjointness $B \cap T = \varnothing$ would make xB, aT and $(A \setminus \{a\}) \cup E_C$ three disjoint members. Thus B is a triple-transversal of every H_a, and therefore a block of each P10. The three actual members aB, $a \in A$, form a sunflower with kernel B. If $c \in B \cap C$, the six prescribed triples in L_c are $t \cup E_A$, $t \in C \setminus \{c\}$ and $E_A \in \binom{A}{2}$. They occupy the same intersecting 18-link component. The point-links of all three points of A contain C_4, whereas every point-link of P10 is C_5 and contains no C_4 subgraph. Thus that component is Q8 with triad A and hubs $C \setminus \{c\}$. Its leaf x forces $B \setminus \{c\} = C \setminus \{c\}$, so $B = C$. For $a \in A, c \in C$, disjoint tails $T \in H_a, V \in H_c$ would make $aT, cV, (A \setminus \{a\}) \cup (C \setminus \{c\})$ three disjoint members. These P10s are cross-intersecting, so each is contained in the triple-transversal catalogue of the other and they coincide. A shared block again supplies three members aT with kernel T. This excludes the degree-two row on arbitrary support. Each of the four degree-three raw rows has singleton 2 at that point for every one of its roots. The three triples of its complete link would consequently be pairwise disjoint, a forbidden sunflower at the point. There is no degree-one raw row. Reconstructing the filtered domain gives 292560 rows and 966 degree profiles, with member degree sum at least 52.

For completeness, the support calculation is a global identity rather than a maximization of unrelated anchors. Put $c(R) = \sum_{v \in R} 1/d(v)$. The verified pointwise row inequality is

$$10000c(R) \le 3400 + 26G_4(R) - 33B_{17a}(R) - 21B_{17b}(R) - 38B_{18}(R) + 9B_{19}(R) + 18T_{16}(R).$$

Here $G_4 = \sum_{v \in R, d(v)=4}(4s_v - 8)$; the high-link corrections are

$$B_{17a} = \sum_{d(v)=17} (7[s_v = 7] - 10[s_v = 10]), \qquad B_{17b} = \sum_{d(v)=17} (8[s_v = 8] - 9[s_v = 9]),$$
$$B_{18} = \sum_{d(v)=18} (8[s_v = 8] - 10[s_v = 10]), \qquad B_{19} = \sum_{d(v)=19} (9[s_v = 9] - 10[s_v = 10]),$$
$$T_{16} = \sum_{d(v)=16} (9[s_v = 9] - 7[s_v = 7]).$$

All displayed sums are over the indicated points of the same R. The B corrections sum to zero across all members, by their complete two-part link decompositions; $\sum_R G_4(R) \le 0$ follows from the triangle-free disjointness graph of a four-triple point-link and Mantel's elementary bound, and $\sum_R T_{16}(R) \le 0$ follows from B4 and isolation of any P9 in a sixteen-link. The positive coefficients 26 and 18 preserve these inequalities after summation. The row check is performed after multiplying by $\operatorname{lcm}(1, \dots, 20)$; its exact margin is $181/85000$. Since $\sum_R c(R) = |\operatorname{supp} \mathcal{F}|$, the upper support bound follows. The pointwise lower bound $37/170$ gives the other bound. The source, finite objects and exact arithmetic contract are listed in `proof/DEPENDENCIES.md`, under *MU42: Neighbourhood bootstrap and separate C5.* □

Proposition 6.2 (Complete F69 degree reduction)**.** *A hypothetical SF4 family of 69 members satisfying H27 has maximum degree 16. Every exact singleton has size at most 8 and every closed neighbourhood has size at most 44.*

Proof. H27 supplies $\mu \ge 42$ directly. The previous proposition gives minimum used degree 4 and support 16–23. The classified maximum-degree cases and their finite covers are as follows. All quantities refer to necessary local domains, not to counts of realized global families.

Maximum degree	Exclusion and final ordinary contradiction
20	P10+P10. One internal certificate and six external certificates force each residual member to use at least two exterior points. A mixed-member cover then has capacity 44, below the required 49. Without a mixed member, each of 49 residual members avoids at least 12 of the 20 cones, giving $12\cdot 49 = 588 > 20\cdot 27 = 540$.
19	P10+P9. All P9 interiors are eliminated before empty P10 is imposed. The mixed cover retains the pure-three P5 exceptions and has capacity 44, below 50. The terminal three-unit deficit gives $547 > 19 \cdot 27 = 513$.
18	All P10+I8 roots are covered by 12 internal certificates and 18 complete extension forests. The P9+P9 root uses the same external identities in both trace systems:20 internal,22 mixed-four and 8 mixed-three exact certificates. Mixed-two has capacity 44 below 51. The joint external capacity of the two complementary P9 traces is 2, giving deficit 22 and $488 > 18 \cdot 27 = 486$.
17	The P10+I7 and P9+I8 alternatives are both covered. The P10 bridge regenerates the 17/42 domain and identifies all 18249 states with two universal finite cores; it imports no F70 family. Twelve P10 and 75 P9 certificates, together with 870 complete extension forests, exclude 17.
≤ 15	All 28 high-neighbourhood row orbits and 35 realized cores are covered. Thirty-three fail another actual anchor; two equivalent remaining cores share one 87-trace system. Its exact certificate has $H = -435, L = -89$, contradicting $L \leq H$. Thus all closed neighbourhoods are at most 43, contradicting the graph lemma below.

Every LP certificate excludes a relaxation containing every actual family in its stated root. For $Az \leq b, Ez = f, l \leq z \leq u$, an integer certificate $y \geq 0, \eta$ gives $q = A^T y + E^T \eta$ and

$$L = \sum_j \min(l_j q_j, u_j q_j) \leq q \cdot z \leq H = y \cdot b + \eta \cdot f.$$

The decisive comparison is $H < L$. Forest verifiers independently reconstruct all children and the reason for each closed leaf. The source-to-model maps keep actual anchors, shared degrees and shared exterior identities; they do not assume that extensions symmetric in the labels exist. The exact source sections, branch counts, files and companion checks are recorded in `proof/DEPENDENCIES.md`, under *Preliminary finite blocks*.

We justify the graph contradiction used in the last row. The disjointness graph G has vertices the members of $\mathcal{F}$, is triangle-free, and has $\alpha(G) \leq 27$. If all neighbourhoods have size at most 43, its minimum degree is at least 26. Jin with $k = 3$ applies because $26 \cdot 11 > 4 \cdot 69$. The cyclic A_3 definition below is carried to Wagner labels by $i \mapsto 3i \pmod 8$: cyclic differences $3, 4, 5$ become $1, 4, 7$. In the Wagner labelling of A_3, let the fibre sizes be a_i and put $b_i = a_{i-1} + a_{i+1} + a_{i+4} \leq 27$. Set $e_i = 27 - b_i$; $\sum e_i = 9$, and the inverse incidence identity gives $a_i = 12 - e_{i-1} - e_i - e_{i+1} \geq 3$. Thus every fibre is occupied and $b_i \geq 26$. But $8 \cdot 26 \leq \sum b_i = 3 \cdot 69 = 207$, impossible.

It remains to state the two finite reductions at maximum 16. The full P9 interior cover has 373 assignments in 79 orbits; the LOW15 bonus excludes 77 and leaves only the empty assignment and the three labelled copies of 0345. NO9 closes these with 60 newly checked forests and 22777 closed leaves, without importing the old 17-degree domain. Consequently no singleton 9 remains; singleton 10 already requires $k \geq 7$ and cannot occur at degree 16. The NO45 cover has 36 high-anchor orbits and 2335 actual cores. Of these,2330 fail another

anchor and five exact systems exclude the rest, so no closed neighbourhood reaches 45. This gives the claimed singleton and neighbourhood bounds. □

7 The fibre bridge forces an R16 point-link

We use the cyclic definition $V(A_k) = \mathbb{Z}/(3k-1)$ and $i \sim j$ when $k \le j - i \le 2k-1$ modulo $3k-1$. Jin provides an image A_4 because the disjointness graph now has minimum degree 25 and $25 \cdot 14 > 5 \cdot 69$. Deleting an unoccupied fibre and applying the explicit edge-preserving deletion map $A_k - v \to A_{k-1}$ yields a surjective image with $k \le 4$. The cases $k = 1, 2$ are impossible: the maximum independent-fibre mass would give respectively $69 \le 2 \cdot 27$ or $2 \cdot 69 \le 5 \cdot 27$. Thus $k = 3$ or 4.

Here are the required deletion maps for $v = 0$; entries list the images of vertices $1, \dots, 3k-2$ in their cyclic order:

k	$A_k - 0 \longrightarrow A_{k-1}$
2	$(0, 0, 1, 1)$
3	$(0, 1, 1, 2, 3, 3, 4)$
4	$(0, 7, 6, 6, 5, 4, 3, 3, 2, 1)$.

Checking the cyclic edge differences verifies every edge. Rotation gives the map for any deleted vertex. If the resulting image has another empty fibre, repeat; k strictly decreases. The displayed maps supply the entire deletion argument without an additional archived witness.

Let a_i be the fibre masses and $b_i = \sum_{j \sim i} a_j$. Neighbourhoods of vertices of A_k are independent, so $b_i \le 27$. Surjectivity and the minimum degree give $b_i \ge 25$. With $r_i = b_i - 25$, the inverse fibre identity is

$$r_i \in \{0, 1, 2\}, \qquad \sum_i r_i = 25 - 6k, \qquad a_i = 6 + r_i + r_{i+k} + r_{i+2k-1}.$$

The formula is an identity for this cyclic incidence matrix; it does not assume that all permissible edges between fibres are present.

Proposition 7.1 (Every degree16 point has an R16 link)**.** *Under the conclusions of Proposition 6.2, every point of degree 16 in $\mathcal{F}$ has an R16 complete point-link.*

Proof. Fix such a point x, and let t_i count its star members in fibre i. Set $I = \{i : t_i > 0\}$ and $J = \{j : N_{A_k}(j) \cap I = \varnothing\}$. Every actual member of a fibre in J avoiding x meets all star members of x, hence is a full four-transversal of L_x. These members are distinct, and their number is $B(x) = \sum_{j \in J}(a_j - t_j)$. Because two star members cannot be adjacent in the disjointness graph, every occupied star fibre satisfies

$$\sum_i t_i = 16, \qquad \sum_{j \sim i} t_j \le r_i \quad (i \in I).$$

The second inequality counts absent permissible edges of any one star member; its budget is $b_i - 25 = r_i$, which can be 2. It is not assumed to be a matching. If I is independent, choose a cyclic window W of k consecutive indices with $I \subseteq W$. Such a window is independent and is the neighbourhood of a vertex of A_k, so $I \subseteq W \subseteq J$ and $\sum_{j \in W} a_j \ge 25$. Since $\sum_{j \in J} t_j = 16$, it follows that $B(x) = \sum_{j \in J} a_j - 16 \ge 9$. The remaining finite domain consists of all positive star vectors on nonindependent supports subject to the displayed inequalities and $t_i \le a_i$. A nonisolated coordinate has $t_i \le 2$, since an adjacent occupied coordinate has budget at most 2; isolated coordinates retain their entire possible interval. No point-star vector is lost.

The complete census and its new set-based reconstruction give:

Image	Dihedral fibre profiles	Nonindependent star vectors	$B = 1$ vectors
A_3	70	1293	52
A_4	1	0	0

All vectors have $B \geq 1$. For every one of the 52 exceptional vectors, in 23 profiles, a certified index i satisfies

$$\sum_{j \sim i} t_j = 1, \qquad a_i - t_i \geq 9.$$

This census is reconstructed by `companion/terminal/check_f69_bridge.py`; its output lists all 52 witness vectors and indices.

Suppose L_x is not R16. A root has at least eight disjoint triples; by the singleton bound these are exactly eight, and B4 gives two intersecting parts of size 8. The separated parts are P8 or Q8, with disjoint supports whenever a P8 occurs and with at most shared leaves when both are Q8. A full four-transversal uses two points in each P8 support, or two points in each of the disjoint five-point Q8 cores. Consequently it is unique. Only P8s whose two omissions meet in two points have a two-cover; the 15 such P8s form one orbit, represented by $P^* = \mathcal{P} \setminus \{012, 015\}$. The quasi-transversal lemma and table in the preceding foundations section allow at most one four-set avoiding a prescribed triple and meeting the other fifteen. The complete 80-case check covers P^*/P^*, $P^*/Q8$, and $Q8/Q8$ with 0,1,2 shared leaves; it has 54 nonempty quasi-transversal sets and maximum cardinality 1. Arbitrary exterior points cannot occur because the four available positions are already exhausted by the two covers.

Thus $B(x) = 1$, and x realizes one of the 52 exceptional vectors. For its witness fibre i, there is exactly one star member A in an adjacent fibre. Every one of the at least nine members of fibre i avoiding x meets the other fifteen star members automatically, since their fibres are nonadjacent. The unique full transversal belongs to J, whereas $i \notin J$, so these nine members are distinct from it. None can also meet A, since that would be a second full transversal. They therefore avoid A and are nine distinct quasi-transversals of L_x relative to the same triple $A \setminus \{x\}$. This contradicts the upper bound 1. Hence L_x is R16. □

8 A complete R16 point-link gives a local upper bound 61

Lemma 8.1 (Crowding bound)**.** *In an SF4 family with $\mu \geq 41$ and $\Delta \leq 18$, the three largest exact singleton counts of any one actual anchor sum to at most 26.*

Proof. Each singleton tail is intersecting SF3 and has size at most 10; all anchor centres avoid every tail support. A violation contains actual three-colour subfamilies of sizes $(10, 10, 7)$, $(10, 9, 8)$ or $(9, 9, 9)$. They are rainbow-sunflower-free, since their lifts have distinct centres. Normalize the first P10/P9 on six points. Every second support consists of a subset of these six and consecutively renamed fresh points. Testing every labelled P10 and omission gives 768,7680,7680 positions, in 18,59,411 orbits, respectively. Only automorphisms of the first colour and fresh-point permutations are factored out.

For a third tail, its intersection with the first two supports determines all rainbow intersections. Every trace of size 0–3 is tested against all pairs from the first two colours. In the $(10, 10, 7)$ regime,15 of the 18 first-pair orbits have an actual anchor with certified neighbourhood at most 40. The cover capacities subtract the twenty already present members from global point/pair/triple/four-set capacities 18,6,2,1. In two of the remaining orbits all permitted tails are internal and no seven-subset is intersecting. In the last they are blocks on disjoint P10 supports and an intersecting third family must lie in one support, yielding cross-intersecting stars of 10+7.

For $(10, 9, 8)$, one first-pair orbit is already cross-intersecting, two have neighbourhood covers 33, and the remaining 56 admit exactly 90 third P8/Q8 records, each containing an

actual cross-intersecting pair. For $(9,9,9)$, three first-pair orbits are cross-intersecting; the other 408 admit exactly 20 third P9 records, again with such a pair. All support identifications, including fresh Q8 leaves, are retained.

The cross-star theorem used here has the following precise guard: both centres are outside both tail supports, both tail families are intersecting SF3, the two tail families are cross-intersecting, and all lifted members are actual members of the same SF4. For total size at least 16, there is an actual root with $|N[R]| \le d(x) + d(y) - d(xy) + 4 \le 40$. Its arbitrary-support reduction extracts 10+6,9+7, or 8+8 tails. P10/P9 triple transversals are internal; an 8-tail has a unique two-cover or none, and at most two triples using that two-cover with new points. Thus all 433 normalized pairs are covered on a derived finite support. Their resource certificates cover all 21567 useful traces; the residual capacity is 4 in 420 cases, 2 in 12 cases, and 0 in one case. The bound counts members through a centre once as $d(x) + d(y) - d(xy)$, and charges every remaining member to a triple or four-set resource after subtracting its occurrences in the sixteen known members. This proves the quoted root inequality.

All three regimes therefore contradict $\mu \ge 41$. The finite objects and independent set-based checkers are `companion/terminal/crowding/certificates/crowding26.json.gz`, `companion/terminal/crowding/code/verify.py`, `companion/terminal/cross15/results/cross16_certificate.json` and `companion/terminal/cross15/src/verify.py`. The written reductions above justify their application on arbitrary support; a finite count alone would not do so. □

Theorem 8.2 (R61)**.** *Let $\mathcal{F}$ be SF4 with $\mu \ge 41$ and $\Delta \le 18$. If a complete point-link is R16, then $|\mathcal{F}| \le 61$.*

Proof. Use Theorem 5.1 to label this complete link by four hubs $K = \{0,1,2,3\}$, peripheral triads D, E, and centre x. There are sixteen members through x, including four central members $R_i = \{x\} \cup (K \setminus \{i\})$. Each hub i has codegree $d(xi) = 6$. A point of global degree 17 or 18 cannot have such a neighbour: its link separates into intersecting parts of size at least seven on disjoint supports, and each such part has point degree at most five. Thus all four hubs have global degrees at most 16, even though the ambient cap is 18.

Among members avoiding x, use exact traces on the same K: singleton counts s_i, pair counts p_{ij}, triple counts t_i omitting i, and four-set count q. Put

$$P = \sum p_{ij}, \quad T = \sum t_i, \quad E_* = \sum s_i + P + T + q, \qquad d_i = 6 + s_i + \sum_{j \ne i} p_{ij} + T - t_i + q.$$

The empty trace is unrestricted. Exact root identities and capacities are

$$0 \le s_i \le 10, \quad 0 \le p_{ij} \le 3, \quad t_i, q \in \{0,1\}, \quad t_i + q \le 1, \quad p_{ij} + t_\ell \le 3 \quad (\ell \notin \{i,j\}),$$
$$|N[R_i]| = 16 + E_* - s_i \ge 41, \qquad d_i \le 16.$$

For $i \ne j$, the exact singleton of j at R_i has size $u_{ij} = s_j + p_{ij}$. Hence $u_{ij} \le 10$ and $\sum_{j \ne i} u_{ij} \le 26$ by Lemma 8.1. If $u_{ij} \ge 8$, B4 makes the complementary part of L_j intersecting. It contains the six triples through x; an intersecting SF3 with a degree-six point has exactly six members. Therefore

$$u_{ij} \ge 8 \quad \Longrightarrow \quad d_j - u_{ij} = 6.$$

If $d_j = 16$, its complete link is either 10+6 or R16, because x has degree six there. In 10+6, the opposite P10 avoids both triangles of the six-star, giving $u_{ij} = 10$ for all $i \ne j$. In R16 the triangle $K \setminus \{j\}$ is either the other-hub triangle or a peripheral triad; all three singleton values are then 3 or 7. Its used pairs have codegree two, so each opposite global triple has codegree two as well. The full necessary guard is

$$d_j = 16 \Longrightarrow (u_{ij} : i \ne j) \in \{(3,3,3), (7,7,7), (10,10,10)\},$$

and in the first two alternatives $t_i + q = 1$ for $i \neq j$. No other 16-link is lost: the separated 9+7 and 8+8 parts have point degree at most five, with shared Q8 leaves having degree at most two.

There are $4^6 \cdot 2^4 \cdot 2 = 131072$ choices of (p, t, q). For each, enumerate all four singleton counts satisfying the exact degree and neighbourhood inequalities. A second construction enumerates the total E_* and its bounded compositions, using

$$E_* \geq \left\lceil \frac{100 - P - T - q}{3} \right\rceil, \qquad E_* \leq 40 - P - 2T - 3q.$$

These inequalities follow by summing the four root-neighbourhood and degree constraints; they do not constrain exterior labels. The exact reconstruction gives 21652 profiles before the singleton/B4 guards, 111 after them, and 11 after the complete-link guard. The eleven records are exactly the following labelled forms.

Form	s	p	t	q	Hub degrees	Records
A	Permutations of $(8, 8, 9, 9)$	all 0	all 0	0	$(14, 14, 15, 15)$	6
B	$(7, 7, 7, 7)$	2 on a perfect matching	all 0	0	$(15, 15, 15, 15)$	3
C	$(7, 7, 7, 7)$	all 0	all 1	0	$(16, 16, 16, 16)$	1
D	$(6, 6, 6, 6)$	all 1	all 0	1	$(16, 16, 16, 16)$	1

The finite proof is `companion/terminal/cap61/certificates/core_certificate.json` with its complete reconstruction in `companion/terminal/cap61/src/verify.py`.

In form B choose a matching edge ij. The singleton of j at R_i has nine tails, hence is P9, but i occurs in exactly two of them. All positive point degrees of P9 are 4 or 5. In form D each hub-link is R16 with $K \setminus \{j\}$ peripheral, whereas the actual member K would insert that forbidden peripheral triad. In form C take the actual member $\{1, 2, 3, z\}$ furnished by $t_0 = 1$. Its reading in L_1 forces $z \in D$: the peripheral pair 23 can be completed only by the centre x or its partner hub, and $z \neq x$. The same member read in L_2 forces $z \in E$. Since $D \cap E = \varnothing$, this is impossible. The outside label is kept identical in both links.

Only the six form-A records remain. For a hub i with $s_i = 9$, $E_* = 34$ and $|N[R_i]| = 16 + 34 - 9 = 41$ exactly. The complement $D(R_i)$ is intersecting. Its complete i-link consists of the nine pure singleton tails, hence is exactly P9: all other nonempty hub traces are zero, and all six members through xi are removed with $N[R_i]$.

We prove the required ordinary bound for this complement. If an intersecting SF4 family $\mathcal{I}$ has a complete point-link $H = P9$ or $P10$ at a, put $W = \operatorname{supp} H$ and $h = |H|$. Every member avoiding a covers H, so uses at least three points of W. Let b count internal four-sets and e count members with a three-point trace on W. If $h = 10$, every internal four-set contains two P10 blocks, whose remaining triple capacities give $e + 2b \leq 10$ and $|\mathcal{I}| = 10 + b + e \leq 20$. If $H = P10 \setminus \{T\}$, every internal four-set contains at least one block of H, so $e_H + b \leq 9$. Its only other possible triple traces are T and $W \setminus T$, which are disjoint. If both occur, intersection of the resulting members forces their single exterior points to coincide, and each class has size at most one. If only one occurs it has size at most two by triple capacity. Therefore $e_T + e_{W \setminus T} \leq 2$ and $|\mathcal{I}| \leq 9 + 9 + 2 = 20$. This argument subtracts the same internal occurrences before adding external capacities. It requires a complete P9/P10 point-link and intersection of $\mathcal{I}$, both already established here. Consequently $|\mathcal{F}| = |N[R_i]| + |D(R_i)| \leq 41 + 20 = 61$. □

9 Conclusion and finite evidence boundary

Proof of the upper bound 68. Suppose an SF4 family has at least 69 members and take a 69-member subfamily. It remains SF4. H27 gives $\mu \geq 42$, and the universal triple bound gives $\Delta \leq 20$. Proposition 6.2 leaves maximum degree 16, so a degree 16 point exists. Proposition 7.1

makes its complete link R16. Theorem 8.2 applies because $42 \geq 41$ and $16 \leq 18$, and gives $69 \leq 61$, a contradiction. Thus every SF4 family has size at most 68. □

The contradiction uses the complete local R61 theorem, including its crowding lemma, after the nine preliminary finite blocks and the fibre bridge. The classification and maximality of R16 supply the high-link decompositions. The separate C5 finite lemma remains an explicit input; the complete route and its acyclic dependency order are given in the companion. The local R61 conclusion does not imply a global bound of 61.

10 The classical lower construction

Partition nine points into three groups U_1, U_2, U_3 of size three, and take all quadruples formed from a pair in each of two groups. This gives $3 \cdot 3 \cdot 3 = 27$ members, denoted $\mathcal{D}_{27}$.

Proposition 10.1. *$\mathcal{D}_{27}$ is intersecting and three-sunflower-free. Two copies on disjoint supports give a three-sunflower-free family of size 54.*

Proof. Two members use two of the three groups, so they share a group; their two-element subsets in that three-point group intersect. Suppose three distinct members formed a sunflower. A group cannot occur in exactly two of them: the two active pairs intersect, whereas an inactive member intersects that group in no point. A group occurring in all three members must use the same pair in all three. Indeed, two equal pairs and one different pair yield unequal pairwise intersections, and the three distinct pairs have three different singleton intersections. Group occurrence counts are consequently $0, 1, 3$. Their sum is six, since each member uses two groups. On three groups the only possibility is $3 + 3 + 0$; the three members then have identical pairs in the two active groups and are equal, a contradiction. For two disjoint copies, a triple using both copies has two intersecting members in one copy and a third member disjoint from them. Its pairwise intersections cannot all coincide. □

11 The remaining extremal question

The proof excludes 69 members and every larger size by heredity. It does not exclude 68 members or establish that the classical 54-member construction is optimal. H27 gives only $\mu \geq 41$ at cardinality 68, so the $\mu \geq 42$ trace and fibre domains cannot be transferred automatically. Determining $M(4, 3)$ remains open.

12 Computational evidence and reproducibility

Each finite lemma has two obligations: a written reduction places every candidate in the enumerated domain, and a checker verifies that entire domain using necessary constraints. Symmetry normalization, complete root coverage and sound monotone pruning are parts of the proof.

The companion supplies nine preliminary blocks and the final structural checks, including C5 and both transported degree-17 cores. It contains no optimizer dependency. Model coefficients and certificate inequalities use integer or rational arithmetic. The full route is `scripts/verify.py` with mode `complete`; mode `quick` checks only explicit constructions and local tables. The exact commands and finite coverage are given in `REPRODUCE.md` and `proof/DEPENDENCIES.md`, relative to the repository root. The paper and companion are available at https://github.com/aconsciousfractal/Four-Uniform-Three-Sunflower-Bound.

The verifier requires a fresh receipt directory, checks the input manifest and exact suite identity, and compares every scientific result with its declared semantic outcome and coverage

counters. It runs ordinary and optimized Python and requires matching normalized results. Failure, timeout or incomplete coverage cannot produce a complete PASS. The verification route is exercised from an allowlisted separate copy; a Python filesystem audit boundary restricts reads to the package, receipts and interpreter. This is not an operating-system sandbox.

The small construction checker tests the 54-member witness on all 24 804 triples, the 45 P8 omission pairs and 60 P10 automorphisms. Across five separated link models it checks 80 near-transversal cases: 54 are nonempty, each with one member. A positive R16 control has seven complete transversals, showing why the separation hypothesis is essential.

13 Scope of verification

H27 is imported from Proposition 6.7 of the cited preprint version; its original exhaustive search is not re-executed here. Jin's precise strict homomorphism theorem is stated through Nikiforov's Theorem A. The companion verifies the named finite computations, rather than a proof-assistant formalization of the entire theorem. Agreement between Python modes does not provide independence from shared mathematical assumptions. The comparison with the published endpoint 83 refers specifically to the cited preprint version; no claim of priority is made.

Use of computational assistance. Generative AI tools assisted proof development, software preparation and manuscript review. They are not authors; responsibility for the manuscript and its claims rests with the named author.

References

[1] P. Erdős and R. Rado, Intersection theorems for systems of sets, *Journal of the London Mathematical Society* **35** (1960), 85–90. doi:10.1112/jlms/s1-35.1.85.

[2] H. L. Abbott and D. Hanson, On finite Delta-systems, *Discrete Mathematics* **8** (1974), 1–12. doi:10.1016/0012-365X(74)90103-4.

[3] H. L. Abbott, D. Hanson and N. Sauer, Intersection theorems for systems of sets, *Journal of Combinatorial Theory, Series A* **12** (1972), 381–389. doi:10.1016/0097-3165(72)90103-3.

[4] E. Axante, C. Budala, D. Chitic, B. Dumitru and M. Nacu, *Sunflower-Free Uniform Families: Recursive Constructions and Explicit Bounds*, preprint (2026). arXiv:2609.06175v1.

[5] G. Jin, Triangle-Free Graphs with High Minimal Degrees, *Combinatorics, Probability and Computing* **2**(4) (1993), 479–490. doi:10.1017/S0963548300000845.

[6] V. Nikiforov, *Chromatic number and minimum degree of K_r-free graphs*, preprint (2010). arXiv:1001.2070v1.